The Microcosm Principle and Concurrency in Coalgebras

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1-slide review of coalgebra/coinduction

abstraction, genericity, (joy)

Theory of ***coalgebras***

= "***categorical*** *theory of state-based systems*"

	categorically	
system	coalgebra	$\begin{array}{c} FX \\ \uparrow \\ X \end{array}$
	morphism of coalgebras	$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ \uparrow & & \uparrow \\ X & \xrightarrow[f]{} & Y \end{array}$
behavior	***coinduction*** (via final coalgebra)	$\begin{array}{ccc} FX & \dashrightarrow & FZ \\ c\uparrow & & \cong\uparrow \text{final} \\ X & \underset{\mathsf{beh}(c)}{\dashrightarrow} & Z \end{array}$

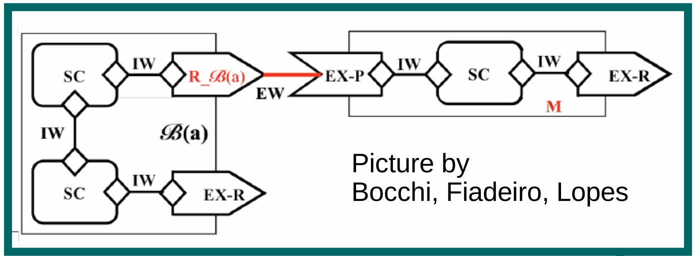
in Sets : bisimilarity

in Kleisli : trace semantics

[Hasuo,Jacobs,Sokolova]

Concurrency

- is about **parallel composition *C* || *D***
  - running ***C*** and ***D*** at the same time
  - with *communication/synchronization* between ***C*** and ***D***
- is **everywhere**
  - computer networks
  - multi-core processors
  - modular, component-based design of complex systems



Picture by
Bocchi, Fiadeiro, Lopes

- is **hard to get right**
  - e.g. so easy to get into *deadlocks*
  - cf. Edward Lee. *Making Concurrency Mainstream.* Invited talk at CONCUR 2006. http://ptolemy.eecs.berkeley.edu/presentations/main.htm

Compositionality

Behavior of *C* || *D*

is determined by

behavior of *C* and **behavior of *D***

- Enables ***compositional*** verification of complex systems
- Conventional presentation:

$$\boldsymbol{C \sim C', \quad D \sim D' \quad \Rightarrow \quad C \,||\, D \sim C' \,||\, D'}$$

- **~** : process/observational/behavioral equivalence
  - bisimilarity, trace equivalence, etc.
- *“bisimilarity is a congruence”*

Compositionality in coalgebras

- **Final coalgebra semantics** as "process semantics".

$$\begin{array}{ccc} FX & \dashrightarrow & FZ \\ {\scriptstyle c}\uparrow & & \uparrow{\scriptstyle \cong\ \mathsf{final}} \\ X & \underset{\mathsf{beh}(c)}{\dashrightarrow} & Z \end{array}$$

- "Coalgebraic compositionality"

$$\mathbf{beh}\left(\begin{array}{c} FX \\ c\uparrow \\ X \end{array} \,{\color{red}\|}\, \begin{array}{c} FY \\ d\uparrow \\ Y \end{array}\right) \quad = \quad \mathbf{beh}\left(\begin{array}{c} FX \\ c\uparrow \\ X \end{array}\right) \,{\color{blue}\|}\, \mathbf{beh}\left(\begin{array}{c} FY \\ d\uparrow \\ Y \end{array}\right)$$

- Two different || !
  - || : **$\mathbf{Coalg}_F$ x $\mathbf{Coalg}_F$ → $\mathbf{Coalg}_F$** on *coalgebras*
  - || : ***Z* x *Z* → *Z*** on *states*

Nested algebraic structures: *the microcosm principle*

$$\begin{array}{ccccc} \mathrm{Coalg}_{\boldsymbol{F}} & \times & \mathrm{Coalg}_{\boldsymbol{F}} & \xrightarrow{\|} & \mathrm{Coalg}_{\boldsymbol{F}} \\ \boldsymbol{Z} & \times & \boldsymbol{Z} & \xrightarrow{\|} & \boldsymbol{Z} \end{array}$$

with

$$\begin{pmatrix} \boldsymbol{F}\boldsymbol{Z} \\ \cong\big\uparrow \text{final} \\ \boldsymbol{Z} \end{pmatrix} \in \mathrm{Coalg}_{\boldsymbol{F}}$$

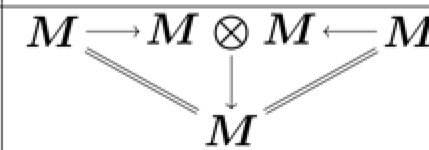
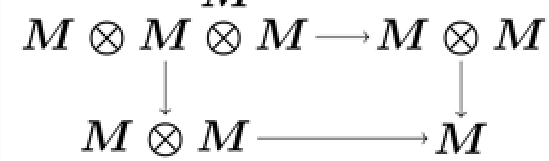
- The same “algebraic structure”
  - ***operations*** (binary ||)
  - ***equations*** (e.g. associativity of ||)
- is carried by
  - the **category Coalg_F** and
  - its **object** $Z \in$ **Coalg_F**

  in a nested manner!
- “Microcosm principle” (Baez & Dolan)

Microcosm in macrocosm

The microcosm principle

- You may have seen it
  - "a monoid is in a monoidal category"

monoidal cat. $\mathbb{C}$		monoid $M \in \mathbb{C}$
$\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$	mult.	$M \otimes M \xrightarrow{m} M$
$I \in \mathbb{C}$	unit	$I \xrightarrow{e} M$
$I \otimes X \cong X \cong X \otimes I$	unit law	
$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$	assoc. law	

  - Notice:
    - The "inner" structure depends on the "outer" one

- We identify (probably) the first CS example

Formalizing the microcosm principle

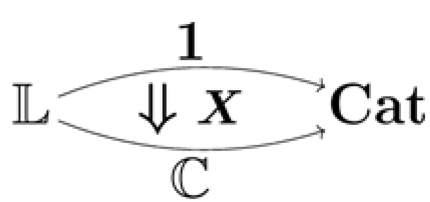
What do we mean exactly by the "microcosm principle"?

- When a category ***L*** presents an algebraic theory *(Lawvere theory)*,
- Its **(set-theoretic) model** is a FP-preserving functor

$$\mathbb{L} \xrightarrow{X} \mathbf{Sets}$$

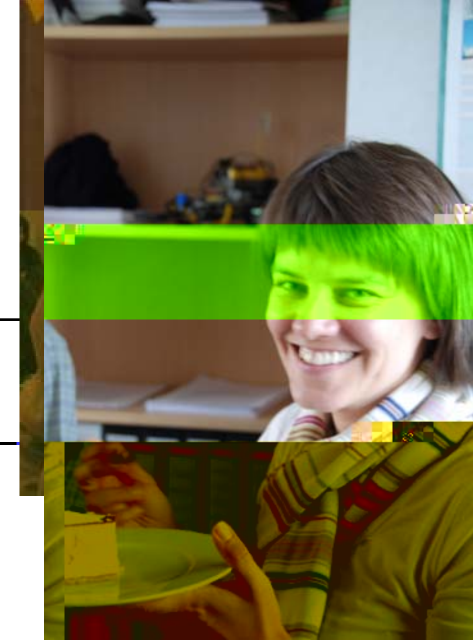
$$\left(\mathbf{2} \xrightarrow{\mathbf{m}} \mathbf{1}\right) \longmapsto \left(X^{\mathbf{2}} \xrightarrow{[\mathbf{m}]} X\right)$$

- How about a **nested model** as in the microcosm principle?
  ➔ Our answer: a lax natural transformation



Outline

- Microcosm principle for concurrency (|| and
  - || and || essentially arise from ***"synchronization"*** natural transformation



$$\textbf{sync} : FX \otimes FY \to F(X \otimes Y)$$

- The microcosm principle syntactically
  - Algebraic structure is syntactically presented as (Σ, *E*)
  - … (Ana can tell you more!)
- The microcosm principle 2-categorically
  - (Common) alg. str. is presented by a ***Lawvere theory***
  - Applications:
    - generic compositionality theorem
    - generic soundness theorem

Part I:

Parallel composition of coalg. via *sync* nat. trans.

Parallel composition of coalgebras

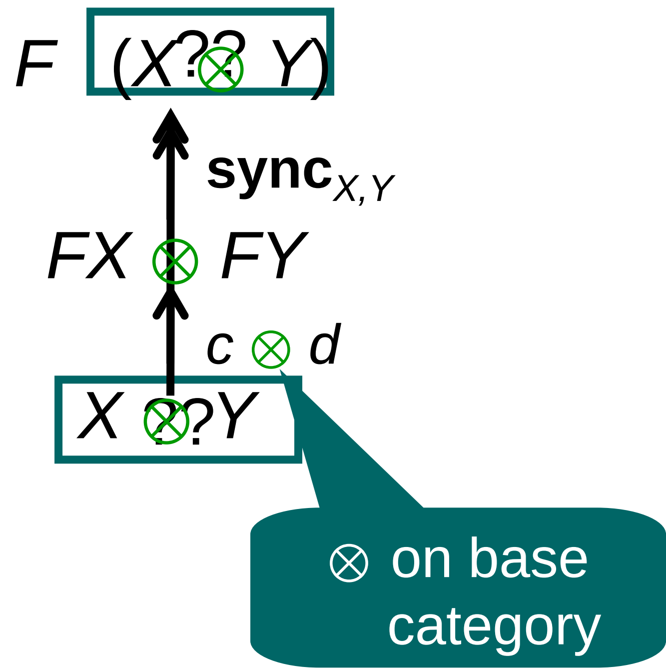
$$\mathrm{beh}\left(\begin{pmatrix} FX \otimes FY \\ \uparrow c \otimes d \\ X \otimes Y \end{pmatrix}\right) = \mathrm{beh}\left(\begin{pmatrix} FX \\ \uparrow c \\ X \end{pmatrix}\right) \parallel \mathrm{beh}\left(\begin{pmatrix} FY \\ \uparrow d \\ Y \end{pmatrix}\right)$$

- $\parallel$: bifunctor **Coalg**$_F$ **x Coalg**$_F$ → **Coalg**$_F$
  ➔ usually denoted by $\otimes$ (tensor)
- **<u>Theorem</u>** If
  - the base category ***C*** has associative tensor
    $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$
  - and $F : \mathbf{C} \to \mathbf{C}$ comes with natural transformation
    $\mathbf{sync}_{X,Y} : FX \otimes FY \to F(X \otimes Y)$
  - then we have $\otimes$ **: Coalg**$_F$ **x Coalg**$_F$ → **Coalg**$_F$

Parallel composition arises from **sync**

Parallel composition of coalgebras

$$\begin{pmatrix} FX \\ c\uparrow \\ X \end{pmatrix} \otimes \begin{pmatrix} FY \\ d\uparrow \\ Y \end{pmatrix} \quad =$$



- Different **sync** yield different $\otimes$

Examples of **sync** : $FX \otimes FY \to F(X \otimes Y)$

C = **Sets**, $F = P_{fin}(\Sigma \times \_)$ for LTS

Cartesian product as ⊗

- **CSP-style** (Hoare) $a.P \parallel a.Q \xrightarrow{a} P \parallel Q$

$$\begin{array}{ccc} \mathcal{P}_{\text{fin.}}(\Sigma \times X) \times \mathcal{P}_{\text{fin.}}(\Sigma \times Y) & \xrightarrow{\text{sync}_{X,Y}} & \mathcal{P}_{\text{fin.}}(\Sigma \times (X \times Y)) \\ (S,T) & \longmapsto & \{\, (a,(x,y)) \mid (a,x) \in S \wedge (a,y) \in T \,\} \end{array}$$

- **CCS-style** (Milner) $a.P \parallel \overline{a}.Q \xrightarrow{\tau} P \parallel Q$

  Assuming $\Sigma = \{a, a', \dots\} + \{\overline{a}, \overline{a'}, \dots\} + \{\tau\}$

$$\begin{array}{ccc} \mathcal{P}_{\text{fin.}}(\Sigma \times X) \times \mathcal{P}_{\text{fin.}}(\Sigma \times Y) & \xrightarrow{\text{sync}_{X,Y}} & \mathcal{P}_{\text{fin.}}(\Sigma \times (X \times Y)) \\ (S,T) & \longmapsto & \{\, (\tau,(x,y)) \mid (a,x) \in S \wedge (\overline{a},y) \in T \,\} \end{array}$$

Compositionality result

Compositionality result

- **Theorem**
  Given that
  - ***C*** has tensor $\otimes$
  - F has $\mathbf{sync}_{X,Y} : FX \otimes FY \to F(X \otimes Y)$
  - there is a final coalgebra $Z \to FZ$

  we have ***compositionality***

$$\mathbf{beh}\begin{pmatrix} FX & & FY \\ c\uparrow & \otimes & d\uparrow \\ X & & Y \end{pmatrix} = \mathbf{beh}\begin{pmatrix} FX \\ c\uparrow \\ X \end{pmatrix} \,\|\, \mathbf{beh}\begin{pmatrix} FY \\ d\uparrow \\ Y \end{pmatrix}$$

- **"Compositionality for free"**
- It follows: ***C ~ C', D ~ D' ➔ C* || *D ~ C'* || *D'***
- Proof By finality
- We shall generalize this to an ***arbitrary*** (single-sorted) algebraic theory

Part II:

2-categorical formulation of the microcosm principle

Microcosm principle (Baez & Dolan)

- The same algebraic theory
- interpreted both on **C** and on $X \in$ **C**
  - **C** : ***outer*** model
  - $X \in$ **C** : ***inner*** model
- Examples:
  - monoid in a monoidal category
  - final coalgebra in $\mathbf{Coalg}_F$ with $\otimes$

What is microcosm principle, mathematically?

Setting

- 2-categorical
  - 2-categories: categories in categories
  - suitable for microcosm structures, i.e. algebras in algebras

Lawvere theory ***L***

- *Categorical presentation* of an algebraic specification/theory

- **Definition**

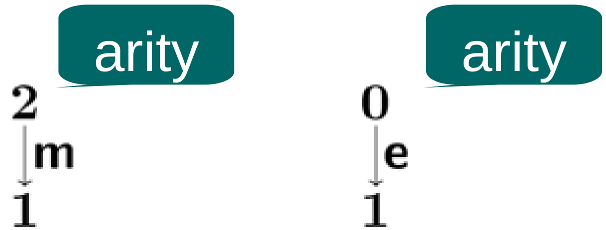
  A ***Lawvere theory*** ***L*** is a small category s.t.

  - ***L***’s objects are natural numbers
  - ***L*** has finite products

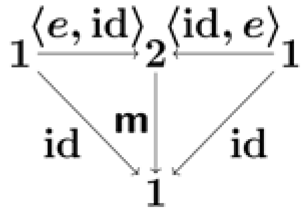
Lawvere theory

is a **category** ***L*** that has

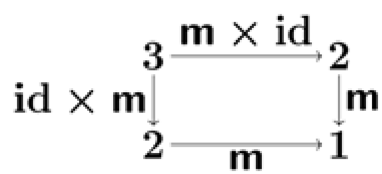
- **Arrows** for operations



- **Commutativity** for equations

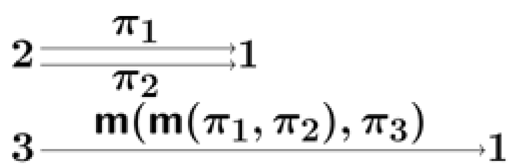


~ unit law



~ assoc. law

- Other arrows:
  - projections
  - (composed) terms



Models for Lawvere theory

- Cf. A **(set-theoretic) model** is a FP-preserving functor

$$\mathbb{L} \xrightarrow{X} \mathbf{Sets}$$

$$\begin{array}{ccc} \mathbf{2} & & X^{\mathbf{2}} \\ \downarrow \mathbf{m} & \longmapsto & \downarrow [\mathbf{m}] \\ \mathbf{1} & & X \end{array}$$

  - *A set with **L**-structure*
  - "Functorial semantics"

- How about the microcosm principle: ***L***-algebraic structures on
  - ***C*** : ***outer*** model
  - *X* ∈ ***C*** : ***inner*** model

Outer model: *L*-category

- Outer model =

  a <u>category</u> with **L**-structure ➔ “***L*-category**”

- <u>***L*-category**</u>: an FP-preserving functor

$$\mathbb{L} \xrightarrow{\ \mathbb{C}\ } \mathbf{Cat}$$

$$\begin{array}{c} \mathbf{2} \\ \downarrow \mathrm{m} \\ \mathbf{1} \end{array} \quad \longmapsto \quad \begin{array}{c} \mathbb{C}^{\mathbf{2}} \\ \downarrow [\mathrm{m}] \\ \mathbb{C} \end{array}$$

  - In fact, a ***pseudo-***functor (equations are up-to-iso)

Inner model: ***L***-object

Definition

An ***L*-object** in an ***L***-category $\mathbb{L} \xrightarrow{\mathbb{C}} \mathbf{Cat}$

is a **lax** natural transformation

Outer model

$\mathbf{1}$

$\mathbb{L}$

compatible with product

Recall: $\dfrac{X \in \mathbb{C}}{\mathbf{1}}$

Inner alg. structure occurs from ***lax naturality***

Components

$X_0 : 1 \xrightarrow{!} 1$

$X_1 : 1 \xrightarrow{X} \mathbb{C}$

$X_2 : 1 \xrightarrow{(X,X)} \mathbb{C}^2$

$\vdots$

naturality

In $\mathbb{L}$: $\mathbf{2} \xrightarrow{\mathbf{m}} \mathbf{1}$

In $\mathbf{Cat}$: $\mathbf{1} \xrightarrow{(X,X)} \mathbb{C}^2$, $\mathbb{C}^2 \xrightarrow{\otimes} \mathbb{C}$, $\mathbf{1} = \mathbf{1}$, $\mathbf{1} \xrightarrow{X} \mathbb{C}$

$$X \otimes X \xrightarrow{X_{\mathbf{m}}} X \quad \text{in } \mathbb{C}$$

Facts

***lax L*-functor**: lax natural trans. $\mathbb{L} \underset{\mathbb{C}}{\overset{\mathbb{C}}{\Downarrow F}} \mathbf{Cat}$

lax naturality:

In $\mathbb{L}$: $\mathbf{2} \xrightarrow{\mathbf{m}} \mathbf{1}$

In $\mathbf{Cat}$:

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{(F,F)} & \mathbb{C}^2 \\ \otimes \downarrow & \swarrow & \downarrow \otimes \\ \mathbb{C} & \xrightarrow[F]{} & \mathbb{C} \end{array}$$

$$FX \otimes FY \xrightarrow{\mathsf{sync}_{X,Y}} F(X \otimes Y) \quad \text{in } \mathbb{C}$$

lax *L*-functor

= functor with **sync**.

- ***L***-
- ***C***: ***L***-category
  - *F*: ***C*** → ***C***, lax ***L***-functor
    ➔ **Coalg**$_F$ is an ***L***-category
  - *F*: ***C*** → ***C***, **oplax** ***L***-functor
    ➔ **Alg**$_F$ is an ***L***-category
  - $X \in$ ***C*** , ***L***-object
    ➔ ***C***/*X* is an ***L***-category
  - Final object is an ***L***-object

F : lax, *G* : oplax
➔ ***Inserter*** Ins(*F*, *G*) is an ***L***-category

Generic compositionality result

- **Theorem**
  Given that
  - ***C*** is an ***L***-category
  - *F* : ***C*** → ***C*** is a lax ***L***-functor
  - there is a final coalgebra *Z* → *FZ*

  the functor

$$\mathbf{Coalg}_F \xrightarrow{\quad \mathsf{beh} \quad} \mathbb{C}/Z$$

$$\begin{pmatrix} FX \\ c\uparrow \\ X \end{pmatrix} \longmapsto \left(X \xrightarrow{\mathsf{beh}(c)} Z\right)$$

by coinduction

$$\begin{array}{ccc} FX & \dashrightarrow & FZ \\ c\uparrow & & \cong\uparrow \text{final} \\ X & \underset{\mathsf{beh}(c)}{\dashrightarrow} & Z \end{array}$$

  is a (strict) ***L*-functor**.

- This subsumes the previous compositionality result

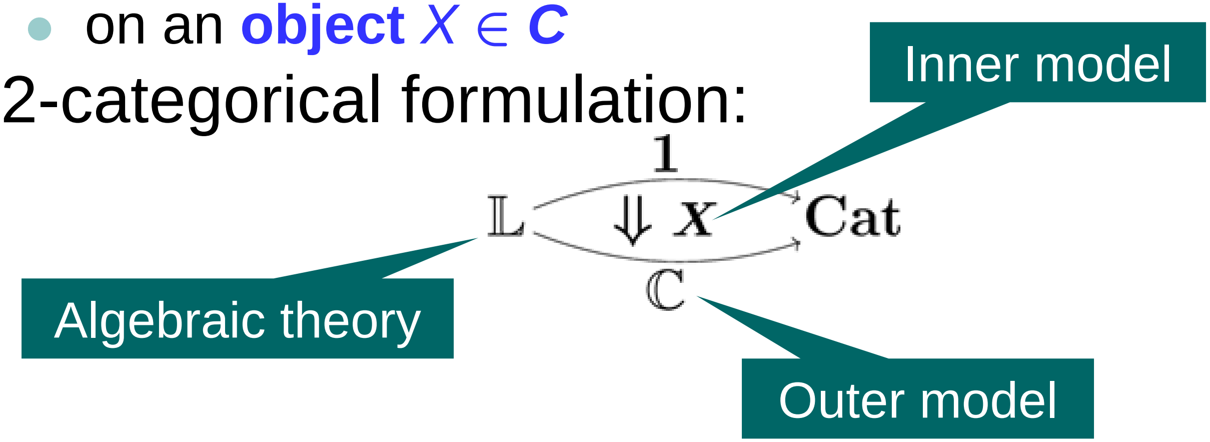
$$\mathsf{beh}\begin{pmatrix} FX \\ c\uparrow \\ X \end{pmatrix} \otimes \begin{pmatrix} FY \\ d\uparrow \\ Y \end{pmatrix} = \mathsf{beh}\begin{pmatrix} FX \\ c\uparrow \\ X \end{pmatrix} \,\|\, \mathsf{beh}\begin{pmatrix} FY \\ d\uparrow \\ Y \end{pmatrix}$$

Related work: bialgebras

- Related to the study of ***bialgebraic structures*** [Turi-Plotkin, Bartels, Klin, …]
  - Algebraic structures on coalgebras
- In the current work:
  - ***Equations***, not only *operations*, are also an integral part
  - Algebraic structures are ***nested***, ***higher-dimensional***

Conclusion

- **<u>Microcosm principle</u>** :
  same algebraic structure
  - on a **category *C*** and
  - on an **object** *X* ∈ ***C***
- 2-categorical formulation:



- ***Concurrency in coalgebras*** as a CS example
- Preprint available:
  http://www.cs.ru.nl/~ichiro

Thank you for your attention!

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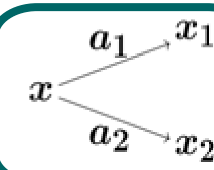
Behavior by coinduction: example

Take $F = \mathcal{P}_{\text{fin}}(\Sigma \times \_)$ in **Sets.**

- System as coalgebra:

$$\begin{array}{ccc} \mathcal{P}_{\text{fin}}(\Sigma \times X) & \qquad & \{(a_1, x_1),\ (a_2, x_2)\} \\ c \uparrow & & \uparrow \\ X & & x \end{array}$$



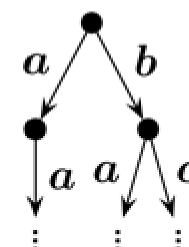
- Behavior by coinduction:

$$\begin{array}{ccc} FX & \dashrightarrow & FZ \\ c \uparrow & & \cong \uparrow \text{final} \\ X & \dashrightarrow & Z \\ & \text{beh}(c) & \end{array}$$

$x \longmapsto$ (process graph of x)

the set of
- finitely branching
- edges labeled with Σ
- infinite-depth

trees, such as



- in **Sets:** ***bisimilarity***
- in certain Kleisli categories: ***trace equivalence*** [Hasuo,Jacobs,Sokolova,CMCS'06]

Examples of **sync** : $FX \otimes FY \to F(X \otimes Y)$

- **Note:**

  Asynchronous/interleaving compositions don't fit in this framework

  - such as $a.P \parallel Q \xrightarrow{a} P \parallel Q$
  - We have to use, instead of *F*,

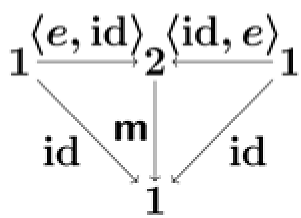
    the ***cofree comonad*** on *F*

Lawvere theory

- Presentation of an algebraic theory as a category:
  - objects: 0, 1, 2, 3, … "***arities***"
  - arrows: "***terms*** (in a context)"

$2 \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} 1$	$2 \overset{x_1, x_2 \vdash x_1}{\underset{x_1, x_2 \vdash x_2}{\rightrightarrows}} 1$	*projections*
$2 \xrightarrow{\mathbf{m}} 1$	$2 \xrightarrow{x_1, x_2 \vdash \mathbf{m}(x_1, x_2)} 1$	*operation*
$3 \xrightarrow{\mathbf{m}(\mathbf{m}(\pi_1, \pi_2), \pi_3)} 1$	$3 \xrightarrow{x_1, x_2, x_3 \vdash m(m(x_1, x_2), x_3)} 1$	*composed term*

  - commuting diagrams are understood as "***equations***"



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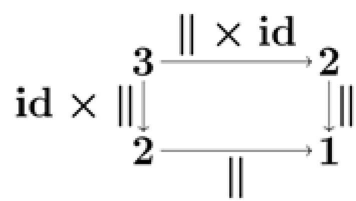

~ assoc. law

  - arises from
    - (single-sorted) algebraic specification (Σ, E) as the *syntactic category*
    - FP-sketch

Outline

- In a coalgebraic study of ***concurrency***,
- ***Nested*** algebraic structures
  - on a **category *C*** and
  - on an **object** $X \in$ ***C***

  arise naturally (**microcosm principle**)
- Our contributions:
  - Syntactic formalization of microcosm principle
  - 2-categorical formalization with Lawvere theories
  - Application to coalgebras:
    - generic compositionality theorem

Generic soundness result

- A Lawvere theory ***L*** is for
  - operations, and
  - ***equations*** (e.g. associativity, commutativity)
- **Coalg**$_F$ is an ***L***-category
  ➔ Parallel composition $\otimes$ is automatically **<u>associative</u>** (for example)
  - Ultimately, this is due to the ***coherence condition*** on the **lax *L***-functor *F*
- **<u>Possible application</u>** :
  Study of *syntactic formats* that ensure associativity/commutativity (future work)