Coalgebraic components in a many-sorted microcosm

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Outline

Hughes' **arrow**, **Freyd category**, … from functional programming

Universal coalgebra in CS

components (state-based machines)	as	coalgebras
their behaviors	by	final coalgebra
calculus on them	as	algebraic structure

SOS by bialgebras

compositionality results

The Microcosm Principle

[Baez-Dolan][Hasuo-Jacobs-Sokolova, FoSSaCS'08]

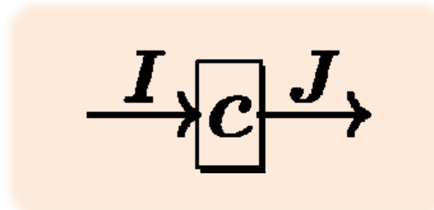
what's new: many-sorted, pseudo algebra

Behavioral view on component calculi

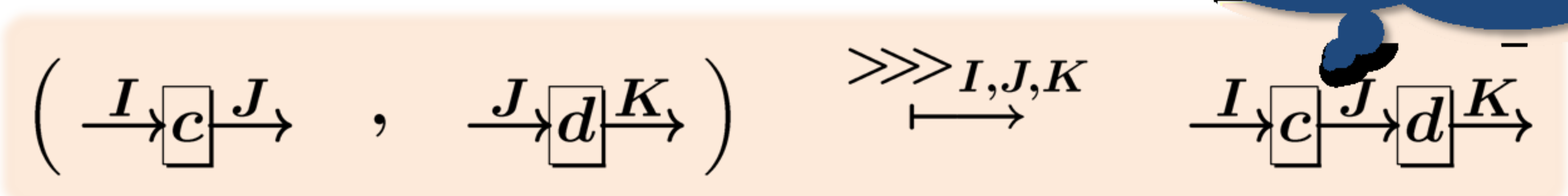
Part 1

Component calculus

- **component**
  - state-based machine
  - with I/O interfaces



- **component calculus**
  - algebraic structure on components



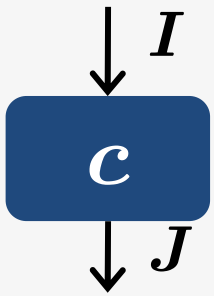
  - compose components to build a bigger system

Component calculus: background

- **Modular design**
  - Against fast-growing complexity of systems
  - Brings order to otherwise messed-up design process

- **Middle-ware layer**
  - You won’t code everything from scratch, but
  - buy software components from other vendors,
  - which you compose

Components as coalgebras [Barbosa, PhD thesis]



as

$$\big(T(J \times X)\big)^I \overset{c}{\longleftarrow} X$$

in **Sets**

$$I \times X \xrightarrow{c} T(J \times X)$$

T: a monad for effect (cf. func. programming)

$\mathcal{P}$ (powerset)	non-determinism
$1 +$ _	exception
$(S \times$ _$)^S$	global state
$\mathcal{D}X$ = { d: $X \to [0,1]$ | $\sum_x d(x) = 1$ }	probability

Component calculus = algebra on $\mathbf{Coalg}_F$

binary operation

$$\left(\begin{array}{c} (J \times X)^I \\ c \uparrow \\ X \end{array} , \begin{array}{c} (K \times Y)^J \\ d \uparrow \\ Y \end{array} \right) \longmapsto \begin{array}{c} (K \times (X \times Y))^I \\ c \ggg d \uparrow \\ X \times Y \end{array}$$

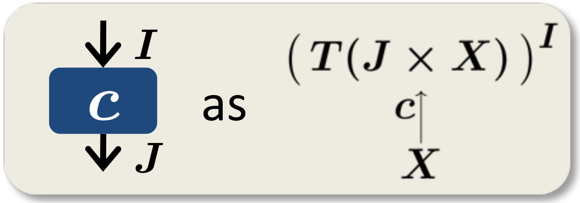
- no effect, for simplicity
- signature: $F_{I,J} = (J \times \_)^I$

➔ ***Mealy machines***

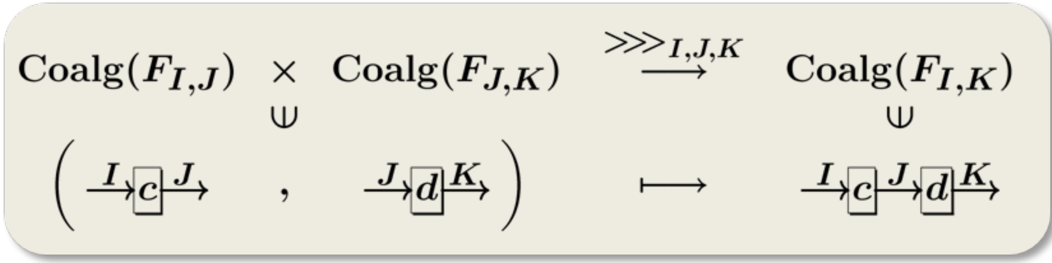
Behavioral view

<u>Summary</u>

- component as coalgebra

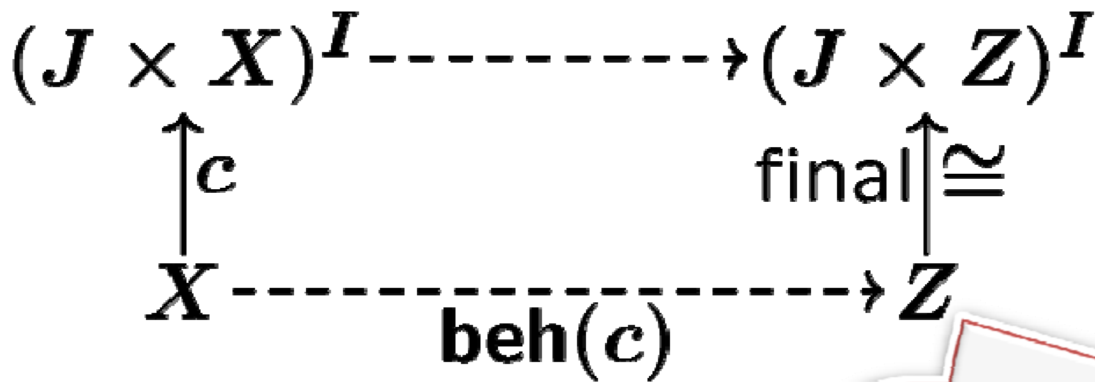


- component calculus as algebra on $\mathbf{Coalg}_F$



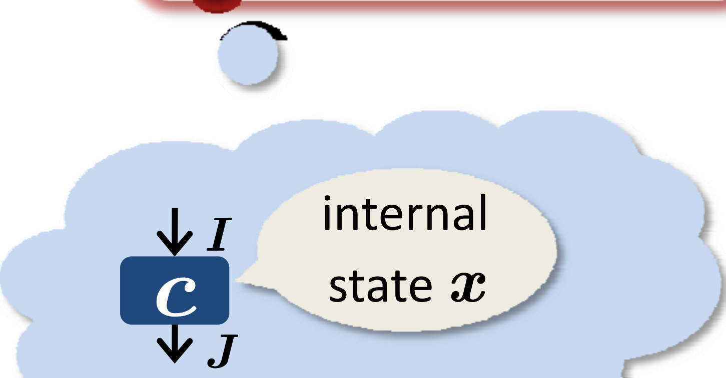
- **Behavior** of components?
  - compositionality of calculus
  - coalgebraic view

Behavior by coinduction

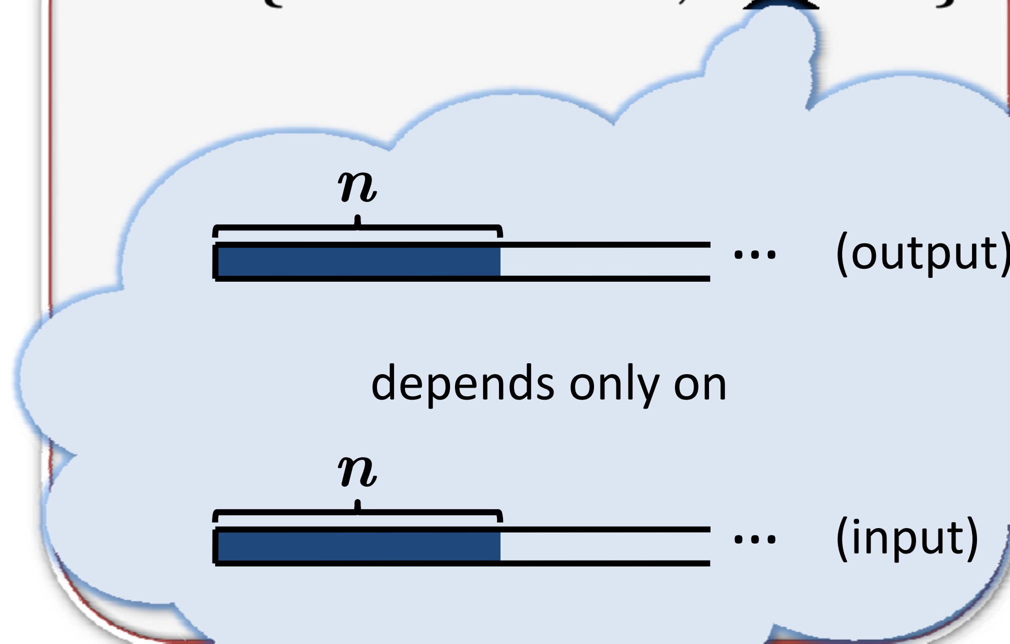


That is,

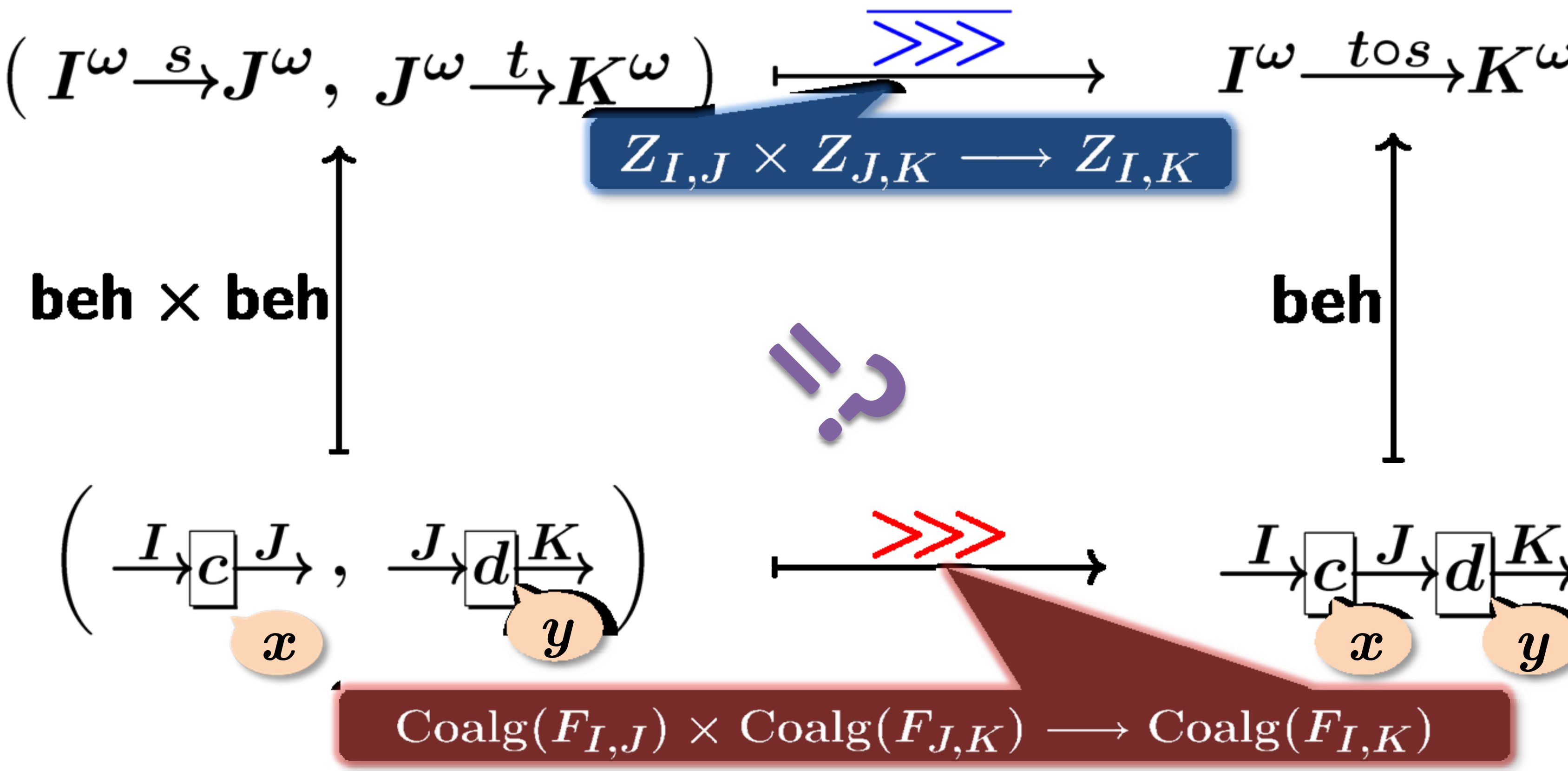
$$\mathbf{beh}(c)(x) = (I^\omega \to J^\omega)$$



$$Z = \{t : I^\omega \to J^\omega, \text{ causal}\}$$



Another “sequential composition”



- Two “sequential composition” operators
- Are they “compatible”? ➔ **compositionality**

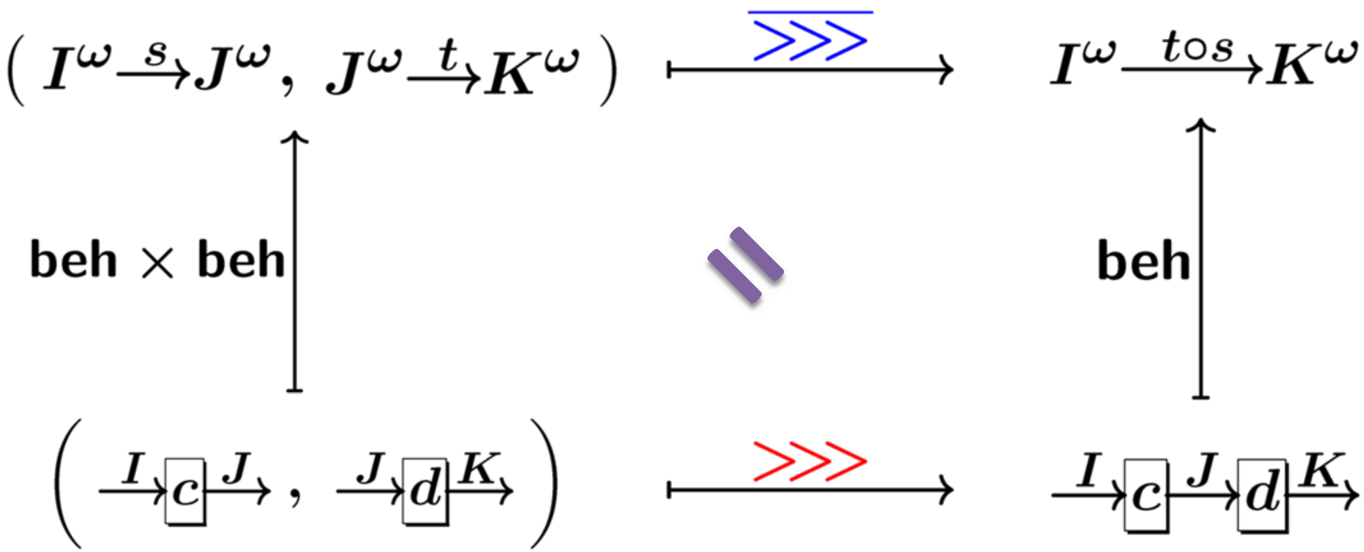
One arises from the other

$$\begin{pmatrix} F_{I,J} Z_{I,J} \\ \uparrow \zeta_{I,J} \\ Z_{I,J} \end{pmatrix} \ggg \begin{pmatrix} F_{J,K} Z_{J,K} \\ \uparrow \zeta_{J,K} \\ Z_{J,K} \end{pmatrix}$$

$$\begin{array}{ccc} Z_{I,J} \times Z_{J,K} & \overset{}{\underset{\ggg}{\dashrightarrow}} & Z_{I,K} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (\, I^{\omega} \xrightarrow{s} J^{\omega},\; J^{\omega} \xrightarrow{t} K^{\omega} \,) & \longmapsto & (\, I^{\omega} \xrightarrow{t \circ s} K^{\omega} \,) \end{array}$$

by coinduction **definition principle**

Compositionality



- i.e.

$$\mathsf{beh}\begin{pmatrix} F_{I,J}X & & F_{J,K}Y \\ \uparrow c & \ggg & \uparrow d \\ X & & Y \end{pmatrix} = \mathsf{beh}\begin{pmatrix} F_{I,J}X \\ \uparrow c \\ X \end{pmatrix} \overline{\ggg}\ \mathsf{beh}\begin{pmatrix} F_{J,K}Y \\ \uparrow d \\ Y \end{pmatrix}$$

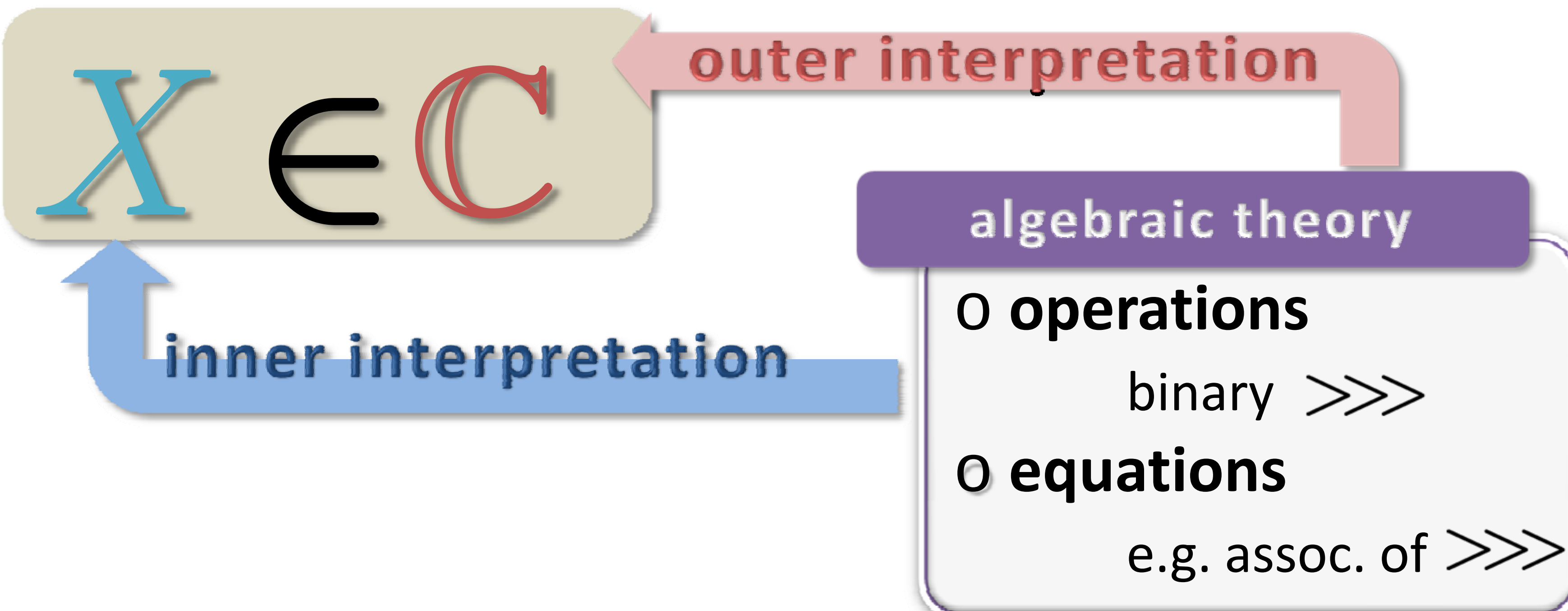
- by coinduction **proof principle**

Nested algebraic structure:
the microcosm principle

$$\begin{array}{ccccc} \mathbf{Coalg}_F & \times & \mathbf{Coalg}_F & \xrightarrow{>\!>\!>} & \mathbf{Coalg}_F \\ Z & \times & Z & \xrightarrow{>\!>\!>} & Z \end{array}$$

with

$$\begin{pmatrix} FZ \\ \cong \uparrow \text{final} \\ Z \end{pmatrix} \in \mathbf{Coalg}_F$$



Microcosm in macrocosm

We name this principle the **microcosm principle**, after the theory, common in pre-modern correlative cosmologies, that every feature of the microcosm (e.g. the human soul) corresponds to some feature of the macrocosm.

John Baez & James Dolan
Higher-Dimensional Algebra III:
n-Categories and the Algebra of Opetopes
Adv. Math. 1998

The microcosm principle: a retrospective

Baez-Dolan
Adv. Math. 1998

- formalization for alg. str. as **opetopes**
- for use in homotopy theory, n-categories

Hasuo-Jacobs-Sokolova
FoSSaCS 2008

- formalization for alg. str. as **Lawvere theories**
- example: parallel composition of coalgebras

Current work
what's new?

- many-sorted → components (varying I/O types)
- pseudo algebraic structure

Hasuo 2009, preprint

- for full GSOS → "2-dimensional GSOS"

Main result: compositionality

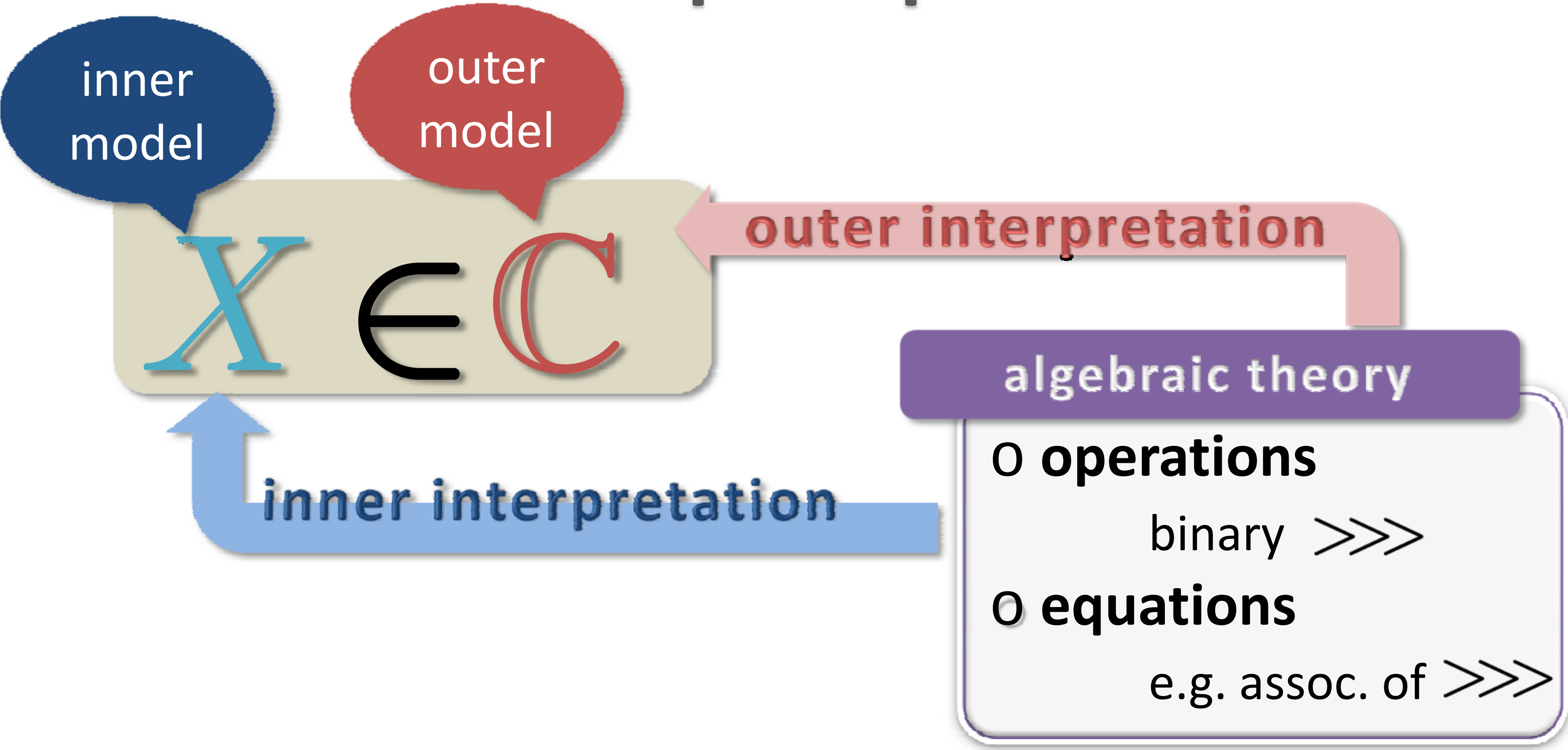
We study three component calculi: $\mathbf{PLTh}$, $\mathbf{ArrTh}$, $\mathbf{MArrTh}$. For each of them:

1. We introduce algebraic structure on $\mathbf{Coalg}_F$
   - such as $\ggg$
2. from which algebraic structure on Z canonically arises
   - such as $\overline{\ggg}$
   - by coinduction (def. principle)
3. relating the two, we have the compositionality property
   - by coinduction (proof principle)

The microcosm principle

Part 2

The microcosm principle

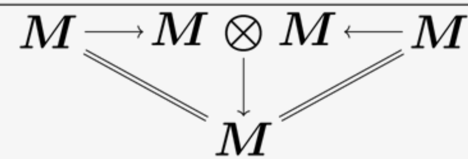
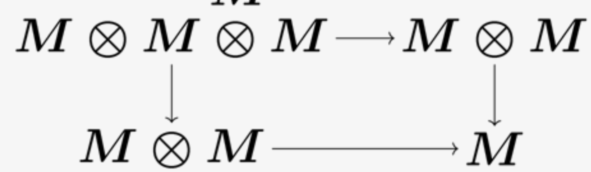


Examples

- components and their behaviors
- monoid in a monoidal category

Microcosm principle in [MacLane, CWM]

monoid in a monoidal category

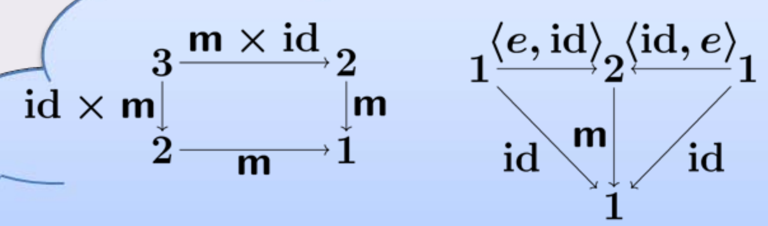
monoidal cat. $\mathbb{C}$		monoid $M \in \mathbb{C}$
$\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ $I \in \mathbb{C}$	mult. unit	$M \otimes M \stackrel{m}{\to} M$ $I \stackrel{e}{\to} M$
$I \otimes X \cong X \cong X \otimes I$	unit law	
$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$	assoc. law	

Lawvere theory $\mathbb{L}$

a category representing an algebraic theory

Lawvere theory

algebraic theory	as category $\mathbb{L}$
operations **m** (binary) **e** (nullary)	as **arrows** $2 \xrightarrow{\mathbf{m}} 1$ $0 \xrightarrow{\mathbf{e}} 1$
equations assoc. of **m** unit law	as **commuting diagrams**



Models for Lawvere theory $\mathbb{L}$

Standard: set-theoretic model

- a set with $\mathbb{L}$-structure ➔ **$\mathbb{L}$-set**

$$\mathbb{L} \xrightarrow{\ X\ } \mathbf{Sets}$$

(product-preserving)

$$\begin{array}{c} 2 \\ \big\downarrow \mathbf{m} \\ 1 \end{array} \quad \longmapsto \quad \begin{array}{c} X^2 \\ \big\downarrow [\mathbf{m}] \\ X \end{array} =: \bullet$$

binary opr. on X

$$\begin{array}{ccc} 3 & \xrightarrow{\mathbf{m}\times\mathrm{id}} & 2 \\ {\scriptstyle \mathrm{id}\times\mathbf{m}}\big\downarrow & \circlearrowright & \big\downarrow{\scriptstyle \mathbf{m}} \\ 2 & \xrightarrow[\mathbf{m}]{} & 1 \end{array} \quad \longmapsto \quad \begin{array}{ccc} X^3 & \xrightarrow{\bullet\times\mathrm{id}} & X^2 \\ {\scriptstyle \mathrm{id}\times\bullet}\big\downarrow & \circlearrowright & \big\downarrow{\scriptstyle \bullet} \\ X^2 & \xrightarrow[\bullet]{} & X \end{array}$$

associativity

what about nested models?

$X \in \mathbb{C}$

Outer model: $\mathbb{L}$-category

outer model

- a **category** with $\mathbb{L}$-structure: **$\mathbb{L}$-category**

$$\mathbb{L} \xrightarrow{\ \mathbb{C}\ } \mathbf{CAT} \qquad \text{(product-preserving)}$$

$$\begin{array}{c} \mathbf{2} \\ \big\downarrow \mathbf{m} \\ \mathbf{1} \end{array} \quad \longmapsto \quad \begin{array}{c} \mathbb{C}^{\mathbf{2}} \\ \big\downarrow [\mathbf{m}] \\ \mathbb{C} \end{array} = \otimes$$

NB. This works only for **strict** algebraic structure

Standard: set-theoretic model

- a set with $\mathbb{L}$-structure ➔ **$\mathbb{L}$-set**

$$\mathbb{L} \xrightarrow{\ X\ } \mathrm{Sets} \qquad \text{product-preserving}$$

$$\begin{array}{c} \mathbf{2} \\ \big\downarrow \mathbf{m} \\ \mathbf{1} \end{array} \quad \longmapsto \quad \begin{array}{c} X^{\mathbf{2}} \\ \big\downarrow [\mathbf{m}] \\ X \end{array}$$

binary opr. on X

Inner model: $\mathbb{L}$-object [cf. Benabou]

Definition

Given an $\mathbb{L}$-category $\mathbb{C}$,

an **$\mathbb{L}$-object** X in it

is a lax natural transformation compatible with products.

inner alg. str. by mediating 2-cells

components

$X_0 : 1 \xrightarrow{!} 1$

$X_1 : 1 \xrightarrow{X} \mathbb{C}$

$X_2 : 1 \xrightarrow{(X,X)} \mathbb{C}^2$

$\vdots$

X: carrier obj.

$$\frac{X \in \mathbb{C}}{1 \xrightarrow{X} \mathbb{C}}$$

lax naturality

In $\mathbb{L}$: $\mathbf{2} \xrightarrow{\mathsf{m}} \mathbf{1}$

In CAT:

$$\begin{array}{ccc} 1 & \xrightarrow{(X,X)} & \mathbb{C}^2 \\ \| & & \downarrow \otimes \\ 1 & \xrightarrow{X} & \mathbb{C} \end{array}$$

$$\overline{\overline{}}$$

$X \otimes X \xrightarrow{X_{\mathsf{m}}} X$ in $\mathbb{C}$

Compositionality theorem

Assume

- $\mathbb{C}$ is an $\mathbb{L}$-category
- $F : \mathbb{C} \to \mathbb{C}$ is a lax $\mathbb{L}$-functor
- there is a final coalgebra $Z \to FZ$

1. $\mathbf{Coalg}_F$ is an $\mathbb{L}$-category
2. $Z \to FZ$ is an $\mathbb{L}$-object
3. the **behavior** functor

$$\mathbf{Coalg}_F \xrightarrow{\mathbf{beh}} \mathbb{C}/Z$$

$$\begin{pmatrix} FX \\ c\uparrow \\ X \end{pmatrix} \longmapsto \left(X \xrightarrow{\mathbf{beh}(c)} Z\right)$$

[by coinduction

$$\begin{array}{ccc} FX & \dashrightarrow & FZ \\ c\uparrow & & \cong\uparrow\text{final} \\ X & \underset{\mathbf{beh}(c)}{\dashrightarrow} & Z \end{array}$$

]

is a (strict) $\mathbb{L}$-functor

What's new (1): many-sorted

$$\mathrm{Coalg}(F_{I,J}) \times \mathrm{Coalg}(F_{J,K}) \xrightarrow{\ggg_{I,J,K}} \mathrm{Coalg}(F_{I,K})$$

$$\left(\xrightarrow{I}\boxed{c}\xrightarrow{J} \;,\; \xrightarrow{J}\boxed{d}\xrightarrow{K} \right) \longmapsto \xrightarrow{I}\boxed{c}\xrightarrow{J}\boxed{d}\xrightarrow{K}$$

$$Z_{I,J} \times Z_{J,K} \xrightarrow{\overline{\ggg}_{I,J,K}} Z_{I,K}$$

sorts: (I,J), (J,K), etc.

The specification:

$$(I, J) \times (J, K) \xrightarrow{\ggg_{I,J,K}} (I, K) \quad \text{in } \mathbb{L}$$

arity: formal product of sorts

one-sorted case:

$$2 = 1 \times 1 \xrightarrow{\mathbf{m}} 1 \quad \text{in } \mathbb{L}$$

What's new (2): pseudo alg. str.

Equations hold
not up-to equalities,
but up-to *isomorphisms*

$$\mathbb{L} \longrightarrow \mathbf{CAT}$$

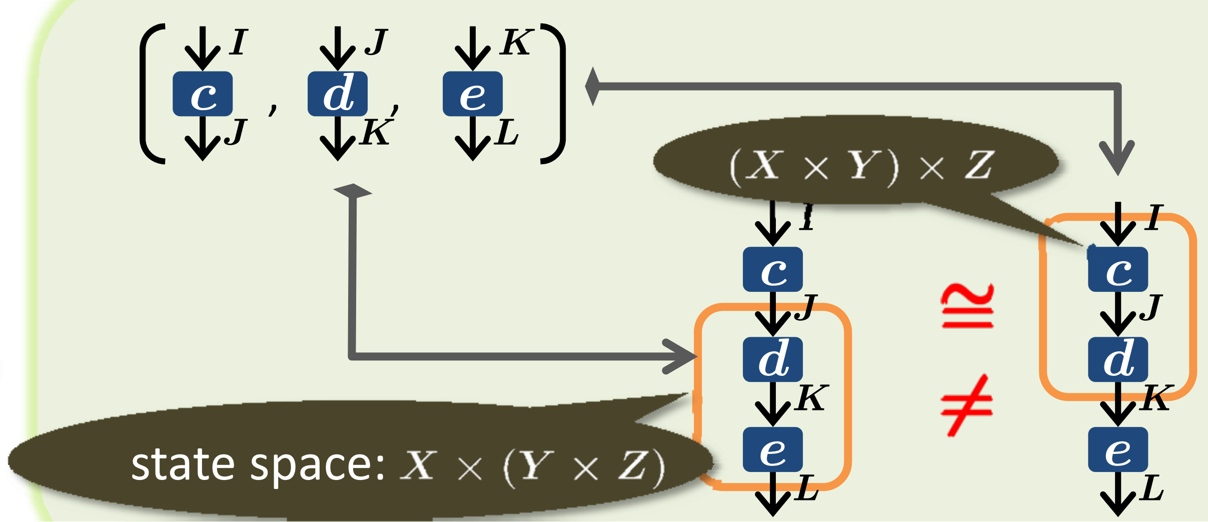
$$\begin{array}{ccc}
(I,J)\times(J,K)\times(K,L) & \xrightarrow{\ggg_{I,J,K}\times \mathrm{id}} & (I,K)\times(K,L) \\
\downarrow {\scriptstyle \mathrm{id}\times\ggg_{J,K,L}} & // & \downarrow {\scriptstyle \ggg_{I,K,L}} \\
(I,J)\times(J,L) & \xrightarrow[\ggg_{I,J,L}]{} & (I,L)
\end{array}
\longmapsto
\begin{array}{ccc}
\mathrm{Coalg}_{F_{I,J}}\times\mathrm{Coalg}_{F_{J,K}}\times\mathrm{Coalg}_{F_{K,L}} & \xrightarrow{\ggg_{I,J,K}\times \mathrm{id}} & \mathrm{Coalg}_{F_{I,K}}\times\mathrm{Coalg}_{F_{K,L}} \\
\downarrow {\scriptstyle \mathrm{id}\times\ggg_{J,K,L}} & \swarrow\!\!\cong & \downarrow {\scriptstyle \ggg_{I,K,L}} \\
\mathrm{Coalg}_{F_{I,J}}\times\mathrm{Coalg}_{F_{J,L}} & \xrightarrow[\ggg_{I,J,L}]{} & \mathrm{Coalg}_{F_{I,L}}
\end{array}$$

assoc.
of $\ggg$

cf. monoidal category

$$X\otimes(Y\otimes Z)\cong(X\otimes Y)\otimes Z$$

- coherence ☹



Functorial semantics

outer model

- a **category** with $\mathbb{L}$-structure: **$\mathbb{L}$-category**

$$\mathbb{L} \xrightarrow{\ \mathbb{C}\ } \mathbf{CAT} \qquad \text{(product-preserving)}$$

$$\begin{array}{c} \mathbf{2} \\ \big\downarrow \mathbf{m} \\ \mathbf{1} \end{array} \quad \longmapsto \quad \begin{array}{c} \mathbb{C}^{2} \\ \big\downarrow [\mathbf{m}] \\ \mathbb{C} \end{array} = \otimes$$

NB. This works only for **strict** algebraic structure

Standard: set-theoretic model

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$$\mathbb{L} \xrightarrow{\ X\ } \mathbf{Sets} \qquad \text{product-preserving}$$

$$\begin{array}{c} \mathbf{2} \\ \big\downarrow \mathbf{m} \\ \mathbf{1} \end{array} \quad \longmapsto \quad \begin{array}{c} X^{2} \\ \big\downarrow [\mathbf{m}] \\ X \end{array}$$

binary opr. on X

Formalizing pseudo alg. str.

- **Lawvere 2-theory** [Power, Lack]
  - explicit isomorphisms, explicit coherence
    - theory of **monoids**:

$$\begin{array}{ccc} \mathbf{3} & \xrightarrow{\mathbf{m}\times\mathbf{id}} & \mathbf{2} \\ {\scriptstyle \mathbf{id}\times\mathbf{m}}\downarrow & /\!/ & \downarrow{\scriptstyle \mathbf{m}} \\ \mathbf{2} & \xrightarrow[\mathbf{m}]{} & \mathbf{1} \end{array} \quad \text{in } \mathbb{L}$$

    - theory of **monoidal categories**:

$$\begin{array}{ccc} \mathbf{3} & \xrightarrow{\mathbf{m}\times\mathbf{id}} & \mathbf{2} \\ {\scriptstyle \mathbf{id}\times\mathbf{m}}\downarrow & \Nearrow \cong & \downarrow{\scriptstyle \mathbf{m}} \\ \mathbf{2} & \xrightarrow[\mathbf{m}]{} & \mathbf{1} \end{array} \quad \text{in } \mathbb{L}$$

  - "2-functorial semantics"
    - model = category with $\mathbb{L}$-structure
    - it is a **product-preserving 2-functor** $\mathbb{L} \to \mathbf{CAT}$
  - not quite a suitable solution for us…

Formalizing pseudo alg. str.

Hasuo, preprint 2009
available on the web

- **Pseudo functorial semantics**

Definition

An $\mathbb{L}$-*category* is a finite-product preserving **pseudo functor** $\mathbb{L} \to \mathbf{CAT}$

– Segal took the same approach in TQFT, **topological quantum field theory**
  - but for **monoidal** (not cartesian/Lawvere) theory $\mathbb{L}$

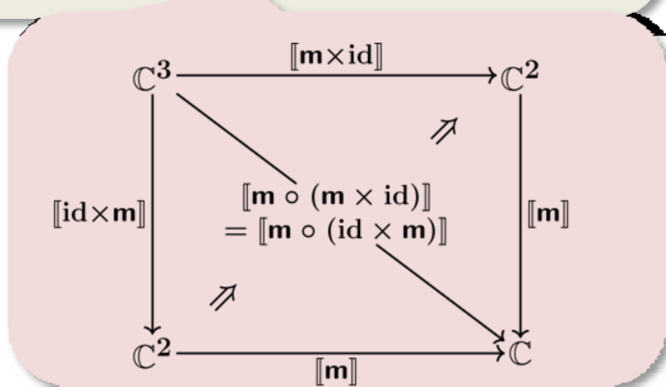
Pseudo functorial semantics

Definition

An $\mathbb{L}$-*category* is a finite-product preserving **pseudo functor** $\mathbb{L} \to \mathbf{CAT}$

- pseudo functor preserves compositions/identities up-to iso:

$$F(g \circ f) \overset{\cong}{\Rightarrow} Fg \circ Ff, \qquad F(\mathrm{id}) \overset{\cong}{\Rightarrow} \mathrm{id}$$



Pseudo functorial semantics

Definition

An $\mathbb{L}$-*category* is a finite-product preserving **pseudo functor** $\mathbb{L} \to \mathbf{CAT}$

subtle, needs fine-tuning

Definition 4.1 ($\mathbb{L}$-category) An $\mathbb{L}$-*category* is a pseudo functor $\mathbb{C} : \mathbb{L} \to \mathbf{CAT}$ which is "FP-preserving" in the following sense:[10]

1. the canonical map $\langle \mathbb{C}\pi_1, \mathbb{C}\pi_2 \rangle : \mathbb{C}(A_1 \times A_2) \to \mathbb{C}(A_1) \times \mathbb{C}(A_2)$ is an isomorphism for each $A_1, A_2 \in \mathbb{L}$;
2. the canonical map $\mathbb{C}(1) \to \mathbf{1}$ is an isomorphism;
3. it preserves identities up-to identity: $\mathbb{C}(\mathrm{id}) = \mathrm{id}$;
4. it preserves pre- and post-composition of identities up-to identity: $\mathbb{C}(\mathrm{id} \circ \mathsf{a}) = \mathbb{C}(a) = \mathbb{C}(\mathsf{a} \circ \mathrm{id})$;
5. it preserves composition of the form $\pi_i \circ \mathsf{a}$ up-to identity: $\mathbb{C}(\pi_i \circ \mathsf{a}) = \mathbb{C}(\pi_i) \circ \mathbb{C}(\mathsf{a})$. Here $\pi_i : A_1 \times A_2 \to A_i$ is a projection.

natural, from an **operadic** point of view

$\mathbb{C}(\boldsymbol{\pi_i} \circ \mathbf{a}) = \mathbb{C}\boldsymbol{\pi_i} \circ \mathbb{C}\mathbf{a}$

$\mathbb{C}(\mathbf{a} \circ \boldsymbol{\pi_i}) \cong \mathbb{C}\mathbf{a} \circ \mathbb{C}\boldsymbol{\pi_i}$

Theorem

$$[\mathbf{MonTh}, \mathbf{CAT}]_{\text{pseudo, prod.-pres.}} \cong \mathbf{BalMonCAT} \quad \text{[Leinster]}$$
$$\simeq \mathbf{MonCAT}$$

Compositionality theorem again

Assume

- $\mathbb{C}$ is an $\mathbb{L}$-category
- $F : \mathbb{C} \to \mathbb{C}$ is a lax $\mathbb{L}$-functor
- there is a final coalgebra $Z \to FZ$

lifts

1. $\mathbf{Coalg}_F$ is an $\mathbb{L}$-category
2. $Z \to FZ$ is an $\mathbb{L}$-object
3. the **behavior** functor

automatic

$$\mathrm{Coalg}_F \xrightarrow{\mathbf{beh}} \mathbb{C}/Z$$

$$\begin{pmatrix} FX \\ c\uparrow \\ X \end{pmatrix} \longmapsto (X \xrightarrow{\mathbf{beh}(c)} Z)$$

by coinduction

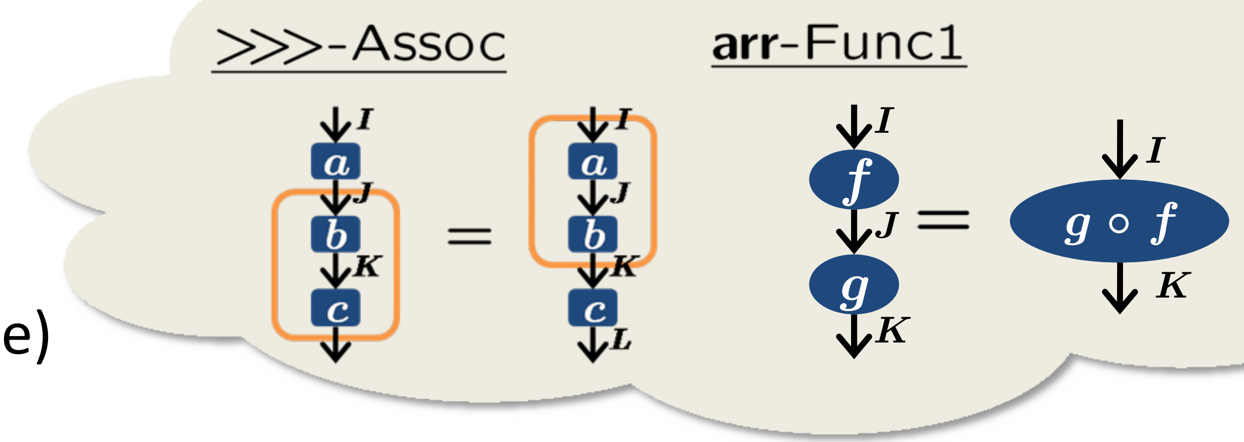
$$\begin{array}{ccc} FX & \dashrightarrow & FZ \\ c\uparrow & & \cong\uparrow\text{final} \\ X & \underset{\mathbf{beh}(c)}{\dashrightarrow} & Z \end{array}$$

is a (strict) L-functor

compositionality results for component calculi

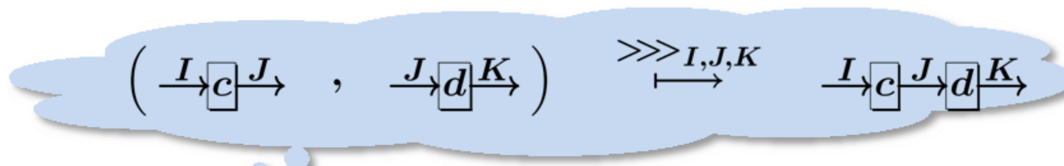
Part 3

The calculi

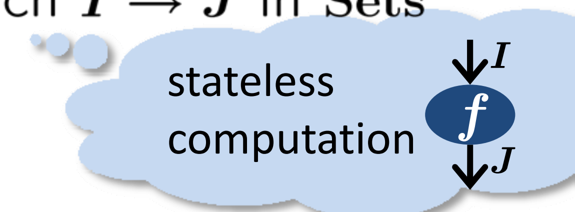


PLTh (PipeLine)

- **sorts** (I,J)
  - where I, J are sets



- **operations**
  - *seq. comp.* $\ggg_{I,J,K} : (I,J) \times (J,K) \longrightarrow (I,K)$
  - *pure function* $\mathbf{arr}\, f : 1 \longrightarrow (I,J)$ for each $I \xrightarrow{f} J$ in Sets

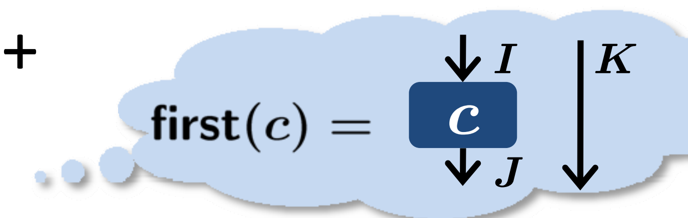


- **equations**

$$a \ggg (b \ggg c) = (a \ggg b) \ggg c \quad (\ggg\text{-Assoc})$$
$$\mathbf{arr}\,(g \circ f) = \mathbf{arr}\, f \ggg \mathbf{arr}\, g \quad (\mathbf{arr}\text{-Func1})$$
$$\mathbf{arr}\,\mathrm{id}_I \ggg a = a = a \ggg \mathbf{arr}\,\mathrm{id}_J \quad (\mathbf{arr}\text{-Func2})$$

The calculi

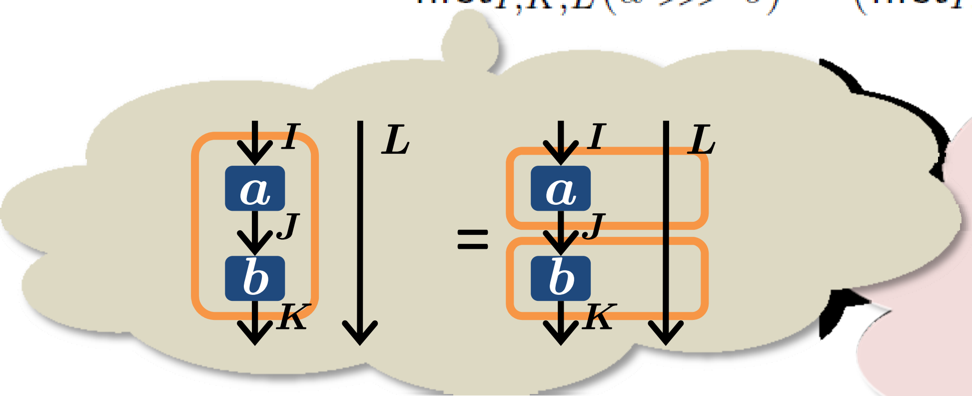
ArrTh (Hughes' *arrow*) $\mathbf{PLTh}$ +

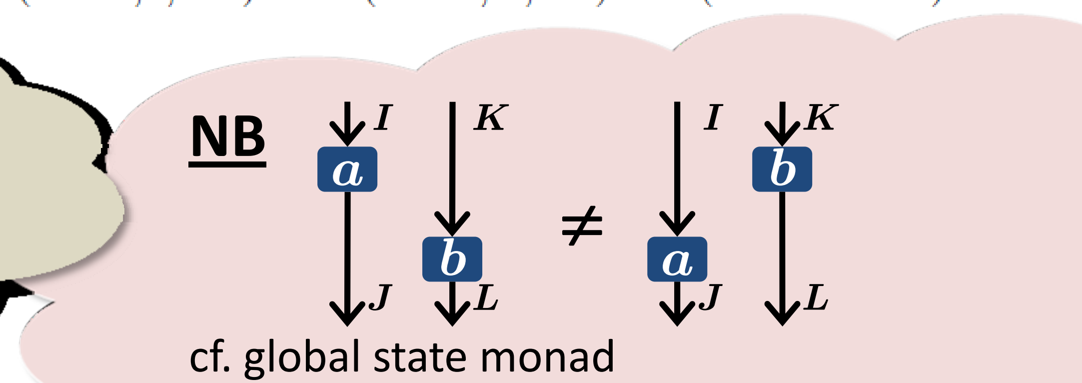


- **operation**
  - *sideline* $\mathsf{first}_{I,J,K} : (I, J) \longrightarrow (I \times K, J \times K)$

- **equations**

$$
\begin{aligned}
\mathsf{first}_{I,J,1}\, a \ggg \mathsf{arr}\,\pi &= \mathsf{arr}\,\pi \ggg a && (\rho\text{-}\mathrm{NAT})\\
\mathsf{first}_{I,J,K}\, a \ggg \mathsf{arr}(\mathrm{id}_J \times f) &= \mathsf{arr}(\mathrm{id}_I \times f) \ggg \mathsf{first}_{I,J,L}\, a && (\mathsf{arr}\text{-}\mathrm{CENTR})\\
(\mathsf{first}_{I,J,K\times L}\, a) \ggg (\mathsf{arr}\,\alpha_{J,K,L}) &= (\mathsf{arr}\,\alpha_{I,K,L}) \ggg \mathsf{first}(\mathsf{first}\, a) && (\alpha\text{-}\mathrm{NAT})\\
\mathsf{first}_{I,J,K}(\mathsf{arr}\, f) &= \mathsf{arr}(f \times \mathrm{id}_K) && (\mathsf{arr}\text{-}\mathrm{PREMON})\\
\mathsf{first}_{I,K,L}(a \ggg b) &= (\mathsf{first}_{I,J,L}\, a) \ggg (\mathsf{first}_{J,K,L}\, b) && (\mathsf{first}\text{-}\mathrm{FUNC})
\end{aligned}
$$





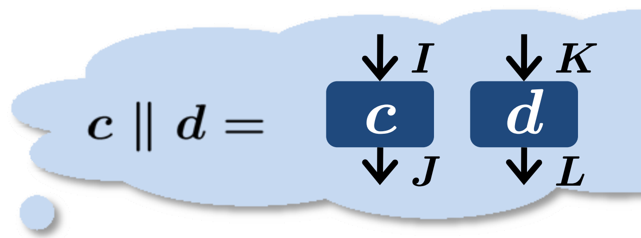
The calculi

PLTh (PipeLine)

ArrTh (Hughes' *arrow*) = **PLTh** + first

MArrTh (Monoidal arrow) = **PLTh** +

- <u>**operation**</u>
  - *(synchronous) parallel composition*



$$\parallel_{I,J,K,L} : (I, J) \times (K, L) \longrightarrow (I \times K, J \times L)$$

- <u>**equations**</u>

$$(a \parallel b) \ggg (c \parallel d) = (a \ggg c) \parallel (b \ggg d) \qquad (\parallel\text{-FUNC1})$$
$$\mathsf{arr}\, \mathrm{id}_I \parallel \mathsf{arr}\, \mathrm{id}_J = \mathsf{arr}\, \mathrm{id}_{I \times J} \qquad (\parallel\text{-FUNC2})$$
$$a \parallel (b \parallel c) \ggg \mathsf{arr}\, \alpha = \mathsf{arr}\, \alpha \ggg (a \parallel b) \parallel c \qquad (\alpha\text{-NAT})$$
$$(a \parallel \mathsf{arr}\, \mathrm{id}_1) \ggg \mathsf{arr}\, \pi = \mathsf{arr}\, \pi \ggg a \qquad (\rho\text{-NAT})$$
$$\mathsf{arr}(f \times g) = \mathsf{arr}\, f \parallel \mathsf{arr}\, g \qquad (\mathsf{arr}\text{-MON})$$
$$(a \parallel b) \ggg \mathsf{arr}\, \gamma = \mathsf{arr}\, \gamma \ggg (b \parallel a) \qquad (\gamma\text{-NAT})$$

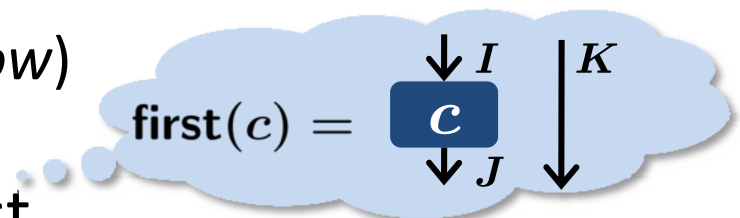
The calculi

PLTH (PipeLine)

$\ggg$, **arr** f

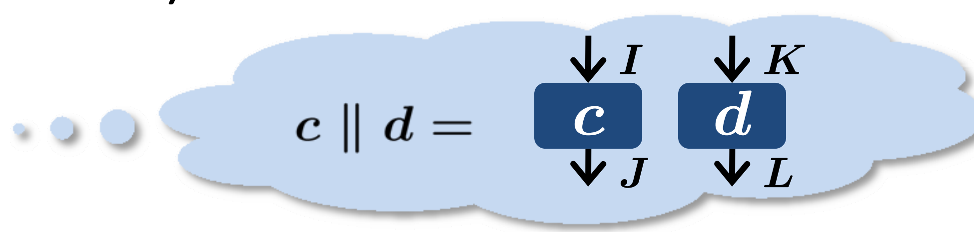
ARRTH (Hughes' *arrow*)

= **PLTh** + first



MARRTH (Monoidal arrow) = **PLTh** +

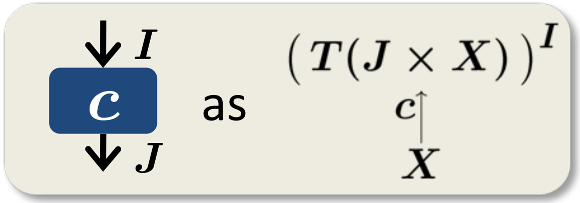
= **PLTh** + $\|$



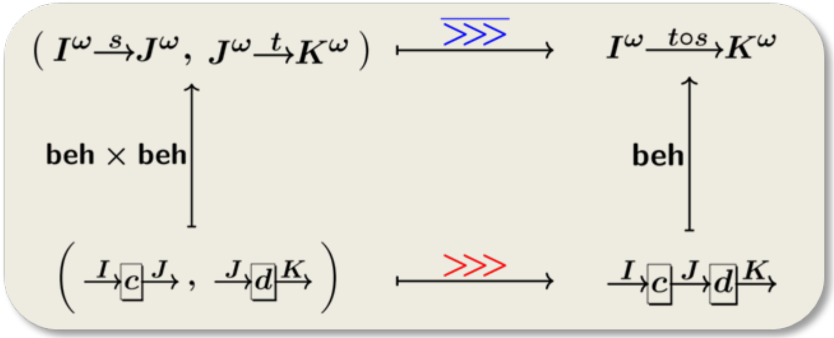
Our goal

For $\mathbb{L} \in \{\mathbf{PLTh}, \mathbf{ArrTh}, \mathbf{MArrTh}\}$,

- "components as coalgebras" constitute a microcosm model for $\mathbb{L}$,



- in particular the **compositionality theorem**



i.e.

$$\mathsf{beh}\begin{pmatrix} F_{I,J}X \\ \uparrow c \\ X \end{pmatrix} \ggg \begin{pmatrix} F_{J,K}Y \\ \uparrow d \\ Y \end{pmatrix} = \mathsf{beh}\begin{pmatrix} F_{I,J}X \\ \uparrow c \\ X \end{pmatrix} \ggg \mathsf{beh}\begin{pmatrix} F_{J,K}Y \\ \uparrow d \\ Y \end{pmatrix}$$

BTW, in functional programming…

- **Kleisli category** $Kl(T)$
  - given a monad T for **effect**
  - the category of T-**effectful computations:**

$$\frac{X \to Y \quad \text{in } \mathcal{K}\ell(T)}{X \to TY \quad \text{in Sets}}$$

- The "hom-model"

$$\mathbb{L} \xrightarrow{\mathcal{K}\ell(T)} \mathbf{Sets}$$

$$(I, K) \longmapsto \mathrm{Hom}_{\mathcal{K}\ell(T)}(I, K)$$

Theorem [Power-Robinson]

1. $Kl(T)$ is a (set-theoretic) model of $\mathbf{PLTh}$
2. T: strong ➔ $Kl(T)$ is a model of $\mathbf{ArrTh}$
3. T: commutative ➔ $Kl(T)$ is a model of $\mathbf{MArrTh}$

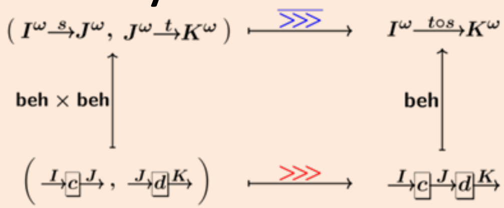
arrow [Hughes]

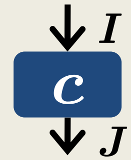
Freyd category [Levy-Power-Thielecke]

"monoidal" arrow

Follow from these outer models:

- inner models
- compositionality





as $\left(T(J \times X)\right)^{I} \overset{c}{\longleftarrow} X$

signature: $F_{I,J} = \left(T(J \times \_)\right)^{I}$

- T: monad
  ➔ a model $\mathbf{PLTh} \xrightarrow{\mathcal{K}\ell(T)} \mathbf{Sets}$

  $(I, J) \longmapsto \mathrm{Hom}_{\mathcal{K}\ell(T)}(I, J)$

- T: strong monad
  ➔ $\mathbf{ArrTh} \xrightarrow{\mathcal{K}\ell(T)} \mathbf{Sets}$

- T: commutative monad
  ➔ $\mathbf{MArrTh} \xrightarrow{\mathcal{K}\ell(T)} \mathbf{Sets}$

Theorem

- T: monad
  ➔ a model $\mathbf{PLTh} \xrightarrow{\mathrm{Coalg}_F} \mathbf{CAT}$

  $(I, J) \longmapsto \mathrm{Coalg}_{F_{I,J}}$

- T: strong monad
  ➔ $\mathbf{ArrTh} \xrightarrow{\mathrm{Coalg}_F} \mathbf{CAT}$

- T: commutative monad
  ➔ $\mathbf{MArrTh} \xrightarrow{\mathrm{Coalg}_F} \mathbf{CAT}$

Proof of the theorem

exploiting the results on $\mathcal{K}l(T)$

Theorem

T: monad

➔ a model $\mathbf{PLTh} \xrightarrow{\mathrm{Coalg}_F} \mathbf{CAT}$

(hence an inner model, compositionality)

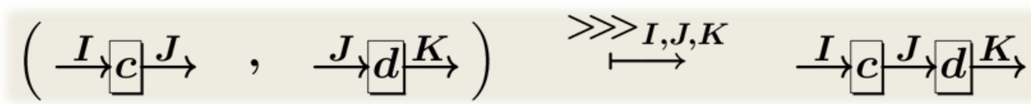
lifting

- a model $\mathbf{PLTh} \xrightarrow{\mathbf{Sets}} \mathbf{CAT}$
- $F : \mathbf{Sets} \longrightarrow \mathbf{Sets}$

lax compatible with $\mathbf{PLTh}$

Lemma

$\mathbf{PLTh} \xrightarrow{\mathbf{Sets}} \mathbf{CAT}$, a model



"carrier set"

$$\mathbf{PLTh} \xrightarrow{\mathbf{Sets}} \mathbf{CAT}$$

$$\begin{array}{c} (I,J)\times(J,K) \\ \downarrow \ggg \\ (I,K) \end{array} \longmapsto \begin{array}{c} \mathbf{Sets}\times\mathbf{Sets} \\ \downarrow \times \\ \mathbf{Sets} \end{array}$$

$$\begin{array}{c} (X,Y) \\ \downarrow \\ X\times Y \end{array}$$

Proof of the theorem

exploiting the results on $Kl(T)$

Theorem

T: monad
➔ a model $\mathbf{PLTh} \xrightarrow{\mathrm{Coalg}_F} \mathbf{CAT}$

(hence an inner model, compositionality)

lifting

- a model $\mathbf{PLTh} \xrightarrow{\mathbf{Sets}} \mathbf{CAT}$
- $F : \mathbf{Sets} \longrightarrow \mathbf{Sets}$

lax compatible with $\mathbf{PLTh}$

Lemma

$F : \mathbf{Sets} \longrightarrow \mathbf{Sets}$, lax $\mathbf{PLTh}$-functor

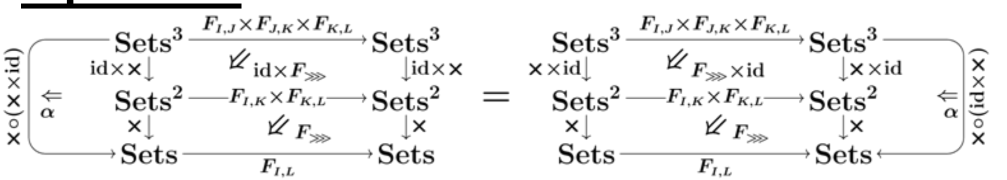
operations

$$\begin{array}{ccc} \mathbf{Sets}\times\mathbf{Sets} & \xrightarrow{F_{I,J}\times F_{J,K}} & \mathbf{Sets}\times\mathbf{Sets} \\ \times\downarrow & \swarrow F_{\ggg} & \downarrow\times \\ \mathbf{Sets} & \xrightarrow[F_{I,K}]{} & \mathbf{Sets} \end{array}$$

$$F_{\ggg} \;:\; F_{I,J}X \times F_{J,K}Y \longrightarrow F_{I,K}(X \times Y)$$

$$\mathrm{Hom}_{\mathcal{K}\ell(T)}(I, J\times X) \times \mathrm{Hom}_{\mathcal{K}\ell(T)}(J, K\times Y) \longrightarrow \mathrm{Hom}_{\mathcal{K}\ell(T)}(I, K\times(X\times Y))$$

equations



$$\begin{array}{ccccc} F_{I,J}X \times (F_{J,K}Y \times F_{K,L}U) & \xrightarrow{\mathrm{id}\times F_{\ggg}} & F_{I,J}X \times F_{J,L}(Y\times U) & \xrightarrow{F_{\ggg}} & F_{I,L}(X\times(Y\times U)) \\ \alpha\downarrow & & & & \downarrow F_{I,L}\alpha \\ (F_{I,J}X \times F_{J,K}Y) \times F_{K,L}U & \xrightarrow[F_{\ggg}\times\mathrm{id}]{} & F_{I,K}(X\times Y)\times F_{K,L}U & \xrightarrow[F_{\ggg}]{} & F_{I,L}((X\times Y)\times U) \end{array}$$

...

an morphism in $\mathbf{PLTh}$, interpreted via $\mathbf{PLTh} \xrightarrow{\mathcal{K}\ell(T)} \mathbf{Sets}$

via equations in $\mathbf{PLTh}$

Future work

- Richer component calculi
  - feedback → traced monoidal structure?
  - delayed/lossy channels → different effect monad $\boldsymbol{T}$?

- Other "algebra in algebra"
  - for component calculi: $\mathbb{L}$-algebra in $\mathbb{L}$-algebra
  - in general: $\mathbb{L}_1$-algebra in $\mathbb{L}_2$-algebra
    - usually: $\mathbb{L}_2$-algebra = "category with finite products"
    - Theory of generalized operads/combinatorial species

      [Leinster] [Fiore-Gambino-Hyland-Winskel] [Curien]

Conclusions

The Microcosm Principle

what's new: many-sorted, pseudo algebra

Hughes' **arrow**, **Freyd category**, … from functional programming

components (state-based machines)	as	coalgebras
their behaviors	by	final coalgebra
calculus on them	as	algebraic structure

compositionality results

Thanks for your attention!

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Conclusion

Compositionality theorem

Assume

- $\mathbb{C}$ is an $\mathbb{L}$-category
- $F : \mathbb{C} \to \mathbb{C}$ is a lax $\mathbb{L}$-functor
- there is a final coalgebra $Z \to FZ$

1. $\mathbf{Coalg}_F$ is an $\mathbb{L}$-category

obtaining

$$\mathrm{Coalg}(F_{I,J}) \times \mathrm{Coalg}(F_{J,K}) \xrightarrow{\ggg_{I,J,K}} \mathrm{Coalg}(F_{I,K})$$

$$\Big(\begin{array}{c}(J \times X)^I \\ c\uparrow \\ X\end{array} \;,\; \begin{array}{c}(K \times Y)^J \\ d\uparrow \\ Y\end{array} \Big) \longmapsto \begin{array}{c}(K \times (X \times Y))^I \\ c \ggg d\uparrow \\ X \times Y\end{array}$$