

MSCS 31/10/2016

adjunction
 \Downarrow
 an extension

Mac Lane
 "Every notion is a Kan extension"
 (concept)

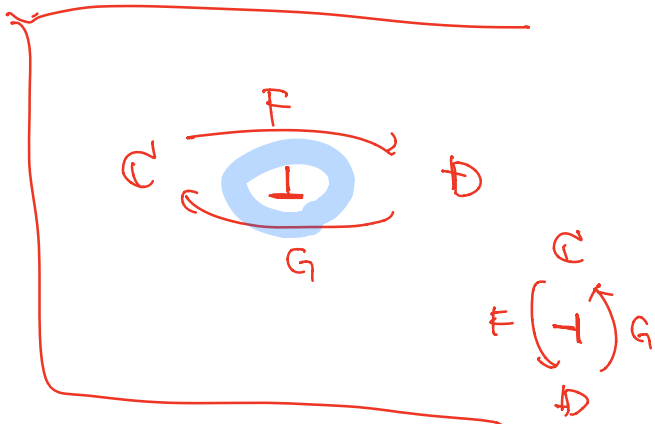
Def.

An adjunction from \mathcal{C} to \mathcal{D}

consists of

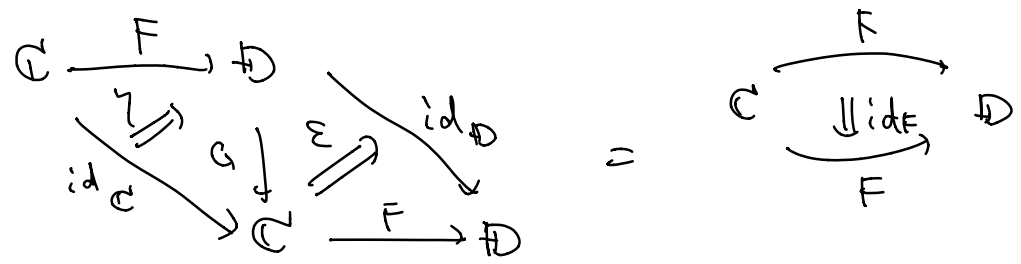
$$\left\{ \begin{array}{l} F: \mathcal{C} \rightarrow \mathcal{D} \\ G: \mathcal{D} \rightarrow \mathcal{C} \\ \eta: id_{\mathcal{C}} \Rightarrow GF \\ \varepsilon: FG \Rightarrow id_{\mathcal{D}} \end{array} \right.$$

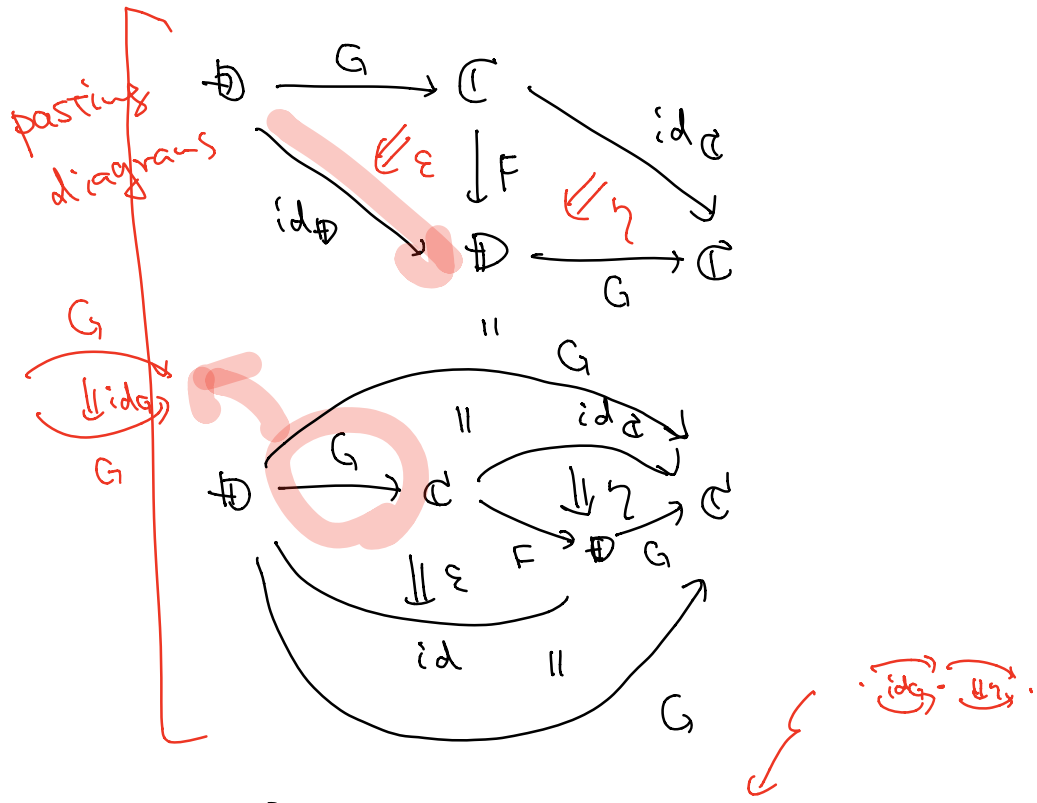
unit η
counit ε



F is a left adjoint to G
 G is a right adjoint to F

s.t.

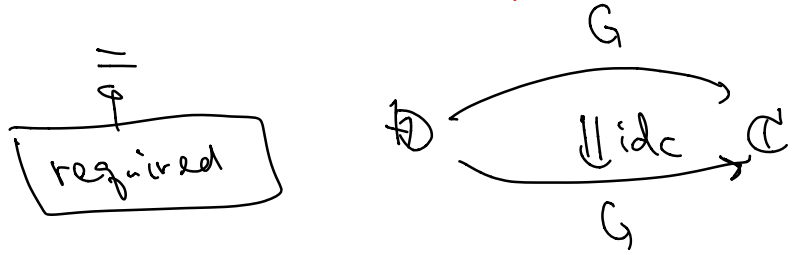


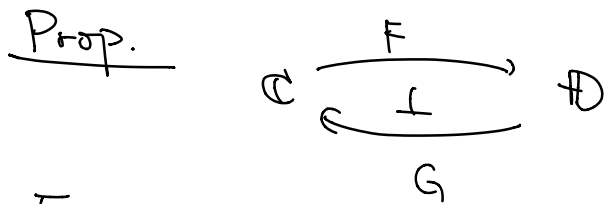


$$= \left(\begin{array}{l} G = id_{\mathbb{C}} \circ G \xrightarrow{\eta \circ id_G} G \circ F \circ G \\ id_G \circ \varepsilon \xrightarrow{\quad} G \circ id_{\mathbb{D}} = G \end{array} \right)$$

$$= (id_G \circ \varepsilon) \bullet (\eta \circ id_G)$$

horizontal composition.
 vert. comp.





F, G, η, ε form an adjunction (for some η, ε)

\Leftrightarrow For each $x \in \mathcal{C}, y \in \mathcal{D}$

we have $\{x \rightarrow GY \text{ in } \mathcal{C}\} \cong \{x \rightarrow Y \text{ in } \mathcal{D}\}$

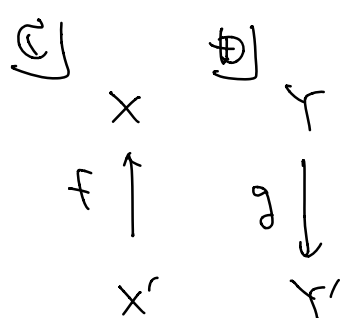
$$\mathcal{C}(x, GY) \cong \mathcal{D}(Fx, Y)$$

Furthermore the correspondence φ is natural in $x \in \mathcal{C}$ and $y \in \mathcal{D}$

"homsets"
↑
arrows
= "homomorphisms"

$$\frac{x \rightarrow GY \text{ in } \mathcal{C}}{Fx \rightarrow Y \text{ in } \mathcal{D}}$$

"is equivalent to"
"is in a bijective correspondence to"



Sets

$$\begin{array}{ccc} \mathcal{C}(x, GY) & \xrightarrow[\varphi_{x,Y}]{\cong} & \mathcal{D}(Fx, Y) \\ \mathcal{C}(f, Gg) = (Gg) \circ f & = & (Gg) \circ (-) \circ f \\ \downarrow & & \downarrow \\ \mathcal{C}(x', GY') & \xrightarrow[\varphi_{x',Y'}]{\cong} & \mathcal{D}(Fx', Y') \end{array}$$

This diagram commutes.

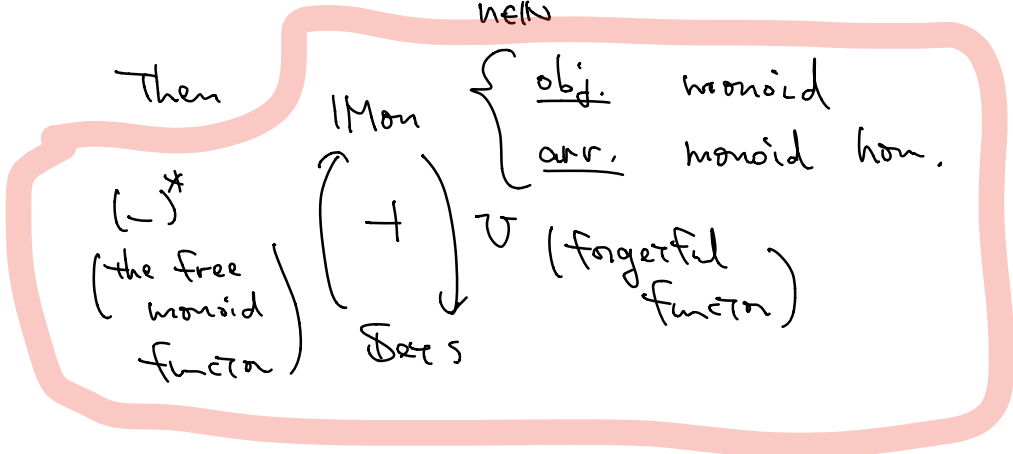
$$\begin{array}{ccc}
 \text{Sets} & \mathcal{C}(X, GR) & \longrightarrow & \mathcal{C}(X', GR') \\
 & \downarrow & & \downarrow \\
 & (X \xrightarrow{R} GR) & \longmapsto & (X' \xrightarrow{R'} GR') \\
 & & & \downarrow \\
 & & & (Gg \rightarrow GR')
 \end{array}$$

Examples

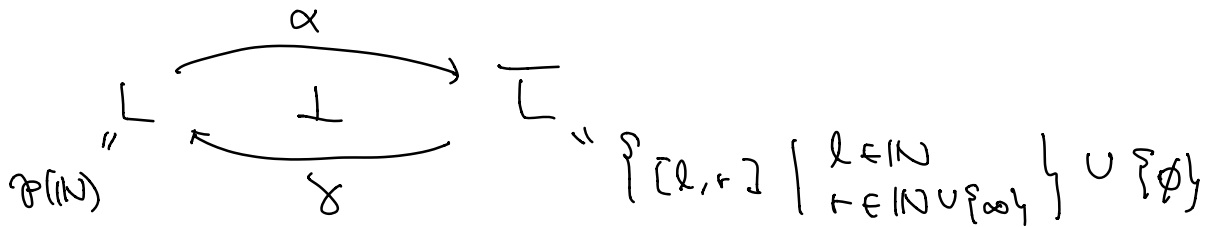
1. S : a set.

The free monoid over S is given

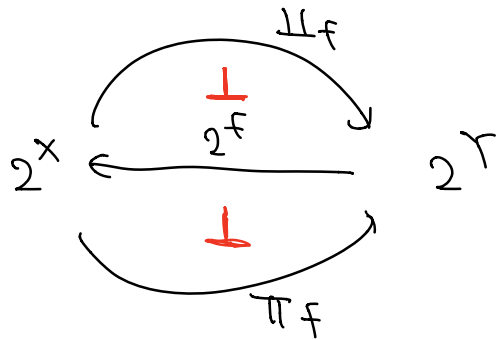
by $S^* = \coprod_{n \in \mathbb{N}} S^n$



2.



3. $f: X \rightarrow Y$ be a function.



$$2^f: 2^Y \rightarrow 2^X$$

$$\varphi \quad Y' \mapsto \{x \in X \mid f(x) \in Y'\}$$

inverse image

$$H_f: 2^X \rightarrow 2^Y$$

$$\varphi \quad X' \mapsto \{f(x) \mid x \in X'\}$$

direct image

$$\Pi_f: 2^X \rightarrow 2^Y$$

$$X' \mapsto \{y \mid \forall x \in X. (f(x) = y \Rightarrow x \in X')\}$$

You have to show

$$H_f X' \subseteq Y' \text{ in } 2^Y$$

$$X' \subseteq (2^f)(Y') \text{ in } 2^X$$

For each

$$X' \in 2^X$$

$$Y' \in 2^Y$$

That is,

$$H_f X' \subseteq Y'$$

$$\Leftrightarrow X' \subseteq (2^f)(Y')$$

For Example 1, we need, for each

set $S \in \text{Sets}$ and a monoid $M \in (\text{Mon}, (M, e, \cdot))$

$$\begin{array}{ccc}
 \mathcal{S}^* & \longrightarrow & M \quad \text{in Mon.} \\
 & & \text{(a monoid hom.)} \\
 \hline
 \mathcal{S} & \longrightarrow & \underline{UM} \quad \text{in Sets.} \\
 & & \uparrow \\
 & & \text{the underlying} \\
 & & \text{set of } M
 \end{array}$$

[↓] Given a monoid hom.

$$f: \mathcal{S}^* \longrightarrow M,$$

we construct

$$\begin{array}{ccccc}
 \mathcal{S} & \hookrightarrow & \mathcal{S}^* & \xrightarrow{f} & M \\
 \mathcal{X} & \hookrightarrow & \mathcal{X} & \xrightarrow{\quad} & f(\mathcal{X}) \\
 & & \uparrow & & \\
 & & \text{a word of length 1} & &
 \end{array}$$

[↑] Given $\mathcal{S} \xrightarrow{g} UM$ in Sets, we define

$$\begin{array}{ccc}
 \mathcal{S}^* & \xrightarrow{\hat{g}} & M \\
 a_1 a_2 \dots a_n & \longmapsto & \left(\dots (g(a_1) \cdot g(a_2)) \dots \right) g(a_n) \\
 (a_i \in \mathcal{S}) & & \\
 \varepsilon & \longmapsto & e
 \end{array}$$

M's monoid structure

S^* : the free monoid over S
 S a set

Let's make S into a monoid ...
 with the least possible effort

Exercise

- Show that $[I]$ and $[J]$ are mutually inverse
- Show the naturality of $[I]$ or $[J]$

Def. A natural isomorphism is

a natural transformation α , as in

$$\begin{array}{ccc} & F & \\ & \xrightarrow{\quad} & \\ C & \xrightarrow{\alpha} & D \\ & \xleftarrow{\quad} & \\ & G & \end{array}$$

where $\alpha_x : Fx \rightarrow Gx$ is an isomorphism in D , for each $x \in C$.

Lemma. In the above situation, let

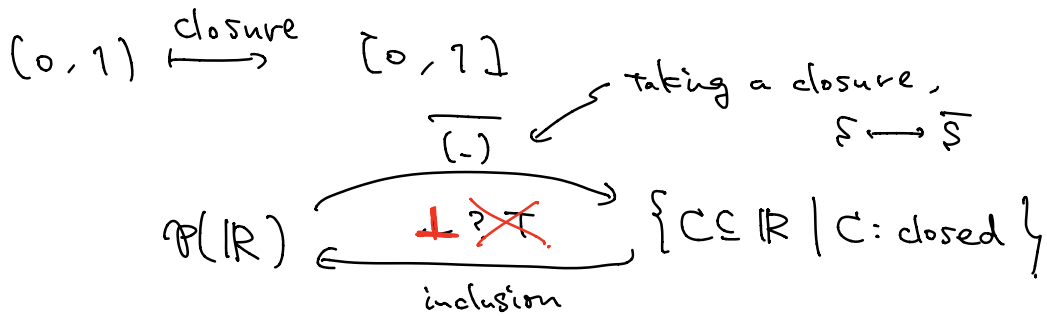
$$\begin{array}{ccc} & F & \\ & \xrightarrow{\quad} & \\ C & \xrightarrow{\alpha^{-1}} & D \\ & \xleftarrow{\quad} & \\ & G & \end{array}$$

be defined by

$(\alpha^{-1})_x := (\alpha_x)^{-1} : Gx \rightarrow Fx$

← the inverse of α_x in D

Then the naturality of α^{-1} is automatic.



$S \in \mathbb{R} \quad C: \text{closed}$

$$\checkmark \frac{\bar{S} \subseteq C}{S \subseteq C} \quad \Leftrightarrow \quad \frac{C \subseteq \bar{S}}{C \subseteq S} \quad \begin{matrix} C = [0, 1] \\ S = (0, 1) \end{matrix}$$

Prop. (ct'd)

In the previous situation,

[\Leftarrow] We can construct (co) units by:

The adjoint transpose of id_{FX}

$$\begin{array}{c} X \xrightarrow{\eta_x} GFX \text{ in } \mathcal{C} \\ \hline FX \xrightarrow{\text{id}_{FX}} FX \text{ in } \mathcal{D} \end{array}$$

$$\left[\begin{array}{l} \eta: \text{id} \Rightarrow GF \\ \varepsilon: FG \Rightarrow \text{id} \end{array} \right]$$

$$\left[\begin{array}{l} \eta_x: X \rightarrow GFX \text{ } \begin{matrix} \mathcal{C} \\ x \in \mathcal{C} \end{matrix} \\ \varepsilon_A: FGA \rightarrow A \text{ } \begin{matrix} \mathcal{D} \\ A \in \mathcal{D} \end{matrix} \end{array} \right]$$

$$\begin{array}{c} FGA \xrightarrow{\varepsilon_A} A \text{ in } \mathcal{D} \\ \hline GA \xrightarrow{\text{id}_{GA}} GA \text{ in } \mathcal{C} \end{array}$$

Rep.]

[We then have to check that these ✓
 η, ε satisfy the required eq. axioms...
 and that η, ε are indeed natural

Rep. 1a

\square \Rightarrow Given F, G, η, ε ,
 how do we get
 $\mathcal{C}(X, GA) \cong \mathcal{D}(FX, A)$
 there is, $\frac{X \rightarrow GA}{FX \rightarrow A} ?$

$$\begin{array}{c}
 \text{[I]} \quad X \xrightarrow{f} GA \text{ in } \mathcal{C} \\
 \hline
 \begin{array}{ccc}
 & F: \mathcal{C} \rightarrow \mathcal{D} & \\
 FX \xrightarrow{Ff} FGA \text{ in } \mathcal{D} & & FGA \xrightarrow{\varepsilon_A} A \text{ in } \mathcal{D} \\
 \hline
 FX \xrightarrow{\varepsilon_A \circ (Ff)} A \text{ in } \mathcal{D} & &
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 \text{[II]} \quad FX \xrightarrow{g} A \text{ in } \mathcal{D} \\
 \hline
 \begin{array}{ccc}
 X \xrightarrow{\eta_X} GFX \text{ in } \mathcal{C} & & GFX \xrightarrow{Gg} GA \text{ in } \mathcal{C} \\
 \hline
 X \xrightarrow{(Gg) \circ \eta_X} GA \text{ in } \mathcal{C} & &
 \end{array}
 \end{array}$$

Rep. 2b

[Now we have to check that
[[I] [II] are mutually inverse ...

[and the naturality of $[\mathbb{I}] [\mathbb{J}]$ Rep. 2a

