

Report Assignments

(Due : the beginning of the lect.)
Mon. 2016.11.7

1a) In the last proof, in the construction
of $\eta: \text{id} \Rightarrow GF$

from $(C(x, GA) \xrightarrow[\varphi_{x,A}]{} D(Fx, A))_{x \in C, A \in A}$

prove that the family

$$(\eta_x: x \longrightarrow GFX)_{x \in C}$$

indeed satisfies the naturality requirement,
and hence constitutes a natural transformation.

Proof.) We have to show the following
commutativity.

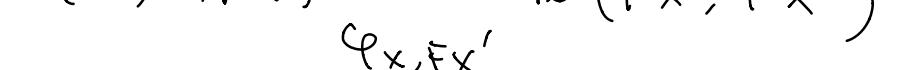
$$\begin{array}{ccc}
 \text{C} & & \text{C} \\
 x & & x \xrightarrow{\eta_x} Gfx \\
 \downarrow f & & \downarrow f \\
 x' & & x' \xrightarrow{\eta_{x'}} Gfx'
 \end{array}$$

~~aim~~

Towards this goal we use the following naturality of the bijective correspondence.

$$\begin{array}{ccc}
 \text{C} & \xrightarrow{\quad \text{Sects} \quad} & \text{FD}(fx, Fx) \\
 \downarrow id_x & \downarrow Ff & \downarrow \varphi_{x,fx} \\
 x & \xrightarrow{\quad C(id_x, GFF) \quad} & \text{FD}(id_{fx}, FF) \\
 x' & \xrightarrow{\quad C(x, GFx') \quad} & \text{FD}(Fx, Fx') \\
 & \xrightarrow{\quad \varphi_{x,fx'} \quad} &
 \end{array}$$

// naturality
 of φ



Therefore

Sets

$$\begin{array}{c} \mathbb{C}(x, GFx) \leftarrow D(Fx, Fx) \\ \Downarrow \quad \Downarrow \\ \mathbb{C}_{x, Fx}^{-1}(id_{Fx}) \leftarrow \mathbb{D}_{x, Fx}(id_{Fx}) \end{array}$$

\Downarrow

$$\begin{array}{c} \mathbb{C}_{x, Fx}^{-1}(id_{Fx}) \leftarrow \mathbb{D}(Fx, Fx) \\ \Downarrow \quad \Downarrow \\ id_{Fx} : Fx \rightarrow Fx \end{array}$$

\Downarrow

$$\begin{array}{c} \mathbb{C}_{x, Fx}^{-1}(id_{Fx}) \leftarrow \mathbb{D}(id, FF) \\ \Downarrow \quad \Downarrow \\ (GFF) \circ \gamma_X \end{array}$$

\Downarrow

$$\begin{array}{c} \mathbb{C}_{x, Fx}'(FF) \leftarrow id \\ \Downarrow \quad \Downarrow \\ id_{Fx} \end{array}$$

\Downarrow

$$\mathbb{D}(id, FF)$$

Thus we've shown

$$(GFF) \circ \gamma_x = \varphi_{x,fx'}^{-1} (FF). \quad \text{---(1)}$$

Here we invoke on another naturality square:

<u>C</u>	<u>D</u>	<u>Sets</u>
x'	FX'	$\mathbb{C}(x', GFX') \xrightarrow[\cong]{\varphi} \mathbb{D}(FX', FX')$
$\uparrow f$	$\downarrow id_{FX'}$	$\downarrow \mathbb{C}(f, id)$
x	FX'	$\mathbb{C}(x, GFX') \xrightarrow[\cong]{\varphi_{x,fx'}} \mathbb{D}(Fx, Fx')$

Hence

$$\begin{array}{ccc}
 \gamma_{x'} & \xleftarrow[\cong]{\varphi^{-1}} & id_{FX'} \\
 \downarrow \mathbb{C}(f, id) & & \downarrow \mathbb{C}(FF, id) \\
 \gamma_{x' \circ f} & \parallel & \\
 & & \varphi_{x,fx'}^{-1} (FF) \xleftarrow[\cong]{\varphi^{-1}} FF
 \end{array}$$

Thus we have

$$\gamma_{x' \circ f} = \varphi_{x,fx'}^{-1} (FF) \quad \text{---(2)}$$

Combining ①, ② yield

$$(GFF) \circ \gamma_x = \gamma_{x'} \circ f$$

that is the desired commutativity. □

1b Show that the same γ, ε as above satisfy the equational axioms in the def. of adjunction.

in
1a

Proof. We shall first show

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow \gamma \quad G \downarrow & \nearrow id & \\ \mathcal{C} & \xrightarrow{\varepsilon} & \mathcal{D} \\ \downarrow F & & \end{array} \stackrel{\text{aim}}{=} \begin{array}{ccc} \mathcal{C} & \xrightleftharpoons[F]{\text{II}} & \mathcal{D} \\ \downarrow & & \end{array} - \text{(*)}$$

that is, for each $x \in \mathcal{C}$,

$$FX \xrightarrow{F\gamma_x} FGFX \xrightarrow{id} FX$$

for $\alpha: F \Rightarrow G$, a nat. trans.
 $\alpha_x: FX \Rightarrow GX$ is its
X-component

Convince yourself
that $\xrightarrow{F\gamma_x} \downarrow \varepsilon_{Fx}$ is indeed
the X-component
of the LHS of (*)

Since

$$\mathcal{D}(FX, FX) \xrightarrow[\cong]{\varphi_{FX, FX}} \mathcal{C}(x, GFX)$$

is an isomorphism in Sets (i.e. a bijection),

it suffices to show that

$$\varphi_{x, Fx}^{-1} (\text{id}_{Fx}) = \varphi_{x, Fx}^{-1} (\varepsilon_{Fx} \circ F\eta_x) \quad (**)$$

Now

$$\begin{aligned}
 (\text{LHS}) &= \left(x \xrightarrow{\eta_x} GFx \right) \quad (\text{by def. of } \eta) \\
 (\text{RHS}) &= \underbrace{\varphi_{x, Fx}^{-1} \left(Fx \xrightarrow{F\eta_x} FGFx \xrightarrow{\varepsilon_{Fx}} FX \right)}_{\substack{\text{def. of } \varphi^{-1} \\ \text{def. of } \varepsilon}} \\
 &= \overline{\eta_x} \quad \boxed{\substack{\mathbb{C}(x, GFx) \\ \uparrow \varphi^{-1} \\ \mathbb{D}(Fx, Fx)}}
 \end{aligned}$$

$$\varphi_{GFx, Fx}(\text{id}_{GFx})$$

naturality of φ , that is,

$$\begin{array}{ccc}
 \mathbb{C} & \xrightarrow{\text{sets}} & \mathbb{C}(GFx, GFx) \xrightarrow{\varphi_{GFx, Fx}} \mathbb{D}(FGFx, Fx) \\
 \uparrow \eta_x & & \downarrow \mathbb{C}(\eta_x, \text{id}) \qquad \mathbb{C}(F\eta_x, \text{id}) \\
 x & & \mathbb{C}(x, GFx) \xrightarrow{\varphi_{x, Fx}} \mathbb{D}(Fx, Fx)
 \end{array}$$

from which it follows that

$$\text{id}_{GFx} \mapsto \varphi(\text{id}_{GFx}) \stackrel{\text{def. of } \varepsilon}{=} \varepsilon_{Fx}$$

$$\boxed{\mathbb{C}(\eta_x, \text{id}) \qquad \mathbb{C}(F\eta_x, \text{id})}$$

$$\eta_x = \varphi^{-1}(\varepsilon_{Fx} \circ F\eta_x) \xleftarrow{\varphi_{x, Fx}^{-1}} \varepsilon_{Fx} \circ F\eta_x$$

claim

Therefore we have $(\text{LHS}) = (\text{RHS})$ in $(**)$.

The other eq. axiom is proved similarly.

2a Let $(F, G, \eta, \varepsilon)$ form an adjunction and define

$$\varphi_{x,A} : \mathcal{C}(x, GA) \longrightarrow \mathcal{D}(Fx, A)$$

by

$$(x \xrightarrow{\eta} \overset{L}{\mathcal{C}}_{GA}) \longmapsto (Fx \xrightarrow{F\eta} FGA \xrightarrow{\varepsilon_A} A)$$

Then φ is natural in X and A .

Proof.

We have to show the following.

C D Sets

$$\begin{array}{ccc} x & A & \mathcal{C}(x, GA) \xrightarrow{\varphi} \mathcal{D}(Fx, A) \\ \uparrow f & \downarrow g & \downarrow \\ x' & A' & \mathcal{C}(x', GA') \xrightarrow{\varphi} \mathcal{D}(Fx', A') \end{array}$$

\cong_{aim} $\downarrow \mathcal{D}(Ff, g)$

That is, for each $f : x \rightarrow GA$,

$$\begin{array}{ccc} f & \longmapsto & \varepsilon_A \circ Ff \\ & \downarrow & \downarrow \\ & & g \circ \varepsilon_{A'} \circ Ff \circ FFf \\ & & \cong_{\text{aim}} \\ (Gg \circ f \circ f) & \longmapsto & (Fx' \xrightarrow{F(Gg \circ f)} FGA' \xrightarrow{\varepsilon_{A'}} A') \end{array}$$

Now we have

$$\begin{array}{ccccc}
 Fx' & \xrightarrow{Ff} & Fx & \xrightarrow{Fr} & FGA \\
 & & \downarrow FGg & \nearrow \varepsilon_A & \text{naturality} \\
 & & & & \downarrow g \\
 & & FGA' & \xrightarrow{\varepsilon_{A'}} & A'
 \end{array}$$

Therefore

$$g \circ \varepsilon_A \circ Fr \circ Ff = \varepsilon_{A'} \circ FGg \circ Fr \circ Ff$$

and this proves the claim. \square

[2b] In the above setting of [2a],

$\varphi_{x,A}$ is an isomorphism.

Proof. We define

$$\psi_{x,A} : D(Fx, A) \longrightarrow C(x, GA)$$

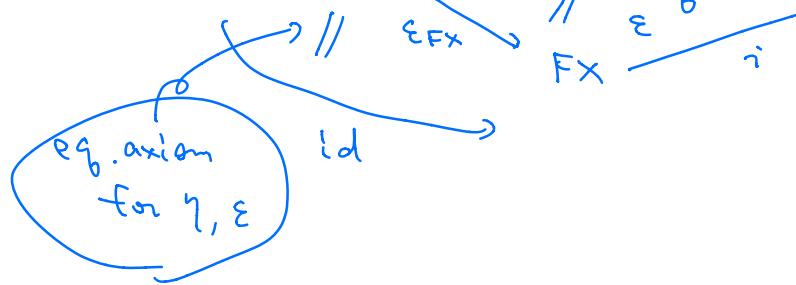
$$(Fx \xrightarrow{i} A) \longmapsto (x \xrightarrow{\gamma_x} GFx \xrightarrow{G_i} GA).$$

We shall show that $\varphi_{x,A}$ and $\psi_{x,A}$ are inverse to each other.

$$\boxed{\varphi \circ \psi = id}$$

$$(\varphi \circ \psi)(i) = \varphi(G_i \circ \gamma_x)$$

$$= \left(Fx \xrightarrow{F\eta_x} FGfx \xrightarrow{FGi} FGA \xrightarrow{\epsilon_A} A \right)$$



$$= i.$$

$\boxed{\psi \circ \varphi = \text{id}}$

Similar. \square