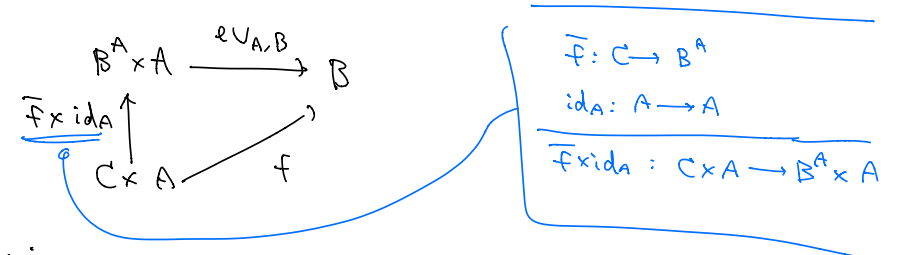
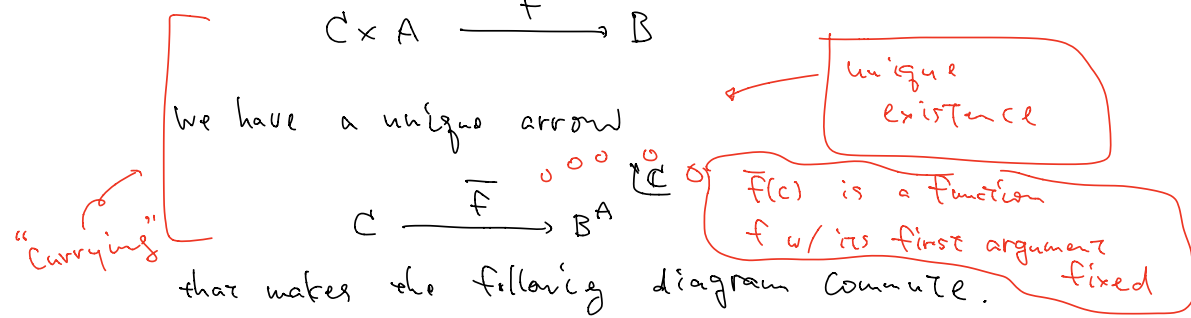
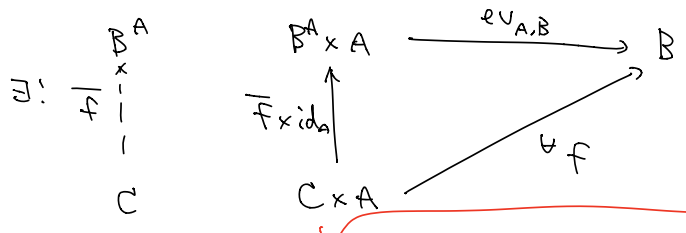


Def. \mathcal{C} : a category $A, B \in \mathcal{C}$
 w/ binary products \times
 The exponential from A to B is a pair
 $(B^A \in \mathcal{C}, B^A \times A \xrightarrow{ev_{A,B}} B)$

such that:
 for each arrow $f: C \rightarrow B$ in \mathcal{C} ,
 there exists a unique arrow $\bar{f}: C \rightarrow B^A$ in \mathcal{C} such that $f \circ \bar{f} \circ \text{ids} = f$.



Briefly:



Def. A Cartesian closed category (CCC)

- is a category \mathcal{C} with
- a terminal (final) object 1 ,
 - binary products \times , and
 - exponentials for all $A, B \in \mathcal{C}$.

Def. $1 \in \mathcal{C}$ is terminal if, for $\forall x \in \mathcal{C}$, $x \dashrightarrow 1$

(0-ary product, the limit for the empty diagram)

all finite products

It turns out:

Typed λ -cal.
 CCC_s

Thm. In a CCC \mathcal{C} , we have an adjunction

$$\mathcal{C} \begin{array}{c} \xrightarrow{A \times (-)} \\ \perp \\ \xleftarrow{(-)^A} \end{array} \mathcal{C}$$

for each $A \in \mathcal{C}$.

- Adjunction

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} \mathcal{D}$$

left right

$$\frac{Fx \rightarrow Y \in \mathcal{D}}{x \rightarrow Gy \in \mathcal{C}}$$

Q1

Is this \perp ? \top ?

Q2

What are units and counits?

Q3

Triangular equalities?

Q1 if $(-)^A \dashv Ax -$

we'd get

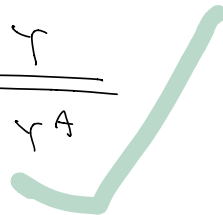
$$\frac{x^A \rightarrow Y}{x \rightarrow AxY}$$



if $Ax - \dashv (-)^A$

we'd get

$$\frac{AxX \rightarrow Y}{x \rightarrow Y^A}$$



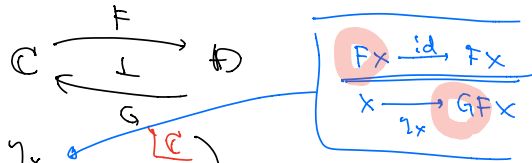
[\Downarrow] given $AxX \xrightarrow{f} Y$
 we get $x \xrightarrow{\bar{f}} Y^A$

[\Uparrow] Given $g: x \rightarrow Y^A$

$$AxX \xrightarrow{id_A \times g} AxY^A \xrightarrow{ev_{A,Y}} Y$$

These correspondences are mutually inverse: one proves it via the uniqueness of \bar{f} .

- In an adjunction



the unit is $(X \xrightarrow{\eta_x} GFX)_{x \in C}$ natural in X

the counit is $(FGA \xrightarrow{\epsilon_A} A)_{A \in D}$ natural in A

Q2



The counit

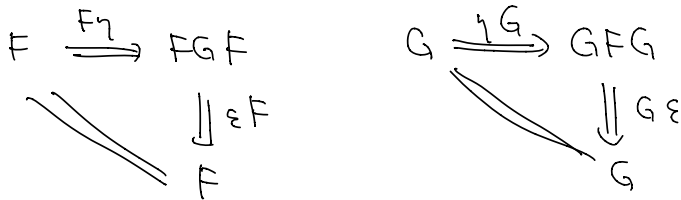
$$A \times X^A \xrightarrow{\epsilon_{A,X}} X$$

The unit

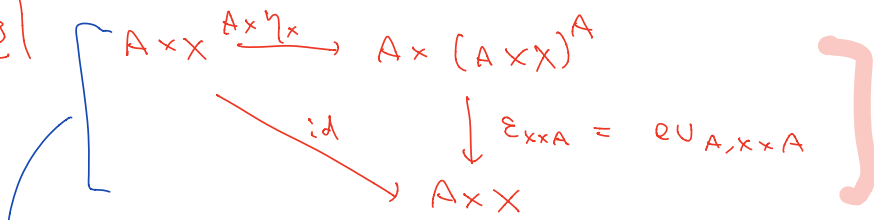
$$X \xrightarrow{\eta_x} (A \times X)^A$$

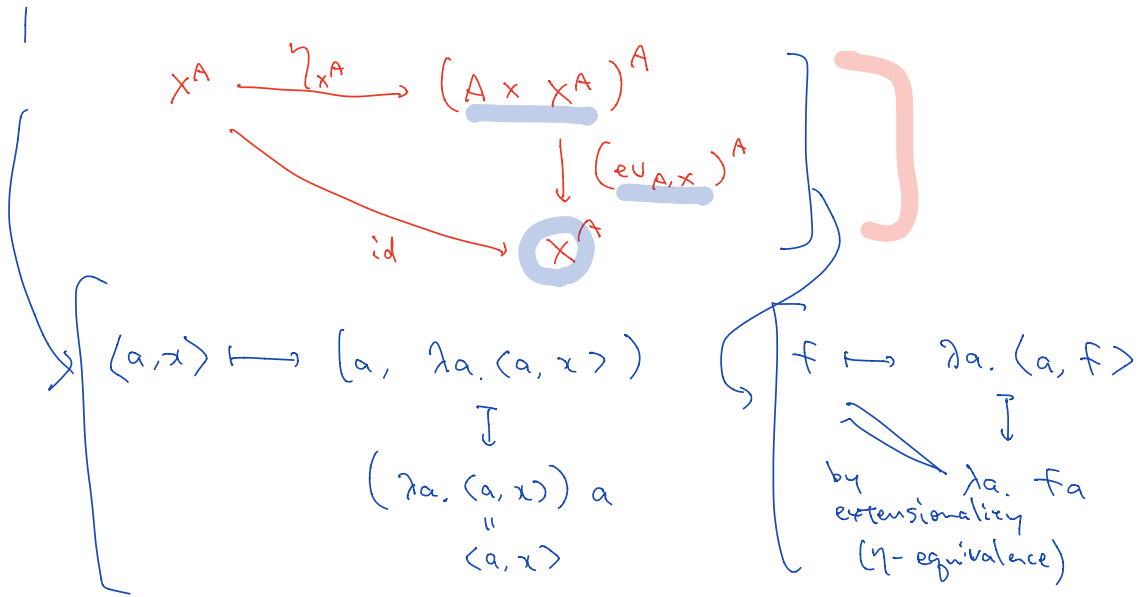
$x \mapsto \lambda a. \langle a, x \rangle$

- Triangular equalities.



Q3





Types

$$\tau ::= \text{L} \mid \tau \rightarrow \tau \mid \tau \times \tau$$

base type

Terms

Typed terms

$$t ::= x \mid \langle \tau, \tau \rangle \mid \pi_1 \tau \mid \pi_2 \tau \mid \lambda x. \tau \mid \tau \tau$$

$$\frac{}{\Gamma, x : \tau \vdash x : \tau} \text{ (Var)}$$

$$\frac{\Gamma \vdash \tau : \tau \quad \Gamma \vdash u : \nu}{\Gamma \vdash \langle \tau, u \rangle : \tau \times \nu} \text{ (x-intro)}$$

$$\frac{\Gamma \vdash v : \tau \times \nu}{\Gamma \vdash \pi_1(v) : \tau} \text{ (x-elim 1)}$$

$$\frac{\Gamma \vdash v : \tau \times \nu}{\Gamma \vdash \pi_2(v) : \nu} \text{ (x-elim 2)}$$

$$\frac{\Gamma, x : \nu \vdash \tau : \tau}{\Gamma \vdash \lambda x. \tau : \nu \rightarrow \tau} \text{ (\(\rightarrow\)-intro)}$$

$$\frac{\Gamma \vdash t : V \rightarrow T \quad \Gamma \vdash u : V}{\Gamma \vdash tu : T} (\rightarrow \text{elim.})$$

Def. - α -equivalence.

$$\lambda x.x \equiv \lambda y.y$$

deemed syntactically equal

$$\lambda x.t \equiv \lambda y.t[y/x]$$

variable binder

all the occurrences of x here gets "bound"

• Capture-avoiding substitution.

$$t[u/x] \quad \text{ooo} \quad (\lambda x.t)u \rightarrow_p t[u/x]$$

* replace all the free occurrences of x in t with u

* in such a way that unwanted and coincidental "capture of variables" is avoided

(by suitably renaming bound variables)

e.g. $(\lambda y.y)[x/y] \equiv \lambda y.y$

the same as $\lambda z.z$

$$\underline{(\lambda z. z y) [z^x / y]} \equiv \lambda u. u(z x)$$

|||

$$(\lambda u. u y) [z^x / y]$$

key Rename b'ld vars
to fresh ones

Red. rules

$$\bullet (\lambda x. \tau) u \longrightarrow_{\beta} \tau[u/x]$$

$$\pi_1 \langle t, u \rangle \longrightarrow_{\beta} t$$

$$\pi_2 \langle t, u \rangle \longrightarrow_{\beta} u$$

$$\bullet =_{\beta} \frac{t \longrightarrow u}{t =_{\beta} u}$$

$$t =_{\beta} u$$

$$\frac{}{t =_{\beta} t}$$

$$\frac{t =_{\beta} u \quad u =_{\beta} v}{t =_{\beta} v}$$

$$\frac{t =_{\beta} u \quad u =_{\beta} v}{t =_{\beta} v}$$

Congruence
(= context-closed
equiv.)

Context
closure

$$\frac{t =_{\beta} u}{t v =_{\beta} u v}$$

$$t v =_{\beta} u v$$

$$u t =_{\beta} v t$$

$$\lambda x. t =_{\beta} \lambda x. u$$

$$\langle t, u \rangle =_{\beta} \langle u, v \rangle$$

$$\langle u, t \rangle =_{\beta} \langle u, u \rangle$$

$$\pi_1 t =_{\beta} \pi_1 u$$

$$\pi_2 t =_{\beta} \pi_2 u$$

Lem. (subject reduction)

$$\left. \begin{array}{l} \Gamma \vdash t : T \\ t \rightarrow_{\beta} u \end{array} \right\} \Rightarrow \Gamma \vdash u : T$$

It follows:

$$\left. \begin{array}{l} \Gamma \vdash t : T \\ t =_{\beta} u \end{array} \right\} \Rightarrow \Gamma \vdash u : T$$

Hence it makes sense to
consider typed equality judgments

$$\Gamma \vdash t =_{\beta} u : T$$

meaning

$$\Gamma \vdash t : T, \quad \Gamma \vdash u : T$$

$$\text{and } t =_{\beta} u$$

$$\begin{array}{c} \cdot \quad =_{\eta} \quad * \\ \uparrow \\ \text{extensionality} \end{array} \quad \frac{\vdash t : \nu \rightarrow \sigma}{\vdash t =_{\eta} \lambda x^{\nu}. t x : \nu \rightarrow \sigma}$$

$$* \quad \frac{\vdash v : \nu \times \sigma}{\vdash v =_{\eta} (\pi_1 v, \pi_2 v) : \nu \times \sigma}$$

$$\vdash v =_{\eta} (\pi_1 v, \pi_2 v) : \nu \times \sigma$$

* And the rules that ensure $=_{\eta}$ is a congruence

- $\equiv_{\beta\eta}$: the congruence
gen. by \equiv_{β} and \equiv_{η} .