MSCS Report 1.

1. In $\mathbf{Sets}$, an arrow $f: A \to B$ is
   - mono iff injective
   - epi iff surjective.

Proof

Let us first recall the definitions.

$\boxed{f : \text{mono}}$

$\forall X \in \mathbf{Sets}, \forall g, g' : X \to A, (f \circ g = f \circ g' \Rightarrow g = g')$

$$X \underset{g'}{\overset{g}{\rightrightarrows}} A \xrightarrow{f} B$$

$\boxed{f : \text{epi.}}$

$\forall Y \in \mathbf{Sets}, \forall h, h' : B \to Y, (h \circ f = h' \circ f \Rightarrow h = h')$

$$A \xrightarrow{f} B \underset{h'}{\overset{h}{\rightrightarrows}} Y$$

$\boxed{f : \text{injective}}$

$\forall a, a' \in A, (f(a) = f(a') \Rightarrow a = a')$

$\boxed{f : \text{surjective}}$

$\forall b \in B, \exists a \in A. \ f(a) = b.$

[f: mono $\Rightarrow$ f: injective] Assume f: mono. (Aim: $a = a'$)

Suppose we have $a, a' \in A$ such that $f(a) = f(a')$.

Because elements of A are in 1-to-1 correspondence with arrows from $1 = \{*\}$ to A, this is equivalent

to: we have $\bar{a}, \bar{a}' : 1 \to A$ such that
$f \circ \bar{a} = f \circ \bar{a}'$. ($\bar{a}(*) := a$, $\bar{a}'(*) := a'$)

$$1 \underset{\bar{a}'}{\overset{\bar{a}}{\rightrightarrows}} A \xrightarrow{f} B$$

Because f: mono by assumption, $\bar{a} = \bar{a}'$, which is equivalent to: $a = a'$.

(So $\boxed{f : \text{injective}}$ is a special case of $\boxed{f : \text{mono}}$ where $X \in \mathbf{Sets}$ is instanciated by 1 (a singleton).)

[f: injective $\Rightarrow$ f: mono]. Assume f: injective.

Suppose we have $X \in \mathbf{Sets}$ and $g, g' : X \to A$ such that $f \circ g = f \circ g'$. (<u>Aim</u>: $g = g'$)

For every $x \in X$, $f(g(x)) = f(g'(x))$ by $f \circ g = f \circ g'$.
Since f: injective by assumption, for every $x \in X$, $g(x) = g'(x)$. So, $g = g'$.

[f: epi. $\Rightarrow$ f: surjective] Assume f: epi.

Suppose we have $b \in B$. (<u>Aim</u>: $\exists a \in A.\ f(a) = b$).
Consider $2 = \{0, 1\} \in \mathbf{Sets}$ and $\bar{b}_0, \bar{b}_1 : B \to 2$ defined as

$$\bar{b}_0(b') = 0$$

$$\bar{b}_1(b') = \begin{cases} 0 & (b' \neq b) \\ 1 & (b' = b). \end{cases}$$

Then, clearly $\bar{b}_0 \neq \bar{b}_1$. From this, using the contrapositive of $\boxed{f : \text{epi}}$, it follows that

$$\bar{b}_0 \circ f \neq \bar{b}_1 \circ f .$$

So there must be some $a \in A$ such that

$$\overline{b_0}(f(a)) \neq \overline{b_1}(f(a)).$$

By the definition of $\overline{b_0}$ and $\overline{b_1}$, it follows that

$f(a) = b$.

[f: surjective $\Rightarrow$ f: epi.] Assume f: surjective.

Suppose we have $Y \in \mathbf{Sets}$ and $h, h' : B \longrightarrow Y$ such that $h \circ f = h' \circ f$. (Aim: $h = h'$).

For every $b \in B$, because f: surjective there exists $a \in A$ such that $f(a) = b$. For such a,

$$h(f(a)) = h'(f(a))$$

by the hypothesis. So $h(b) = h'(b)$ for every $b \in B$, showing $h = h'$. $\square$

2. $\mathbb{C}$: category

$X, Y \in \mathbb{C}$.

A product of X and Y, if it exists, is unique up-to a canonical isomorphism.

Proof

Suppose $X \xleftarrow{p_1} P \xrightarrow{p_2} Y$ and $X \xleftarrow{q_1} Q \xrightarrow{q_2} Y$ are products of X and Y. We proceed as follows.

Step. 1. Construct a canonical arrow $m : P \to Q$ using the universality of Q.

Step. 2. Construct a canonical arrow $n : Q \to P$ using the universality of P.

Step. 3. Prove $n \circ m = 1_P$ using the universality of P.

Step. 4. Prove $m \circ n = 1_Q$ using the universality of Q.

So m and n $(= m^{-1})$ constitute the canonical isomorphism which ensures $P \cong Q$.

[Step. 1.]

$$\begin{tikzcd}
 & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, dashed, "m"] & \\
X & & Y \\
 & Q \arrow[ul, "q_1"] \arrow[ur, "q_2"'] &
\end{tikzcd}$$

← a cone

← a <u>universal</u> cone

[Step. 2]

$$\begin{tikzcd}
 & Q \arrow[dl, "q_1"'] \arrow[dr, "q_2"] \arrow[dd, dashed, "n"] & \\
X & & Y \\
 & P \arrow[ul, "p_1"] \arrow[ur, "p_2"'] &
\end{tikzcd}$$

← a cone

← a <u>universal</u> cone

[Step. 3] Both diagrams

$$\begin{tikzcd}
 & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[d, "m"] & \\
X & Q \arrow[l, "q_1"'] \arrow[r, "q_2"] \arrow[d, "n"] & Y \\
 & P \arrow[ul, "p_1"] \arrow[ur, "p_2"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "1_P"] & \\
X & & Y \\
 & P \arrow[ul, "p_1"] \arrow[ur, "p_2"'] &
\end{tikzcd}$$

← a cone →

← a <u>universal</u> cone →

Commute. So $n \circ m = 1_P$.

[Step. 4]. Similar to Step. 3. $\square$