Coalgebraic Trace Semantics for Probabilistic Systems

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Introduction The authors introduced in [1] the technique of *coalgebraic trace semantics* for the powerset monad $\mathcal{P}$. There the initial F-algebra $\alpha : FA \stackrel{\cong}{\Rightarrow} A$ in **Sets** gives rise to the final coalgebra in the category **Rel** of sets and relations.

The category **Rel** is also described as the Kleisli category $\mathbf{Sets}_{\mathcal{P}}$ for the powerset monad $\mathcal{P}$. In this work we show that the analogous result holds for what we call the *distribution monad* $\mathcal{D}$, instead of $\mathcal{P}$. The monad $\mathcal{D}$ is defined as $\mathcal{D}X = \{d : X \to [0,1] \mid \sum_{x \in X} d(x) \le 1\}$. It has the same monad structure as the (standard) distribution monad.

The proof for $\mathcal{D}$ does not follow trivially from the one for $\mathcal{P}$. However we notice similar constructions they have in common. Hopefully the current work will cast a light over the essence underlying these two different settings, and lead to a result with more generality (e.g. for monads other than $\mathcal{P}, \mathcal{D}$).

Technical result and example The endofunctors F that we consider are constructed inductively by: $F ::= \mathrm{id} \mid \Sigma \mid F \times F \mid \coprod_{i \in I} F_i$, where Σ is a constant functor. This family of functors is large enough to contain many interesting examples, including the list functor $X^* = \coprod_{n<\omega} X^n$. Notice that a functor F thus constructed preserves ω-colimits: hence the initial F-algebra is obtained as the colimit $\{\alpha_n : F^n 0 \to A\}$ of the initial sequence.

Theorem 1 *Let $\alpha : FA \stackrel{\cong}{\Rightarrow} A$ be the initial F-algebra, and $c : X \to \mathcal{D}FX$ be a coalgebra (both in* **Sets***). Then there exists a unique arrow trace_c that makes the following diagram in $\mathbf{Sets}_{\mathcal{D}}$ commute.*

$$\begin{array}{ccc} FX & \xrightarrow{F\mathsf{trace}_c} & FA \\ {\scriptstyle c}\uparrow & & \uparrow{\scriptstyle \eta_{FA} \circ \alpha^{-1}} \\ X & \xrightarrow[\mathsf{trace}_c]{} & A \end{array} \tag{1}$$

We sketch the construction of the map trace_c. In the first place, to lift a functor F in **Sets** to a functor in $\mathbf{Sets}_{\mathcal{D}}$, we use a distributive law $\lambda : F\mathcal{D} \Rightarrow \mathcal{D}F$. This is constructed inductively on F.[1] With λ we can define the *n-th composition*

[1] In fact, for a functor F under consideration we can construct a distributive law $FT \Rightarrow TF$ for any *commutative* monad T ($\mathcal{D}$ is commutative). Most notably, for $F = F_1 \times F_2$ we use the *double strength* [2] of T.

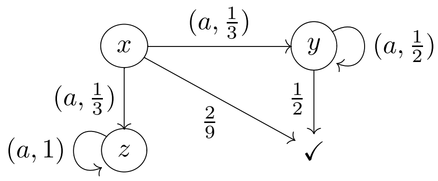
$c^n : X \to \mathcal{D}F^n X$ of a coalgebra c, whose one step corresponds to n successive steps of c.

We use a construction which might be called the "contravariant distribution functor": for a mono $m : X \rightarrowtail Y$, the map $\overline{\mathcal{D}}m : \mathcal{D}Y \to \mathcal{D}X$ is defined by $[(\overline{\mathcal{D}}m)(d)](x) = (d \circ m)(x)$. The *n-th trace* $\mathsf{trace}_c^n : X \to \mathcal{D}A$ is defined as the following composite, where $?_X : 0 \to X$ is the unique arrow.

$$X \xrightarrow{c^n} \mathcal{D}F^n X \xrightarrow{\overline{\mathcal{D}}F^n ?_X} \mathcal{D}F^n 0 \xrightarrow{\mathcal{D}\alpha_n} \mathcal{D}A$$

The n-th trace gives the distribution over the behavior which terminate within n steps. Now the trace $\mathsf{trace}_c : X \to \mathcal{D}A$ is defined as the limit of the n-th trace, that is, for $a \in \operatorname{Im} \alpha_n$, $[\mathsf{trace}_c(x)](a) = [\mathsf{trace}_c^n(x)](a)$. We have shown that trace_c is indeed well-defined, and that it is the unique arrow which makes the diagram (1) commute.

Example 2 (Lists) Consider the functor $F = 1 + \Sigma \times -$. The initial F-algebra $[\mathtt{nil}, \mathtt{cons}] : 1 + \Sigma \times \Sigma^* \xrightarrow{\cong} \Sigma^*$ consists of the *lists* over Σ. The following is an example of a coalgebra $c : X \to \mathcal{D}FX$.



The behavior of the state x is: it transits to y outputting a with the probability of 1/3, the same to z, and it terminates with the probability of 2/9. The remaining 1/9 is best understood as the probability x gets into *deadlock*.

By Theorem 1 we obtain $\mathsf{trace}_c : X \to \mathcal{D}\Sigma^*$ via finality. The distribution $\mathsf{trace}_c(x)$ is such that: $\langle\rangle \mapsto 2/9$ and $a^n \mapsto 1/(3 \cdot 2^n)$. Out of the remaining 4/9, 1/9 is the probability that x gets into deadlock at the first transition, and 1/3 is the probability that x goes to z and keep outputting a without termination (*livelock*). The n-th trace trace_c^n is the restriction of trace_c to the lists of at most length n.

References

1. I. Hasuo and B. Jacobs. Context-free languages via coalgebraic trace semantics. In *Conference on Algebra and Coalgebra in Computer Science (CALCO 2005)*, to appear.
2. B. Jacobs. Semantics of weakening and contraction. *Ann. Pure & Appl. Logic*, 69(1):73–106, 1994.