**Def.** $\mathbb{C}$: a category w/ binary products $\times$, $A, B \in \mathbb{C}$

The exponential from $A$ to $B$ is a pair $\subseteq \mathbb{C}$

$$\left( B^A \in \mathbb{C},\quad B^A \times A \xrightarrow{ev_{A,B}} B \right)$$

such that:

for each arrow (a function with two arguments) $\subseteq \mathbb{C}$

$$C \times A \xrightarrow{f} B$$

we have a unique arrow (unique existence) $\subseteq \mathbb{C}$ ($\bar{f}(c)$ is a function $f$ w/ its first argument fixed)

$$C \xrightarrow{\bar{f}} B^A$$

"Currying"

that makes the following diagram commute.

$$\begin{tikzcd}
B^A \times A \arrow[r, "ev_{A,B}"] & B \\
C \times A \arrow[u, "\bar{f} \times id_A"] \arrow[ur, "f"'] &
\end{tikzcd}$$

$\bar{f} : C \to B^A$

$id_A : A \to A$

$\bar{f} \times id_A : C \times A \to B^A \times A$

Briefly: $\subseteq \mathbb{C}$

$$\begin{tikzcd}
B^A & B^A \times A \arrow[r, "ev_{A,B}"] & B \\
C \arrow[u, dashed, "\exists!\, \bar{f}"] & C \times A \arrow[u, "\bar{f} \times id_A"] \arrow[ur, "\forall f"'] &
\end{tikzcd}$$

**Def.** $1 \in \mathbb{C}$ is terminal if, for $\forall x \in \mathbb{C}$, $x \overset{\exists!}{\dashrightarrow} 1$

0-ary product, the limit for the empty diagram

**Def.** A Cartesian closed category (CCC) is a category $\mathbb{C}$ with

- a terminal (final) object $1$,
- binary products $\times$, and
- exponentials for all $A, B \in \mathbb{C}$.

(terminal object and binary products: all finite products)

It turns out:

$$\frac{\text{Typed } \lambda\text{-cal.}}{\text{CCCs}}$$

---

**Thm.** In a CCC $\mathbb{C}$, we have an adjunction

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "A\times(-)"] \arrow[r, phantom, "\perp"] & \mathbb{C} \arrow[l, bend left, "(-)^A"] \end{tikzcd}$$

for each $A \in \mathbb{C}$.

---

- Adjunction

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

left / right

$$\frac{FX \longrightarrow Y \quad \text{in } \mathbb{D}}{X \longrightarrow GY \quad \text{in } \mathbb{C}}$$

$\boxed{Q1}$ Is this $\perp$? $\top$?

$\boxed{Q2}$ What are units and counits?

$\boxed{Q3}$ Triangular equalities?

$\boxed{Q1}$ if $(-)^A \dashv A\times -$

we'd get

$$\cancel{\frac{X^A \longrightarrow Y}{X \longrightarrow A\times Y}}$$

if $A\times - \dashv (-)^A$

we'd get

$$\frac{A\times X \longrightarrow Y}{X \longrightarrow Y^A}$$

[↓] given $A\times X \xrightarrow{f} Y$

we get $X \xrightarrow{\bar{f}} Y^A$

[↑] Given $g: X \longrightarrow Y^A$

$$A\times X \xrightarrow{id_A \times g} A\times Y^A \xrightarrow{ev_{A,Y}} Y$$

These correspondences are mutually inverse; one proves it via the uniqueness of $\overline{f}$.

- In an adjunction
$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

the unit is $\left( X \xrightarrow{\eta_X} \overset{\mathbb{C}}{GFX} \right)_{X \in \mathbb{C}}$ natural in $X$

$$\frac{FX \xrightarrow{id} FX}{X \xrightarrow{\eta_X} GFX}$$

the counit is $\left( FGA \xrightarrow{\varepsilon_A} \overset{\mathbb{D}}{A} \right)_{A \in \mathbb{D}}$ natural in $A$

$$\frac{FGA \xrightarrow{\varepsilon_A} A}{GA \xrightarrow{id} GA}$$

$\boxed{Q2}$

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "A\times(-)"] \arrow[r, phantom, "\perp"] & \mathbb{C} \arrow[l, bend left, "(-)^A"] \end{tikzcd}$$

The counit

$$A \times X^A \xrightarrow{ev_{A,X}} X$$

The unit

$$X \xrightarrow{\eta_X} (A \times X)^A$$

$$x \longmapsto \lambda a. \langle a, x \rangle$$

- Triangular equalities.

$$\begin{tikzcd} F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, equal] & FGF \arrow[d, Rightarrow, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, Rightarrow, "\eta G"] \arrow[dr, equal] & GFG \arrow[d, Rightarrow, "G\varepsilon"] \\ & G \end{tikzcd}$$

$\boxed{Q3}$

$$\left[ \begin{tikzcd} A \times X \arrow[r, "A\times\eta_X"] \arrow[dr, "id"'] & A \times (A \times X)^A \arrow[d, "\varepsilon_{X\times A} = ev_{A, X\times A}"] \\ & A \times X \end{tikzcd} \right]$$

$$
\begin{tikzcd}
X^A \arrow[r, "\eta_{X^A}"] \arrow[dr, "id"'] & (A \times X^A)^A \arrow[d, "(ev_{A,X})^A"] \\
 & X^A
\end{tikzcd}
$$

$$\langle a, x\rangle \longmapsto (a,\ \lambda a.\langle a, x\rangle) \longmapsto (\lambda a.\langle a, x\rangle)\, a = \langle a, x\rangle$$

$$f \longmapsto \lambda a.\langle a, f\rangle \longmapsto \lambda a.\, fa = f$$

by extensionality ($\eta$-equivalence)

## Types

$$T ::= \underset{\substack{\varphi \\ \text{base type}}}{\iota} \;\Big|\; T \to T \;\Big|\; T \times T$$

## Terms

$$t ::= x \mid \langle t, t \rangle \mid \pi_1 t \mid \pi_2 t \mid \lambda x. t \mid t\, t$$

## Typed terms

$$\frac{}{\Gamma, x : T \vdash x : T}\ (\text{Var})$$

$$\frac{\Gamma \vdash t : T \qquad \Gamma \vdash u : U}{\Gamma \vdash \langle t, u \rangle : T \times U}\ (\times\text{-intro})$$

$$\frac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_1(v) : T}\ (\times\text{-elim } 1)$$

$$\frac{\Gamma \vdash v : T \times U}{\Gamma \vdash \pi_2(v) : U}\ (\times\text{-elim } 2)$$

$$\frac{\Gamma, x : U \vdash t : T}{\Gamma \vdash \lambda x^{U}. t : U \to T}\ (\to\text{-intro})$$

$$\frac{\Gamma \vdash t : U \to T \qquad \Gamma \vdash u : U}{\Gamma \vdash tu : T} \ (\to\text{elim.})$$

$\lambda x . x \equiv \lambda y . y$

<u>Def.</u> · $\alpha$-equivalence. ∘∘○

deemed syntactically equal

$$\underbrace{\lambda x}_{\text{variable binder}} . \underbrace{t}_{} \equiv \lambda y . t[y/x]$$

all the occurrences of $x$ here gets "bound"

· Capture-avoiding substitution.

$t[u/x]$ ∘∘∘ in $(\lambda x . t)u \longrightarrow_\beta t[u/x]$

* replace all the <u>free</u> occurrences of $x$ in $t$ with $u$

* in such a way that unwanted and coincidental "capture of variables" is avoided

(by suitably renaming bound variables)

e.g. $\underbrace{(\lambda y . y)}_{\text{the same as } \lambda z . z}[x/y] \equiv \lambda y . y$

$$\underline{(\lambda z.\, z\, y)\,[zx/y]} \equiv \lambda u.\, u(zx)$$

$$\equiv$$

$$(\lambda u.\, u y)\,[zx/y]$$

Key: Rename b'd vars to fresh ones

## Red. rules

- $$(\lambda x.\, t)\, u \longrightarrow_\beta t[u/x]$$
$$\pi_1 \langle t, u\rangle \longrightarrow_\beta t$$
$$\pi_2 \langle t, u\rangle \longrightarrow_\beta u$$

- $=_\beta$
$$\frac{t \longrightarrow u}{t =_\beta u}$$

$$\frac{}{t =_\beta t} \qquad \frac{t =_\beta u}{u =_\beta t} \qquad \frac{t =_\beta u \quad u =_\beta v}{t =_\beta v}$$

Congruence (= context-closed eq. rel.)

Context closure

$$\frac{t =_\beta u}{\begin{array}{c} t v =_\beta u v \\ v t =_\beta v u \\ \lambda x.\, t =_\beta \lambda x.\, u \\ \langle t, v\rangle =_\beta \langle u, v\rangle \end{array}}$$

$$\langle v, t\rangle =_\beta \langle v, u\rangle$$
$$\pi_1 t =_\beta \pi_1 u$$
$$\pi_2 t =_\beta \pi_2 u$$

**Lem.** (subject reduction)

$$\left.\begin{array}{l}\Gamma \vdash t : T \\ t \to_\beta u\end{array}\right\} \Rightarrow \Gamma \vdash u : T$$

It follows:

$$\left.\begin{array}{l}\Gamma \vdash t : T \\ t =_\beta u\end{array}\right\} \Rightarrow \Gamma \vdash u : T$$

Hence it makes sense to consider typed equality judgments

$$\Gamma \vdash t =_\beta u : T$$

meaning $\Gamma \vdash t : T$, $\Gamma \vdash u : T$

and $t =_\beta u$

$=_\eta$

extensionality

* $$\frac{\Gamma \vdash t : \upsilon \to \sigma}{\Gamma \vdash t =_\eta \lambda x^\upsilon . t\, x : \upsilon \to \sigma}$$

* $$\frac{\Gamma \vdash v : \upsilon \times \sigma}{\Gamma \vdash v =_\eta \langle \pi_1 v, \pi_2 v \rangle : \upsilon \times \sigma}$$

* And the rules that ensure $=_\eta$ is a congruence

- $\equiv_{\beta\eta}$ : the congruence gen. by $\equiv_\beta$ and $\equiv_\eta$.