MSCS 31/10/2016

adjunction

$\Downarrow$

an extension

Mac Lane

"Every notion (concept) is a Kan extension"

Def.

An *adjunction* from $\mathbb{C}$ to $\mathbb{D}$

consists of

$$F : \mathbb{C} \longrightarrow \mathbb{D}$$

$$G \quad \mathbb{D} \longrightarrow \mathbb{C}$$

$$\text{unit} \quad \eta : \mathrm{id}_{\mathbb{C}} \Longrightarrow GF$$

$$\text{counit} \quad \varepsilon : FG \Longrightarrow \mathrm{id}_{\mathbb{D}}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{C} \arrow[d, bend right, "F"'] \arrow[d, phantom, "\dashv"] \\ \mathbb{D} \arrow[u, bend right, "G"'] \end{tikzcd}$$

F is a left adjoint to G

G is a right adjoint to F

s.t.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F"] \arrow[dr, "\mathrm{id}_{\mathbb{C}}"'] & \mathbb{D} \arrow[d, "G"] \arrow[dr, "\mathrm{id}_{\mathbb{D}}"] \arrow[dl, phantom, "\eta \Rightarrow"] & \\ & \mathbb{C} \arrow[r, "F"'] \arrow[ur, phantom, "\varepsilon \Rightarrow"] & \mathbb{D} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_F"] & \mathbb{D} \end{tikzcd}$$

pasting diagrams

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "G"] \arrow[dr, "\mathrm{id}_{\mathbb{D}}"'] & \mathbb{C} \arrow[d, "F"] \arrow[dr, "\mathrm{id}_{\mathbb{C}}"] \arrow[dl, phantom, "\Leftarrow\varepsilon"] & \\
& \mathbb{D} \arrow[r, "G"'] \arrow[ur, phantom, "\Leftarrow\eta"] & \mathbb{C}
\end{tikzcd}$$

$\|$

$$\begin{tikzcd}
\mathbb{D} \arrow[r, bend left, "G"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_G"] & \mathbb{C}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "G"] \arrow[rrr, bend left=50, "G"] \arrow[rr, bend right=40, "\mathrm{id}"'] \arrow[rrr, bend right=70, "G"'] & \mathbb{C} \arrow[r, "F"'] \arrow[rr, bend left, "\mathrm{id}_{\mathbb{C}}"] \arrow[d, phantom, "\Downarrow\varepsilon"] & \mathbb{D} \arrow[r, "G"'] \arrow[u, phantom, "\Downarrow\eta"] & \mathbb{C} \\
& {} & {} &
\end{tikzcd}$$

$$\begin{tikzcd}
\cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\mathrm{id}_G"] & \cdot \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\eta"] & \cdot
\end{tikzcd}$$

$$= \left(G = \mathrm{id}_{\mathbb{C}} \circ G \xRightarrow{\eta \circ \mathrm{id}_G} G \circ F \circ G \xRightarrow{\mathrm{id}_G \circ \varepsilon} G \circ \mathrm{id}_{\mathbb{D}} = G \right)$$

$$= (\mathrm{id}_G \circ \varepsilon) \bullet (\eta \circ \mathrm{id}_G)$$

horizontal composition

vert. comp.

$=$

$\boxed{\text{required}}$

$$\begin{tikzcd}
\mathbb{D} \arrow[r, bend left, "G"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_c"] & \mathbb{C}
\end{tikzcd}$$

Prop.

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

F, G, η, ε form an adjunction (for some η, ε)

$\iff$ • For each $X \in \mathbb{C}$, $Y \in \mathbb{D}$

we have $\{X \to GY \text{ in } \mathbb{C}\}$ $\qquad$ $\{X \to Y \text{ in } \mathbb{D}\}$

$$\underline{\mathbb{C}(X, GY)} \underset{\varphi}{\overset{\cong}{\longrightarrow}} \underline{\mathbb{D}(FX, Y)}$$

"homsets"

• Furthermore the correspondence φ is natural in $X \in \mathbb{C}$ and $Y \in \mathbb{D}$

φ arrows = "homomorphisms"

$$\frac{X \longrightarrow GY \quad \text{in } \mathbb{C}}{FX \longrightarrow Y \quad \text{in } \mathbb{D}}$$

"is equivalent to"
"is in a bijective correspondence to"

$\underline{\mathbb{C}}$ $\qquad$ $\underline{\mathbb{D}}$ $\qquad$ $\underline{\text{Sets}}$

$$\begin{tikzcd} X & Y \arrow[d, "g"] & \mathbb{C}(X, GY) \arrow[r, "\cong", "\varphi_{X,Y}"'] \arrow[d, "{\mathbb{C}(f, Gg) = (Gg)\circ(\_)\circ f}"'] & \mathbb{D}(FX, Y) \arrow[d, "{\mathbb{D}(Ff, g)}"] \\ X' \arrow[u, "f"] & Y' & \mathbb{C}(X', GY') \arrow[r, "\cong", "\varphi_{X',Y'}"'] & \mathbb{D}(FX', Y') \end{tikzcd}$$

This diagram commutes.

$$\mathbf{Sets}\ \big]\ \mathbb{C}(X, GY) \longrightarrow \mathbb{C}(X', GY')$$

$$\left(X \xrightarrow{h} GY \ \text{ in } \mathbb{C} \right) \longmapsto \left(X' \xrightarrow{f} X \xrightarrow{h} GY \xrightarrow{Gg} GY' \right)$$

Examples

1. Σ : a set.

The <u>free monoid</u> over Σ is given by $\Sigma^* = \coprod_{n \in \mathbb{N}} \Sigma^n$

Then

$$\mathbf{Mon} \begin{cases} \underline{\text{obj.}} & \text{monoid} \\ \underline{\text{arr.}} & \text{monoid hom.} \end{cases}$$

$$\begin{array}{c} (-)^* \\ \left(\begin{array}{c}\text{the free} \\ \text{monoid} \\ \text{functor}\end{array}\right) \end{array} \ \begin{tikzcd} \mathbf{Mon} \arrow[d, bend left, "U"] \arrow[d, phantom, "\dashv"] \\ \mathbf{Sets} \arrow[u, bend left] \end{tikzcd} \ U \left(\begin{array}{c}\text{forgetful} \\ \text{functor}\end{array}\right)$$

2.

$$\begin{tikzcd} L \arrow[r, bend left, "\alpha"] \arrow[r, phantom, "\perp"] & I \arrow[l, bend left, "\gamma"] \end{tikzcd}$$

$L = \mathcal{P}(\mathbb{N})$, $I = \left\{ [l, r] \;\middle|\; \begin{array}{l} l \in \mathbb{N} \\ r \in \mathbb{N} \cup \{\infty\} \end{array} \right\} \cup \{\emptyset\}$

3. $f : X \to Y$ be a function.

$$\begin{tikzcd}[column sep=huge]
2^X \arrow[r, bend left=60, "\coprod_f"] \arrow[r, bend right=60, "\Pi_f"'] & 2^Y \arrow[l, "2^f"' description] \arrow[l, phantom, shift right=4ex, "\perp"] \arrow[l, phantom, shift left=4ex, "\perp"]
\end{tikzcd}$$

- $2^f : 2^Y \longrightarrow 2^X$
  $Y' \mapsto \{x \in X \mid f(x) \in Y'\}$
  inverse image

- $\coprod_f : 2^X \longrightarrow 2^Y$
  $X' \mapsto \{f(x) \mid x \in X'\}$
  direct image

- $\Pi_f : 2^X \longrightarrow 2^Y$

$$X' \longmapsto \{y \mid \forall x \in X. (f(x) = y \Rightarrow x \in X')\}$$

You have to show

$$\coprod_f X' \xrightarrow{\subseteq} Y' \text{ in } 2^Y$$

$$X' \xrightarrow{\subseteq} (2^f)(Y') \text{ in } 2^X$$

For each $X' \in 2^X$, $Y' \in 2^Y$

That is,

$$\coprod_f X' \subseteq Y'$$

$$\iff X' \subseteq (2^f)(Y')$$

For Example 1, we need, for each set $S \in \mathbf{Sets}$ and a monoid $M \in \mathbf{Mon}$, $(M, e, \cdot)$

$$\frac{S^* \longrightarrow M \quad \text{in } \mathbf{Mon}. \quad \text{(a monoid hom.)}}{S \longrightarrow \underline{UM} \quad \text{in } \mathbf{Sets}.}$$

the underlying set of M

[I] Given a monoid hom.

$$f : S^* \longrightarrow M,$$

we construct

$$\begin{array}{ccccc} S & \hookrightarrow & S^* & \xrightarrow{f} & M \\ x & \longmapsto & x & \longmapsto & f(x) \end{array}$$

a word of length 1

[II] Given $S \xrightarrow{g} UM$ in Sets, we define

M's monoid structure

$$\begin{array}{ccc} S^* & \xrightarrow{\hat{g}} & M \\ a_1 a_2 \ldots a_n & \longmapsto & (\cdots(g(a_1) \cdot g(a_2)) \cdot \ \cdots \ ) g(a_n) \\ (a_i \in S) & & \\ \varepsilon & \longmapsto & e \end{array}$$

S^* : the free monoid over S (a set)

Let's make S into a monoid ... with the least possible effort

Exercise

- Show that $[\downarrow]$ $[\uparrow]$ are mutually inverse
- Show the naturality of $[\downarrow]$ or $[\uparrow]$

Def. A natural isomorphism is a natural transformation α, as in

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathbb{D} \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd},$$

where $\alpha_X : FX \longrightarrow GX$ $(\mathbb{D})$ is an isomorphism in $\mathbb{D}$, for each $X \in \mathbb{C}$.

Lem. In the above situation, let

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathbb{D} \arrow[from=U, to=D, phantom, "\Uparrow\alpha^{-1}"] \end{tikzcd}$$

be defined by

the inverse of α_X in $\mathbb{D}$

$$(\alpha^{-1})_X := (\alpha_X)^{-1} : GX \longrightarrow FX$$

Then the naturality of α^{-1} is automatic.

$[0,1) \xmapsto{\text{closure}} [0,1]$

taking a closure,
$S \longmapsto \overline{S}$

$$\begin{tikzcd}
\mathcal{P}(\mathbb{R}) \arrow[r, bend left, "\overline{(-)}"] \arrow[r, phantom, "\perp ? \cancel{\top}"] & \{C \subseteq \mathbb{R} \mid C : \text{closed}\} \arrow[l, bend left, hook', "\text{inclusion}"]
\end{tikzcd}$$

$S \subseteq \mathbb{R}$ $\quad$ C : closed

$$\frac{\overline{S} \subseteq C}{S \subseteq C} \quad \text{or} \quad \cancel{\frac{C \subseteq \overline{S}}{C \subseteq S}}$$

$C = [0,1]$
$S = (0,1)$

Prop. (cont'd)

In the previous situation,

[$\Leftarrow$] We can construct (co)units by:

$$\begin{bmatrix} \eta : \mathrm{id} \Rightarrow GF \\ \varepsilon : FG \Rightarrow \mathrm{id}. \end{bmatrix}$$

$\{\eta_X : X \longrightarrow GFX\}_{X \in \mathbb{C}}$

$\{\varepsilon_A : FGA \longrightarrow A\}_{A \in \mathbb{D}}$

The adjoint transpose of id_{FX}

$$\frac{X \xrightarrow{\eta_X} GFX \text{ in } \mathbb{C}}{FX \xrightarrow{\mathrm{id}_{FX}} FX \text{ in } \mathbb{D}}$$

$$\frac{FGA \xrightarrow{\varepsilon_A} A \text{ in } \mathbb{D}}{GA \xrightarrow{\mathrm{id}_{GA}} GA \text{ in } \mathbb{C}}$$

Rep. 7_b

We then have to check that these

η, ε satisfy the required eq. axioms...
and that η, ε are indeed natural

Rep. 1a

[$\Rightarrow$]. $\boxed{Q}$ Given F, G, η, ε,

how do we get

$$\mathbb{C}(X, GA) \cong \mathbb{D}(FX, A)$$

that is, $\dfrac{X \longrightarrow GA}{FX \longrightarrow A}$?

[J] $\dfrac{X \xrightarrow{f} GA \text{ in } \mathbb{C}}{\dfrac{FX \xrightarrow{Ff} FGA \text{ in } \mathbb{D} \qquad FGA \xrightarrow{\varepsilon_A} A \text{ in } \mathbb{D}}{FX \xrightarrow{\varepsilon_A \circ (Ff)} A \quad \text{in } \mathbb{D}}}$ $F : \mathbb{C} \to \mathbb{D}$

[I] $\dfrac{FX \xrightarrow{g} A \text{ in } \mathbb{D}}{\dfrac{X \xrightarrow{\eta_X} GFX \text{ in } \mathbb{C} \qquad GFX \xrightarrow{Gg} GA \text{ in } \mathbb{C}}{X \xrightarrow{(Gg) \circ \eta_X} GA \quad \text{in } \mathbb{C}.}}$

Rep. 2b

Now we have to check that

[J] [I] are mutually inverse ...

and the naturality of $[I][J]$ Rep.2a