

MSCS Report 3.

①

1. Formulate and prove the following statement.

A right adjoint preserves limits.

Formulation. (Only a few students did this properly!)

Suppose

$$\mathbb{C} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} \mathbb{D}$$

is an adjunction, and

$$D: \mathbb{I} \longrightarrow \mathbb{D}$$

be a functor (diagram). Assume there exists a

limit $(\lim D \in \mathbb{D}, (\pi_i: \lim D \rightarrow D_i)_{i \in \mathbb{I}})$ of D in \mathbb{D} .

Then, G preserves the limit $(\lim D, (\pi_i: \lim D \rightarrow D_i)_{i \in \mathbb{I}})$ in the sense that

$$(G(\lim D) \in \mathbb{C}, (G\pi_i: G(\lim D) \rightarrow G D_i)_{i \in \mathbb{I}})$$

is a limit of $G \circ D: \mathbb{I} \rightarrow \mathbb{C}$ in \mathbb{C} .

Remark.

Most of the answers formulated the condition

" G preserves the limit $(\lim D, (\pi_i: \lim D \rightarrow D_i)_{i \in \mathbb{I}})$ "

as

$$G(\lim D) \cong \lim (G \circ D)$$

But you also need the condition on cones! (That is, a

limiting cone $(\pi_i)_{i \in \mathbb{I}}$ is mapped to a limiting cone $(G\pi_i)_{i \in \mathbb{I}}$ by G .)

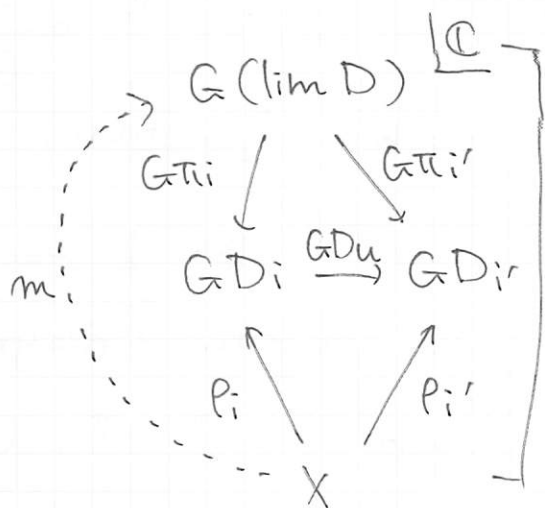
Proof

(2)

First, $(G(\lim D) \in \mathcal{C}, (G\pi_i : G(\lim D) \rightarrow G D_i)_{i \in I})$ is indeed a cone over $G \circ D : I \rightarrow \mathcal{C}$ thanks to the functoriality of G .

Now suppose $(X \in \mathcal{C}, (p_i : X \rightarrow G D_i)_{i \in I})$ is a cone over $G \circ D$.

Aim: There exists a unique morphism $m : X \rightarrow G(\lim D)$ in \mathcal{C} s.t. for all $i \in I$, $p_i = G\pi_i \circ m$.



By the adjointness $F \dashv G$, we obtain a pair

$$(FX \in \mathcal{D}, (\hat{p}_i : FX \rightarrow D_i)_{i \in I})$$

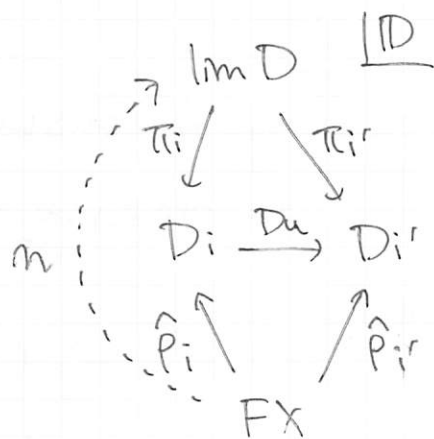
where \hat{p}_i corresponds to p_i . This pair is a cone over $D : I \rightarrow \mathcal{D}$ by the naturality of adjoint isomorphism (check!).

So by the universality of limits,

we obtain a morphism $n : FX \rightarrow \lim D$ in \mathcal{C}

which is unique one s.t.

$$\hat{p}_i = \pi_i \circ n \quad (\forall i \in I).$$



Using the adjointness again, we obtain a morphism

$$m : X \rightarrow G(\lim D) \text{ in } \mathcal{C}$$

corresponding to n , and by the naturality of adjoint iso.

this m is the unique one s.t. $p_i = G\pi_i \circ m$ ($\forall i \in I$). (check!) \square

2. In an adjunction

$$\mathbb{C} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} \mathbb{D}$$

(3)

G is faithful iff every component of the counit ε is epi.

Proof

First let us restate the conditions.

G is faithful

$$\Leftrightarrow \forall D, D' \in \mathbb{D}, G_{D,D'} : \mathbb{D}(D, D') \rightarrow \mathbb{C}(GD, GD') \\ (D \xrightarrow{f} D') \mapsto (GD \xrightarrow{Gf} GD')$$

is injective.

Every component of ε is epi.

$$\Leftrightarrow \forall D \in \mathbb{D}, \varepsilon_D : FG D \rightarrow D \text{ is epi.}$$

$$\Leftrightarrow \forall D, D' \in \mathbb{D}, \mathbb{D}(\varepsilon_D, D') : \mathbb{D}(D, D') \rightarrow \mathbb{D}(FG D, D') \\ (D \xrightarrow{f} D') \mapsto (FG D \xrightarrow{\varepsilon_D} D \xrightarrow{f} D')$$

is injective.

Now recall that the adjunction $F \dashv G$ defines

$\forall C \in \mathbb{C}, \forall D' \in \mathbb{D}$, a bijection

$$\varphi_{C,D'} : \mathbb{C}(C, GD') \xrightarrow{\cong} \mathbb{D}(FC, D')$$

$$(C \xrightarrow{g} GD') \mapsto (FC \xrightarrow{Fg} FG D' \xrightarrow{\varepsilon_{D'}} D')$$

In particular, (when $C = GD$)

$\forall D, D' \in \mathbb{D}$, a bijection

$$\varphi_{GD,D'} : \mathbb{C}(GD, GD') \xrightarrow{\cong} \mathbb{D}(FG D, D')$$

$$(GD \xrightarrow{g} GD') \mapsto (FG D \xrightarrow{Fg} FG D' \xrightarrow{\varepsilon_{D'}} D').$$

The key observation is :

For all $D, D' \in \mathcal{D}$, the following commutes :

$$\begin{array}{ccc}
 & G_{D,D'} & \mathbb{C}(G_D, G_{D'}) \\
 \mathbb{D}(D, D') & \nearrow & \cong \downarrow \varphi_{G_D, D'} \\
 & \mathbb{D}(\varepsilon_D, D') & \mathbb{D}(FG_D, D')
 \end{array}$$

This is because

$$\begin{array}{ccc}
 & G_{D,D'} & (G_D \xrightarrow{Gf} G_{D'}) \\
 (D \xrightarrow{f} D') & \nearrow & \downarrow \varphi_{G_D, D'} \\
 & \mathbb{D}(\varepsilon_D, D') & (FG_D \xrightarrow{FGf} FG_{D'} \xrightarrow{\varepsilon_{D'}} D') \\
 & & \parallel \text{ (by naturality of } \varepsilon \text{)} \\
 & & (FG_D \xrightarrow{\varepsilon_D} D \xrightarrow{f} D')
 \end{array}$$

Because $\varphi_{G_D, D'}$ is bijective, it follows that

$G_{D,D'}$ is injective iff $\mathbb{D}(\varepsilon_D, D')$ is injective.

Allowing $D, D' \in \mathcal{D}$ to vary, we obtain the required result. \square