

MSCS Report 1.

①

1. In Sets, an arrow $f: A \rightarrow B$ is
- = mono iff injective
 - = epi iff surjective.

Proof

Let us first recall the definitions.

f : mono

$$\forall X \in \text{Sets}, \forall g, g': X \rightarrow A, (f \circ g = f \circ g' \Rightarrow g = g')$$

$$X \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{g'} \end{array} A \xrightarrow{f} B$$

f : epi.

$$\forall Y \in \text{Sets}, \forall h, h': B \rightarrow Y, (h \circ f = h' \circ f \Rightarrow h = h')$$

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{h'} \end{array} Y$$

f : injective

$$\forall a, a' \in A, (f(a) = f(a') \Rightarrow a = a')$$

f : surjective

$$\forall b \in B, \exists a \in A. f(a) = b.$$

[f : mono \Rightarrow f : injective] Assume f : mono. (Aim: $a = a'$)

Suppose we have $a, a' \in A$ such that $f(a) = f(a')$.

Because elements of A are in 1-to-1 correspondence

with arrows from $1 = \{*\}$ to A , this is equivalent

to: we have $\bar{a}, \bar{a}' : 1 \rightarrow A$ such that

$$f \circ \bar{a} = f \circ \bar{a}' \quad (\bar{a}(\ast) := a, \bar{a}'(\ast) := a')$$

$$1 \begin{array}{c} \xrightarrow{\bar{a}} \\ \xrightarrow{\bar{a}'} \end{array} A \xrightarrow{f} B$$

Because f : mono by assumption, $\bar{a} = \bar{a}'$, which is equivalent to: $a = a'$.

(So f : injective is a special case of f : mono where $X \in \mathbf{Sets}$ is instantiated by 1 (a singleton).)

[f : injective $\Rightarrow f$: mono]. Assume f : injective.

Suppose we have $X \in \mathbf{Sets}$ and $g, g' : X \rightarrow A$ such that $f \circ g = f \circ g'$. (Aim: $g = g'$)

For every $x \in X$, $f(g(x)) = f(g'(x))$ by $f \circ g = f \circ g'$. Since f : injective by assumption, for every $x \in X$.

$$g(x) = g'(x). \text{ So, } g = g'.$$

[f : epi. $\Rightarrow f$: surjective] Assume f : epi.

Suppose we have $b \in B$. (Aim: $\exists a \in A. f(a) = b$).

Consider $2 = \{0, 1\} \in \mathbf{Sets}$ and $\bar{b}_0, \bar{b}_1 : B \rightarrow 2$ defined as

$$\bar{b}_0(b') = 0$$

$$\bar{b}_1(b') = \begin{cases} 0 & (b' \neq b) \\ 1 & (b' = b). \end{cases}$$

Then, clearly $\bar{b}_0 \neq \bar{b}_1$. From this, using the contrapositive of f : epi., it follows that

$$\bar{b}_0 \circ f \neq \bar{b}_1 \circ f.$$

So there must be some $a \in A$ such that

$$\bar{b}_0(f(a)) \neq \bar{b}_1(f(a)).$$

By the definition of \bar{b}_0 and \bar{b}_1 , it follows that

$$f(a) = b.$$

[f : surjective $\Rightarrow f$: epi.] Assume f : surjective.

Suppose we have $\mathcal{Y} \in \text{Sets}$ and $h, h': B \rightarrow \mathcal{Y}$ such that $h \circ f = h' \circ f$. (Aim: $h = h'$).

For every $b \in B$, because f : surjective there exists $a \in A$ such that $f(a) = b$. For such a ,

$$h(f(a)) = h'(f(a))$$

by the hypothesis. So $h(b) = h'(b)$ for every $b \in B$, showing $h = h'$. \square

2. \mathcal{C} : category

$$X, Y \in \mathcal{C}.$$

A product of X and Y , if it exists, is unique up-to a canonical isomorphism.

Proof 1

Suppose $X \xleftarrow{p_1} P \xrightarrow{p_2} Y$ and $X \xleftarrow{q_1} Q \xrightarrow{q_2} Y$ are products of X and Y . We proceed as follows.

Step 1. Construct a canonical arrow $m: P \rightarrow Q$ using the universality of Q .

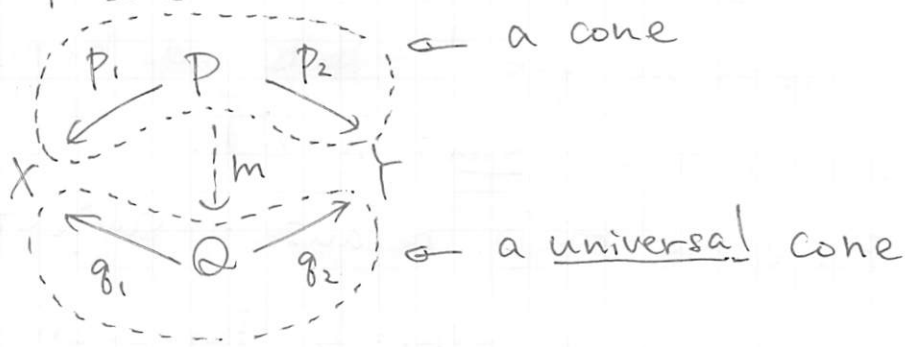
Step 2. Construct a canonical arrow $n: Q \rightarrow P$ using the universality of P .

Step. 3. Prove $h \circ m = \gamma \circ p$ using the universality of P . ④

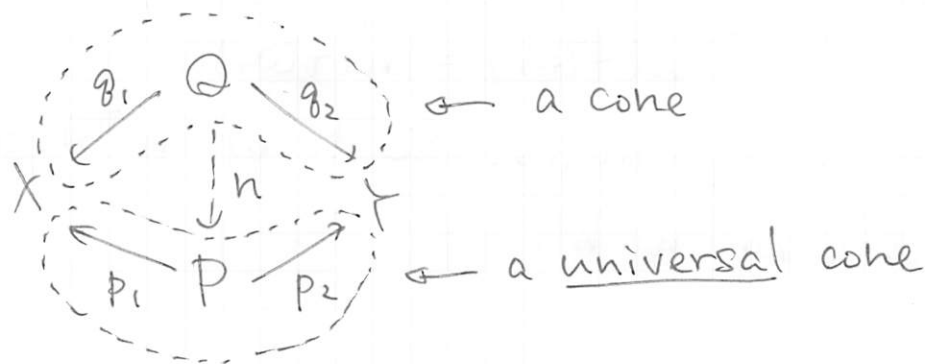
Step. 4. Prove $m \circ h = \gamma \circ q$ using the universality of Q .

So m and $n (= m^{-1})$ constitute the canonical isomorphism which ensures $P \cong Q$.

[Step. 1.]



[Step. 2.]



[Step. 3.] Both diagrams



Commute. So $h \circ m = \gamma \circ p$.

[Step. 4.] Similar to Step. 3.

