MSCS 23/1/2017

Goal

- Let CCCs to model typed λ-calculus

[w/ finite products $1, \times$
exponentials
$A^B \times A \xrightarrow{ev} B$ $\boxed{\mathbb{E}}$]

Types
(λ-) terms
$=_{\beta\eta}$
$(\lambda x. s)\, t \to_\beta s[t/x]$

Lem

Each derivable type judgment

$$\Gamma \vdash t : T$$

has a unique derivation tree.

[trees by typing rules like ($\to$-intro) ($\to$-elim) ...]

[**Proof.** By looking at the typing rules, induction on the construction of the term t. $\boxed{\mathbb{E}}$]

[Our syntactic choice $\lambda x^T. s$ is useful here.]

Def. Let $\mathbb{C}$ be a CCC.

Let $\mathcal{V}$ be an assignment of objects of $\mathbb{C}$ to each atomic type ι. (called valuation)

($\mathcal{V}(\iota) \in \mathbb{C}$ for each ι)

We define $[\![\tau]\!]_{\mathcal{V}} \in \mathbb{C}$, for each type τ,

the interpretation of τ

$\tau ::= \iota \mid \tau \times \tau \mid \tau \to \tau$

in the following inductive manner.

$$[\![\iota]\!]_{\mathcal{V}} := \mathcal{V}(\iota)$$

$$[\![\tau \times \upsilon]\!]_{\mathcal{V}} := [\![\tau]\!]_{\mathcal{V}} \times [\![\upsilon]\!]_{\mathcal{V}}$$

syntactic type constructor

product in $\mathbb{C}$

$$[\![\tau \to \upsilon]\!]_{\mathcal{V}} := [\![\upsilon]\!]_{\mathcal{V}}^{[\![\tau]\!]_{\mathcal{V}}}$$

exponential in $\mathbb{C}$.

$\underline{\text{Def.}}$ (interpretation of (typed) terms)

Let $\underbrace{x_1 : T_1, \ldots, x_n : T_n}_{\Gamma} \vdash t : T$

be a derivable type judgment.

We define its $\underline{\text{interpretation}}$

$$[\![T_1]\!] \times \cdots \times [\![T_n]\!] \xrightarrow{[\![x_1:T_1,\ldots,x_n:T_n \vdash t:T]\!]} [\![T]\!] \quad \boxed{\mathcal{C}}$$

by the interpretation of its unique derivation tree. The latter, in turn, is defined as follows $\underline{\text{by induction}}$.

[on the construction of derivation trees, or on the height of deriv. trees.]

$\boxed{\text{Base case}}$

The judgment $\Gamma \vdash t : T$ is derived by (Var). In this case we necessarily have

$\Gamma \vdash t : T$ is of the form

$\Gamma', x : T \vdash \underline{x} : T$

t is a variable!

and the derivation tree is of the form

$$\Pi = \left[\frac{}{\Gamma', x : T \vdash x : T} (\mathrm{Var}) \right]$$

Its interpretation is

the 2nd proj. for categorical product.

$\underline{\mathbb{C}}$

$$\underbrace{[\![\Gamma']\!]}_{\varphi} \times [\![T]\!] \xrightarrow{[\![\Pi]\!] := \pi_2} [\![T]\!]$$

like

$[\![T_1]\!] \times \cdots \times [\![T_n]\!]$

if $\Gamma' = (x_1 : T_1, \ldots, x_n : T_n)$

$\boxed{\text{Step case 1}}$ The last rule applied is ($\times$-intro), that is,

we want to interpret a deriv. rule of the form

$$\Pi = \left[\frac{\begin{matrix} \Pi' \\ \Gamma \vdash s : S \end{matrix} \qquad \begin{matrix} \Pi'' \\ \Gamma \vdash u : U \end{matrix}}{\Gamma \vdash \langle s, u \rangle : S \times U} (\times\text{-intro}) \right]$$

By induction, we have already defined the interpretations

$$[\![\Gamma]\!] \xrightarrow{[\![\pi']\!]} [\![S]\!] \quad \lfloor \mathbb{C} \qquad \text{and}$$

$$[\![\Gamma]\!] \xrightarrow{[\![\pi'']\!]} [\![U]\!] \quad \lfloor \mathbb{C}$$

We define $[\![\pi]\!]$ by

$$[\![\Gamma]\!] \xrightarrow[{[\![\pi]\!] \,:=\, \langle [\![\pi']\!], [\![\pi'']\!] \rangle}]{} [\![S \times U]\!] \underset{\text{by def.}}{=} [\![S]\!] \times [\![U]\!]$$

e.g. tupling

```latex
\begin{tikzcd}
 & & A \\
C \arrow[urr, "f"] \arrow[rr, dashed, "\langle f, g\rangle"] \arrow[drr, "g"'] & & A \times B \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\
 & & B
\end{tikzcd}
```

Step Cases 2, 3

The last rule applied is $(\times\text{-elim } 1)$
$(\times\text{-elim } 2)$

(skip ... Use projections)

$\boxed{\text{Step Case 4}}$ The last rule is $(\to\text{-intro})$.

In this case

$$\Pi = \left[\ \dfrac{\overbrace{\Gamma, x : \upsilon \vdash s : \varsigma}^{\Pi'}}{\Gamma \vdash \lambda x^{\upsilon}.\, s : \upsilon \to \varsigma}\,(\to\text{-intro})\ \right]$$

By induction we have obtained

$$[\![\Pi']\!] : [\![\Gamma]\!] \times [\![\upsilon]\!] \longrightarrow [\![\varsigma]\!] \quad \text{in } \mathbb{C}$$

We define

$$[\![\Pi]\!] : [\![\Gamma]\!] \longrightarrow [\![\upsilon \to \varsigma]\!] \overset{\text{by def.}}{=} [\![\varsigma]\!]^{[\![\upsilon]\!]}$$

← exponential in $\mathbb{C}$

by

- using the adjunction

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "(-)\times[\![\upsilon]\!]"] \arrow[r, phantom, "\perp"] & \mathbb{C} \arrow[l, bend left, "(-)^{[\![\upsilon]\!]}"] \end{tikzcd}$$

that is,

$$\frac{X \times [\![U]\!] \longrightarrow Y \quad \text{in } \mathbb{C}}{X \longrightarrow Y^{[\![U]\!]}}$$

- in other words:

$$\begin{tikzcd}
{[\![S]\!]^{[\![U]\!]}} \text{ in } \mathbb{C} & & {[\![S]\!]^{[\![U]\!]} \times [\![U]\!]} \arrow[rr, "ev"] & & {[\![S]\!]} \text{ in } \mathbb{C} \\
{[\![\Gamma]\!]} \arrow[u, dashed, "\exists!\,[\![\pi]\!]"] & & {[\![\Gamma]\!] \times [\![U]\!]} \arrow[u, "{[\![\pi]\!] \times [\![U]\!]}"] \arrow[urr, "{[\![\pi']\!]}"'] & &
\end{tikzcd}$$

precisely $id_{[\![U]\!]}$

$\boxed{\text{Step Case 5}}$

$$\Pi = \left[\ \dfrac{\dfrac{\Pi'}{\Gamma \vdash s : U \to S} \qquad \dfrac{\Pi''}{\Gamma \vdash u : U}}{\Gamma \vdash su : S}\ (\to\text{-elim}) \right]$$

By induction we have got

$$[\![\pi']\!] : [\![\Gamma]\!] \longrightarrow [\![U \to S]\!] \text{ in } \mathbb{C}$$

$$= \text{ by def. } [\![S]\!]^{[\![U]\!]}$$

$$\llbracket \pi'' \rrbracket : \llbracket \Gamma \rrbracket \longrightarrow \llbracket U \rrbracket$$

We combine these to define $\llbracket \pi \rrbracket$. Specifically:

Tupling

$$\llbracket \Gamma \rrbracket \xrightarrow{\langle \llbracket \pi' \rrbracket, \llbracket \pi'' \rrbracket \rangle} \llbracket S \rrbracket^{\llbracket U \rrbracket} \times \llbracket U \rrbracket \xrightarrow{\ \mathrm{ev}\ } \llbracket S \rrbracket \qquad \boxed{\mathbb{C}}$$

By the way $\llbracket \Gamma \rrbracket$ is defined by

$$\begin{cases} \llbracket \underline{\quad} \rrbracket = 1 \\ \quad \Gamma = (\ ) \text{ empty} \\ \\ \llbracket x : T, \Gamma' \rrbracket = \llbracket T \rrbracket \times \llbracket \Gamma' \rrbracket \end{cases}$$

Intermission Structural induction.

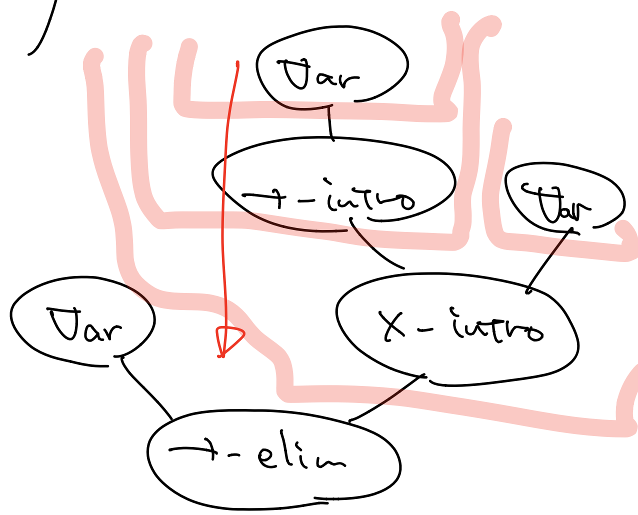
Usual induction (for $n \in \mathbb{N}$)

Tree!!

"successor"

$$n = \left(\begin{array}{l} 0 \\ | \\ \left.\begin{array}{c} S \\ | \\ S \\ | \\ \vdots \\ | \\ S \end{array}\right\} n \text{ times} \end{array} \right)$$

Structural induction



Thm. (Soundness)

Let $\mathbb{C}$ be a CCC.

Assume

$$\left.\begin{array}{l} \Gamma \vdash t : T \\ \Gamma \vdash u : T \end{array}\right\} \text{ are derivable, and}$$

moreover

$$\Gamma \vdash t =_{\beta\eta} u : T$$

Then

statement

$$\begin{tikzcd}[column sep=huge] [\![\Gamma]\!] \arrow[r, shift left=2, "{[\![\Gamma \vdash t : T]\!]}"] \arrow[r, shift right=2, "{[\![\Gamma \vdash u : T]\!]}"'] \arrow[r, phantom, "="] & [\![T]\!] \end{tikzcd} \qquad \mathbb{C}$$

[Therefore, if $[\![\Gamma \vdash t : T]\!] \neq [\![\Gamma \vdash u : T]\!]$ in some CCC $\mathbb{C}$, then $t \neq_{\beta\eta} u$]

In the proof we rely on the following lemma.

Lem. (Substitution lemma)

Assume

$$\left.\begin{array}{c} \Gamma, x : U \vdash t : T \\ \Gamma \vdash u : U \end{array}\right\} \text{ are derivable.}$$

Then

- $\Gamma \vdash t[u/x] : T$ is derivable, and
- in any CCC $\mathbb{C}$,

$$[\![\Gamma \vdash t[u/x] : T]\!]^{\mathbb{C}} = \left([\![\Gamma]\!] \xrightarrow{\langle \mathrm{id}_{[\![\Gamma]\!]},\, [\![\Gamma \vdash u : U]\!] \rangle} [\![\Gamma]\!] \times [\![U]\!] \xrightarrow{[\![\Gamma, x : U \vdash t : T]\!]} [\![T]\!] \right)$$

Proof. By induction on the derivation of $\Gamma, x : U \vdash t : T$ $\square$

Proof of soundness

By induction on the derivation of $=_{\beta\eta}$

$\boxed{\text{Base case 1}}$

If $\Gamma \vdash t =_{\beta\eta} u : T$ is of the form

$$\Gamma \vdash (\lambda x^{\cancel{U} V}. s)\, \cancel{u}\, v =_{\beta\eta} s[\cancel{u}\, v / x] : T$$

Let

$$f_1 := \left([\![\Gamma]\!] \times [\![\cancel{U}\, V]\!] \xrightarrow{[\![\Gamma, x : \cancel{U}\, V \vdash s : T]\!]} [\![T]\!] \right)$$

$$f_2 := \left([\![\Gamma]\!] \xrightarrow{[\![\Gamma \vdash v : V]\!]} [\![V]\!] \right)$$

Then

$$[\![\Gamma \vdash (\lambda x^V. s)\, v : T]\!]$$

$$\underset{\text{by def.}}{=} \left([\![\Gamma]\!] \xrightarrow{\langle f_1^{\wedge}, f_2 \rangle} [\![T]\!]^{[\![V]\!]} \times [\![V]\!] \xrightarrow{ev} [\![T]\!] \right)$$

$$\frac{[\![\Gamma]\!] \times [\![V]\!] \xrightarrow{f_1} [\![T]\!]}{[\![\Gamma]\!] \xrightarrow{f_1^{\wedge}} [\![T]\!]^{[\![V]\!]}}$$

$$\left(A \xrightarrow{\langle f, g \rangle} B \times C \right) = \left(A \xrightarrow{\langle id_A, g \rangle} A \times C \xrightarrow{f \times id_C} B \times C \right)$$

@ Derive this from the universality of products

$$= \left([\![\Gamma]\!] \xrightarrow{\langle id_{[\![\Gamma]\!]},\, f_2 \rangle} [\![\Gamma]\!] \times [\![U]\!] \xrightarrow{\hat{f_1} \times id_{[\![U]\!]}} [\![T]\!]^{[\![U]\!]} \times [\![U]\!] \xrightarrow{ev} [\![T]\!] \right)$$

$$\frac{A \xrightarrow{f} B \qquad C \xrightarrow{g} D}{A \times C \xrightarrow{f \times g} B \times D}$$

$$\overset{\star}{=} \left([\![\Gamma]\!] \xrightarrow{\langle id_{[\![\Gamma]\!]},\, f_2 \rangle} [\![\Gamma]\!] \times [\![U]\!] \xrightarrow{f_1} [\![T]\!] \right)$$

$$= [\![\Gamma \vdash s[v/x] : T]\!]$$

by Subst. Lem.

Here $\star$ follows from

$$\begin{tikzcd} {[\![T]\!]^{[\![U]\!]}} & & {[\![T]\!]^{[\![U]\!]} \times [\![U]\!]} \arrow[r, "ev"] & {[\![T]\!]} \\ {[\![\Gamma]\!]} \arrow[u, "\hat{f_1}"'] & & {[\![\Gamma]\!] \times [\![U]\!]} \arrow[u, "\hat{f_1} \times id_{[\![U]\!]}"'] \arrow[ur, "f_1"'] & \end{tikzcd}$$

The other cases are skipped.

$\square$