MSCS Report 3.

1. Formulate and prove the following statement.

A right adjoint preserves limits.

$\boxed{\text{Formulation.}}$ (Only a few students did this properly!)

Suppose

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

is an adjunction, and

$$D : \mathbb{I} \longrightarrow \mathbb{D}$$

be a functor (diagram). Assume there exists a limit $(\lim D \in \mathbb{D}, (\pi_i : \lim D \to D_i)_{i \in \mathbb{I}})$ of D in $\mathbb{D}$.

Then, G preserves the limit $(\lim D, (\pi_i : \lim D \to D_i)_{i\in\mathbb{I}})$ in the sense that

$$(G(\lim D) \in \mathbb{C}, (G\pi_i : G(\lim D) \longrightarrow GD_i)_{i\in\mathbb{I}})$$

is a limit of $G \circ D : \mathbb{I} \to \mathbb{C}$ in $\mathbb{C}$.

Remark

Most of the answers formulated the condition

"G preserves the limit $(\lim D, (\pi_i : \lim D \to D_i)_{i\in\mathbb{I}})$"

as

"$G(\lim D) \cong \lim (G \circ D)$".

But you also need the condition on cones! (That is, a limiting cone $(\pi_i)_{i\in\mathbb{I}}$ is mapped to a limiting cone $(G\pi_i)_{i\in\mathbb{I}}$ by G.)

Proof

First, $(G(\lim D) \in \mathbb{C}, (G\pi_i : G(\lim D) \to GD_i)_{i \in \mathbb{I}})$ is indeed a cone over $G \circ D : \mathbb{I} \to \mathbb{C}$ thanks to the functoriality of G.

Now suppose $(X \in \mathbb{C}, (\rho_i : X \to GD_i)_{i \in \mathbb{I}})$ is a cone over $G \circ D$.

[<u>Aim</u>: There exists a unique morphism $m : X \to G(\lim D)$ in $\mathbb{C}$ s.t. for all $i \in \mathbb{I}$,

$$\rho_i = G\pi_i \circ m.$$]

$\mathbb{C}$

$$\begin{tikzcd}
 & G(\lim D) \arrow[dl, "G\pi_i"'] \arrow[dr, "G\pi_{i'}"] & \\
GD_i \arrow[rr, "GDu"] & & GD_{i'} \\
 & X \arrow[ul, "\rho_i"] \arrow[ur, "\rho_{i'}"'] \arrow[uu, dashed, bend left=60, "m"] &
\end{tikzcd}$$

<u>By the adjointness</u> $F \dashv G$, we obtain a pair

$$(FX \in \mathbb{D}, (\hat{\rho}_i : FX \to D_i)_{i \in \mathbb{I}})$$

where $\hat{\rho}_i$ corresponds to ρ_i. This pair is a cone over $D : \mathbb{I} \to \mathbb{D}$ by the naturality of adjoint isomorphism (Check!).

So by the universality of limits, we obtain a morphism

$$n : FX \to \lim D \text{ in } \mathbb{C}$$

which is unique one s.t.

$$\hat{\rho}_i = \pi_i \circ n \quad (\forall i \in \mathbb{I}).$$

$\mathbb{D}$

$$\begin{tikzcd}
 & \lim D \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_{i'}"] & \\
D_i \arrow[rr, "Du"] & & D_{i'} \\
 & FX \arrow[ul, "\hat{\rho}_i"] \arrow[ur, "\hat{\rho}_{i'}"'] \arrow[uu, dashed, bend left=60, "n"] &
\end{tikzcd}$$

Using the adjointness again, we obtain a morphism

$$m : X \to G(\lim D) \text{ in } \mathbb{C}$$

corresponding to n, and by the naturality of adjoint iso. this m is the unique one s.t. $\rho_i = G\pi_i \circ m$ $(\forall i \in \mathbb{I})$. (Check!) $\square$

2. In an adjunction

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

G is faithful iff every component of the counit ε is epi.

$\boxed{\text{Proof}}$

First let us restate the conditions.

$\boxed{G \text{ is faithful}}$

$$\iff \forall D, D' \in \mathbb{D},\ G_{D,D'} : \mathbb{D}(D, D') \to \mathbb{C}(GD, GD')$$
$$(D \xrightarrow{f} D') \mapsto (GD \xrightarrow{Gf} GD')$$

is injective.

$\boxed{\text{Every component of } \varepsilon \text{ is epi.}}$

$$\iff \forall D \in \mathbb{D},\ \varepsilon_D : FGD \to D \text{ is epi.}$$

$$\iff \forall D, D' \in \mathbb{D},\ \mathbb{D}(\varepsilon_D, D') : \mathbb{D}(D, D') \to \mathbb{D}(FGD, D')$$
$$(D \xrightarrow{f} D') \mapsto (FGD \xrightarrow{\varepsilon_D} D \xrightarrow{f} D')$$

is injective.

Now recall that the adjunction $F \dashv G$ defines $\forall C \in \mathbb{C}, \forall D' \in \mathbb{D}$, a bijection

$$\varphi_{C,D'} : \mathbb{C}(C, GD') \xrightarrow{\cong} \mathbb{D}(FC, D')$$
$$(C \xrightarrow{g} GD') \mapsto (FC \xrightarrow{Fg} FGD' \xrightarrow{\varepsilon_{D'}} D').$$

In particular, (when $C = GD$) $\forall D, D' \in \mathbb{D}$, a bijection

$$\varphi_{GD,D'} : \mathbb{C}(GD, GD') \xrightarrow{\cong} \mathbb{D}(FGD, D')$$
$$(GD \xrightarrow{g} GD') \mapsto (FGD \xrightarrow{Fg} FGD' \xrightarrow{\varepsilon_{D'}} D').$$

The key observation is :

For all $D, D' \in \mathbb{D}$, the following commutes :

$$\begin{tikzcd}
 & \mathbb{C}(GD, GD') \arrow[d, "\varphi_{GD,D'}", "\cong"'] \\
\mathbb{D}(D, D') \arrow[ur, "G_{D,D'}"] \arrow[r, "\mathbb{D}(\varepsilon_D, D')"'] & \mathbb{D}(FGD, D')
\end{tikzcd}$$

This is because

$$\begin{tikzcd}
 & (GD \xrightarrow{Gf} GD') \arrow[d, mapsto, "\varphi_{GD,D'}"] \\
 & (FGD \xrightarrow{FGf} FGD' \xrightarrow{\varepsilon_{D'}} D') \arrow[d, equal, "\text{(by naturality of } \varepsilon)"] \\
(D \xrightarrow{f} D') \arrow[uur, mapsto, "G_{D,D'}"] \arrow[r, mapsto, "\mathbb{D}(\varepsilon_D, D')"'] & (FGD \xrightarrow{\varepsilon_D} D \xrightarrow{f} D')
\end{tikzcd}$$

Because $\varphi_{GD,D'}$ is bijective, it follows that

$G_{D,D'}$ is injective iff $\mathbb{D}(\varepsilon_D, D')$ is injective.

Allowing $D, D' \in \mathbb{D}$ to vary, we obtain the required result. $\square$