Exercises in Nonstandard Static Analysis of Hybrid Systems

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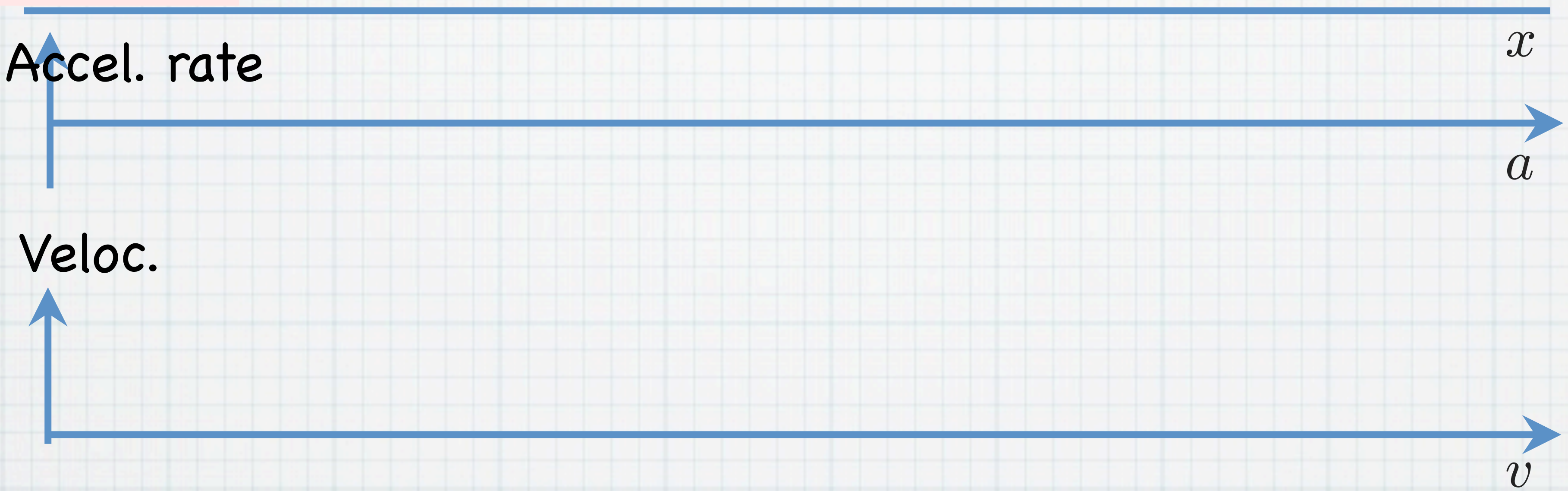
京都大学 KYOTO UNIVERSITY

Hybrid System



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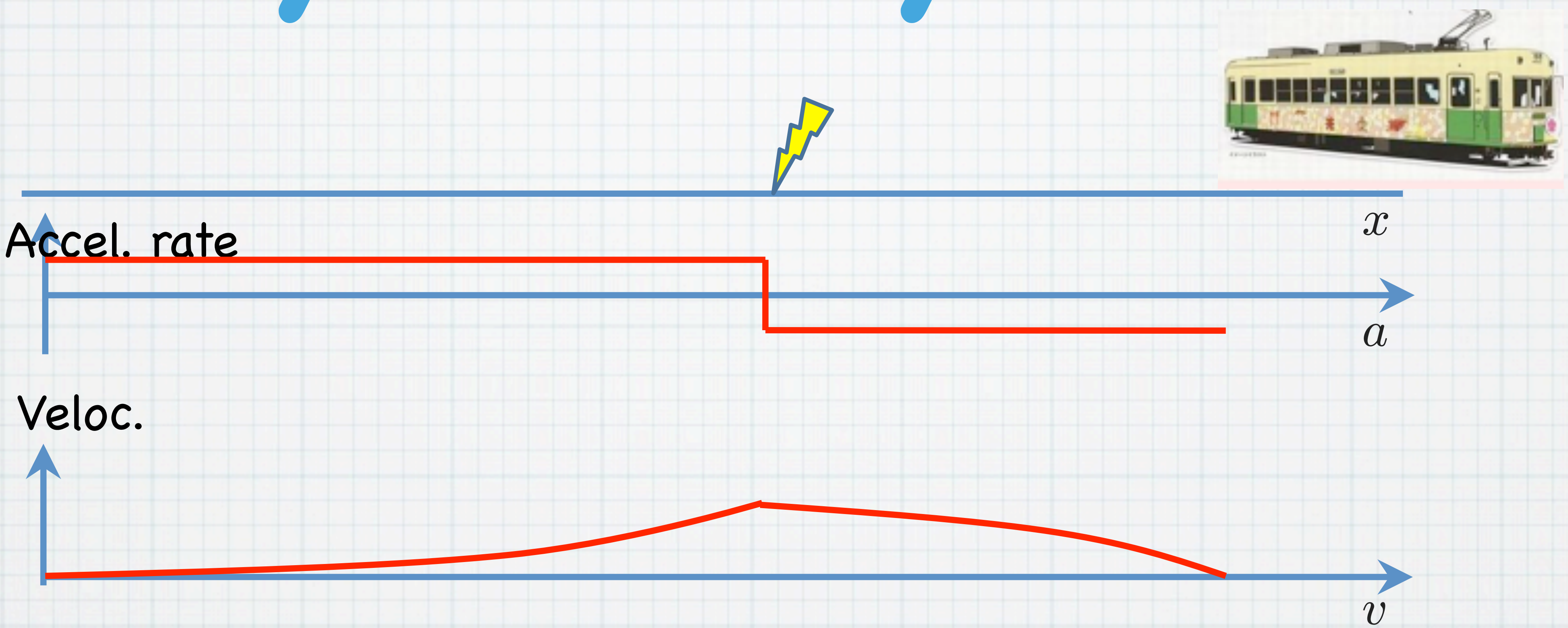

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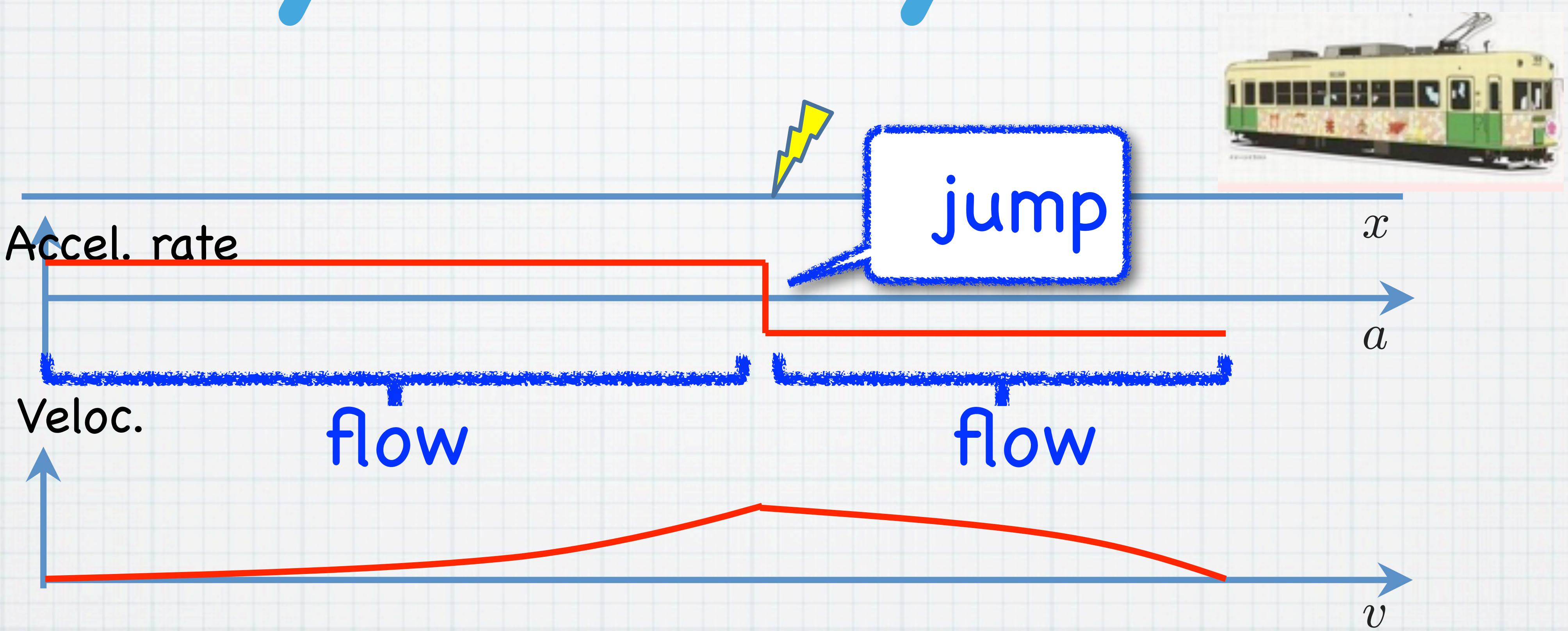

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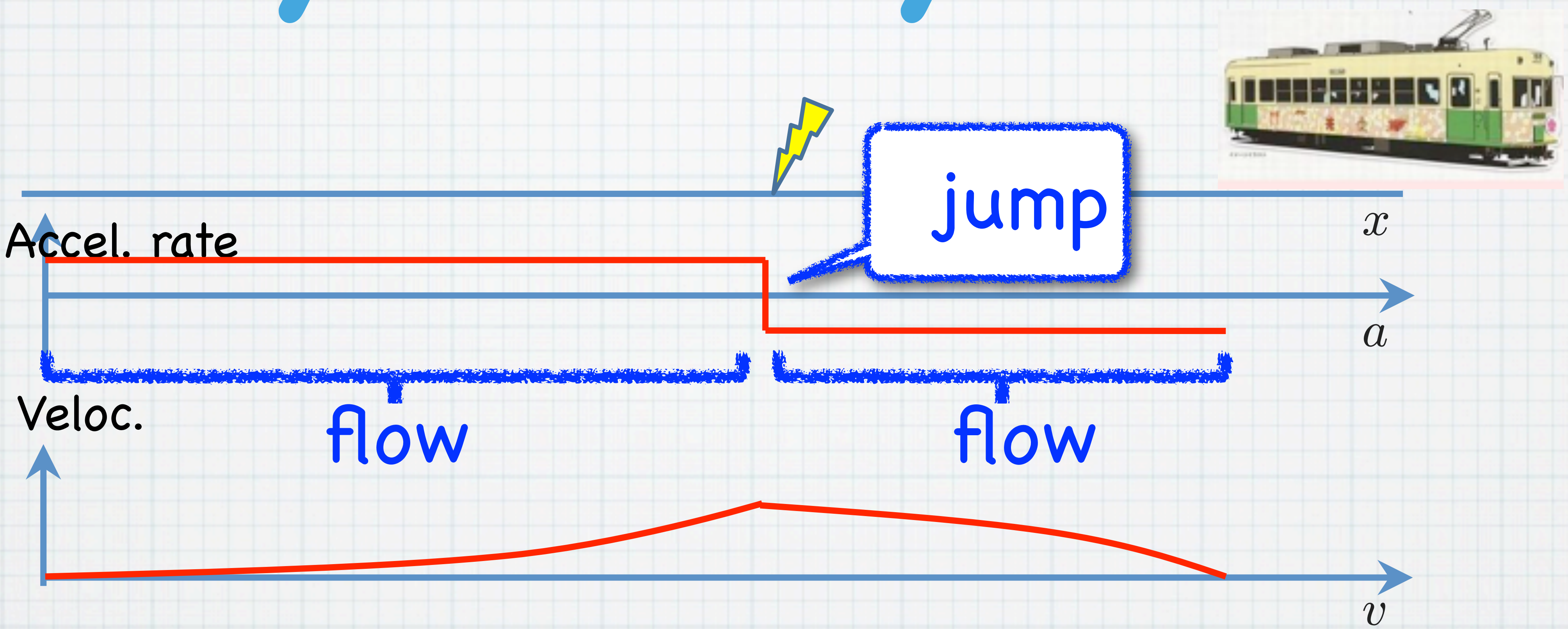
Hybrid System



- Flow & jump
  - Digital control in a physical environment
  - Component of cyber-physical systems

Hybrid System

Discrete

"jump"

and

Continuous

"flow"

Hybrid System

Discrete
"jump"

and

Continuous
"flow"

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Hybrid System

Formal verification
(computer science)

Discrete
"**jump**"

and

Continuous
"**flow**"

Control theory
(applied analysis)

Hybrid System

Formal verification
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Hybrid!

Discrete
"**jump**"

and

Continuous
"**flow**"

Hybrid!

Control theory
(applied analysis)

Hybrid System

Formal verification
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Hybrid!

- Flow?
- With minimal cost?

Discrete
"**jump**"

and

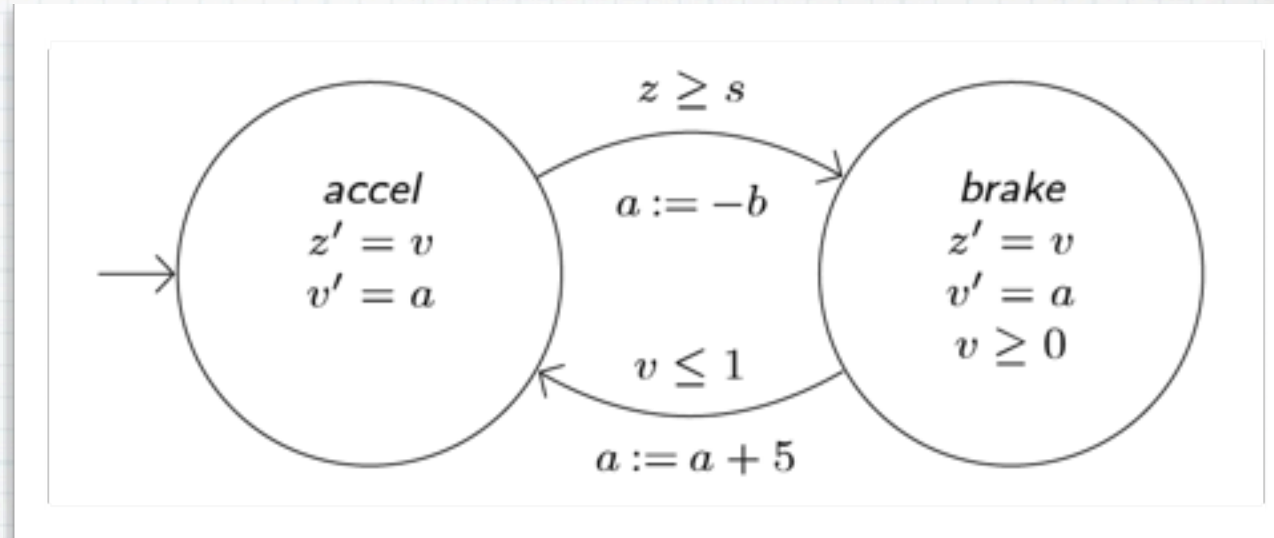
Continuous
"**flow**"

Hybrid!

Control theory
(applied analysis)

Formal Verification Approaches

- Hybrid automata
  [Alur, Henzinger, ...; '90s-]



- Differential dynamic logic
  [Platzer & others, '07-]

$$[\dot{x} = 1 \text{ while } x \leq 3]\varphi$$

- Differential equations, explicitly
  ➜ distinction jump vs. flow

Formal Verification Approaches

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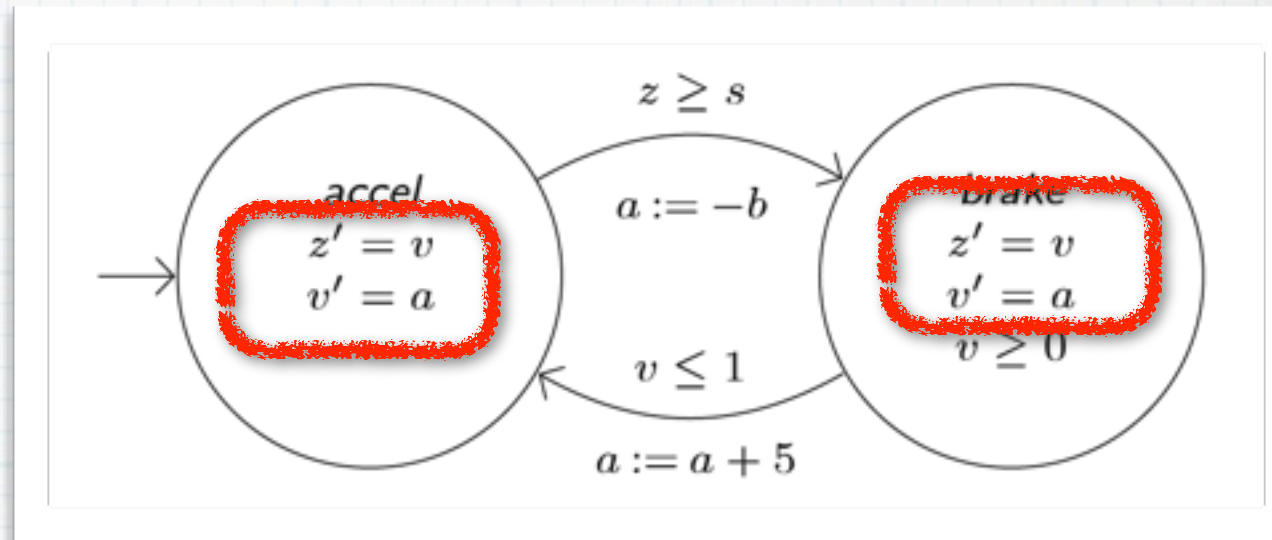

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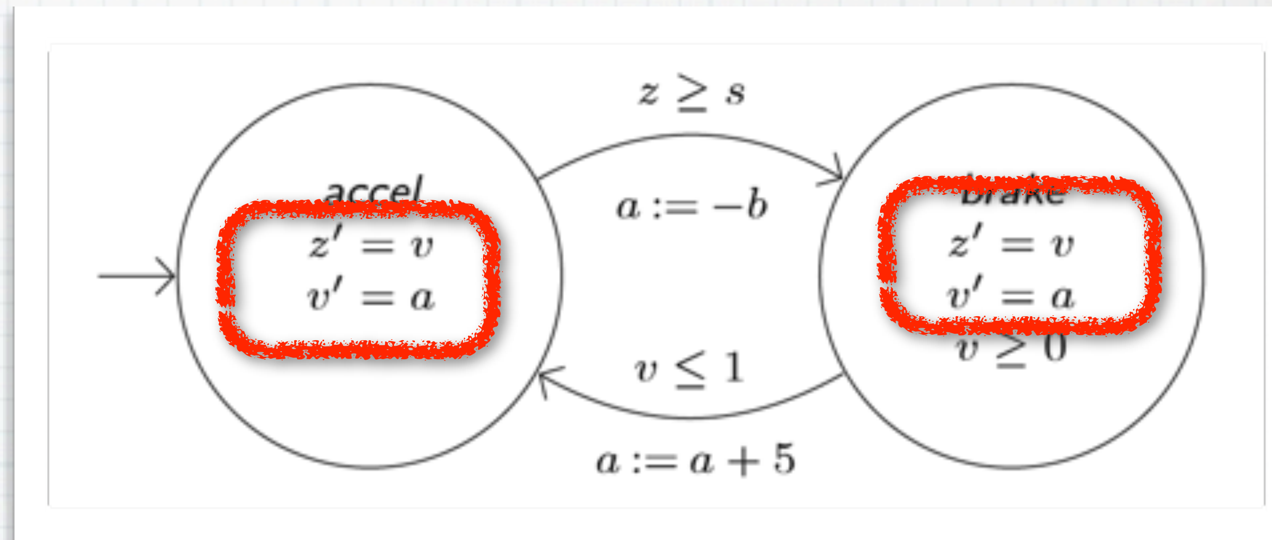

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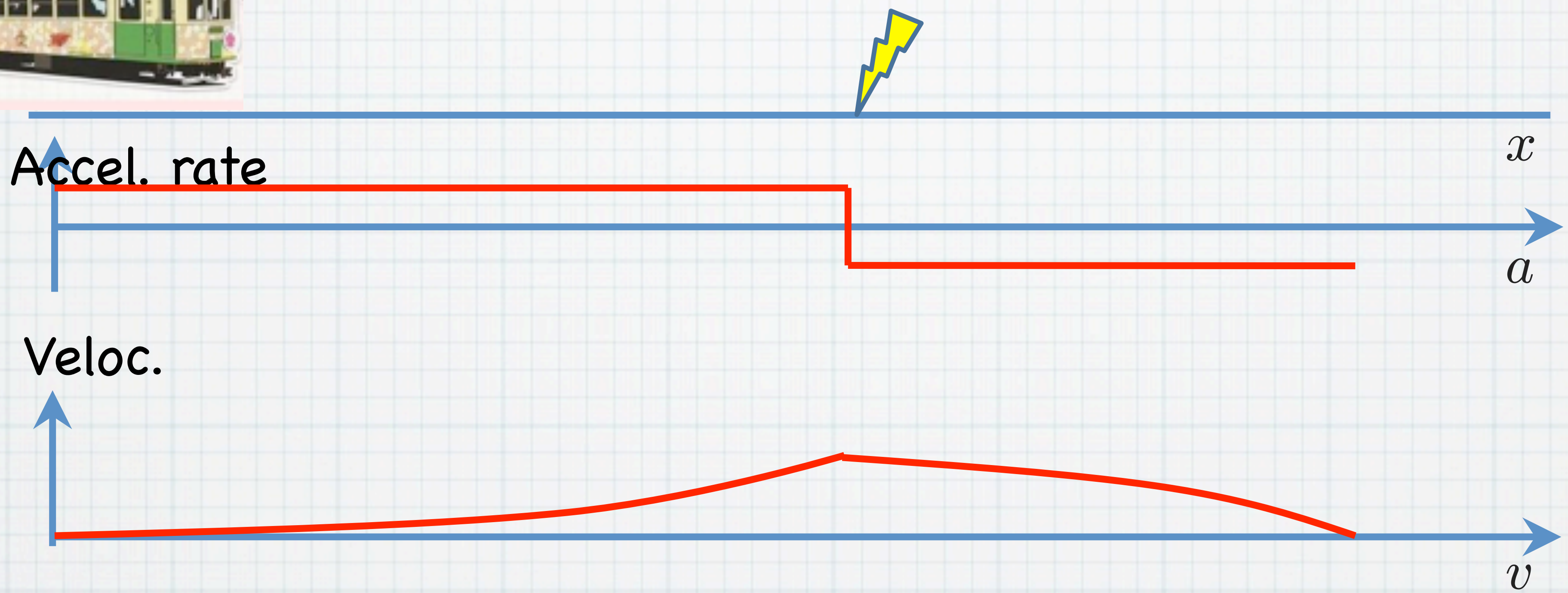
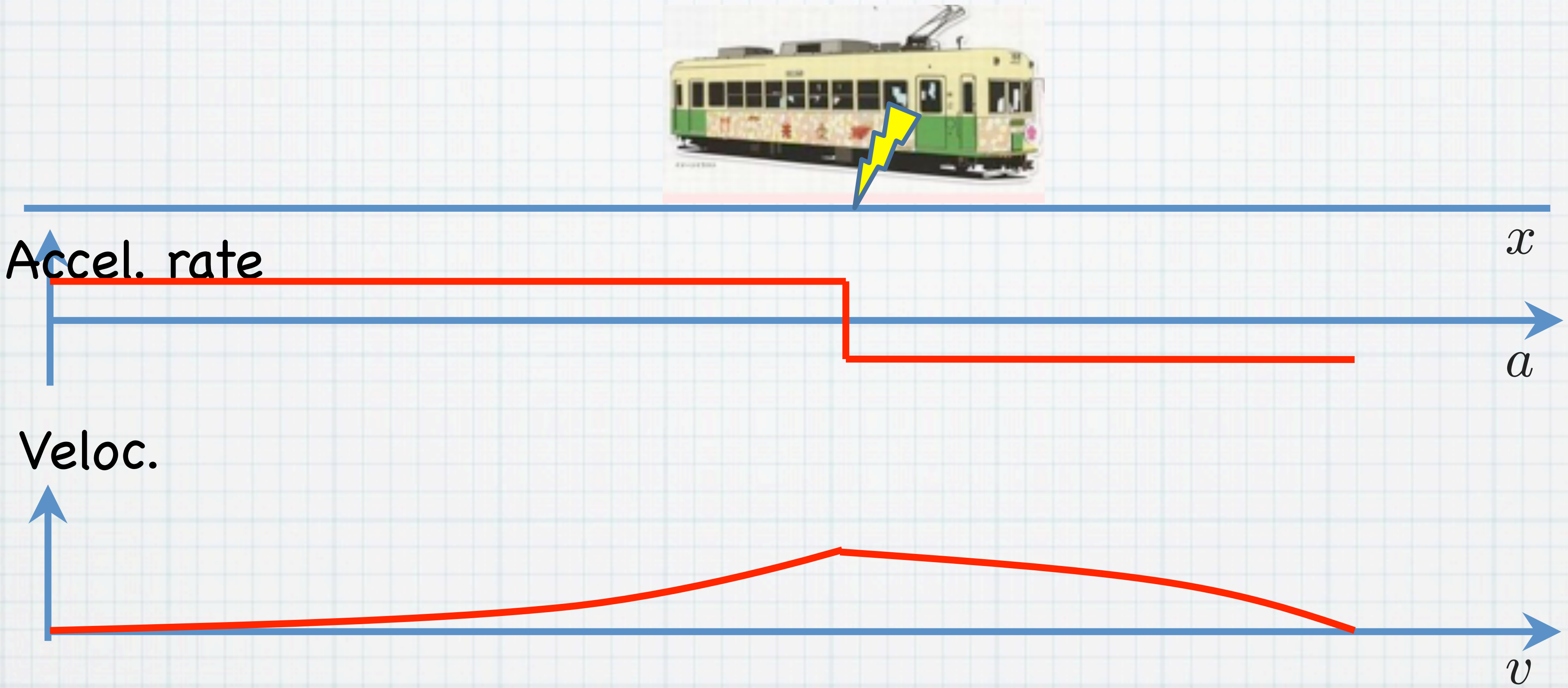


* Flow as infinitely many, infinitely small jumps

"Turn Flow into Jump"



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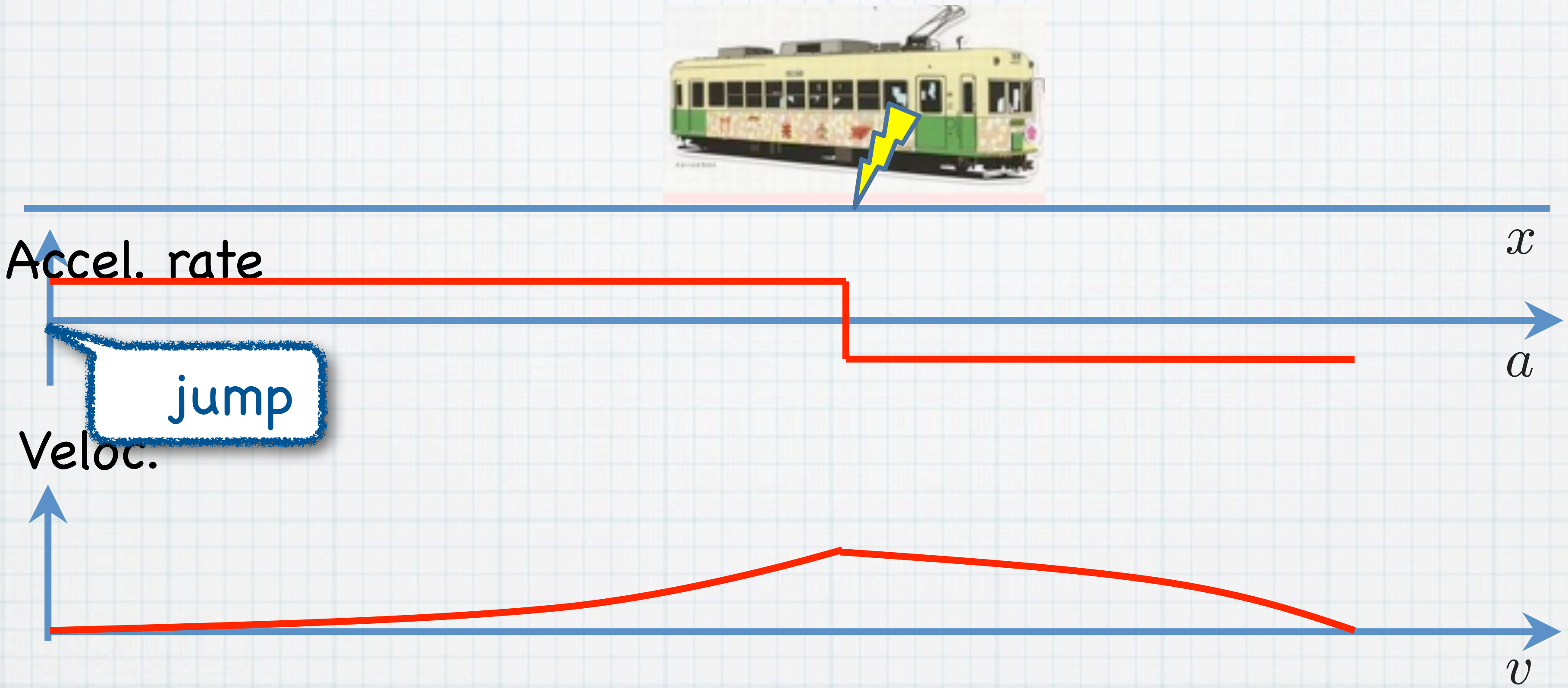

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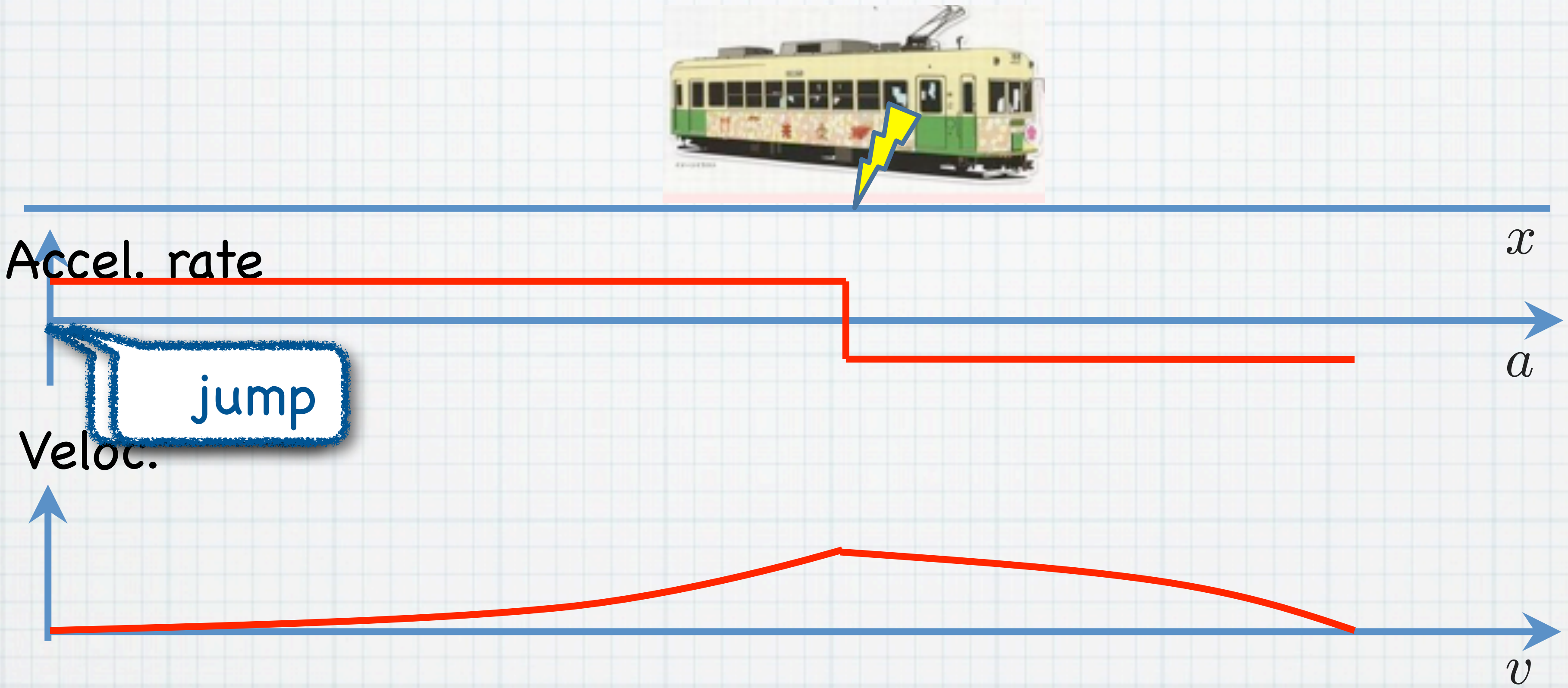

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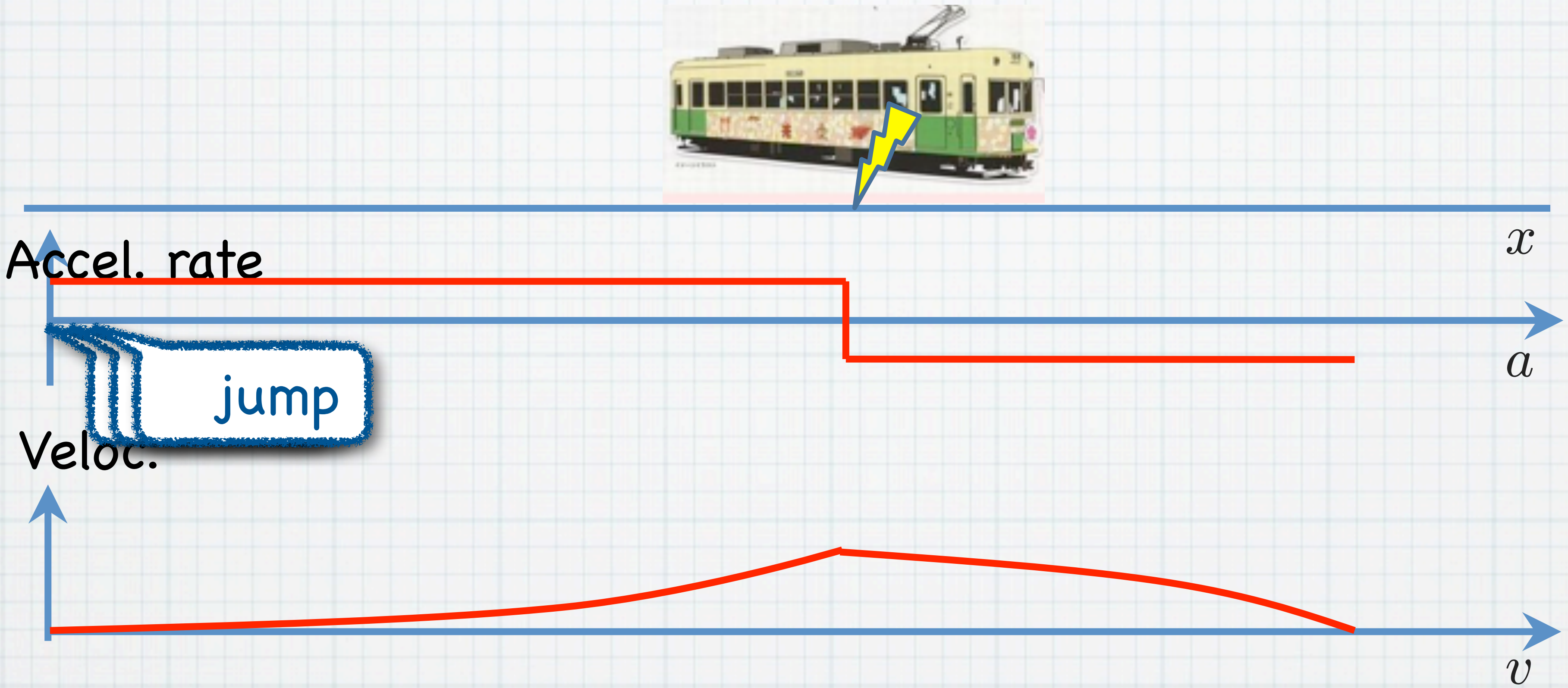

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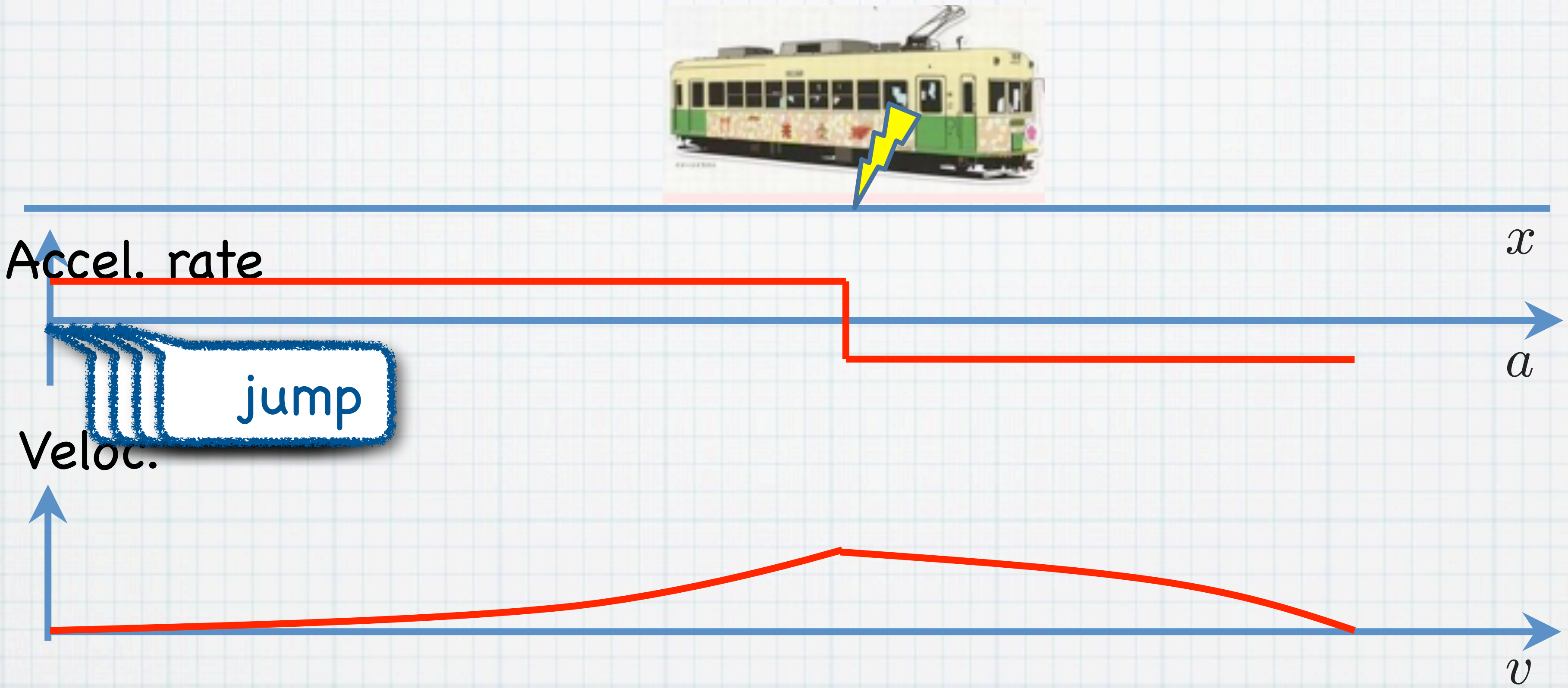

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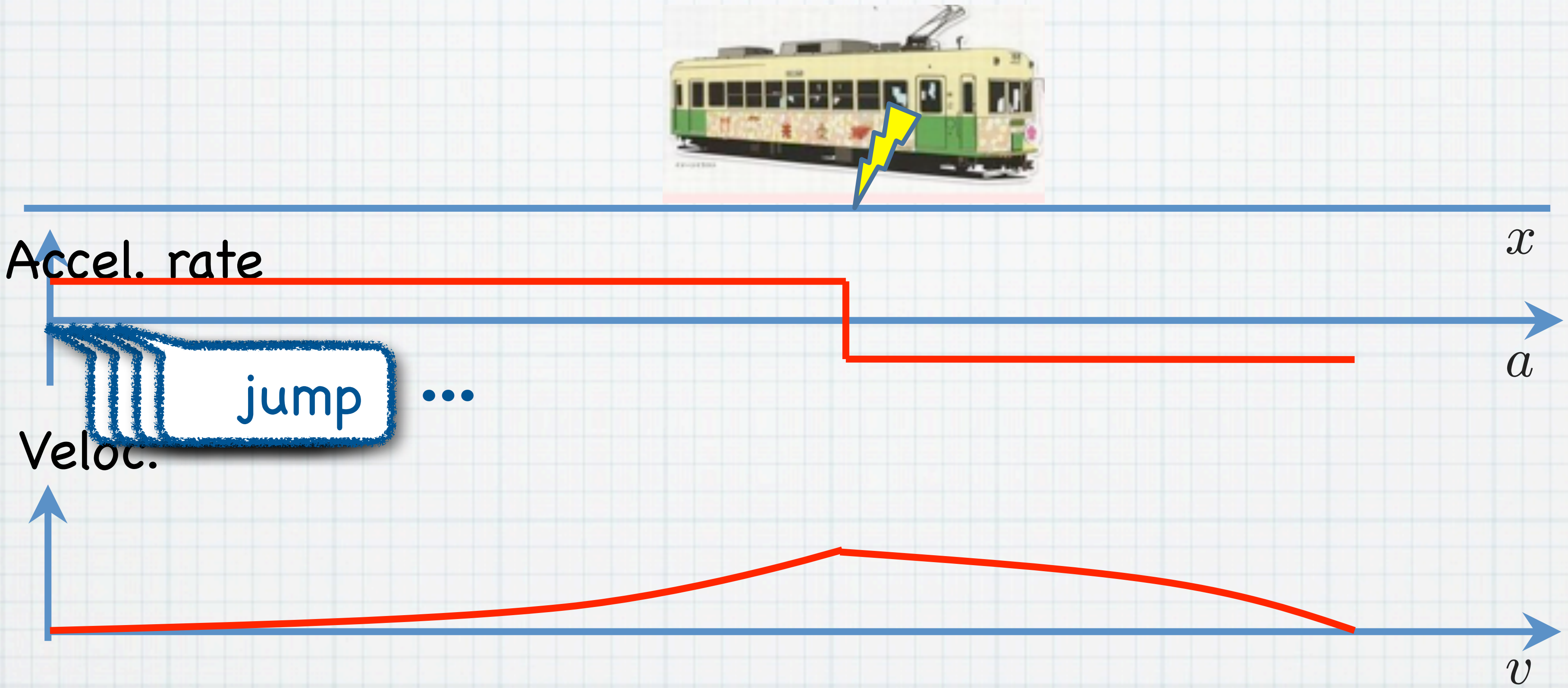

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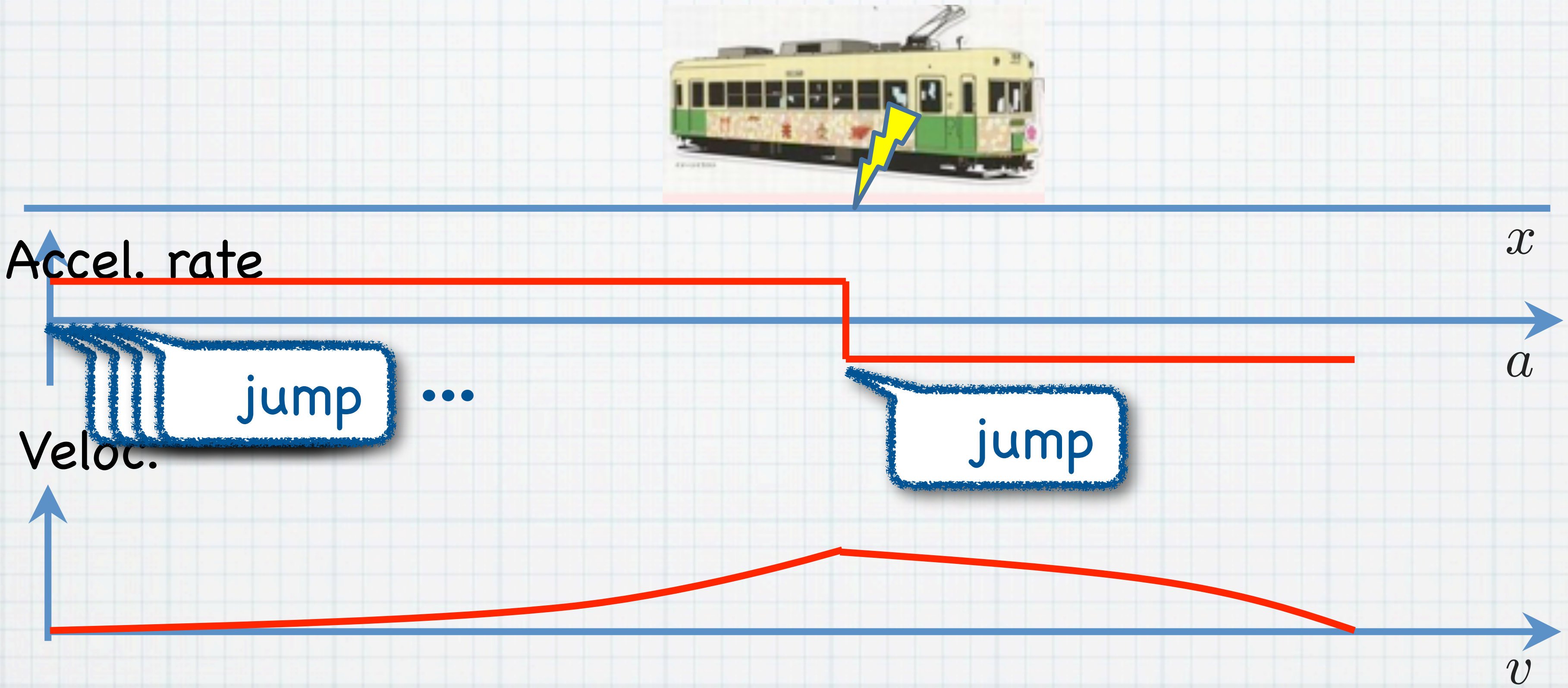

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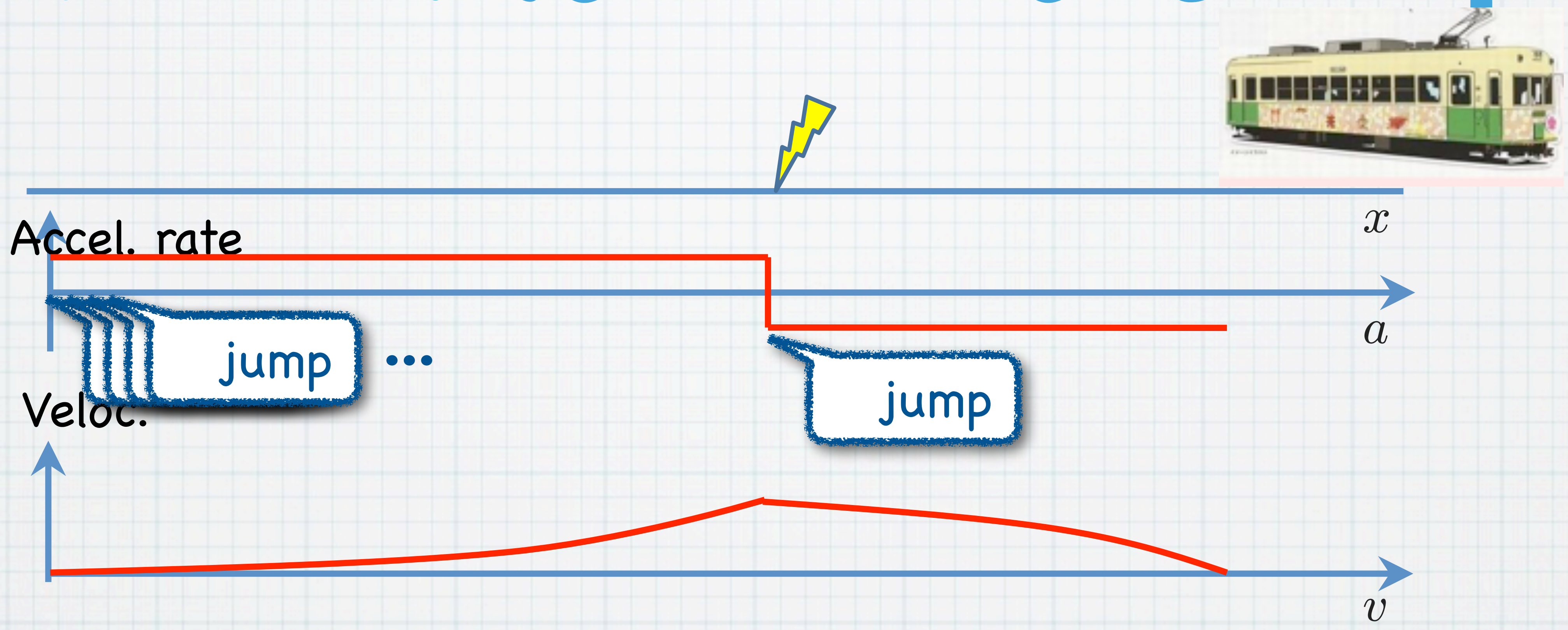

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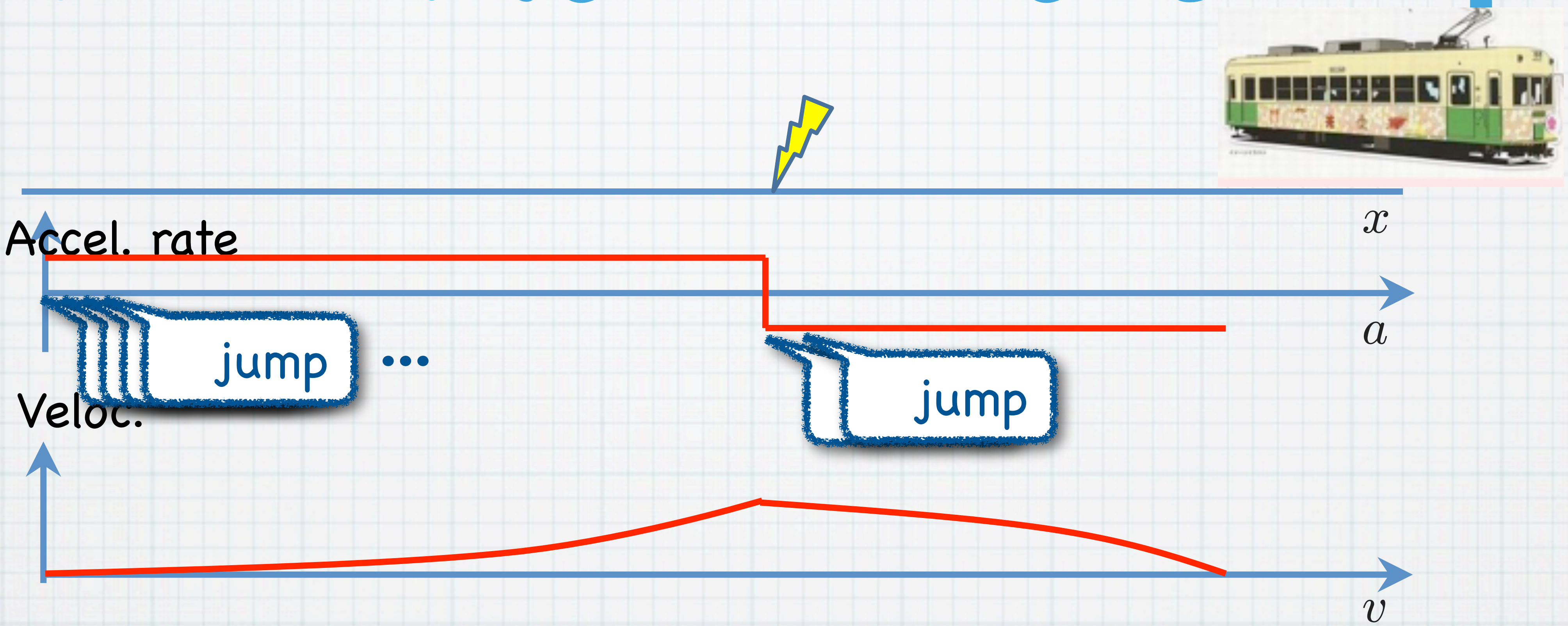

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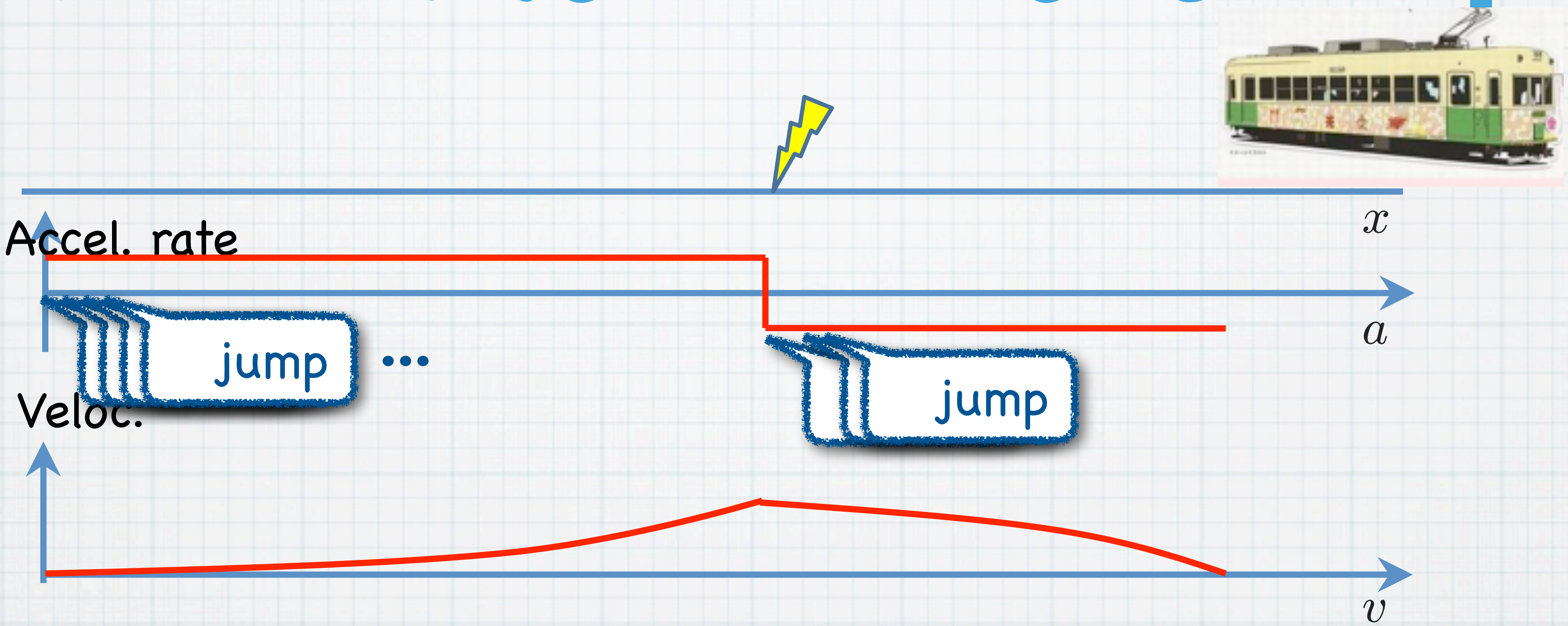

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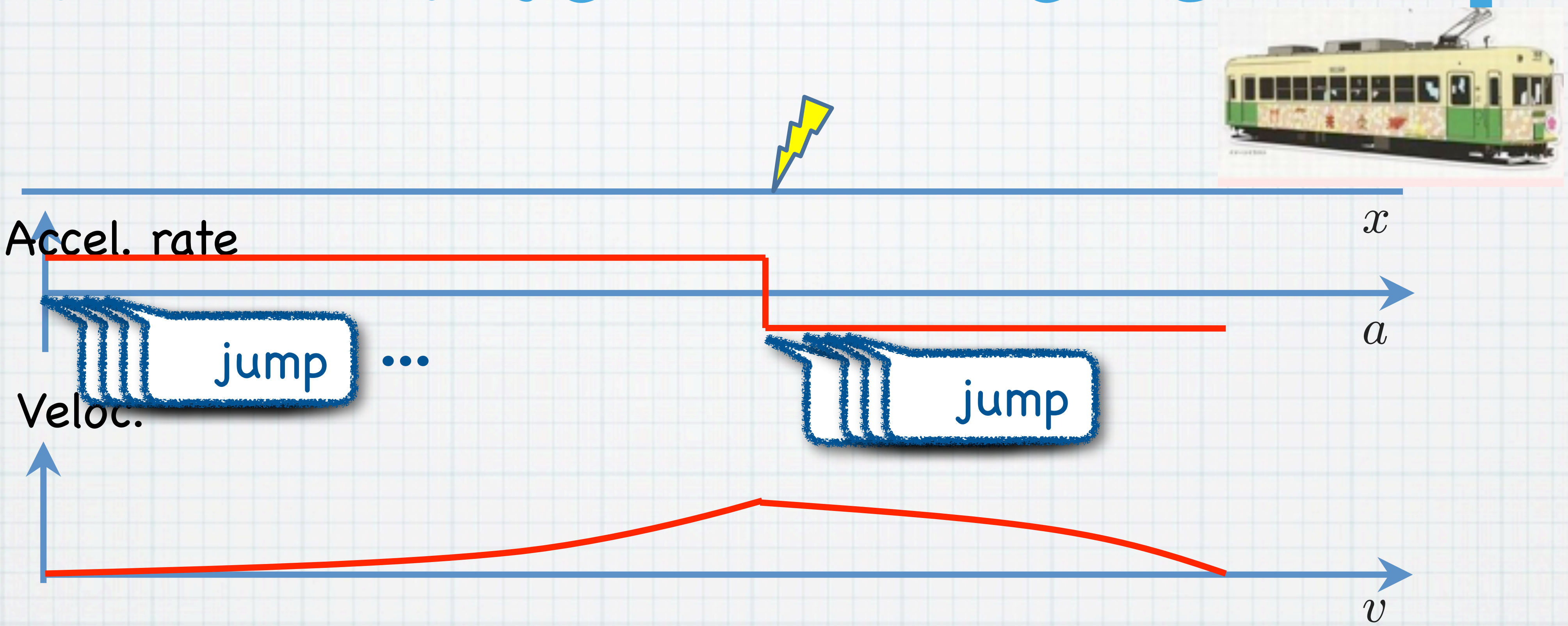

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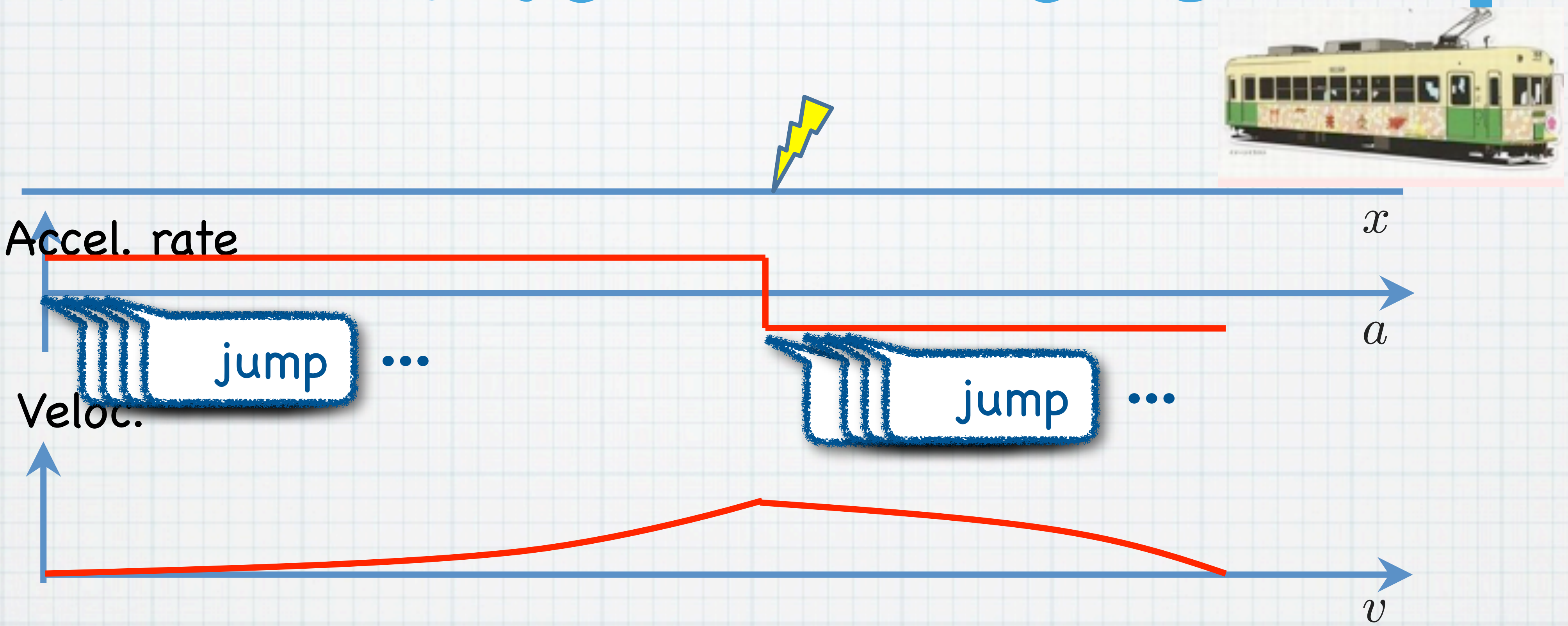

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"Turn Flow into Jump"

$$
\begin{array}{l}
t := 0\ ; \\
\texttt{while}\ (t \leq 1)\ \texttt{do}\ \{ \\
\quad t := t + \mathrm{dt} \\
\}
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    $0 < \mathrm{dt} < r$
    for **any** positive real r

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* Nonstandard analysis!
  [Robinson '60s]

Theoretical Framework [Suenaga&H., ICALP'11]

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While

Programming lang.

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while (t<a) do {
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First-order assertion lang.

$\exists z(x=2*z \land y=3*z)$

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Hoare

Hoare-style program logic

$$\frac{\{A \wedge b\}\, c\, \{A\}}{\{A\}\ \texttt{while}\ b\ \texttt{do}\ c\, \{A \wedge \neg b\}}$$

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Rigorous semantics by nonstandard analysis

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While^{dt}	**Assn**^{dt}	**Hoare**^{dt}
Programming lang.	First-order assertion lang.	Hoare-style program logic

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Rigorous semantics by nonstandard analysis

- **Hoare**^{dt} : sound and relatively complete

Static Analysis

Nonstandard Analysis

Nonstandard Static Analysis

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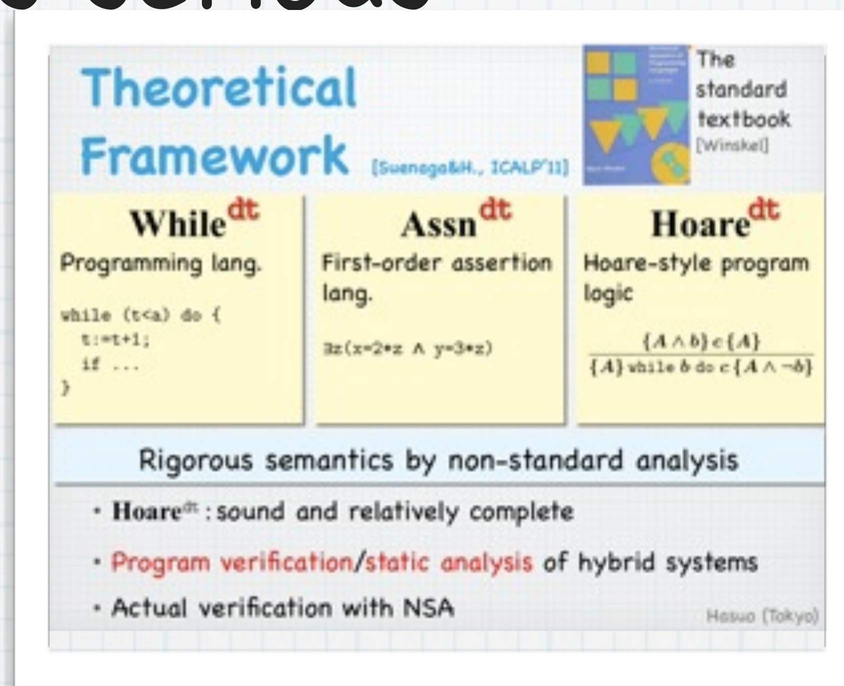
Nonstandard Static Analysis

Nonstandard Static Analysis

* Towards serious use of



Nonstandard Static Analysis

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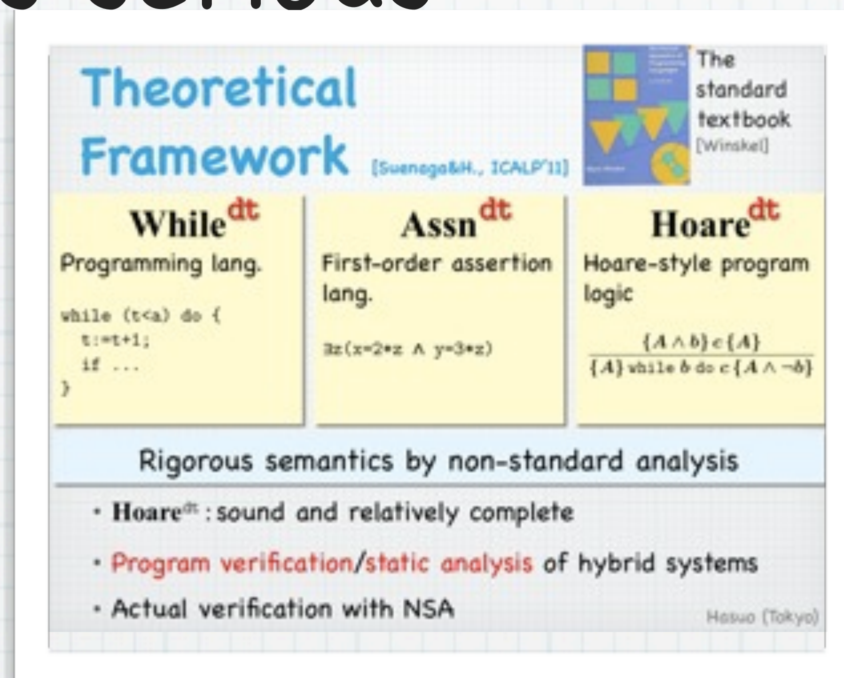

- Static analysis techniques **transferred** to hybrid appl.

Nonstandard Static Analysis

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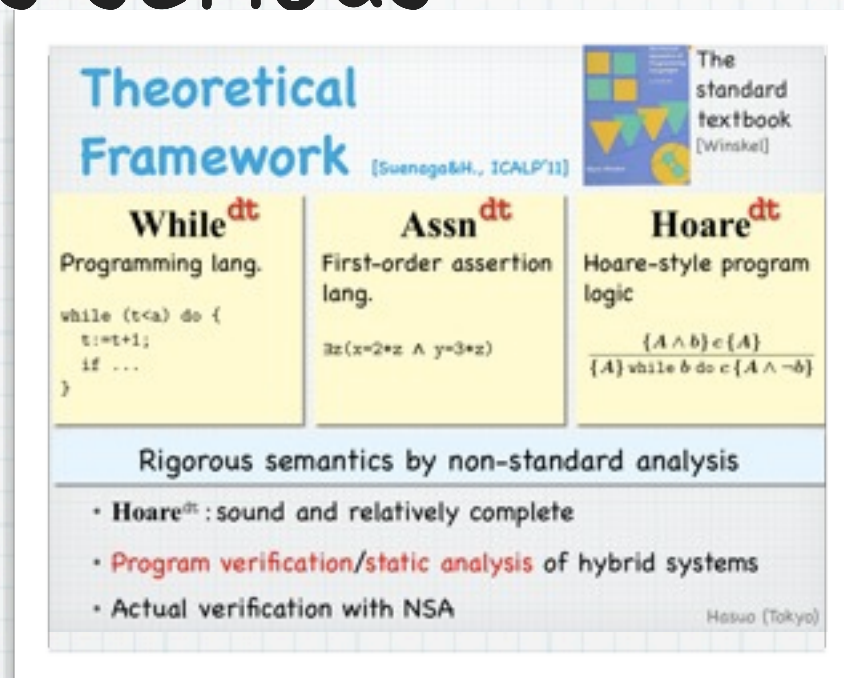

Exactly as they are!

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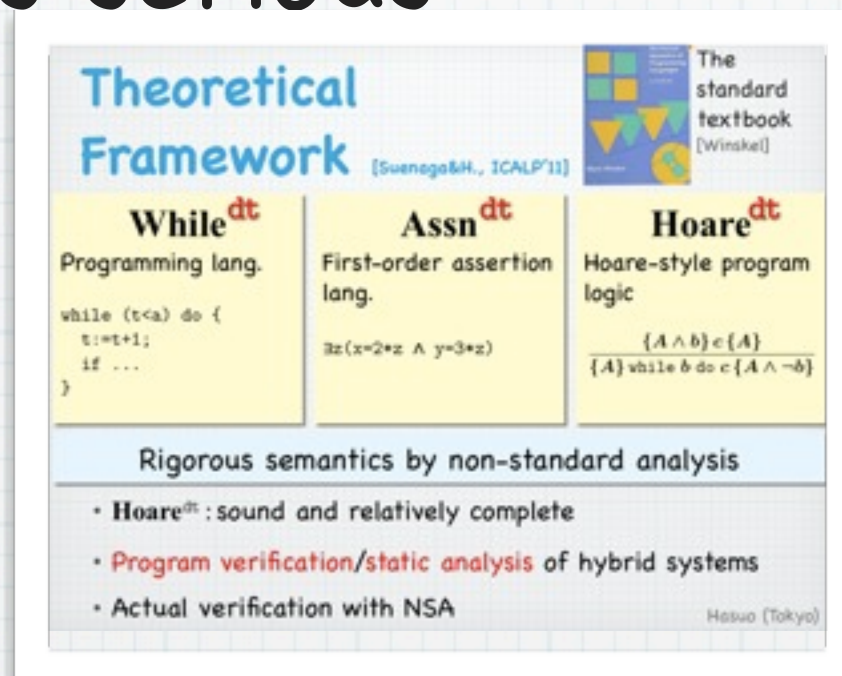
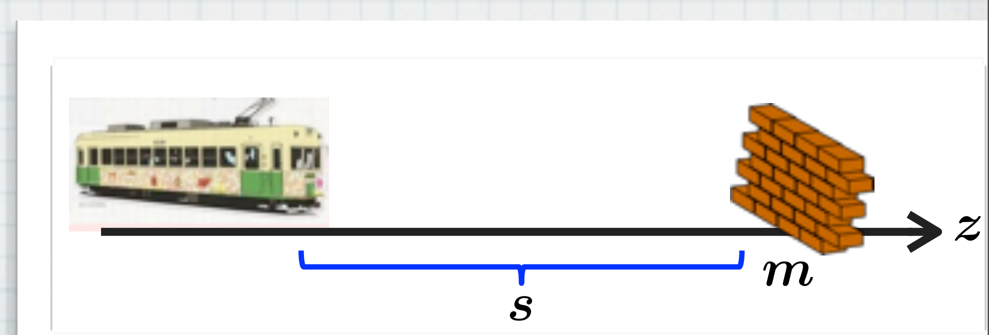


Exactly as they are!

- Static analysis techniques **transferred** to hybrid appl.
- Leading example: ETCS



Prototype Automatic Prover

Hoare^{dt} Analyzer

Prototype Automatic Prover

P

(**While**^{dt} program) →

B

(postcondition)

Hoare^{dt} Analyzer

Prototype Automatic Prover

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(**While**^{dt} program) →

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Hoare^{dt} Analyzer

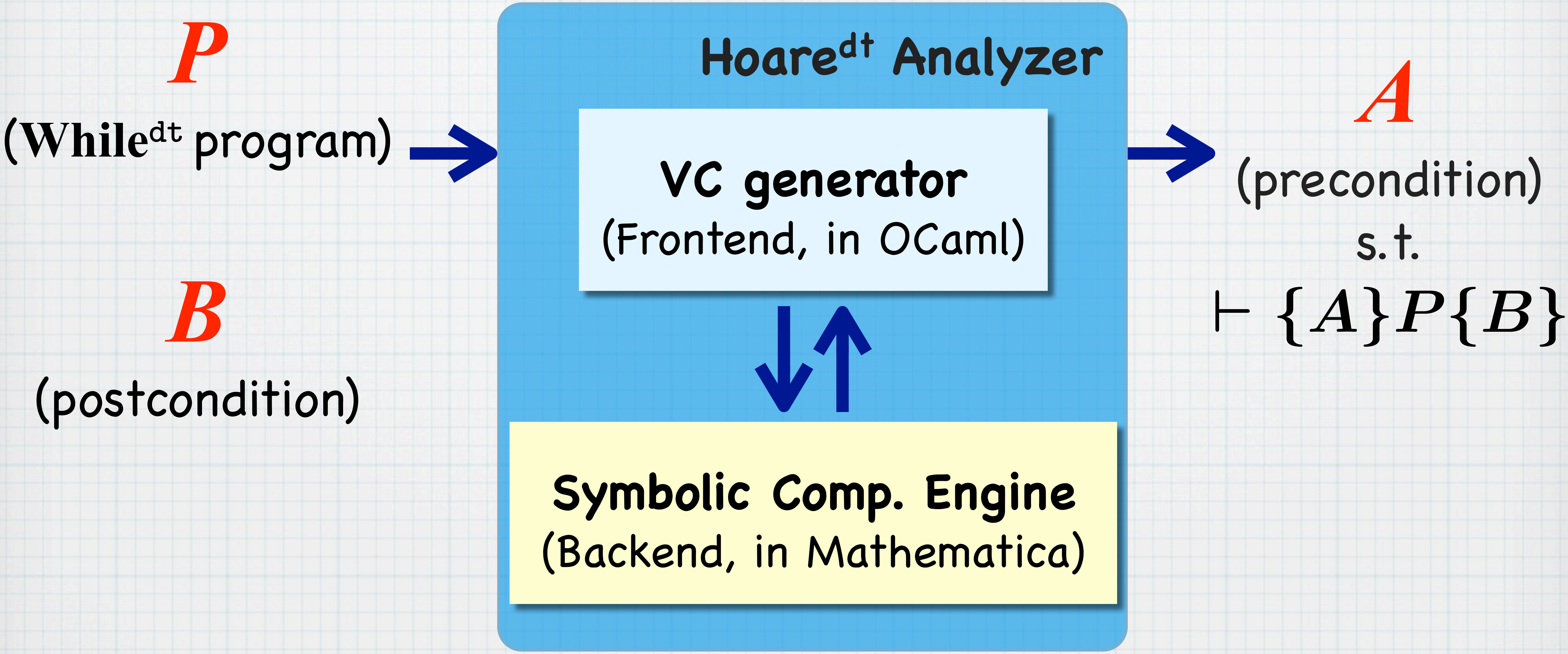
→ A

(precondition)

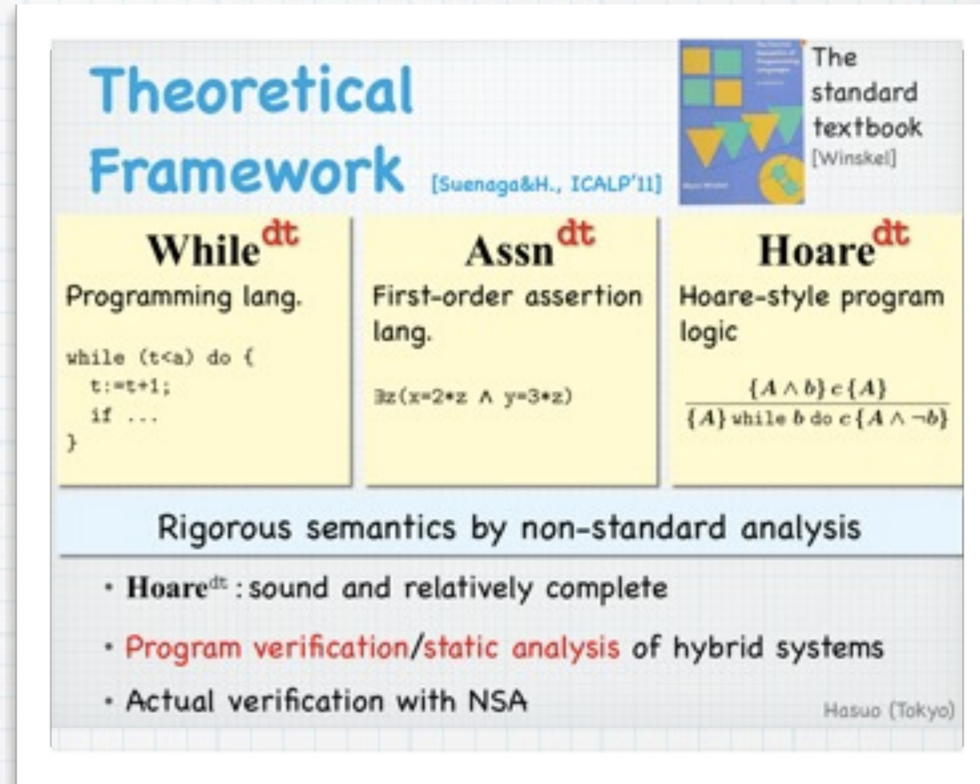
s.t.

$\vdash \{A\}P\{B\}$

Prototype Automatic Prover



Outline



- **Theoretical foundations**
  - $\mathbf{While}^{\mathrm{dt}}$, $\mathbf{Assn}^{\mathrm{dt}}$, $\mathbf{Hoare}^{\mathrm{dt}}$
  - Rigorous semantics via NSA
  - Transfer principle, "sectionwise lemmas"
- **Static analysis techniques, transferred as they are**
  - Phase split [Sharma,Dillig,Dillig,Aiken; CAV'11] [Balakrishnan,Sankaranarayanan,Ivancic,Gupta; EMSOFT'09] [Gopan,Reps; SAS'07]
  - Differential invariant [Platzer,Clarke; CAV'08]
  - ... and more!

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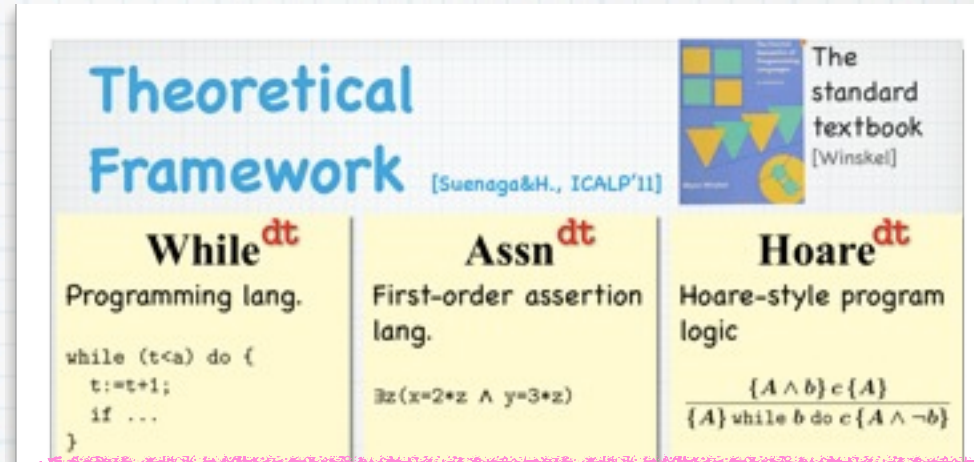

w/ or w/o dt ...

➜ logically "**the same**"

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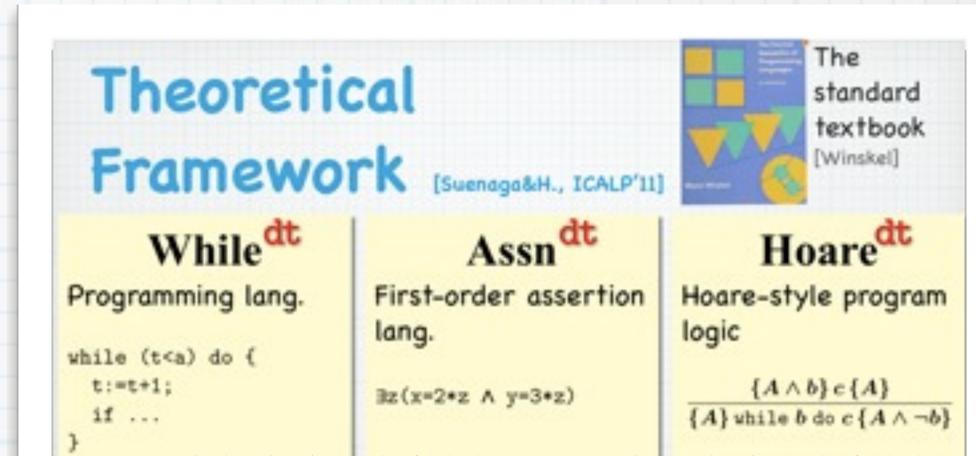

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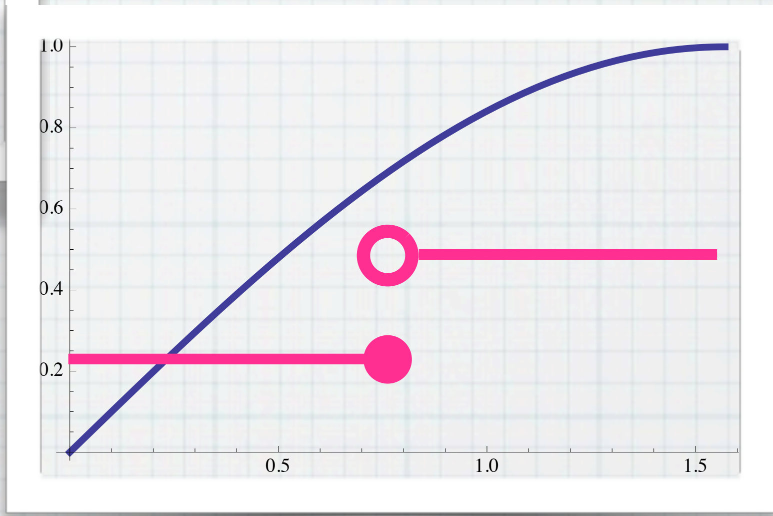
Part I:

Theoretical Foundations

Nonstandard Analysis

- Analysis with an **infinitesimal** ∂, e.g.

$$f \text{ is continuous} \iff \left(\begin{array}{l} |x - x'| \text{ is infinitesimal} \\ \implies |f(x) - f(x')| \text{ is infinitesimal} \end{array} \right)$$



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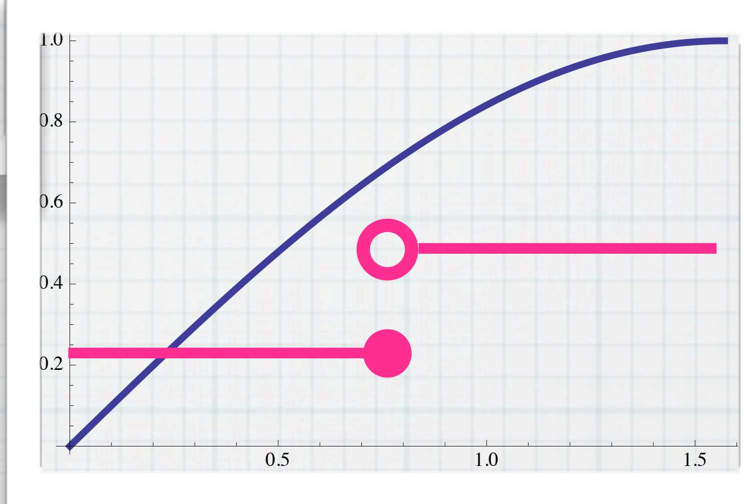

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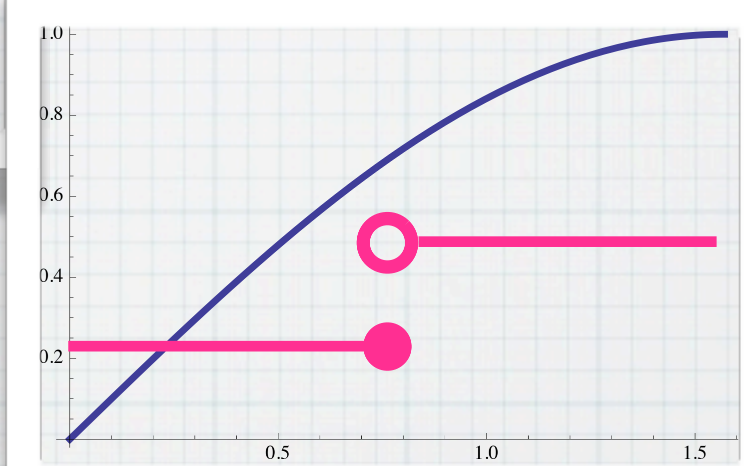

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Logical foundation via an ultrafilter

[Robinson,1960]

Hyperreals

= Reals + Infinitesimals + ...

Defn.

The set of *hyperreal numbers* is

$$^*\mathbb{R} \;:=\; \mathbb{R}^{\mathbb{N}} / \sim_{\mathcal{F}}$$

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Ignore

0th section

1st section

2nd section

Hasuo (Tokyo)

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$$\begin{array}{ll} & \big[\,(a_0, a_1, \ldots)\,\big] \\ + & \big[\,(b_0, b_1, \ldots)\,\big] \\ = & \big[\,(a_0 + b_0, a_1 + b_1, \ldots)\,\big] \end{array}$$

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- Reals are hyperreals

$$\mathbb{R} \hookrightarrow {}^*\mathbb{R}, \qquad r \mapsto [\,(r, r, \dots)\,]$$

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"For sufficiently large i"
"Except for finitely many i"

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* Predicates:
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  "for almost all i"

$$ \big[(a_i)_{i\in\mathbb{N}}\big] < \big[(b_i)_{i\in\mathbb{N}}\big] $$

$\iff \quad a_i < b_i \quad$ "for almost every i"

$\Longleftarrow \quad \{i \in \mathbb{N} \mid a_i \not< b_i\} \quad$ is finite

"For sufficiently large i"
"Except for finitely many i"

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- **Predicates:** sectionwise, "for almost all i"

$$\big[(a_i)_{i\in\mathbb{N}}\big] < \big[(b_i)_{i\in\mathbb{N}}\big]$$
$$\iff \quad a_i < b_i \qquad \text{"for almost every } i\text{"}$$
$$\Longleftarrow \quad \{i \in \mathbb{N} \mid a_i \not< b_i\} \qquad \text{is finite}$$

"For sufficiently large i"
"Except for finitely many i"

Precise defn. is via an ultrafilter $\mathcal{F}$:

$$\big[(a_i)_{i\in\mathbb{N}}\big] < \big[(b_i)_{i\in\mathbb{N}}\big]$$
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Hyperreals

= Reals + Infinitesimals + ...

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Hasuo (Tokyo)

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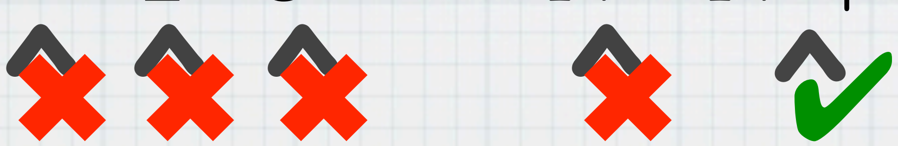

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✗ ✗ ✗ ✗ ✓ ✓

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Ultrafilter

(existence by AC)

Defn.
An *ultrafilter* $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is such that:

1. For each $X \subseteq \mathbb{N}$, exactly one of
   $$X \quad \text{and} \quad \mathbb{N} \setminus X$$
   is in $\mathcal{F}$.
2. $X, Y \in \mathcal{F} \Longrightarrow X \cap Y \in \mathcal{F}$
3. $X \in \mathcal{F}, X \subseteq Y \Longrightarrow Y \in \mathcal{F}$
4. $\emptyset \notin \mathcal{F}$

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Thm. (Transfer Principle)
A: a first-order formula.
$^*A$: its $*$-*transform*. Then

$$\mathbb{R} \models A \iff {}^*\mathbb{R} \models {}^*A .$$

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Same as A, except:
$\forall x \in \mathbb{R}$ in A is
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$\forall x \in {}^*\mathbb{R}$ in $^*A$

$\mathbb{R}$ and $^*\mathbb{R}$ are
"logically the same"

Theoretical Framework

[Suenaga&H., ICALP'11]

The standard textbook [Winskel]

$\mathbf{While}^{dt}$

Programming lang.

```
while (t<a) do {
  t:=t+1;
  if ...
}
```

$\mathbf{Assn}^{dt}$

First-order assertion lang.

```
∃z(x=2*z ∧ y=3*z)
```

$\mathbf{Hoare}^{dt}$

Hoare-style program logic

$$\frac{\{A \wedge b\}\, c\, \{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \{A \wedge \neg b\}}$$

Rigorous semantics by nonstandard analysis

Syntax

$\mathbf{While}^{\mathrm{dt}}$

$$
\begin{array}{llll}
\mathrm{AExp} \ni & a & ::= & x \mid \mathtt{c}_r \mid a_1 \ \mathtt{aop} \ a_2 \mid \mathtt{dt} \\
& & & \text{where } \mathtt{c}_r \text{ is a const. for } r \in \mathbb{R},\ \mathtt{aop} \in \{+, -, \cdot, \hat{}\,, /\} \\
\mathrm{BExp} \ni & b & ::= & \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 \\
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& & & \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c
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$\mathbf{Assn}^{\mathrm{dt}}$

$$
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$$\frac{}{\{A\}\ \mathtt{skip}\ \{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\ A[a/x]\ \}\ x := a\ \{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1 ; c_2\ \{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \{B\}}\ (\textsc{If})$$

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Tokyo)

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While + dt

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$$A \quad ::= \quad \mathtt{true} \mid \mathtt{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \mid \forall x \in {}^*\mathbb{N}.\, A \mid \forall x \in {}^*\mathbb{R}.\, A$$

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$$\mathrm{Cmd} \ni \quad c \quad ::= \quad \mathtt{skip} \mid x := a \mid c_1; c_2 \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c$$

Assn^{dt}

$$A \quad ::= \quad \mathtt{true} \mid \mathtt{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \mid \forall x \in {}^*\mathbb{N}.\, A \mid \forall x \in {}^*\mathbb{R}.\, A$$

Hoare^{dt}

$$\frac{}{\{A\}\ \mathtt{skip}\ \{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\, A[a/x]\,\}\ x := a\ \{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1; c_2\ \{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\ c\ \{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\ c\ \{B'\} \quad \models B' \Rightarrow B}{\{A\}\ c\ \{B\}}\ (\textsc{Conseq})$$

Tokyo)

Syntax

While^{dt}

While + dt

$$\begin{array}{llll} \text{AExp} \ni & a & ::= & x \mid \mathtt{c}_r \mid a_1 \ \mathtt{aop}\ a_2 \mid \mathtt{dt} \\ & & & \text{where } \mathtt{c}_r \text{ is a const. for } r \in \mathbb{R},\ \mathtt{aop} \in \{+, -, \cdot, \hat{}, /\} \\ \text{BExp} \ni & b & ::= & \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 \\ \text{Cmd} \ni & c & ::= & \mathtt{skip} \mid x := a \mid c_1; c_2 \\ & & & \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c \end{array}$$

Assn^{dt}

$$\begin{array}{lll} A & ::= & \mathtt{true} \mid \mathtt{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \mid \\ & & \forall x \in {}^*\mathbb{N}.\, A \mid \forall x \in {}^*\mathbb{R}.\, A \end{array}$$

Hoare^{dt}

$$\frac{}{\{A\}\ \mathtt{skip}\ \{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\,A[a/x]\,\}\ x := a\ \{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1; c_2\ \{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\ c\ \{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\ c\ \{B'\} \quad \models B' \Rightarrow B}{\{A\}\ c\ \{B\}}\ (\textsc{Conseq})$$

Tokyo)

Syntax

While$^{\mathrm{dt}}$

While + dt

$$\begin{array}{llll} \text{AExp} \ni & a & ::= & x \mid \mathtt{c}_r \mid a_1 \,\mathtt{aop}\, a_2 \mid \mathtt{dt} \\ & & & \text{where } \mathtt{c}_r \text{ is a const. for } r \in \mathbb{R},\ \mathtt{aop} \in \{+, -, \cdot, \hat{}, /\} \\ \text{BExp} \ni & b & ::= & \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 \\ \text{Cmd} \ni & c & ::= & \mathtt{skip} \mid x := a \mid c_1; c_2 \\ & & & \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c \end{array}$$

Assn$^{\mathrm{dt}}$

$$\begin{array}{lll} A & ::= & \mathtt{true} \mid \mathtt{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \\ & & \forall x \in {}^*\mathbb{N}.\, A \mid \forall x \in {}^*\mathbb{R}.\, A \end{array}$$

$$\frac{}{\{A\}\,\mathtt{skip}\,\{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\,A[a/x]\,\}\,x := a\,\{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\,c_1\,\{C\} \quad \{C\}\,c_2\,\{B\}}{\{A\}\,c_1; c_2\,\{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\,c_1\,\{B\} \quad \{A \wedge \neg b\}\,c_2\,\{B\}}{\{A\}\,\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\,\{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\,c\,\{A\}}{\{A\}\,\mathtt{while}\ b\ \mathtt{do}\ c\,\{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\,c\,\{B'\} \quad \models B' \Rightarrow B}{\{A\}\,c\,\{B\}}\ (\textsc{Conseq})$$

Tokyo)

Syntax

While + dt

$\mathbf{While}^{\mathtt{dt}}$

$$\begin{array}{llll}
\mathrm{AExp} \ni & a & ::= & x \mid \mathsf{c}_r \mid a_1 \ \mathsf{aop} \ a_2 \mid \mathtt{dt} \\
& & & \text{where } \mathsf{c}_r \text{ is a const. for } r \in \mathbb{R},\ \mathsf{aop} \in \{+, -, \cdot, \hat{}, /\} \\
\mathrm{BExp} \ni & b & ::= & \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 \\
\mathrm{Cmd} \ni & c & ::= & \mathtt{skip} \mid x := a \mid c_1; c_2 \\
& & & \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c
\end{array}$$

$\mathbf{Assn}^{\mathtt{dt}}$

$$\begin{array}{lll}
A & ::= & \mathtt{true} \mid \mathtt{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \\
& & \forall x \in {}^*\mathbb{N}.\ A \mid \forall x \in {}^*\mathbb{R}.\ A
\end{array}$$

$$\frac{}{\{A\}\ \mathtt{skip}\ \{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\ A[a/x]\ \}\ x := a\ \{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1; c_2\ \{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\ c\ \{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\ c\ \{B'\} \quad \models B' \Rightarrow B}{\{A\}\ c\ \{B\}}\ (\textsc{Conseq})$$

Tokyo)

Syntax

While^dt — While + dt

$$\begin{array}{llll}
\mathrm{AExp} \ni & a & ::= & x \mid \mathsf{c}_r \mid a_1 \ \mathsf{aop}\ a_2 \mid \mathsf{dt} \\
& & & \text{where } \mathsf{c}_r \text{ is a const. for } r \in \mathbb{R},\ \mathsf{aop} \in \{+,-,\cdot,\hat{},/\} \\
\mathrm{BExp} \ni & b & ::= & \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 \\
\mathrm{Cmd} \ni & c & ::= & \mathtt{skip} \mid x := a \mid c_1; c_2 \\
& & & \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c
\end{array}$$

Assn^dt — Assn, *-transformed

$$\begin{array}{lll}
A & ::= & \mathtt{true} \mid \mathtt{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \\
& & \forall x \in {}^*\mathbb{N}.\ A \mid \forall x \in {}^*\mathbb{R}.\ A
\end{array}$$

$$\frac{}{\{A\}\ \mathtt{skip}\ \{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\ A[a/x]\ \}\ x := a\ \{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1; c_2\ \{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\ c\ \{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\ c\ \{B'\} \quad \models B' \Rightarrow B}{\{A\}\ c\ \{B\}}\ (\textsc{Conseq})$$

Tokyo)

Syntax

While^{dt}

While + dt

$$\text{AExp} \ni \quad a \quad ::= \quad x \mid \mathtt{c}_r \mid a_1 \;\mathtt{aop}\; a_2 \mid \mathtt{dt}$$

where $\mathtt{c}_r$ is a const. for $r \in \mathbb{R}$, $\mathtt{aop} \in \{+, -, \cdot, \hat{}, /\}$

$$\text{BExp} \ni \quad b \quad ::= \quad \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2$$

$$\text{Cmd} \ni \quad c \quad ::= \quad \mathtt{skip} \mid x := a \mid c_1; c_2 \mid \mathtt{if}\; b \;\mathtt{then}\; c_1 \;\mathtt{else}\; c_2 \mid \mathtt{while}\; b \;\mathtt{do}\; c$$

Assn^{dt}

Assn, *-transformed

$$A \quad ::= \quad \mathtt{true} \mid \mathtt{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \mid \forall x \in {}^*\mathbb{N}.\, A \mid \forall x \in {}^*\mathbb{R}.\, A$$

Hoare^{dt}

$$\frac{}{\{A\}\;\mathtt{skip}\;\{A\}}\;(\textsc{Skip}) \qquad \frac{}{\{\,A[a/x]\,\}\; x := a\;\{A\}}\;(\textsc{Assign})$$

$$\frac{\{A\}\;c_1\;\{C\} \quad \{C\}\;c_2\;\{B\}}{\{A\}\;c_1; c_2\;\{B\}}\;(\textsc{Seq}) \qquad \frac{\{A \wedge b\}\;c_1\;\{B\} \quad \{A \wedge \neg b\}\;c_2\;\{B\}}{\{A\}\;\mathtt{if}\;b\;\mathtt{then}\;c_1\;\mathtt{else}\;c_2\;\{B\}}\;(\textsc{If})$$

$$\frac{\{A \wedge b\}\;c\;\{A\}}{\{A\}\;\mathtt{while}\;b\;\mathtt{do}\;c\;\{A \wedge \neg b\}}\;(\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\;c\;\{B'\} \quad \models B' \Rightarrow B}{\{A\}\;c\;\{B\}}\;(\textsc{Conseq})$$

Tokyo)

Syntax

While^{dt}

While + dt

$$\text{AExp} \ni \quad a \quad ::= \quad x \mid \mathtt{c}_r \mid a_1 \;\mathtt{aop}\; a_2 \mid \mathtt{dt}$$

where $\mathtt{c}_r$ is a const. for $r \in \mathbb{R}$, $\mathtt{aop} \in \{+, -, \cdot, \hat{}, /\}$

$$\text{BExp} \ni \quad b \quad ::= \quad \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2$$

$$\text{Cmd} \ni \quad c \quad ::= \quad \mathtt{skip} \mid x := a \mid c_1 ; c_2 \mid \mathtt{if}\; b \;\mathtt{then}\; c_1 \;\mathtt{else}\; c_2 \mid \mathtt{while}\; b \;\mathtt{do}\; c$$

Assn^{dt}

Assn, *-transformed

$$A \quad ::= \quad \mathtt{true} \mid \mathtt{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \mid \forall x \in {}^*\mathbb{N}.\, A \mid \forall x \in {}^*\mathbb{R}.\, A$$

Hoare^{dt}

$$\frac{}{\{A\}\,\mathtt{skip}\,\{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\,A[a/x]\,\}\; x := a\;\{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\,c_1\,\{C\} \quad \{C\}\,c_2\,\{B\}}{\{A\}\,c_1 ; c_2\,\{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\,c_1\,\{B\} \quad \{A \wedge \neg b\}\,c_2\,\{B\}}{\{A\}\,\mathtt{if}\; b\;\mathtt{then}\; c_1\;\mathtt{else}\; c_2\,\{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\,c\,\{A\}}{\{A\}\,\mathtt{while}\; b\;\mathtt{do}\; c\,\{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\,c\,\{B'\} \quad \models B' \Rightarrow B}{\{A\}\,c\,\{B\}}\ (\textsc{Conseq})$$

Tokyo)

While$^{\mathrm{dt}}$

While + dt

Syntax

AExp $\ni$ a ::= $x \mid \mathsf{c}_r \mid a_1 \;\mathsf{aop}\; a_2 \mid \mathsf{dt}$

where c_r is a const. for $r \in \mathbb{R}$, $\mathsf{aop} \in \{+, -, \cdot, \hat{}, /\}$

BExp $\ni$ b ::= $\mathsf{true} \mid \mathsf{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2$

$$\frac{}{\{A\}\,\mathtt{skip}\,\{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\,A[a/x]\,\}\; x := a\,\{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\,c_1\,\{C\} \quad \{C\}\,c_2\,\{B\}}{\{A\}\,c_1; c_2\,\{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\,c_1\,\{B\} \quad \{A \wedge \neg b\}\,c_2\,\{B\}}{\{A\}\,\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\,\{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\,c\,\{A\}}{\{A\}\,\mathtt{while}\ b\ \mathtt{do}\ c\,\{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\,c\,\{B'\} \quad \models B' \Rightarrow B}{\{A\}\,c\,\{B\}}\ (\textsc{Conseq})$$

Hoare$^{\mathrm{dt}}$

Syntax

While$^{\mathtt{dt}}$

While + dt

AExp $\ni$ a ::= $x \mid \mathtt{c}_r \mid a_1 \,\mathtt{aop}\, a_2 \mid \mathtt{dt}$

where $\mathtt{c}_r$ is a const. for $r \in \mathbb{R}$, $\mathtt{aop} \in \{+, -, \cdot, \wedge, /\}$

BExp $\ni$ b ::= $\mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2$

$$\frac{}{\{A\}\,\mathtt{skip}\,\{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\,A[a/x]\,\}\,x := a\,\{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\,c_1\,\{C\} \quad \{C\}\,c_2\,\{B\}}{\{A\}\,c_1; c_2\,\{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\,c_1\,\{B\} \quad \{A \wedge \neg b\}\,c_2\,\{B\}}{\{A\}\,\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\,\{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\,c\,\{A\}}{\{A\}\,\mathtt{while}\ b\ \mathtt{do}\ c\,\{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\,c\,\{B'\} \quad \models B' \Rightarrow B}{\{A\}\,c\,\{B\}}\ (\textsc{Conseq})$$

Hoare$^{\mathtt{dt}}$

Precisely the same rules

Syntax

While^{dt} — While + dt

$$\begin{array}{llll} \text{AExp} \ni & a & ::= & x \mid \mathtt{c}_r \mid a_1 \;\mathtt{aop}\; a_2 \mid \mathtt{dt} \\ & & & \text{where } \mathtt{c}_r \text{ is a const. for } r \in \mathbb{R},\ \mathtt{aop} \in \{+, -, \cdot, \hat{}, /\} \\ \text{BExp} \ni & b & ::= & \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 \\ \text{Cmd} \ni & c & ::= & \mathtt{skip} \mid x := a \mid c_1; c_2 \\ & & & \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c \end{array}$$

Assn^{dt} — Assn, *-transformed

$$\begin{array}{lll} A & ::= & \mathtt{true} \mid \mathtt{false} \mid A_1 \wedge A_2 \mid \neg A \mid a_1 < a_2 \mid \\ & & \forall x \in {}^*\mathbb{N}.\, A \mid \forall x \in {}^*\mathbb{R}.\, A \end{array}$$

Hoare^{dt} — Precisely the same rules

$$\frac{}{\{A\}\ \mathtt{skip}\ \{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\, A[a/x] \,\}\ x := a\ \{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1; c_2\ \{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\ c\ \{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\ c\ \{B'\} \quad \models B' \Rightarrow B}{\{A\}\ c\ \{B\}}\ (\textsc{Conseq})$$

Tokyo)

Syntax

While$^{\mathrm{dt}}$

While + dt

AExp $\ni$ a ::= $x \mid c_r \mid a_1 \;\mathtt{aop}\; a_2 \mid \mathtt{dt}$

where c_r is a const. for $r \in \mathbb{R}$, $\mathtt{aop} \in \{+, -, \cdot, \hat{}, /\}$

BExp $\ni$ b ::= $\mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2$

Cmd $\ni$ c ::= $\mathtt{skip} \mid x := a \mid c_1; c_2$
$\mid \mathtt{if}\; b \;\mathtt{then}\; c_1 \;\mathtt{else}\; c_2 \mid \mathtt{while}\; b \;\mathtt{do}\; c$

$< a_2 \mid$

Thm.
HOARE$^{\mathrm{dt}}$ rules are sound and relatively complete.

Hoare$^{\mathrm{dt}}$

Precise...

$$\frac{}{\{A\}\,\mathtt{skip}\,\{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{A[a/x]\}\, x := a\,\{A\}}$$

$$\frac{\{A\}\,c_1\,\{C\} \quad \{C\}\,c_2\,\{B\}}{\{A\}\,c_1; c_2\,\{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\,c_1\,\{B\} \quad \{A \wedge \neg b\}\,c_2\,\{B\}}{\{A\}\,\mathtt{if}\; b\;\mathtt{then}\; c_1\;\mathtt{else}\; c_2\,\{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\,c\,\{A\}}{\{A\}\,\mathtt{while}\; b\;\mathtt{do}\; c\,\{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\,c\,\{B'\} \quad \models B' \Rightarrow B}{\{A\}\,c\,\{B\}}\ (\textsc{Conseq})$$

Tokyo)

Denotational Semantics

- Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (t < 1)
  t := t + dt;
```

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (t < 1)
  t := t + dt;
```

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := (0,0,0,...);
while (t < (1,1,1,...))
```

$$\texttt{t := t +} \left(1, \frac{1}{2}, \frac{1}{3}, \dots\right) \ ;$$

Denotational Semantics

- Execute sectionwise and bundle up the outcomes!

0th section	1st section	2nd section	
`t := 0;` `while (t < 1)` $t := t + 1;$	`t := 0;` `while (t < 1)` $t := t + \frac{1}{2};$	`t := 0;` `while (t < 1)` $t := t + \frac{1}{3};$	...

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

0th section	1st section	2nd section	

`t := 0;`
`while (t < 1)`
`t := t + 1 ;`
	`t := 0;`
	`while (t < 1)`
	$t := t + \frac{1}{2}$;
		`t := 0;`
		`while (t < 1)`
		$t := t + \frac{1}{3}$;
			…

t = 1	t = 1	t = 1	…

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

$$\texttt{t := } (0,0,0,\ldots)\texttt{;}$$
$$\texttt{while (t < } (1,1,1,\ldots)\texttt{)}$$
$$\texttt{t := t + } \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right) \texttt{;}$$

$$\texttt{t = } (1,1,1,\ldots)$$

Denotational Semantics

* Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (t < 1)
  t := t + dt;
```

```
t = 1
```

Denotational Semantics

$$[\![x]\!]\sigma := \sigma(x) \qquad [\![\mathtt{c}_r]\!]\sigma := r \quad \text{for each } r \in \mathbb{R}$$

$$[\![a_1 \ \mathtt{aop}\ a_2]\!]\sigma := [\![a_1]\!]\sigma \ \mathtt{aop}\ [\![a_2]\!]\sigma$$

$$[\![\mathtt{dt}]\!]\sigma := \omega^{-1} = \left[\,(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots)\,\right] \qquad [\![\infty]\!]\sigma := \omega = \left[\,(1, 2, 3, \dots)\,\right]$$

$$[\![\mathtt{true}]\!]\sigma := \mathtt{tt} \qquad [\![\mathtt{false}]\!]\sigma := \mathtt{ff}$$

$$[\![b_1 \wedge b_2]\!]\sigma := [\![b_1]\!]\sigma \wedge [\![b_2]\!]\sigma \qquad [\![\neg b]\!]\sigma := \neg([\![b]\!]\sigma)$$

$$[\![a_1 < a_2]\!]\sigma := [\![a_1]\!]\sigma < [\![a_2]\!]\sigma$$

$$[\![\mathtt{skip}]\!]\sigma := \sigma \qquad [\![x := a]\!]\sigma := \sigma\left[\,x \mapsto [\![a]\!]\sigma\,\right] \qquad [\![c_1; c_2]\!]\sigma := [\![c_2]\!]\left(\,[\![c_1]\!]\sigma\,\right)$$

$$[\![\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2]\!]\sigma := \begin{cases} [\![c_1]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathtt{tt} \\ [\![c_2]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathtt{ff} \end{cases}$$

$$[\![\mathtt{while}\ b\ \mathtt{do}\ c]\!]\sigma := \Big(\,[\![\,(\mathtt{while}\ b\ \mathtt{do}\ c)|_i\,]\!](\sigma|_i)\,\Big)_{i \in \mathbb{N}}$$

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Denotational Semantics

$$[\![x]\!]\sigma := \sigma(x) \qquad [\![\mathtt{c}_r]\!]\sigma := r \quad \text{for each } r \in \mathbb{R}$$

$$[\![a_1 \ \mathtt{aop}\ a_2]\!]\sigma := [\![a_1]\!]\sigma \ \mathtt{aop}\ [\![a_2]\!]\sigma$$

$$[\![\mathtt{dt}]\!]\sigma := \omega^{-1} = \left[\,(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots)\,\right] \qquad [\![\infty]\!]\sigma := \omega = \left[\,(1, 2, 3, \dots)\,\right]$$

$$[\![\mathtt{true}]\!]\sigma := \mathrm{tt} \qquad [\![\mathtt{false}]\!]\sigma := \mathrm{ff}$$

$$[\![b_1 \wedge b_2]\!]\sigma := [\![b_1]\!]\sigma \wedge [\![b_2]\!]\sigma \qquad [\![\neg b]\!]\sigma := \neg([\![b]\!]\sigma)$$

$$[\![a_1 < a_2]\!]\sigma := [\![a_1]\!]\sigma < [\![a_2]\!]\sigma$$

$$[\![\mathtt{skip}]\!]\sigma := \sigma \qquad [\![x := a]\!]\sigma := \sigma\left[\,x \mapsto [\![a]\!]\sigma\,\right] \qquad [\![c_1; c_2]\!]\sigma := [\![c_2]\!]\left(\,[\![c_1]\!]\sigma\,\right)$$

$$[\![\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2]\!]\sigma := \begin{cases} [\![c_1]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{tt} \\ [\![c_2]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{ff} \end{cases}$$

$$[\![\mathtt{while}\ b\ \mathtt{do}\ c]\!]\sigma := \Big(\,[\![\,(\mathtt{while}\ b\ \mathtt{do}\ c)|_i\,]\!](\sigma|_i)\,\Big)_{i \in \mathbb{N}}$$

Sectionwise definition

Hasuo (Tokyo)

Denotational Semantics

$$
\begin{aligned}
[\![x]\!]\sigma &:= \sigma(x) \\
[\![a_1 \ \mathtt{aop}\ a_2]\!]\sigma &:= [\![a_1]\!]\sigma \ \mathtt{aop}\ [\![a_2]\!]\sigma \\
[\![\mathtt{dt}]\!]\sigma &:= \omega^{-1} = \left[\, (1, \tfrac{1}{2}, \tfrac{1}{3}, \dots)\, \right]
\end{aligned}
$$

$$
\begin{aligned}
[\![\mathtt{true}]\!]\sigma &:= \mathrm{tt} \\
[\![b_1 \wedge b_2]\!]\sigma &:= [\![b_1]\!]\sigma \wedge [\![b_2]\!]\sigma \\
[\![a_1 < a_2]\!]\sigma &:= [\![a_1]\!]\sigma < [\![a_2]\!]\sigma
\end{aligned}
$$

$[\![\mathtt{fal}$

$[\![\neg b$

Def.

The ***i*-*th section*** of a $\mathrm{W{\scriptsize HILE}}^{\mathrm{dt}}$ expression e is

$$
e|_i \quad :\equiv \quad e\left[\, \frac{1}{i+1} \,/\, \mathtt{dt} \,\right] .
$$

$$
[\![\mathtt{skip}]\!]\sigma := \sigma \qquad [\![x := a]\!]\sigma := \sigma\left[\, x \mapsto [\![a]\!]\sigma \,\right] \qquad [\![c_1; c_2]\!]\sigma := [\![c_2]\!]\left([\![c_1]\!]\sigma \right)
$$

$$
[\![\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2]\!]\sigma := \begin{cases} [\![c_1]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{tt} \\ [\![c_2]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{ff} \end{cases}
$$

$$
[\![\mathtt{while}\ b\ \mathtt{do}\ c]\!]\sigma := \Big([\![\, (\mathtt{while}\ b\ \mathtt{do}\ c)|_i \,]\!](\sigma|_i) \Big)_{i \in \mathbb{N}}
$$

Sectionwise definition

Denota

$$\left[\begin{array}{l} t := 0\,; \\ \texttt{while}\ (t \le 1)\ \texttt{do} \\ \quad t := t + \mathtt{dt} \end{array}\right] \overset{i\text{-th section}}{\longmapsto} \left[\begin{array}{l} t := 0\,; \\ \texttt{while}\ (t \le 1)\ \texttt{do} \\ \quad t := t + \frac{1}{i+1} \end{array}\right]$$

Def.
The *i-th section* of a WHILE$^{\mathrm{dt}}$ expression e is

$$e|_i \;:\equiv\; e\left[\,\frac{1}{i+1}\,/\,\mathtt{dt}\,\right]\,.$$

$$[\![x]\!]\sigma \;:=\; \sigma(x)$$
$$[\![a_1\ \mathtt{aop}\ a_2]\!]\sigma \;:=\; [\![a_1]\!]\sigma\ \mathtt{aop}\ [\![a_2]\!]\sigma$$
$$[\![\mathtt{dt}]\!]\sigma \;:=\; \omega^{-1} = \left[\,(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots)\,\right]$$

$$[\![\mathtt{true}]\!]\sigma \;:=\; \mathrm{tt} \qquad [\![\mathtt{fal}$$
$$[\![b_1 \wedge b_2]\!]\sigma \;:=\; [\![b_1]\!]\sigma \wedge [\![b_2]\!]\sigma \qquad [\![\neg b$$
$$[\![a_1 < a_2]\!]\sigma \;:=\; [\![a_1]\!]\sigma < [\![a_2]\!]\sigma$$

$$[\![\mathtt{skip}]\!]\sigma \;:=\; \sigma \qquad [\![x := a]\!]\sigma \;:=\; \sigma\left[\,x \mapsto [\![a]\!]\sigma\,\right] \qquad [\![c_1; c_2]\!]\sigma \;:=\; [\![c_2]\!]\left([\![c_1]\!]\sigma\right)$$

$$[\![\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2]\!]\sigma := \begin{cases} [\![c_1]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{tt} \\ [\![c_2]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{ff} \end{cases}$$

$$[\![\mathtt{while}\ b\ \mathtt{do}\ c]\!]\sigma := \Big(\, [\![\, (\mathtt{while}\ b\ \mathtt{do}\ c)|_i \,]\!](\sigma|_i) \,\Big)_{i \in \mathbb{N}}$$

Sectionwise definition

Hasuo (Tokyo)

“Sectionwise Lemmas”

Sectionwise Execution Lemma. For any expr. $\boldsymbol{e}$ and $\boldsymbol{i} \in \mathbb{N}$,

$$[\![\boldsymbol{e}]\!]\boldsymbol{\sigma} = \left[\,\left(\,[\![\boldsymbol{e}|_{\boldsymbol{i}}]\!](\boldsymbol{\sigma}|_{\boldsymbol{i}})\,\right)_{\boldsymbol{i}\in\mathbb{N}}\,\right] \ .$$

Sectionwise Satisfaction Lemma. For any hyperstate $\boldsymbol{\sigma}$ and an $\textsc{Assn}^{\mathrm{dt}}$ formula $\boldsymbol{\varphi}$:

$$\boldsymbol{\sigma} \models \boldsymbol{\varphi} \quad \Longleftrightarrow \quad \boldsymbol{\sigma}|_{\boldsymbol{i}} \models \boldsymbol{\varphi}|_{\boldsymbol{i}} \quad \text{for almost every } \boldsymbol{i}.$$

“Sectionwise Lemmas”

Sectionwise Execution Lemma. For any expr. $\boldsymbol{e}$ and $\boldsymbol{i} \in \mathbb{N}$,

$$[\![e]\!] \boldsymbol{\sigma} = \Big[\Big([\![e|_i]\!] (\boldsymbol{\sigma}|_i) \Big)_{\boldsymbol{i} \in \mathbb{N}} \Big] \ .$$

“Whole semantics is sectionwise”

Sectionwise Satisfaction Lemma. For any hyperstate $\boldsymbol{\sigma}$ and an $\mathrm{Assn}^{\mathrm{dt}}$ formula $\boldsymbol{\varphi}$:

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Hasuo (Tokyo)

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Sectionwise Execution Lemma. For any expr. e and $i \in \mathbb{N}$,

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"Whole semantics is sectionwise"

Sectionwise Satisfaction Lemma. For any hyperstate σ and an $\textsc{Assn}^{\mathrm{dt}}$ formula φ:

$$\sigma \models \varphi \quad \iff \quad \sigma|_i \models \varphi|_i \quad \text{for almost every } i.$$

"Łos' Theorem"

"Sectionwise Lemmas"

Lem. (Sectionwise validity of Hoare triples)

$$\models \{A\}c\{B\} \quad \iff$$

$$\models \{A|_i\}\, c|_i\, \{B|_i\} \text{ for almost every } i.$$

"Sectionwise Lemmas"

Lem. (Sectionwise validity of Hoare triples)

$$\models \{A\}c\{B\} \iff \models \{A|_i\}\, c|_i\, \{B|_i\} \text{ for almost every } i.$$

Interface for **transferring** static analysis techniques

Q. Is a $\mathbf{While}^{\mathrm{dt}}$ program executable?

* A. Not exactly.
  * A **modeling** language
    * Not numerical approx., but **exact** modeling

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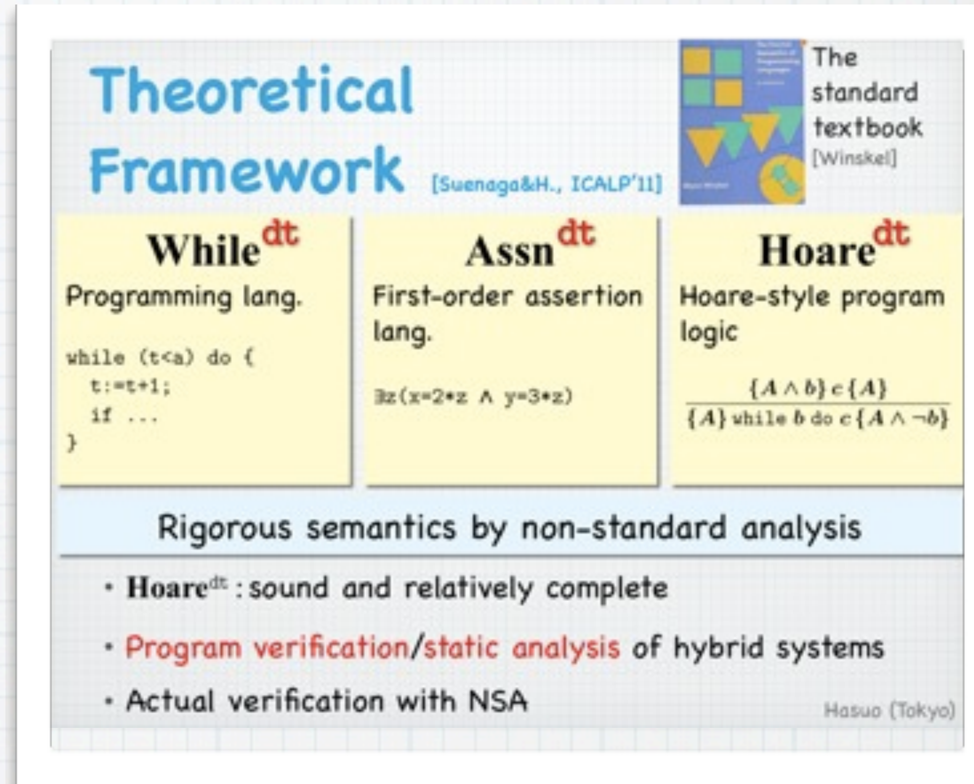
Q. Is a $\mathbf{While}^{\mathrm{dt}}$ program executable?

- A. Not exactly.
  - A **modeling** language
    - Not numerical approx., but **exact** modeling
  - Static analysis ➜ **no need to execute!**
    - Mathematical semantics suffices

Outline



Suenaga & H., ICALP'11

- **Theoretical foundations**
  - **While^{dt}, Assn^{dt}, Hoare^{dt}**
  - Rigorous semantics via NSA
  - Transfer principle, "sectionwise lemmas"

H. & Suenaga, CAV'12

- **Static analysis techniques, transferred as they are**
  - Phase split [Sharma,Dillig,Dillig,Aiken; CAV'11] [Balakrishnan,Sankaranarayanan,Ivancic,Gupta; EMSOFT'09] [Gopan,Reps; SAS'07]
  - Differential invariant [Platzer,Clarke; CAV'08]
  - ... and more!

Outline

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* **Theoretical foundations**
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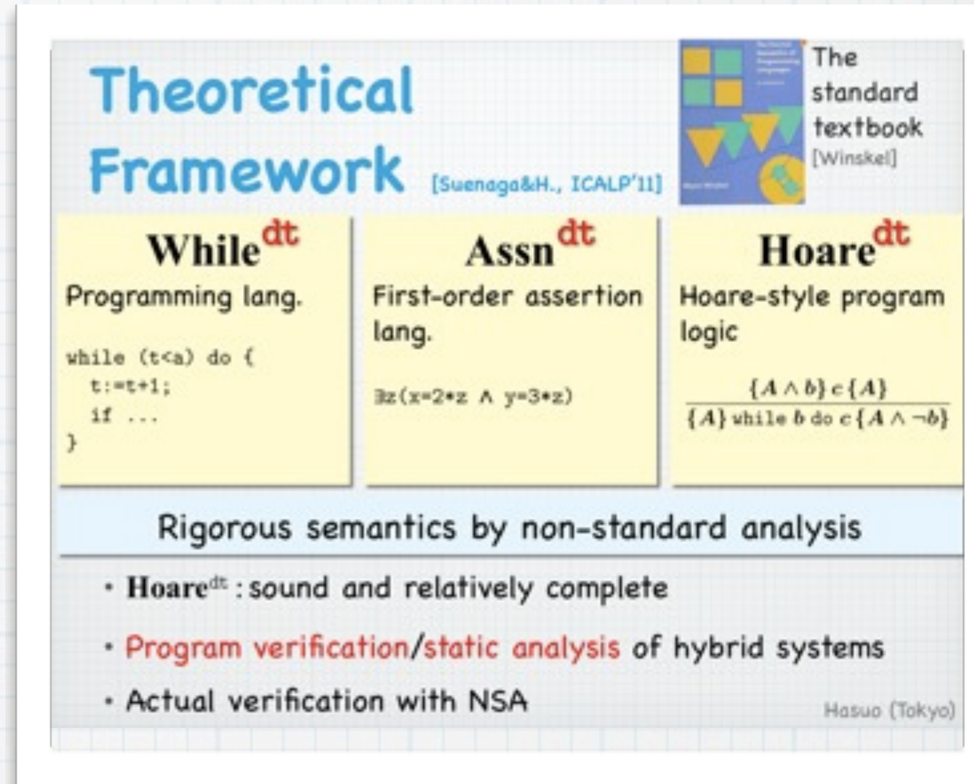

Done

H. & Suenaga, CAV'12

* **Static analysis techniques, transferred as they are**
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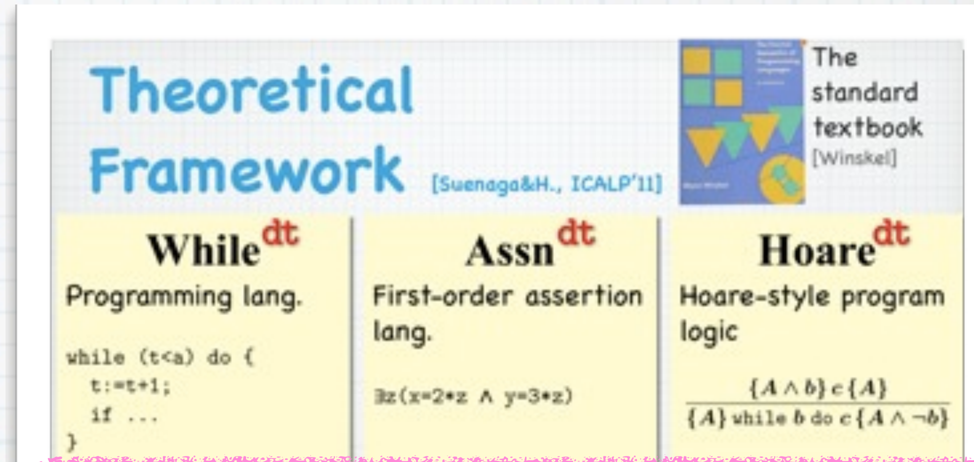
Suenaga & H., ICALP'11



- **Theoretical foundations**
  - **While**^{dt}**, Assn**^{dt}**, Hoare**^{dt}
  - Rigorous semantics via NSA
  - Transfer principle, "sectionwise lemmas"

w/ or w/o dt ...

➜ logically "**the same**"

Done

H. & Suenaga, CAV'12

- **Static analysis techniques, transferred as they are**
  - Phase split [Sharma,Dillig,Dillig,Aiken; CAV'11] [Balakrishnan,Sankaranarayanan,Ivancic,Gupta; EMSOFT'09] [Gopan,Reps; SAS'07]
  - Differential invariant [Platzer,Clarke; CAV'08]
  - ... and more!

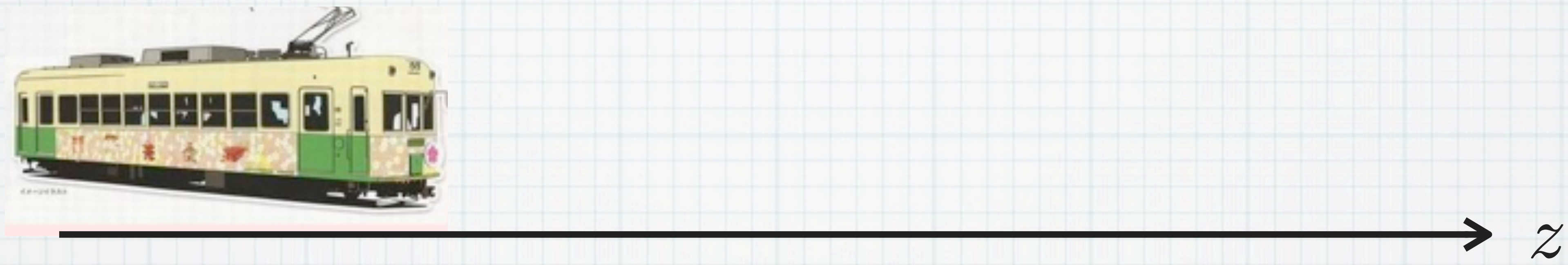
Part II:

Exercises in Nonstandard Static Analysis

Exercise 1.1

$$
\begin{array}{l}
\texttt{while}\ t < \varepsilon\ \texttt{do}\ \{ \\
\qquad t := t + \mathrm{dt}\,; \\
\qquad v := v + a \cdot \mathrm{dt}\,; \\
\qquad z := z + v \cdot \mathrm{dt} \\
\}
\end{array}
$$

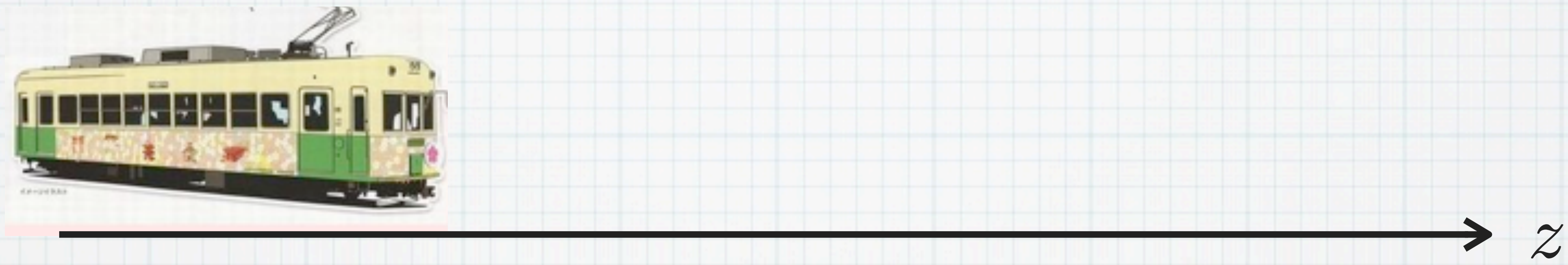
Exercise 1.1




```
while t < ε do {
    t := t + dt ;
    v := v + a · dt ;
    z := z + v · dt
}
```

```
while v > 0 do {
    t := 0 ;
    if m − z < s then a := −b else a := a₀ ;
    while t < ε do {
        t := t + dt ;
        v := v + a · dt ;
        z := z + v · dt
    }}
```

Exercise 1.1



$\texttt{while } v > 0 \texttt{ do } \{$

$\quad t := 0\,;$

$\quad \texttt{if } m - z < s \texttt{ then } a := -b \texttt{ else } a := a_0\,;$

$\quad \texttt{while } t < \varepsilon \texttt{ do } \{$

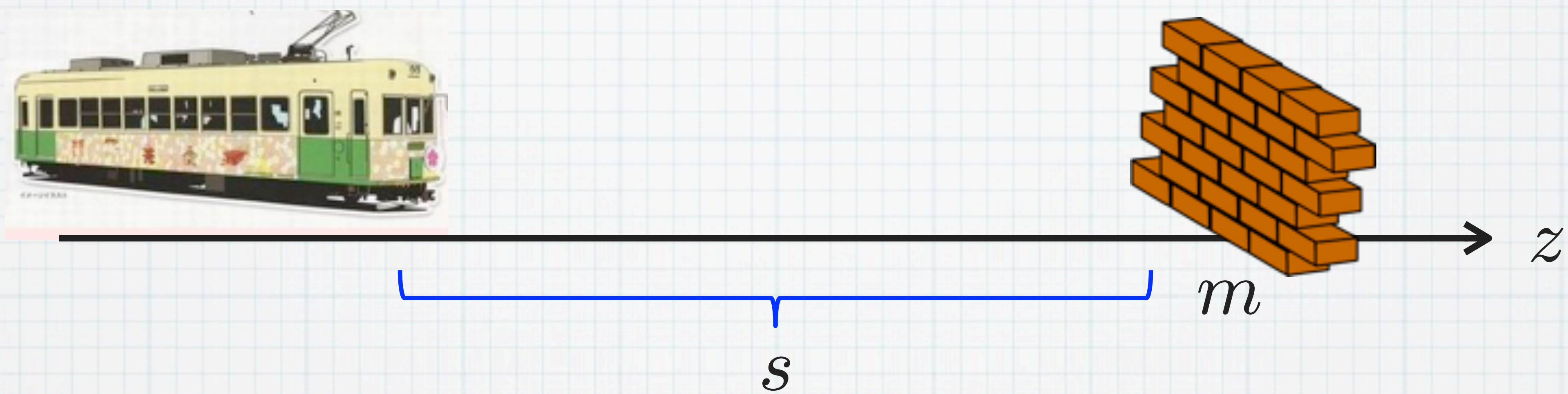
$\qquad t := t + \mathrm{dt}\,;$

$\qquad v := v + a \cdot \mathrm{dt}\,;$

$\qquad z := z + v \cdot \mathrm{dt}$

$\quad \}\}$

Exercise 1.1



$\texttt{while}\ v > 0\ \texttt{do}\ \{$

$\quad t := 0\,;$

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$\qquad t := t + \mathrm{dt}\,;$

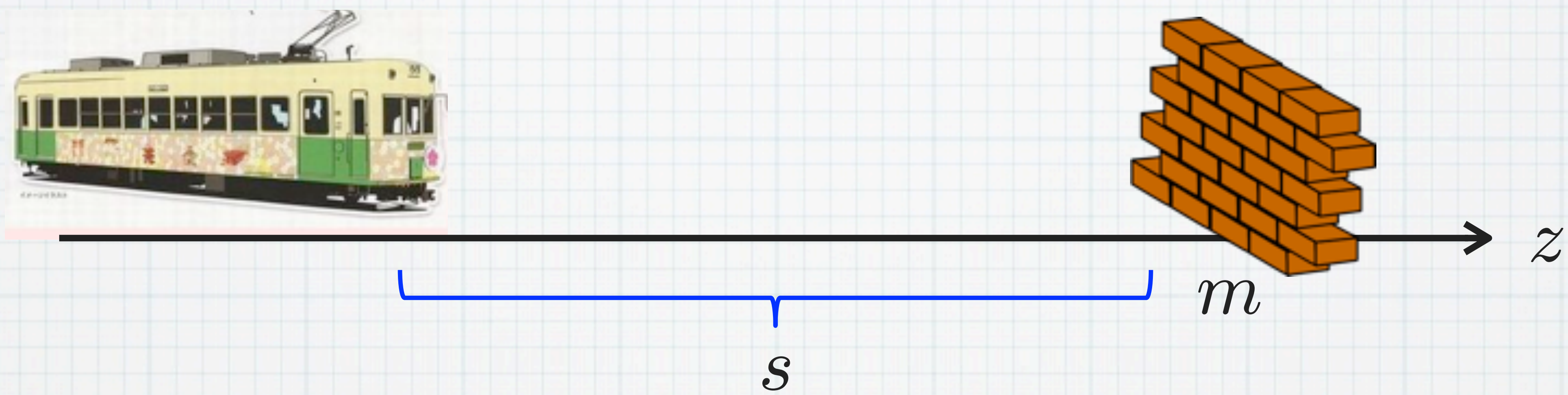
$\qquad v := v + a \cdot \mathrm{dt}\,;$

$\qquad z := z + v \cdot \mathrm{dt}$

$\quad \}\}$

Exercise 1.1

(Tiny) fragment of
Euro. Train Ctrl. Sys. (ETCS)




```
while v > 0 do {
    t := 0 ;
    if m − z < s then a := −b else a := a₀ ;
    while t < ε do {
        t := t + dt ;
        v := v + a · dt ;
        z := z + v · dt
}}
```

ETCS_0

Exercise 1.1

(Tiny) fragment of
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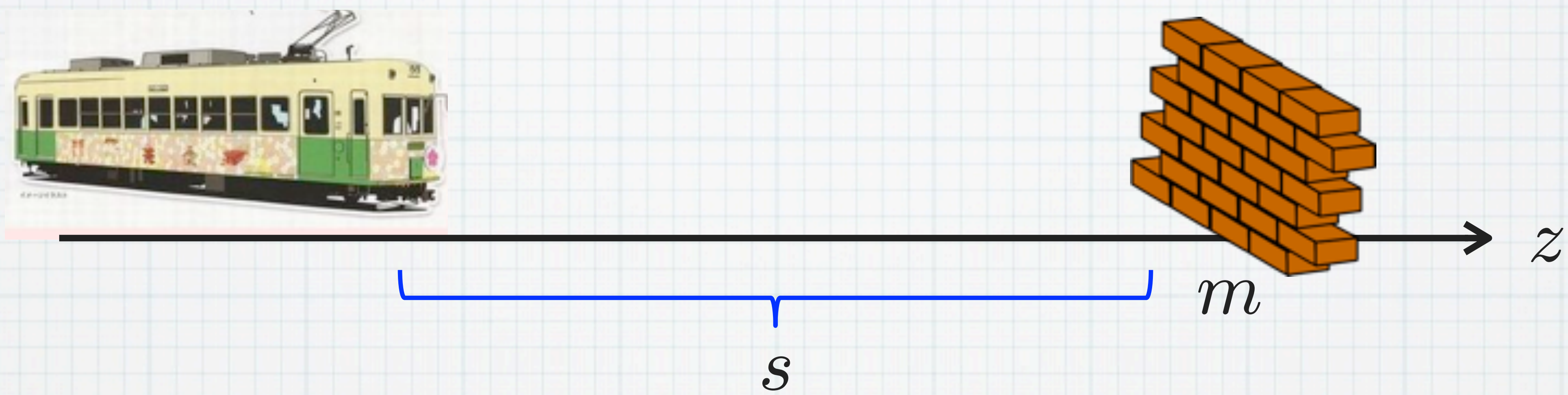

$$
\begin{array}{l}
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\quad t := 0\,; \\
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\quad \texttt{while } t < \varepsilon \texttt{ do } \{ \\
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\qquad v := v + a \cdot \mathrm{dt}\,; \\
\qquad z := z + v \cdot \mathrm{dt} \\
\quad \}\}
\end{array}
$$

ETCS_0

Q. Find A s.t. $\models \{A\}\,\mathrm{ETCS}_0\,\{z < m\}$

Exercise 1.1

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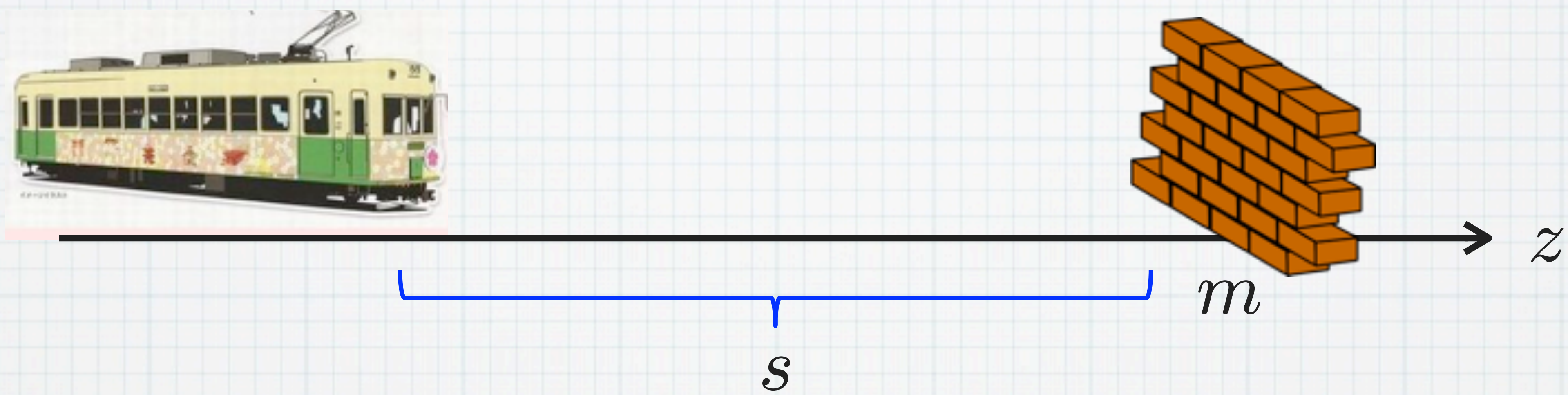


```
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        t := t + dt ;
        v := v + a · dt ;
        z := z + v · dt
}}
```

ETCS_0

s: big enough
b: big enough
a_0: small enough
…

Q. Find A s.t. $\models \{A\}\,\mathrm{ETCS}_0\,\{z < m\}$

```
while (v > 0) {
  if m - z < s
    then a := -b
    else a := a0;
  t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a * dt;
    t := t + dt }}

{z < m}
```

```
while (v > 0) {
  if m - z < s
    then a := -b
    else a := a0;
  t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a * dt;
    t := t + dt }}

{z < m}
```

```
while (v > 0) {
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    then a := -b
    else a := a0;
  t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a * dt;
    t := t + dt }}

{z < m}
```

```
while (v > 0 && m - z >= s) {
  a := a0;    t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }};
while (v > 0 && m - z < s) {
  a := -b;    t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v - b * dt;
    t := t + dt }}

{z < m}
```

```
while (v > 0) {
  if m - z < s
    then a := -b
    else a := a0;
  t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a * dt;
    t := t + dt }}

{z < m}
```

```
while (v > 0 && m - z >= s) {
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  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }};
while (v > 0 && m - z < s) {
  a := -b;   t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v - b * dt;
    t := t + dt }}

{z < m}
```

accel.

brake

```
while (v > 0) {
  if m - z < s
    then a := -b
    else a := a0;
  t := 0;
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    z := z + v * dt;
    v := v + a * dt;
    t := t + dt }}

{z < m}
```

```
while (v > 0 && m - z >= s) {
  a := a0;    t := 0;
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    v := v + a0 * dt;
    t := t + dt }};
while (v > 0 && m - z < s) {
  a := -b;    t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v - b * dt;
    t := t + dt }}

{z < m}
```

accel.

brake

Startegy 1 **"Phase split"**

[Sharma,Dillig,Dillig,Aiken; CAV'11]
[Balakrishnan,Sankaranarayanan,Ivancic,Gupta; EMSOFT'09] [Gopan,Reps; SAS'07]

Phase Split

(Standard Ver., for While & Hoare)

[Sharma,Dillig,Dillig,Aiken; CAV'11]

Defn.
The set of *holed commands* $\mathbf{Cmd}_{[\_]}$ is:

$$\mathbf{Cmd}_{[\_]} \ni h \;::=\; \mathtt{if}\ [\_]\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid h; c \mid c; h \mid \mathtt{if}\ b\ \mathtt{then}\ h\ \mathtt{else}\ c \mid \mathtt{if}\ b\ \mathtt{then}\ c\ \mathtt{else}\ h$$

For each holed command h, its *pre-hole fragment* $\overline{h}$ is:

$$\overline{\mathtt{if}\ [\_]\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2} :\equiv \mathtt{skip}$$
$$\overline{h; c} :\equiv \overline{h} \qquad \overline{c; h} :\equiv c; \overline{h}$$
$$\overline{\mathtt{if}\ b\ \mathtt{then}\ h\ \mathtt{else}\ c} :\equiv \mathtt{assert}\ b\,; \overline{h}$$
$$\overline{\mathtt{if}\ b\ \mathtt{then}\ c\ \mathtt{else}\ h} :\equiv \mathtt{assert}\ \neg b\,; \overline{h}$$

Lem.
If a Boolean expression $b_s \in \mathbf{BExp}$ satisfies

$$\models \{b_s\}\,\overline{h}\,\{b_c\}\;,\quad \models \{\neg b_s\}\,\overline{h}\,\{\neg b_c\}\;,\quad \text{and} \quad \models \{b_g \wedge b_s\}\,h[b_c]\,\{\neg b_g \vee b_s\}\;,$$

then we have

$$[\![\, \mathtt{while}\ b_g\ \mathtt{do}\ h[b_c] \,]\!] = \left[\!\!\left[\, \begin{array}{l} \mathtt{while}\ (b_g \wedge \neg b_s)\ \mathtt{do}\ h[\mathtt{false}]\,; \\ \mathtt{while}\ (b_g \wedge b_s)\ \mathtt{do}\ h[\mathtt{true}] \end{array} \,\right]\!\!\right] .$$

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The set of *holed commands* $\mathbf{Cmd}_{[\_]}$ is:

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For each holed command h, its *pre-hole fragment* $\overline{h}$ is:

$$\overline{\texttt{if}\;[\_]\;\texttt{then}\;c_1\;\texttt{else}\;c_2} :\equiv \texttt{skip}$$
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Phase Split

(Standard Ver., for While & Hoare)

[Sharma,Dillig,Dillig,Aiken; CAV'11]

$\texttt{while}\;b_{\mathrm{g}}\;\texttt{do}\;\boxed{\ldots(\texttt{if}\ldots)\ldots}$

into $\left[\begin{array}{l}\texttt{while}\;b_{\mathrm{g}}\wedge\neg b_{\mathrm{s}}\;\texttt{do}\;\boxed{\ldots}\;;\\ \texttt{while}\;b_{\mathrm{g}}\wedge b_{\mathrm{s}}\;\texttt{do}\;\boxed{\ldots}\end{array}\right]$

Lem.
If a Boolean expression $b_{\mathrm{s}} \in \mathbf{BExp}$ satisfies

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then we have

$$[\![\,\texttt{while}\;b_{\mathrm{g}}\;\texttt{do}\;h[b_{\mathrm{c}}]\,]\!] = \left[\!\!\left[\begin{array}{l}\texttt{while}\;(b_{\mathrm{g}}\wedge\neg b_{\mathrm{s}})\;\texttt{do}\;h[\texttt{false}]\;;\\ \texttt{while}\;(b_{\mathrm{g}}\wedge b_{\mathrm{s}})\;\texttt{do}\;h[\texttt{true}]\end{array}\right]\!\!\right].$$

Phase Split

(Nonstandard Ver., for $\mathbf{While}^{\mathrm{dt}}$ & $\mathbf{Hoare}^{\mathrm{dt}}$)

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$$\overline{\mathtt{if}\ b\ \mathtt{then}\ h\ \mathtt{else}\ c} :\equiv \mathtt{assert}\ b\ ; \overline{h}$$
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Lem.
If a Boolean expression $b_{\mathrm{s}} \in \mathbf{BExp}$ satisfies

$$\models \{b_{\mathrm{s}}\}\, \overline{h}\, \{b_{\mathrm{c}}\} \ , \quad \models \{\neg b_{\mathrm{s}}\}\, \overline{h}\, \{\neg b_{\mathrm{c}}\} \ , \quad \text{and} \quad \models \{b_{\mathrm{g}} \wedge b_{\mathrm{s}}\}\, h[b_{\mathrm{c}}]\, \{\neg b_{\mathrm{g}} \vee b_{\mathrm{s}}\} \ ,$$

then we have

$$[\![\, \mathtt{while}\ b_{\mathrm{g}}\ \mathtt{do}\ h[b_{\mathrm{c}}] \,]\!] = \left[\!\!\left[\begin{array}{l} \mathtt{while}\ (b_{\mathrm{g}} \wedge \neg b_{\mathrm{s}})\ \mathtt{do}\ h[\mathtt{false}]\ ; \\ \mathtt{while}\ (b_{\mathrm{g}} \wedge b_{\mathrm{s}})\ \mathtt{do}\ h[\mathtt{true}] \end{array}\right]\!\!\right] .$$

Hasuo (Tokyo)

Phase Split
(Nonstandard Ver., for $\mathbf{While}^{\mathrm{dt}}$ & $\mathbf{Hoare}^{\mathrm{dt}}$)

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The set of *holed commands* $\mathbf{Cmd}_{[\_]}$ is:

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For each holed command h, its *pre-hole fragment* $\overline{h}$ is:

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$$\overline{\texttt{if}\ b\ \texttt{then}\ h\ \texttt{else}\ c} :\equiv \texttt{assert}\ b\ ; \overline{h}$$
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Lem.
If a Boolean expression $b_{\mathrm{s}} \in \mathbf{BExp}$ satisfies

$$\models \{b_{\mathrm{s}}\}\ \overline{h}\ \{b_{\mathrm{c}}\}\ , \quad \models \{\neg b_{\mathrm{s}}\}\ \overline{h}\ \{\neg b_{\mathrm{c}}\}\ , \quad \text{and} \quad \models \{b_{\mathrm{g}} \wedge b_{\mathrm{s}}\}\ h[b_{\mathrm{c}}]\ \{\neg b_{\mathrm{g}} \vee b_{\mathrm{s}}\}\ ,$$

then we have

$$[\![\ \texttt{while}\ b_{\mathrm{g}}\ \texttt{do}\ h[b_{\mathrm{c}}]\ ]\!] = \left[\!\!\left[\ \begin{array}{l} \texttt{while}\ (b_{\mathrm{g}} \wedge \neg b_{\mathrm{s}})\ \texttt{do}\ h[\texttt{false}]\ ; \\ \texttt{while}\ (b_{\mathrm{g}} \wedge b_{\mathrm{s}})\ \texttt{do}\ h[\texttt{true}] \end{array}\ \right]\!\!\right] .$$

Phase Split

(Nonstandard Ver., for $\mathbf{While}^{\mathrm{dt}}$ & $\mathbf{Hoare}^{\mathrm{dt}}$)

Defn.
The set of *holed commands* $\mathbf{Cmd}_{[\_]}$ is:

$$\mathbf{Cmd}_{[\_]} \ni h \ ::= \ \mathtt{if}\ [\_]\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid h; c \mid c; h \mid \mathtt{if}\ b\ \mathtt{then}\ h\ \mathtt{else}\ c \mid \mathtt{if}\ b\ \mathtt{then}\ c\ \mathtt{else}\ h$$

For each holed command h, its *pre-hole fragment* $\overline{h}$ is:

$$\overline{\mathtt{if}\ [\_]\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2} :\equiv \mathtt{skip}$$
$$\overline{h; c} :\equiv \overline{h} \qquad \overline{c; h} :\equiv c; \overline{h}$$
$$\overline{\mathtt{if}\ b\ \mathtt{then}\ h\ \mathtt{else}\ c} :\equiv \mathtt{assert}\ b\ ; \overline{h}$$
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Lem.
If a Boolean expression $b_{\mathrm{s}} \in \mathbf{BExp}$ satisfies

$$\models \{b_{\mathrm{s}}\}\,\overline{h}\,\{b_{\mathrm{c}}\}\ , \quad \models \{\neg b_{\mathrm{s}}\}\,\overline{h}\,\{\neg b_{\mathrm{c}}\}\ , \quad \text{and} \quad \models \{b_{\mathrm{g}} \wedge b_{\mathrm{s}}\}\, h[b_{\mathrm{c}}]\, \{\neg b_{\mathrm{g}} \vee b_{\mathrm{s}}\}\ ,$$

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Proof.

$$\models \{b_{\mathrm{s}}\}\,\overline{h}\,\{b_{\mathrm{c}}\}$$
$$\models \{\neg b_{\mathrm{s}}\}\,\overline{h}\,\{\neg b_{\mathrm{c}}\}$$
$$\models \{b_{\mathrm{g}} \wedge b_{\mathrm{s}}\}\, h[b_{\mathrm{c}}]\ \{\neg b_{\mathrm{g}} \vee b_{\mathrm{s}}\}$$

Hasuo (Tokyo)

Phase Split

(Nonstandard Ver., for While^{dt} & Hoare^{dt})

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$$\models \{b_s\}\, \overline{h}\, \{b_c\}$$
$$\models \{\neg b_s\}\, \overline{h}\, \{\neg b_c\}$$
$$\models \{b_g \wedge b_s\}\, h[b_c]\, \{\neg b_g \vee b_s\}$$

$\Leftrightarrow$ sectionwise

$$\models \{b_s|_i\}\, \overline{h}|_i\, \{b_c|_i\}$$
$$\models \{\neg b_s|_i\}\, \overline{h}|_i\, \{\neg b_c|_i\}$$
$$\models \{b_g|_i \wedge b_s|_i\}\, h|_i[b_c|_i]\, \{\neg b_g|_i \vee b_s|_i\}$$

⋮

(for almost all i)

Hasuo (Tokyo)

Phase Split

(Nonstandard Ver., for $\mathbf{While}^{\mathrm{dt}}$ & $\mathbf{Hoare}^{\mathrm{dt}}$)

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If a Boolean expression $b_{\mathrm{s}} \in \mathbf{BExp}$ satisfies

$$\models \{b_{\mathrm{s}}\}\,\overline{h}\,\{b_{\mathrm{c}}\}\ , \quad \models \{\neg b_{\mathrm{s}}\}\,\overline{h}\,\{\neg b_{\mathrm{c}}\}\ , \quad \text{and} \quad \models \{b_{\mathrm{g}} \wedge b_{\mathrm{s}}\}\,h[b_{\mathrm{c}}]\,\{\neg b_{\mathrm{g}} \vee b_{\mathrm{s}}\}\ ,$$

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$$[\![\, \mathtt{while}\ b_{\mathrm{g}}\ \mathtt{do}\ h[b_{\mathrm{c}}] \,]\!] = [\![\, \begin{array}{l} \mathtt{while}\ (b_{\mathrm{g}} \wedge \neg b_{\mathrm{s}})\ \mathtt{do}\ h[\mathtt{false}]\ ; \\ \mathtt{while}\ (b_{\mathrm{g}} \wedge b_{\mathrm{s}})\ \mathtt{do}\ h[\mathtt{true}] \end{array} \,]\!]\ .$$

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$$\models \{b_{\mathrm{s}}|_i\}\,\overline{h|_i}\,\{b_{\mathrm{c}}|_i\}$$
$$\models \{\neg b_{\mathrm{s}}|_i\}\,\overline{h|_i}\,\{\neg b_{\mathrm{c}}|_i\}$$
$$\models \{b_{\mathrm{g}}|_i \wedge b_{\mathrm{s}}|_i\}\,h|_i[b_{\mathrm{c}}|_i]\,\{\neg b_{\mathrm{g}}|_i \vee b_{\mathrm{s}}|_i\}$$

⋮

(for almost all i)

$\Rightarrow$ std. ver.

$$[\![\, \mathtt{while}\ b_{\mathrm{g}}|_i\ \mathtt{do}\ h|_i[b_{\mathrm{c}}|_i] \,]\!] = [\![\, \begin{array}{l} \mathtt{while}\ (b_{\mathrm{g}}|_i \wedge \neg b_{\mathrm{s}}|_i)\ \mathtt{do}\ h|_i[\mathtt{false}]\ ; \\ \mathtt{while}\ (b_{\mathrm{g}}|_i \wedge b_{\mathrm{s}}|_i)\ \mathtt{do}\ h|_i[\mathtt{true}] \end{array} \,]\!]$$

⋮

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(Nonstandard Ver., for While^{dt} & Hoare^{dt})

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The set of *holed commands* $\mathbf{Cmd}_{[\_]}$ is:

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If a Boolean expression $b_s \in \mathbf{BExp}$ satisfies

$$\models \{b_s\}\, \overline{h}\, \{b_c\} \ , \quad \models \{\neg b_s\}\, \overline{h}\, \{\neg b_c\} \ , \quad \text{and} \quad \models \{b_g \wedge b_s\}\, h[b_c]\, \{\neg b_g \vee b_s\} \ ,$$

then we have

$$[\![\texttt{while } b_g \texttt{ do } h[b_c]]\!] = \left[\!\!\left[\begin{array}{l} \texttt{while } (b_g \wedge \neg b_s) \texttt{ do } h[\texttt{false}] \,; \\ \texttt{while } (b_g \wedge b_s) \texttt{ do } h[\texttt{true}] \end{array} \right]\!\!\right] .$$

Proof.

$\models \{b_s\}\, \overline{h}\, \{b_c\}$
$\models \{\neg b_s\}\, \overline{h}\, \{\neg b_c\}$
$\models \{b_g \wedge b_s\}\, h[b_c]\, \{\neg b_g \vee b_s\}$

$\Leftrightarrow$ sectionwise

$\models \{b_s|_i\}\, \overline{h|_i}\, \{b_c|_i\}$
$\models \{\neg b_s|_i\}\, \overline{h|_i}\, \{\neg b_c|_i\}$
$\models \{b_g|_i \wedge b_s|_i\}\, h|_i[b_c|_i]\, \{\neg b_g|_i \vee b_s|_i\}$

⋮

(for almost all i)

$\Rrightarrow$ ... std. ver.

$$[\![\texttt{while } b_g|_i \texttt{ do } h|_i[b_c|_i]]\!] = \left[\!\!\left[\begin{array}{l} \texttt{while } (b_g|_i \wedge \neg b_s|_i) \texttt{ do } h|_i[\texttt{false}] \,; \\ \texttt{while } (b_g|_i \wedge b_s|_i) \texttt{ do } h|_i[\texttt{true}] \end{array} \right]\!\!\right]$$

⋮

sectionwise $\Updownarrow$

$$[\![\texttt{while } b_g \texttt{ do } h[b_c]]\!] = \left[\!\!\left[\begin{array}{l} \texttt{while } (b_g \wedge \neg b_s) \texttt{ do } h[\texttt{false}] \,; \\ \texttt{while } (b_g \wedge b_s) \texttt{ do } h[\texttt{true}] \end{array} \right]\!\!\right]$$

Transferring Static Analysis Strategies

$\models \{b_s\}\, \overline{h}\, \{b_c\}$
$\models \{\neg b_s\}\, \overline{h}\, \{\neg b_c\}$
$\models \{b_g \wedge b_s\}\, h[b_c]\, \{\neg b_g \vee b_s\}$

⇔ sectionwise

$\models \{b_s|_i\}\, \overline{h}|_i\, \{b_c|_i\}$
$\models \{\neg b_s|_i\}\, \overline{h}|_i\, \{\neg b_c|_i\}$
$\models \{b_g|_i \wedge b_s|_i\}\, h|_i[b_c|_i]\, \{\neg b_g|_i \vee b_s|_i\}$

⋮

(for almost all i)

⇒ std. ver.

$[\![\ \texttt{while}\ b_g|_i\ \texttt{do}\ h|_i[b_c|_i]\ ]\!] = \left[\!\!\left[\ \begin{array}{l}\texttt{while}\ (b_g|_i \wedge \neg b_s|_i)\ \texttt{do}\ h|_i[\texttt{false}]\ ;\\ \texttt{while}\ (b_g|_i \wedge b_s|_i)\ \texttt{do}\ h|_i[\texttt{true}]\end{array}\ \right]\!\!\right]$

⋮

sectionwise ⇕

$[\![\ \texttt{while}\ b_g\ \texttt{do}\ h[b_c]\ ]\!] = \left[\!\!\left[\ \begin{array}{l}\texttt{while}\ (b_g \wedge \neg b_s)\ \texttt{do}\ h[\texttt{false}]\ ;\\ \texttt{while}\ (b_g \wedge b_s)\ \texttt{do}\ h[\texttt{true}]\end{array}\ \right]\!\!\right]$

* Doesn't matter what "std. ver." is
* → **modular method** for transfer

```
while (v > 0) {
  if m - z < s
    then a := -b
    else a := a0;
  t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a * dt;
    t := t + dt }}

{z < m}
```

```
while (v > 0 && m - z >= s) {
  a := a0;    t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }};
while (v > 0 && m - z < s) {
  a := -b;    t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v - b * dt;
    t := t + dt }}

{z < m}
```

accel.

brake

Startegy 1 "Phase split"

[Sharma,Dillig,Dillig,Aiken; CAV'11]
[Balakrishnan,Sankaranarayanan,Ivancic,Gupta; EMSOFT'09]
[Gopan,Reps; SAS'07]

```
while (v > 0 && m - z >= s) {
  a := a0;   t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }};
while (v > 0 && m - z < s) {
  a := -b;   t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v - b * dt;
    t := t + dt }}

{z < m}
```

```
while (v > 0 && m - z >= s) {
  a := a0;   t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }};
while (v > 0 && m - z < s) {
  a := -b;   t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v - b * dt;
    t := t + dt }}

{z < m}
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }

{z < m}
```

```
while (v > 0 && m - z >= s) {
  a := a0;   t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }};
while (v > 0 && m - z < s) {
  a := -b;   t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v - b * dt;
    t := t + dt }}

{z < m}
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }

{z < m}
```

Startegies 2,3

"Superfluous guard elim." "Time elapse"

```
while (v > 0 && m - z >= s) {
  a := a0;   t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }};
while (v > 0 && m - z < s) {
  a := -b;   t := 0;
  while (t < eps && v > 0) {
    z := z + v * dt;
    v := v - b * dt;
    t := t + dt }}

{z < m}
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
while (v > 0) {
  a := -b;
```

Startegy 4

"Differential invariant"

[Platzer,Clarke; CAV'08]

Startegies 2,3

"Superfluous guard elim." **"Time elapse"**

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }

{z < m}
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }

{z < m}
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
```

$$\{ (v > 0 \lor m > z) \land (b^2 \mathrm{dt}^2 + 4b\mathrm{dt}v + 8bz + 4v^2 < 8bm \lor b\mathrm{dt}v + 2bz + v^2 \leq 2bm) \}$$

```
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }

{z < m}
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
```

$$\{ \; (v > 0 \lor m > z) \land (b^2 \mathrm{dt}^2 + 4b\mathrm{dt}v + 8bz + 4v^2 < 8bm \lor b\mathrm{dt}v + 2bz + v^2 \leq 2bm) \; \}$$

```
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

Startegy 5

"QE Invariant"

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;    t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
```

$$\left\{ \begin{array}{l} (v > 0 \lor m > z) \land \\ \left(b^2 \mathrm{dt}^2 + 4b\mathrm{dt}v + 8bz + 4v^2 < 8bm \right. \\ \quad \left. \lor\, b\mathrm{dt}v + 2bz + v^2 \leq 2bm\right) \end{array} \right\}$$

```
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
```

$$\{ \; (v > 0 \lor m > z) \land \left(b^2\mathrm{dt}^2 + 4b\mathrm{dt}v + 8bz + 4v^2 < 8bm \lor b\mathrm{dt}v + 2bz + v^2 \leq 2bm\right) \; \}$$

```
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

+ some fwd. propagation

```
{ ... (long fml. with dt) }
while (m - z >= s) {
  a := a0;   t := 0;
  while (t < eps) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }}
{ ... }
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

```
if (v > 0)
  then
    while (m - z >= s) {
      a := a0;   t := 0;
      while (t < eps) {
        z := z + v * dt;
        v := v + a0 * dt;
        t := t + dt }}
  else skip;
```

$$\{ \; (v > 0 \lor m > z) \land (b^2 \mathrm{dt}^2 + 4b\mathrm{dt}v + 8bz + 4v^2 < 8bm \lor b\mathrm{dt}v + 2bz + v^2 \leq 2bm) \; \}$$

```
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

+ some fwd. propagation

```
{ ... (long fml. with dt) }
while (m - z >= s) {
  a := a0;   t := 0;
  while (t < eps) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }}
{ ... }
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

Startegy 6

"Iteration count"

```
{ ... (long fml. with dt) }
while (m - z >= s) {
  a := a0;   t := 0;
  while (t < eps) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }}
{ ... }
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

```
{ ... (long fml. with dt) }
while (m - z >= s) {
  a := a0;   t := 0;
  while (t < eps) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }}
{ ... }
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

long fml. w/o `dt`, whose core is

$$a_0\left(2\varepsilon\sqrt{2a_0(m-s-z_0)+v_0^2}+b\epsilon^2+2m-2s-2z_0\right)$$
$$+2b\varepsilon\sqrt{2a_0(m-s-z_0)+v_0^2}+a_0^2\epsilon^2+v_0^2<2bs$$

```
{ ... (long fml. with dt) }
while (m - z >= s) {
  a := a0;   t := 0;
  while (t < eps) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }}
{ ... }
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

long fml. w/o `dt`, whose core is

$$a_0\left(2\varepsilon\sqrt{2a_0(m-s-z_0)+v_0^2}+b\epsilon^2+2m-2s-2z_0\right)$$
$$+2b\varepsilon\sqrt{2a_0(m-s-z_0)+v_0^2}+a_0^2\epsilon^2+v_0^2<2bs$$

Startegy 7

"Cast to shadow"

(Eliminates `dt`, strengthens the precond.)

```
{ ... (long fml. with dt) }
while (m - z >= s) {
  a := a0;   t := 0;
  while (t < eps) {
    z := z + v * dt;
    v := v + a0 * dt;
    t := t + dt }}
{ ... }
while (v > 0) {
  a := -b;
  z := z + v * dt;
  v := v - b * dt }
```

long fml. w/o dt, whose core is

$$a_0\left(2\varepsilon\sqrt{2a_0(m-s-z_0)+v_0^2}+b\epsilon^2+2m-2s-2z_0\right)$$
$$+2b\varepsilon\sqrt{2a_0(m-s-z_0)+v_0^2}+a_0^2\epsilon^2+v_0^2<2bs$$

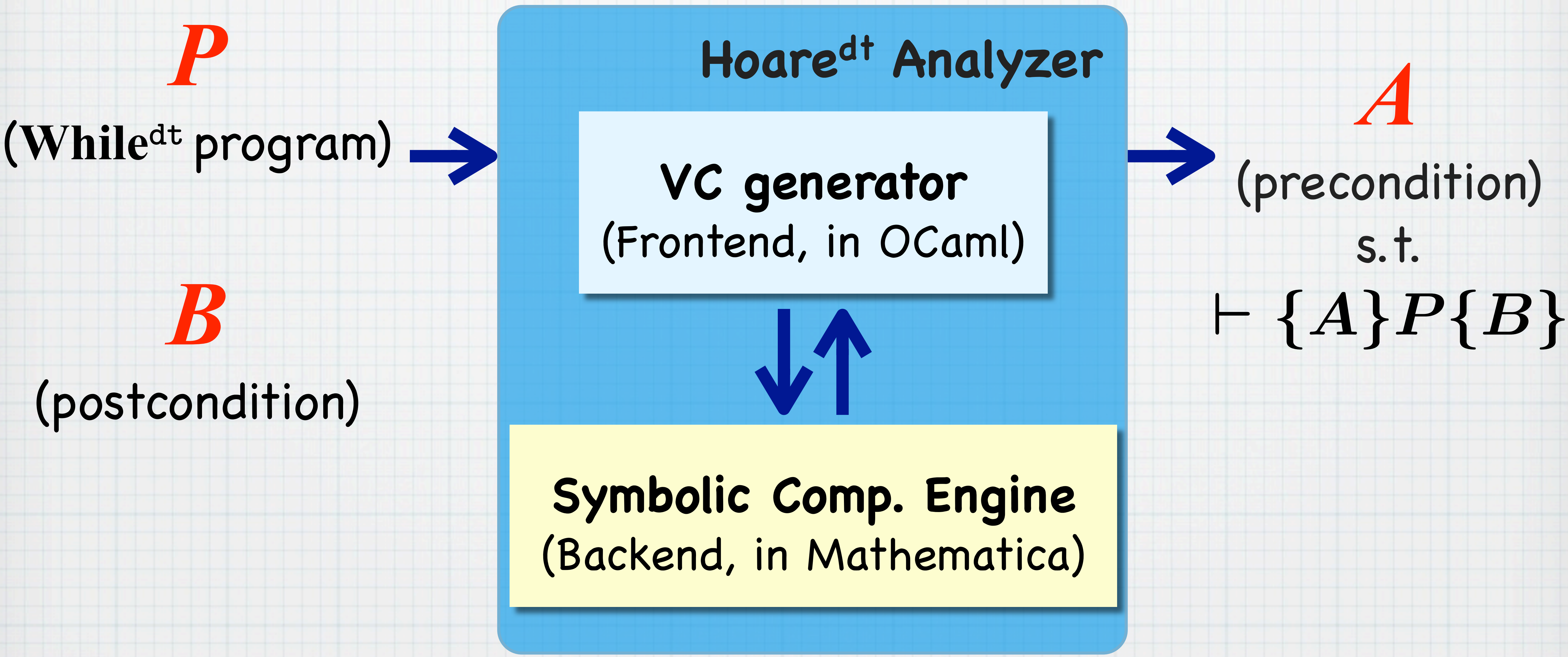
the final outcome

Startegy 7

"Cast to shadow"

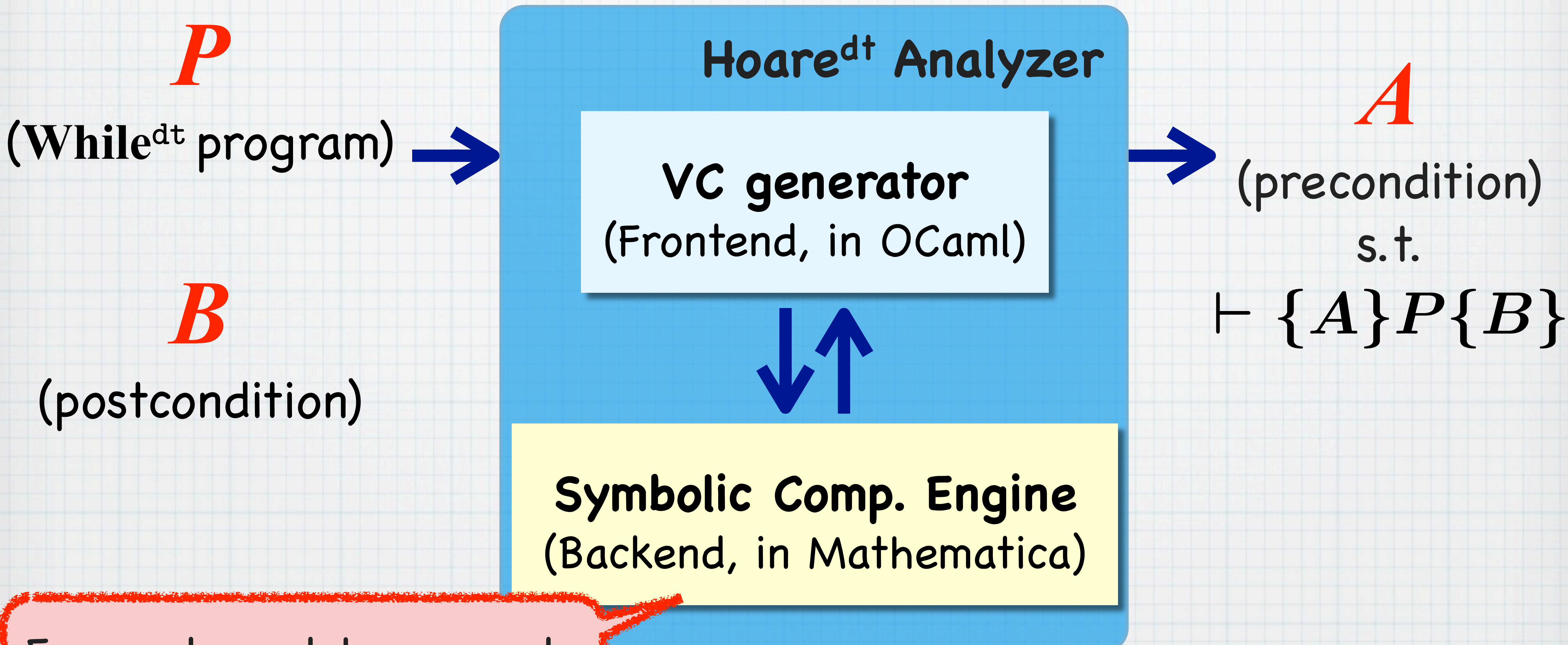
(Eliminates dt, strengthens the precond.)

Prototype Automatic Prover



- Fujitsu HX600 with Quad Core AMD Opteron 2.3GHz CPU, 32GB memory. Mathematica 7.0 for Linux x86 (64-bit)
  - ETCS: 40.96 sec.
  - Bouncing ball: runs with one manual insertion of invariants

Prototype Automatic Prover



For reals, not hyperreals
➜ justified by the **transfer principle**

Core AMD Opteron 2.3GHz CPU, 32GB memory.
x86 (64-bit)

th one manual insertion of invariants

Prototype Automatic Prover

P

(While^{dt} program) →

B

(postcondition)

Hoare^{dt} Analyzer

VC generator
(Frontend, in OCaml)

↓↑

Symbolic Comp. Engine
(Backend, in Mathematica)

→ A

(precondition)

s.t.

$\vdash \{A\}P\{B\}$

Totally symbolic
(crucial for transfer)

For reals, not hyperreals
➜ justified by the **transfer principle**

Core AMD Opteron 2.3GHz CPU, 32GB memory.

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th one manual insertion of invariants

Related Work

- **Deductive verification** of hybrid sys. [Platzer, '10] [Platzer, LICS'12]
  - Automatic prover KeYmaera
- **Static analysis techniques**
  - A **LOT** in CAV, SAS, VMCAI, ...
  - Applied to hybrid systems (w/ diff. eq.)
    [Rodriguez-Carbonell, Tiwari; HSCC'05] [Sankaranarayanan; HSCC'10]
    [Sankaranarayanan, Sipma, Manna; Formal Methods Sys. Design '08]
- Use of **NSA for hybrid systems**
  [Benveniste, Bourke, Caillaud, Pouzet; J. Comput. Syst. Sci. '12]
  [Bliudze, Krob; Fundam. Inform. '09] [Gamboa, Kaufmann; J. Autom. Reason. '01]
- **Continuous techniques applied to discrete appl.**
  [Chaudhuri, Gulwani, Lublinerman, NavidPour; FSE '11]
  - Not contending! Combination?

Conclusions

$\textbf{While}^{\text{dt}}$

Programming lang.

```
while (t<a) do {
  t:=t+1;
  if ...
}
```

$\textbf{Assn}^{\text{dt}}$

First-order assertion lang.

∃z(x=2*z ∧ y=3*z)

$\textbf{Hoare}^{\text{dt}}$

Hoare-style program logic

$$\frac{\{A \wedge b\}\, c\, \{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\, \{A \wedge \neg b\}}$$

Rigorous semantics by non-standard analysis

Conclusions

Nonstandard Static Analysis

$\mathbf{While}^{dt}$

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Rigorous semantics by non-standard analysis

- Tool's effectivity. More heuristics?
- (Any discrete frmwk.)^{dt} ?
- Simulink as stream processing?
- With (explicit) differential equations?

Conclusions

Nonstandard Static Analysis

$\mathbf{While}^{dt}$

Programming lang.

```
while (t<a) do {
  t:=t+1;
  if ...
}
```

$\mathbf{Assn}^{dt}$

First-order assertion lang.

$\exists z(x=2*z \wedge y=3*z)$

$\mathbf{Hoare}^{dt}$

Hoare-style program logic

$$\frac{\{A \wedge b\}\, c\, \{A\}}{\{A\}\ \mathtt{while}\ b\ \mathtt{do}\ c\, \{A \wedge \neg b\}}$$

Rigorous semantics by non-standard analysis

* Tool's effectivity. More heuristics?
* $(\text{Any discrete frmwk.})^{dt}$?
* Simulink as stream processing?
* With (explicit) differential equations?

Thank you for your attention!

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