Generic Weakest Precondition Semantics from Monads Enriched with Order

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CMCS (Grenoble) 2014/4/6

Prologue: Alternating Branching

- (Automata-theoretic) **model checking**
  See e.g. [LNCS 2500]

$$\mathcal{S} \models \psi$$

$$\iff \mathcal{S} \text{ is accepted by } \mathcal{A}_{\psi}$$

$$\iff \text{Player has a winning str. in } \mathcal{G}(\mathcal{S}, \mathcal{A}_{\psi})$$

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Kripke frame

modal μ-fml.

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alternating parity tree automaton

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parity game for acceptance

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Algorithms like [Jurdzinski '98]

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modal μ-fml.

$$\psi :\equiv \nu X.\, \mu Y.\, \big((p \wedge \Diamond X) \vee (q \wedge \Diamond Y) \big)$$

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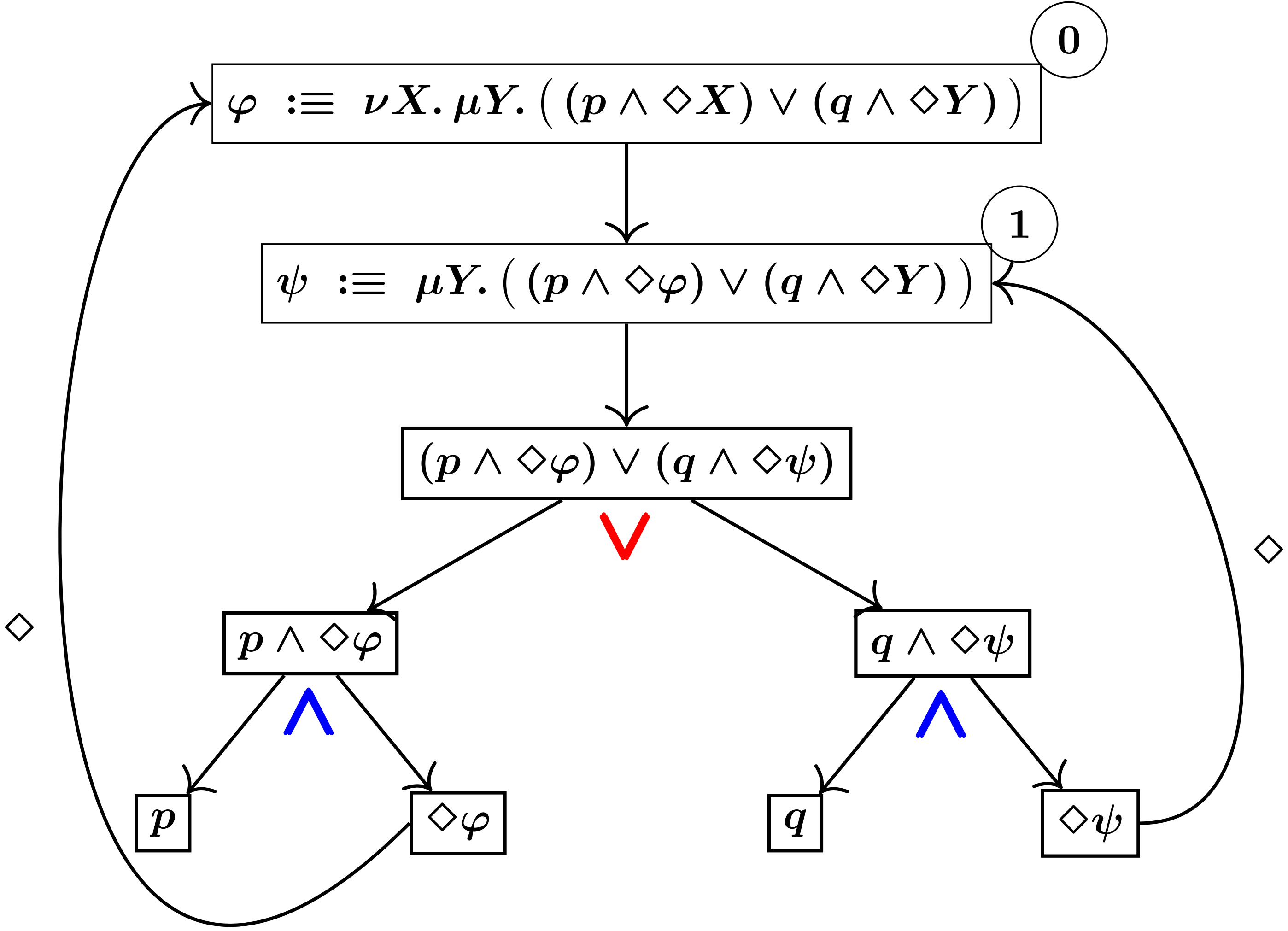
alternating parity tree automaton



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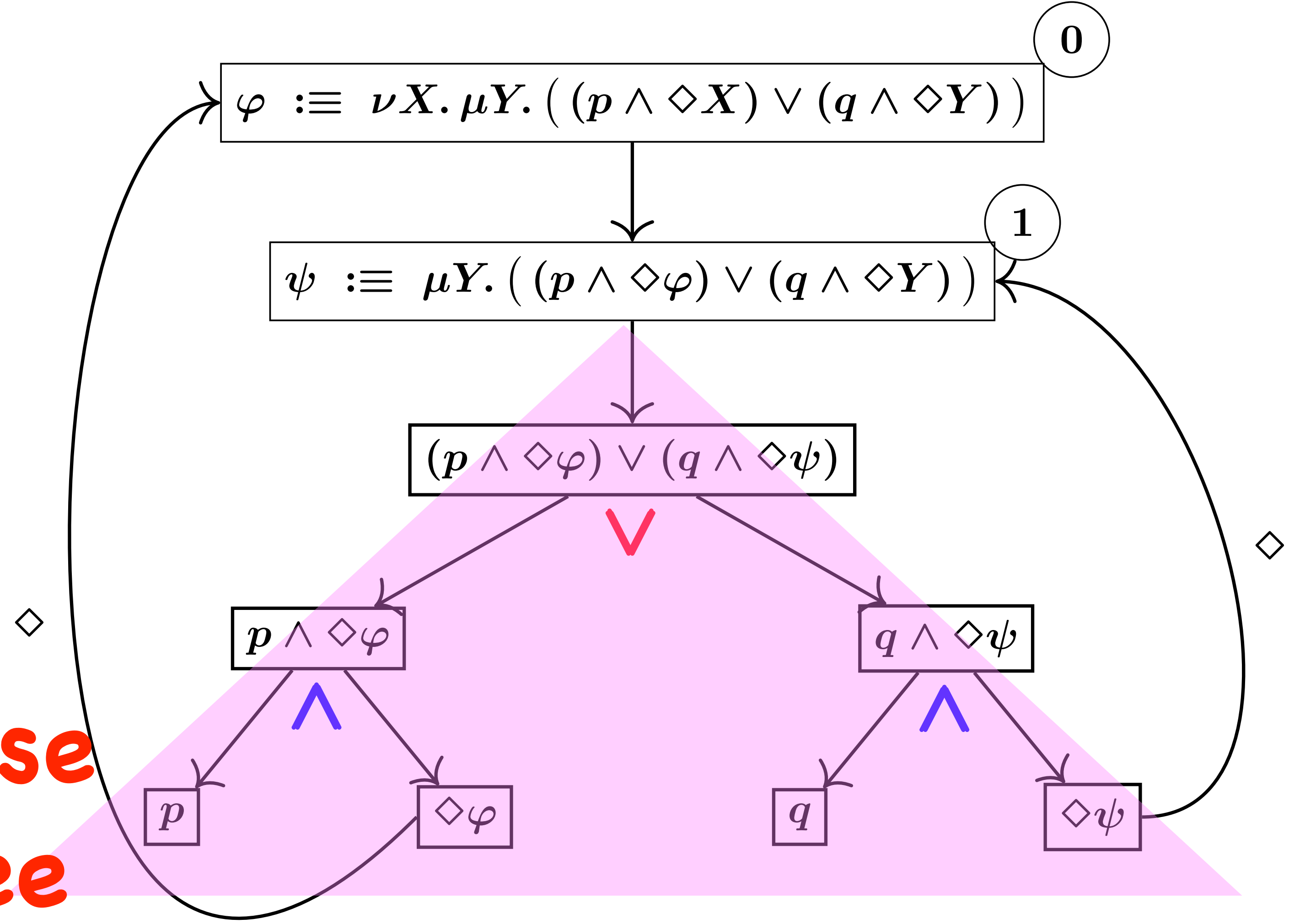

parse tree

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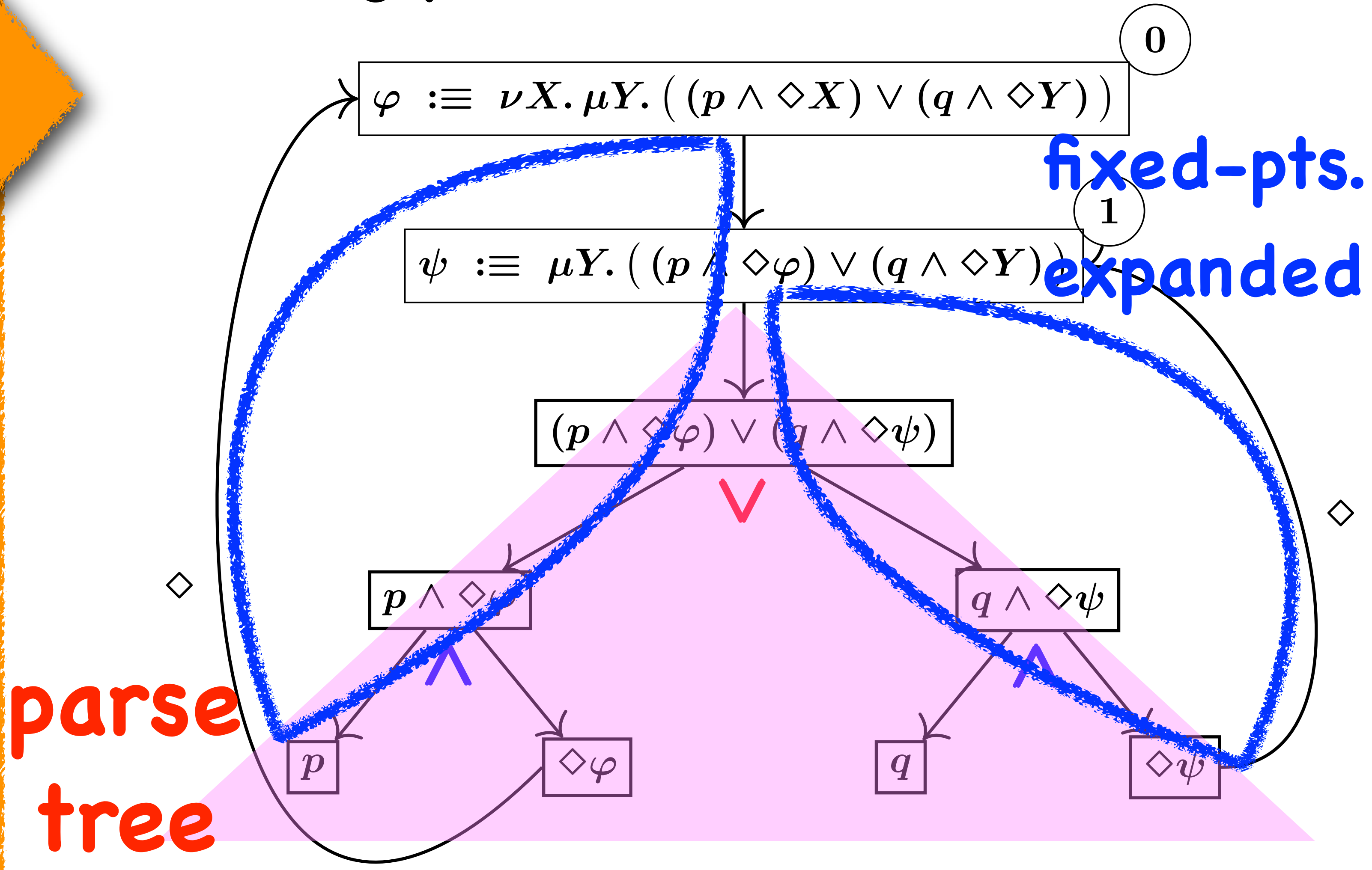

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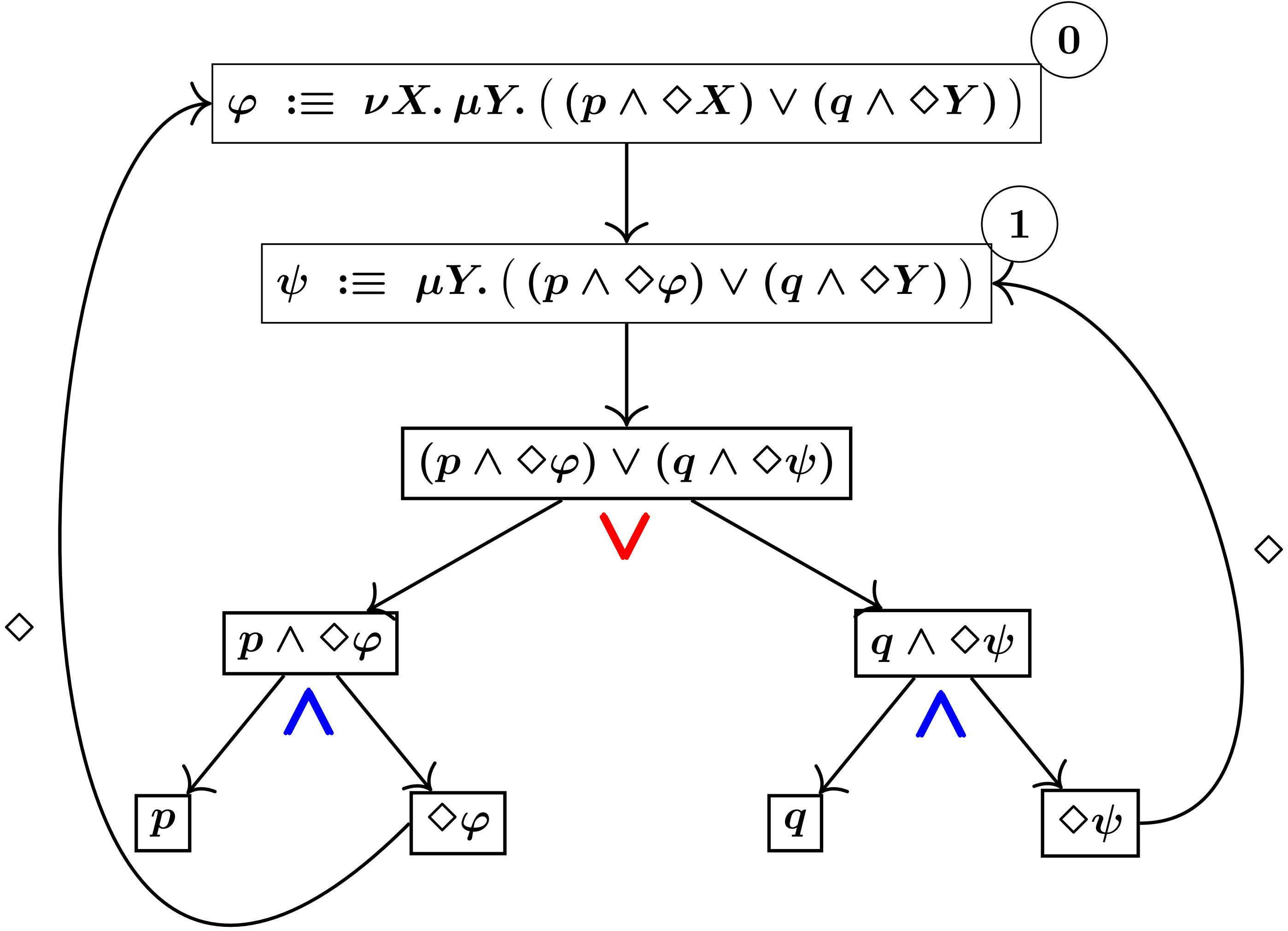

Prologue: Alternating Branching

- Monadic modeling?

$$\begin{tikzcd}
X \arrow[r] & \mathcal{P}\mathcal{P}X \\
\{x\} \arrow[r, mapsto] & \{\,\{x_1, x_2\},\ \{x_3\}\,\}
\end{tikzcd}$$

Prologue: Alternating Branching

- Monadic modeling?

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{P}\mathcal{P}X \\ \{x\} & \longmapsto & \big\{\, \{x_1, x_2\},\ \{x_3\} \,\big\} \end{array}$$

intention: positive DNF

$$(x_1 \wedge x_2) \vee x_3$$

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Q1. How?

Prologue:
Alternating Branching

- Monad multiplication

$$\begin{tikzcd}
 & & \mathcal{P}\mathcal{P}(\mathcal{P}\mathcal{P}\boldsymbol{X}) \arrow[d, "\mu^{\mathcal{P}\mathcal{P}}"] \\
 & \mathcal{P}\mathcal{P}\boldsymbol{X} \arrow[ur] & \mathcal{P}\mathcal{P}\boldsymbol{X} \\
\boldsymbol{X} \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \begin{array}{l} \left(\begin{array}{l} \big((y_1 \wedge y_2) \vee y_3 \big) \\ \wedge (z_1 \wedge z_2) \end{array} \right) \\ \vee (u_1 \wedge (u_2 \wedge u_3)) \end{array} \arrow[d, mapsto] \\
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- Monad multiplica

$$\begin{tikzcd}
 & & \mathcal{PP}(\mathcal{PP}X) \arrow[d, "\mu^{\mathcal{PP}}"] \\
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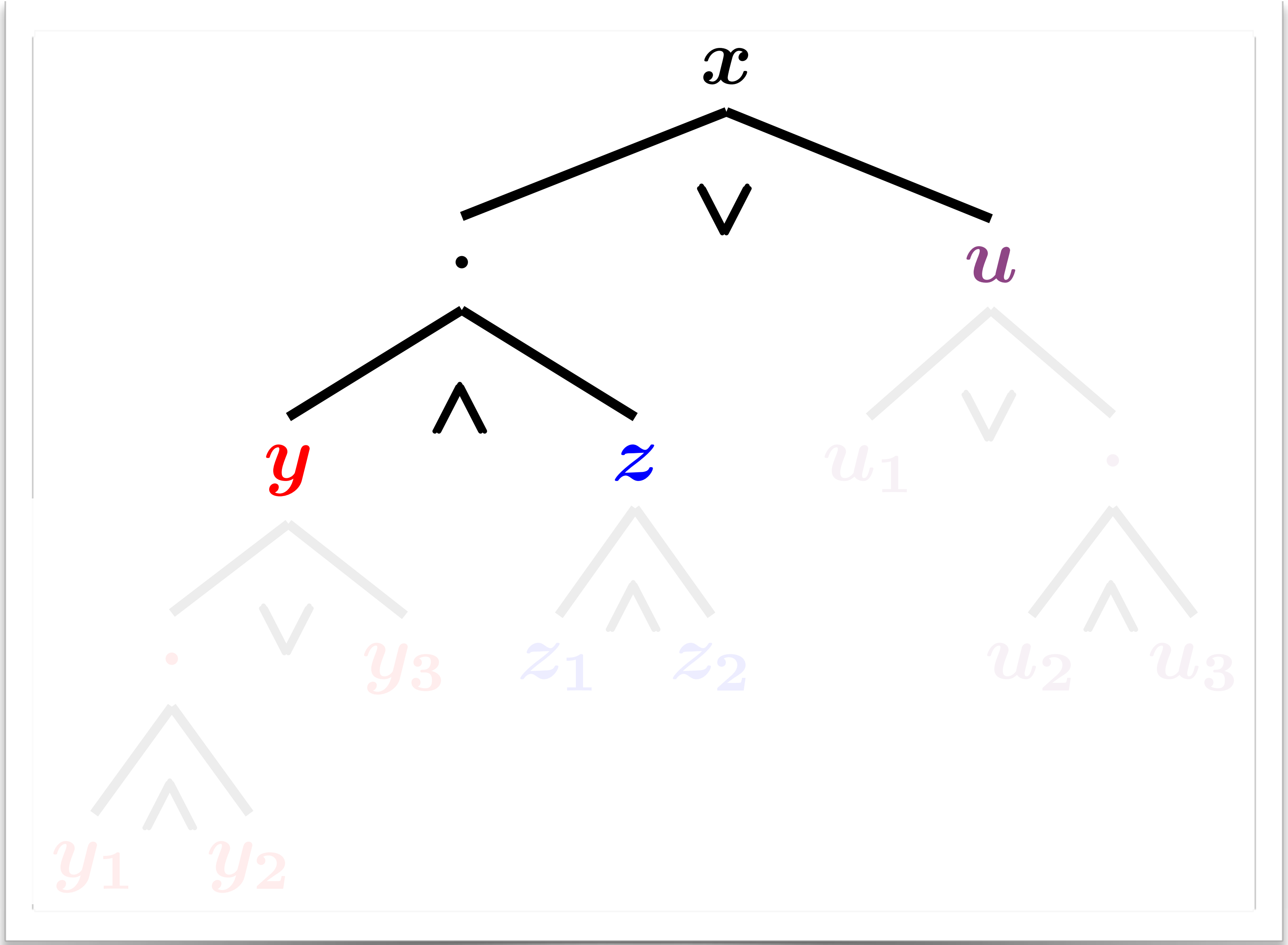

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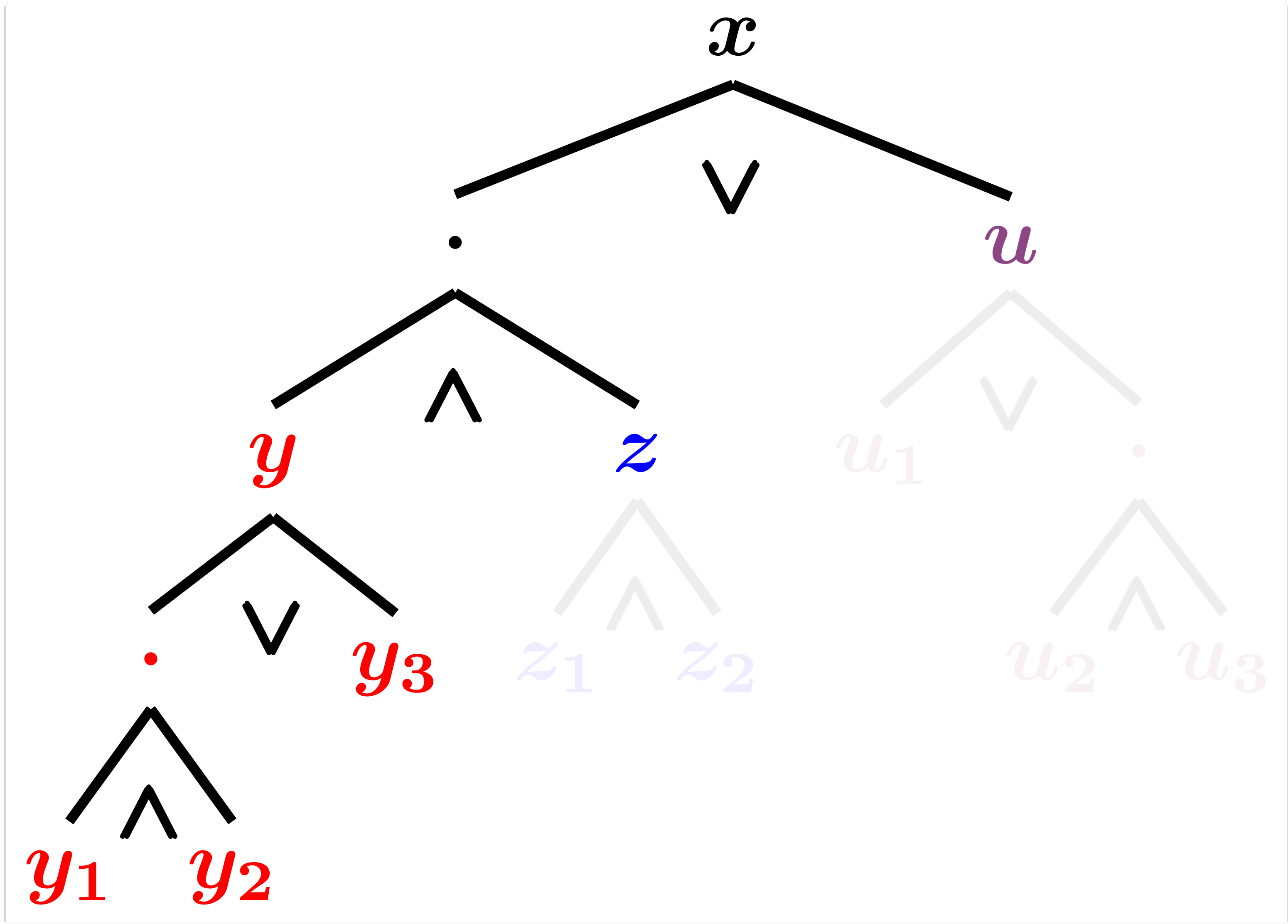

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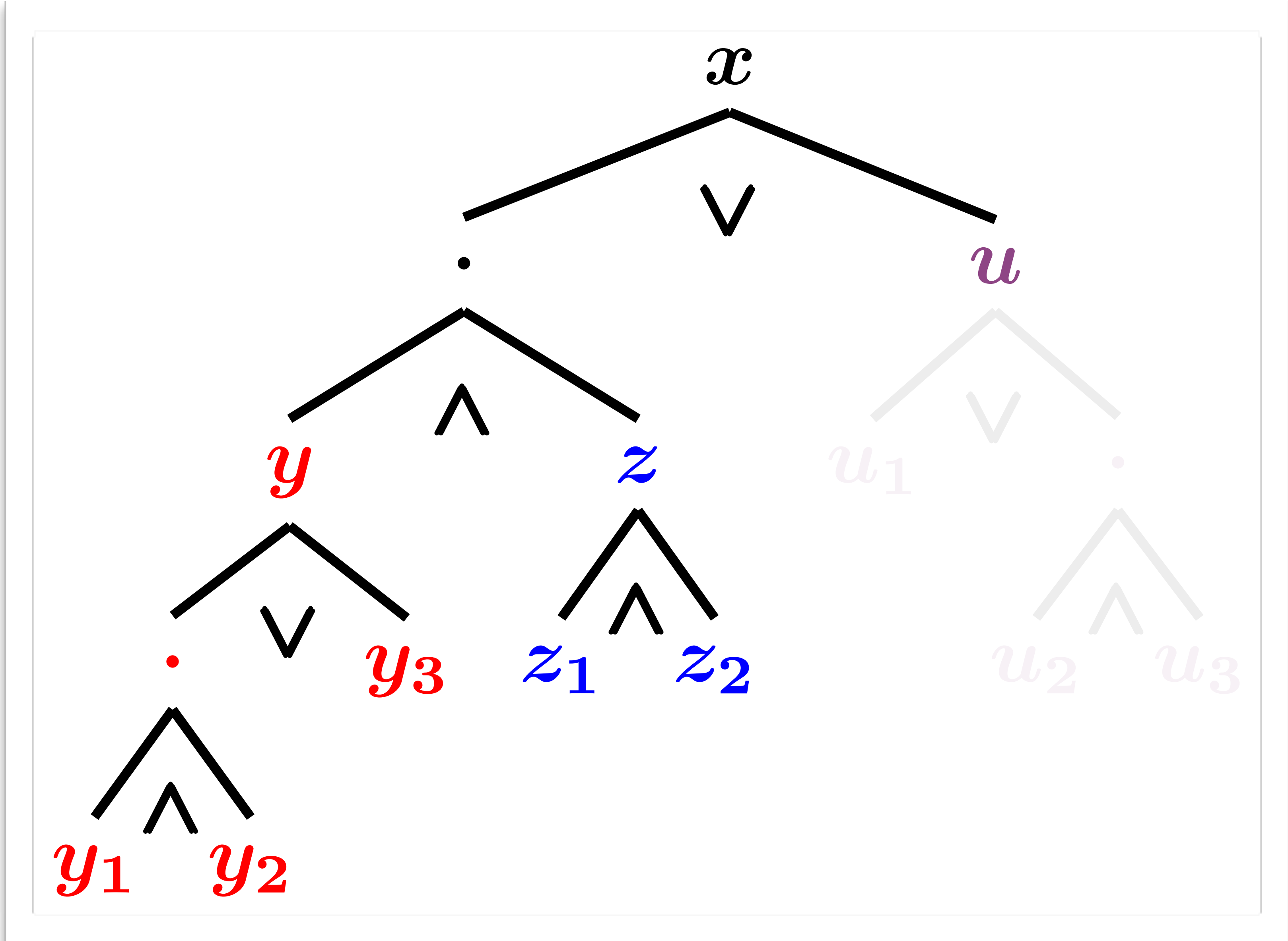

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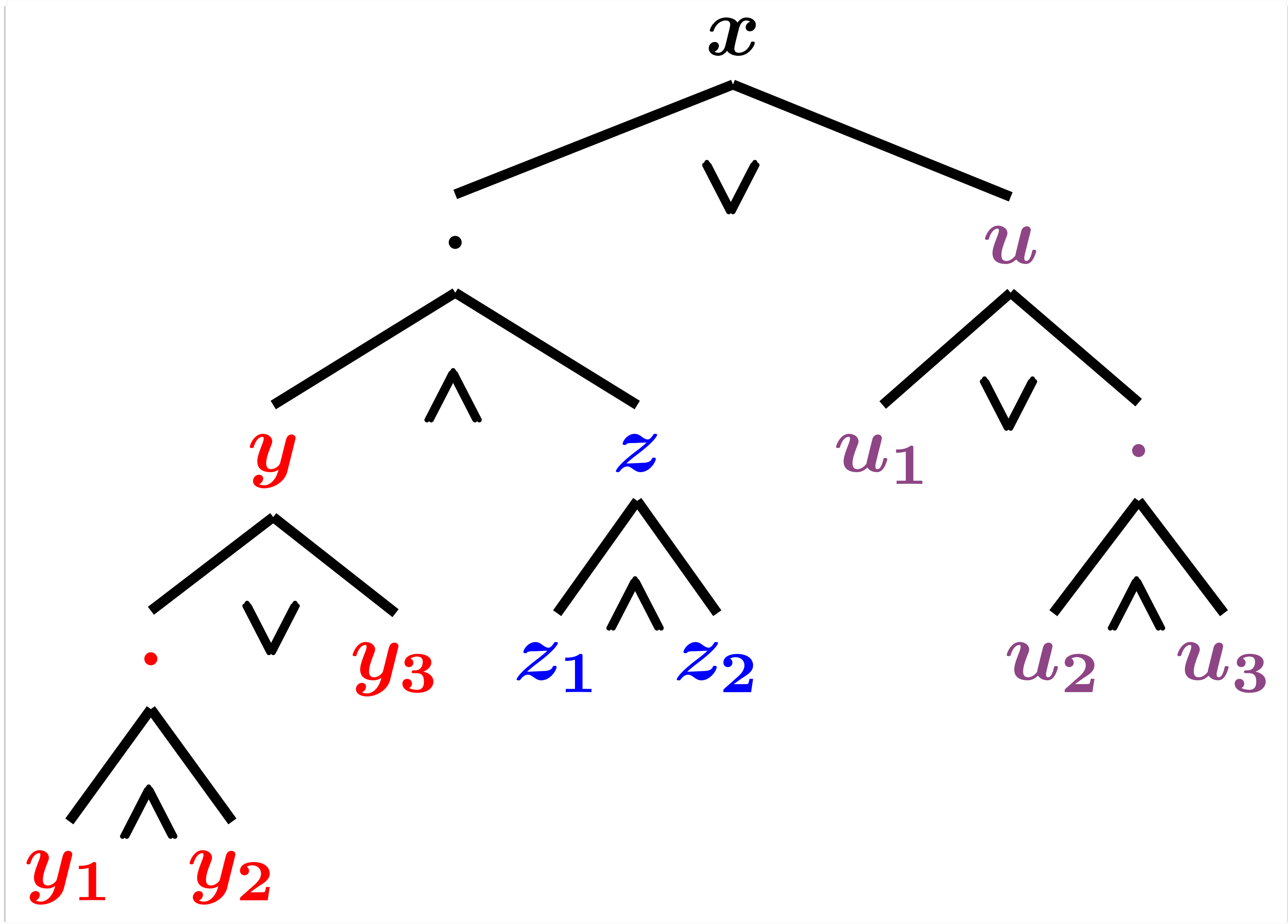

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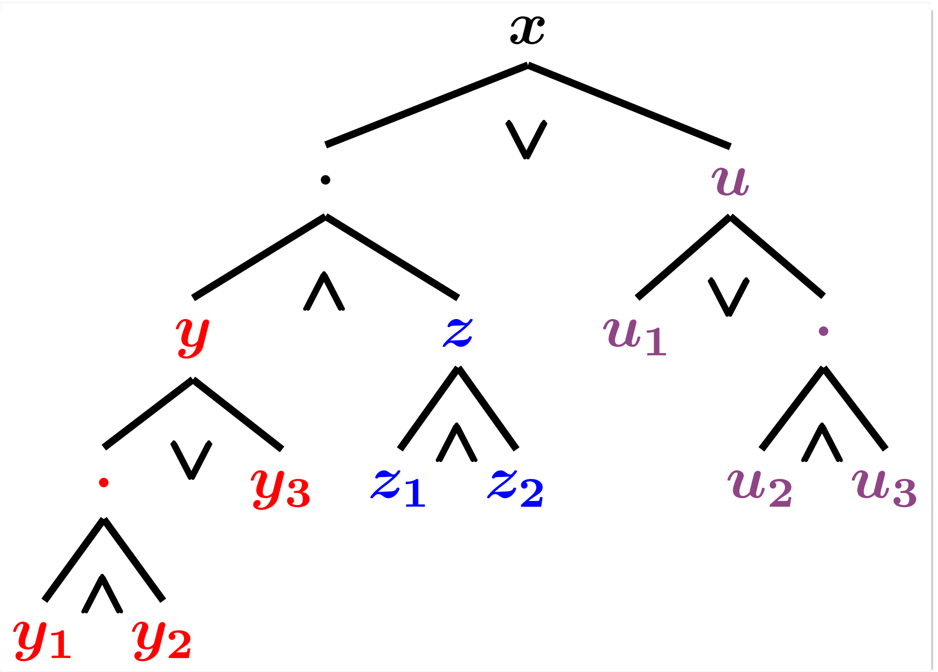

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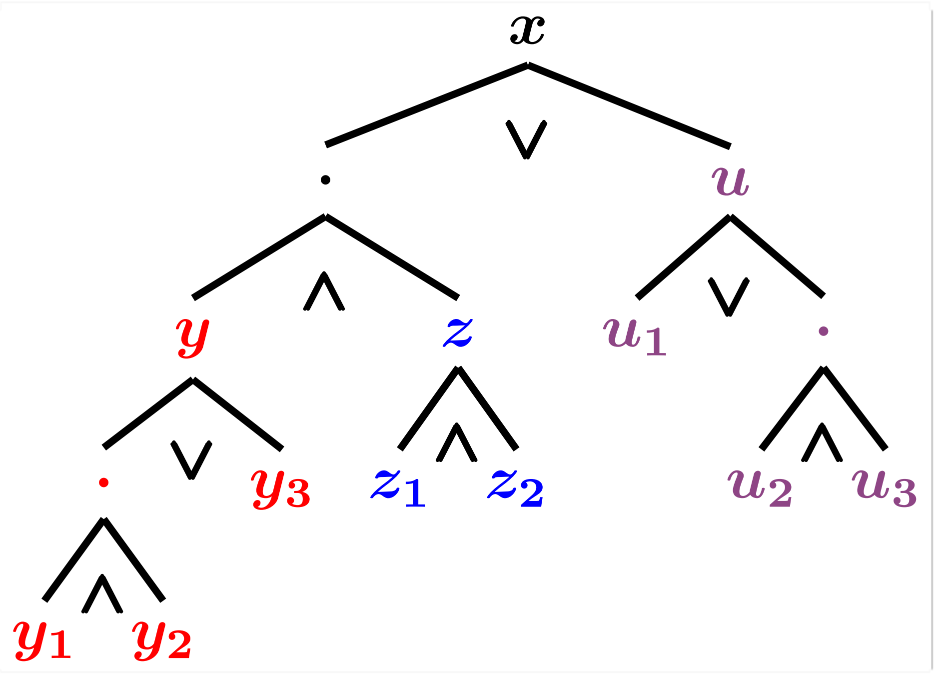

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Q2. How?

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Introduction

Weakest Precondition Semantics

- **Hoare logic** calculus
- **Weakest precond.** calculus

$$\frac{}{\{\,A[a/x]\,\}\; x := a\; \{A\}}\;(\textsc{Assign})$$

$$\frac{\{A\}\; c_1\; \{C\} \quad \{C\}\; c_2\; \{B\}}{\{A\}\; c_1; c_2\; \{B\}}\;(\textsc{Seq})$$

$$\frac{\{A \wedge b\}\; c_1\; \{B\} \quad \{A \wedge \neg b\}\; c_2\; \{B\}}{\{A\}\; \mathtt{if}\; b\; \mathtt{then}\; c_1\; \mathtt{else}\; c_2\; \{B\}}\;(\textsc{If})$$

$$\frac{\{A \wedge b\}\; c\; \{A\}}{\{A\}\; \mathtt{while}\; b\; \mathtt{do}\; c\; \{A \wedge \neg b\}}\;(\textsc{While})$$

$$\begin{aligned}
\mathbf{wp}[\![x := a, \Phi]\!] &\equiv \Phi[a/x] \\
\mathbf{wp}[\![c_1; c_2, \Phi]\!] &\equiv \mathbf{wp}[\![c_1, \mathbf{wp}[\![c_2, \Phi]\!]]\!]
\end{aligned}$$

$$\begin{aligned}
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$\models \{P\}c\{Q\}$ iff

$\models P \Rightarrow \mathbf{wp}[\![c, Q]\!]$

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$$\models \{P\}c\{Q\} \quad \text{iff}$$

$$\models P \Rightarrow \mathbf{wp}[\![c, Q]\!]$$

Advertisement: Hoare logic for **hybrid systems**, by **nonstandard analysis** (esp. transfer principle)

[Suenaga, Hasuo, Sekine, ICALP'11/CAV'12/POPL'13]

Categorical View on Weakest Precondition Semantics

[Jacobs, LICS'13/LMCS'13/...]

$$\begin{tikzcd}
\mathbf{CL}_\wedge & \\
& \mathcal{K}\ell(\mathcal{P})^{\mathrm{op}} \arrow[ul, "\mathbb{P}^{\mathcal{K}\ell}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\left(\begin{array}{c} \mathcal{P}X \overset{\mathsf{wp}(f)}{\longleftarrow} \mathcal{P}Y \\ \{x \mid f(x) \subseteq Q\} \longleftarrow\!\shortmid (Q \subseteq Y) \end{array} \right) & \\
& (\, X \xrightarrow{f} \mathcal{P}Y \,) \arrow[ul, mapsto, "\mathbb{P}^{\mathcal{K}\ell}"]
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Categorical View on Weakest Precondition Semantics

[Jacobs, LICS'13/LMCS'13/...]

$$\begin{tikzcd} \mathbf{CL}_\wedge & \\ & \mathcal{K}\ell(\mathcal{P})^{\mathrm{op}} \arrow[ul, bend right, "\mathbb{P}^{\mathcal{K}\ell}"'] \end{tikzcd}$$

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nondet. “computation”

Categorical View on Weakest Precondition Semantics

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"Every output is in Q"

nondet. "computation"

Categorical View on Weakest Precondition Semantics

[Jacobs, LICS'13/LMCS'13/...]

$$\begin{tikzcd}
\mathbf{CL}_{\wedge} \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] & & (\mathbf{CL}_{\vee})^{\mathrm{op}} \cong \mathcal{EM}(\mathcal{P})^{\mathrm{op}} \arrow[ll, bend left, "\mathbb{P}^{\mathcal{EM}}"] \\
& \mathcal{K}\ell(\mathcal{P})^{\mathrm{op}} \arrow[ul, "\mathbb{P}^{\mathcal{K}\ell}"] \arrow[ur, "K^{\mathrm{op}}"'] &
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& (\,X \overset{f}{\to} \mathcal{P}Y\,) \arrow[ul, mapsto, "\mathbb{P}^{\mathcal{K}\ell}"] \arrow[ur, mapsto, "K^{\mathrm{op}}"'] &
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Categorical View on Weakest Precondition Semantics

[Jacobs, LICS'13/LMCS'13/...]

$$\begin{tikzcd}
\mathbf{CL}_{\wedge} \arrow[rr, bend left=10] \arrow[rr, phantom, "\perp"] & & (\mathbf{CL}_{\vee})^{\mathrm{op}} \cong \mathcal{EM}(\mathcal{P})^{\mathrm{op}} \arrow[ll, bend left=10, "\mathbb{P}^{\mathcal{EM}}"] \\
& \mathcal{K}\ell(\mathcal{P})^{\mathrm{op}} \arrow[ul, bend left=10, "\mathbb{P}^{\mathcal{K}\ell}"] \arrow[ur, bend right=10, "K^{\mathrm{op}}"'] &
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Categorical View on Weakest Precondition Semantics

[Jacobs, LICS'13/LMCS'13/...]

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\end{tikzcd}$$

quantum

Categorical Axiomatics

In the paper in the proceedings...

- Foundation by **order-enriched monad** T
  - That is: $\mathcal{K}\ell(T)$ is $\mathbf{Posets}$-enriched
  - $T(1)$ = {truth values}

    [Kock; Coumans & Jacobs; Cirstea]

- Additional generality
  - **modalities**, like $\Box$ vs. $\Diamond$
  - Two players (cf. Prologue)

Modality
in Predicate Transformers

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What if:

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$\mathsf{wp}_{\Diamond}(f)$

MONADIC FOUNDATION OF WEAKEST PRECONDITION SEMANTICS

➡ 1-player setting

2-player setting

(1-Player) Monadic Preconditions: Overview

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- T : an **order-enriched monad**
  - i.e. $\mathcal{K}\ell(T)$ is $\mathbf{Posets}$-enriched
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- $$\begin{array}{c} \text{a predicate over } X \\ \hline\hline X \longrightarrow T1 \end{array}$$

Order-Enriched Monads

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An *order-enriched monad* $\boldsymbol{T}$ on $\mathbb{C}$ is a monad together with a **Posets**-enriched str. of $\mathcal{K}\ell(\boldsymbol{T})$.

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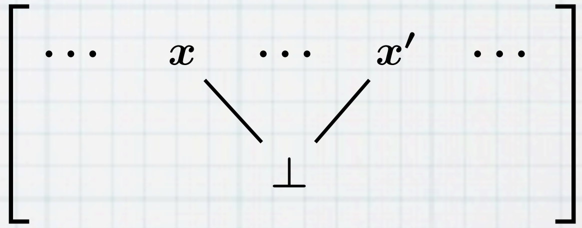
the **lift** monad $\mathcal{L}$	(potential) nontermination	$\mathcal{L}X = \{\perp\} + X$ $=$ 
the **powerset** monad $\mathcal{P}$	nondeterminism	$\mathcal{P}X = \{U \subseteq X\}$
the **subdistribution** monad $\mathcal{D}$	probability	$\mathcal{D}X =$ $\{d\colon X \to [0,1] \mid \sum_{x \in X} d(x) \leq 1\}$

Order-Enriche

Char. by **substitutivity** & **congruence**

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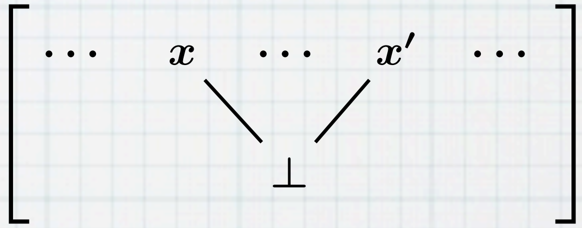
the **lift** monad $\mathcal{L}$	(potential) nontermination	$\mathcal{L}X = \{\bot\} + X$ $=$ 
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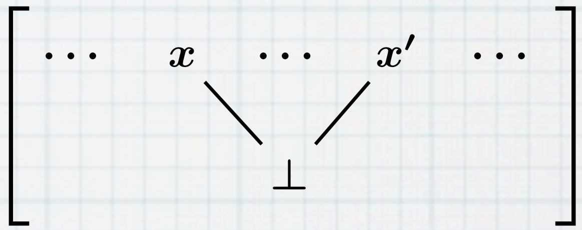
the **lift** monad $\mathcal{L}$	(potential) nontermination	$\mathcal{L}X = \{\bot\} + X$ $=$ 
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$T(1)$ as {truth values}

- Moreover: $$\frac{\text{a predicate over } X}{X \longrightarrow T1}$$

$= \{\bot\} + X$ $= \left[\begin{tikzcd}[column sep=small] \cdots & x \arrow[dr, dash] & \cdots & x' \arrow[dl, dash] & \cdots \\ & & \bot & & \end{tikzcd}\right]$	$\mathcal{L}1 \;=\; \left[\begin{array}{c} (\mathtt{tt} := *) \\ \rotatebox{90}{$\sqsubseteq$} \\ (\mathtt{ff} := \bot) \end{array}\right]$	$\mathbf{Sets}(X, \mathcal{L}1)$ $\cong\; 2^X$
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$T(1)$ as {trut

[0,1]-valued random var.,
“likelihood”

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Weakest Precondition Semantics

$$\text{Given} \quad \begin{cases} \boldsymbol{X} \xrightarrow{\boldsymbol{f}} \boldsymbol{TY} & \text{“branching computation”} \\ \boldsymbol{Y} \xrightarrow{\boldsymbol{q}} \boldsymbol{T1} & \text{“postcondition”} \end{cases}$$

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- multiplication μ^T ? ➜ not necessarily!
- choice of $$\begin{tikzcd} T(T1) \arrow[d, "\tau"] \\ T1 \end{tikzcd}$$

= choice of “**modality**”

Categorical Axiomatics

Defn. A *PT situation* on $\mathbb{C}$ is

$$\Bigl(T, \Omega, \begin{array}{c} T(T\Omega) \\ \downarrow\tau \\ T\Omega \end{array}\Bigr)$$

where

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Prop. A PT situation gives rise to a functor

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- **monotonicity**
- **functoriality:**

$$\mathbf{wp}(f,\ \mathbf{wp}(g, Q)) = \mathbf{wp}(g \circ f, Q)$$

Categorical Axiomatics

Defn. A *PT situation* on $\mathbb{C}$ is

$$\left(\boldsymbol{T}, \boldsymbol{\Omega}, \quad \begin{array}{c} \boldsymbol{T}(\boldsymbol{T\Omega}) \\ \downarrow\boldsymbol{\tau} \\ \boldsymbol{T\Omega} \end{array} \right)$$

where

- $\boldsymbol{T}$ is an order-enriched monad
- $\boldsymbol{\Omega} \in \mathbb{C}$
- $\begin{array}{c} \boldsymbol{T}(\boldsymbol{T\Omega}) \\ \downarrow\boldsymbol{\tau} \\ \boldsymbol{T\Omega} \end{array}$ is an EM-alg. that is "monotone"

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$$\begin{array}{c} \boldsymbol{T}(\boldsymbol{T\Omega}) \xrightarrow{\boldsymbol{\tau}} \boldsymbol{T\Omega} \\ \hline\hline \mathbb{C}(\_\,, \boldsymbol{T\Omega}) \xRightarrow{\boldsymbol{\tau}^{\wedge}} \mathbb{C}(\boldsymbol{T}\_\,, \boldsymbol{T\Omega}) \end{array} \quad \text{Yoneda}$$

Defn.
$\boldsymbol{\tau}$ is *monotone* if $\boldsymbol{\tau}^{\wedge}$ is monotone

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$$\frac{T(T\Omega) \xrightarrow{\tau} T\Omega}{\mathbb{C}(\_\,, T\Omega) \overset{\tau^\wedge}{\Longrightarrow} \mathbb{C}(T\_\,, T\Omega)}\ \text{Yoneda}$$

Defn.
τ is *monotone* if $\tau^\wedge$ is monotone

$$2^{-} \Rightarrow 2^{T(\_)}$$

if $T = \mathcal{P}$ and $\Omega = 1$

➜ predicate lifting!

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"monotone pred. lifting
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- Induces:

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$$\begin{tikzcd} \left(\begin{array}{ccc} \mathbb{C}(X, T1) & \longleftarrow & \mathbb{C}(Y, T1) \\ (\tau \circ Tq \circ f) & \leftarrow\!\shortmid & q \end{array}\right) \\ (\, X \xrightarrow{f} TY \,) \arrow[u, mapsto, "\mathbb{P}^{\mathcal{K}\ell}"] \end{tikzcd}$$

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\end{tikzcd}$$

Limitations

$$\begin{tikzcd}
\left(\begin{array}{c}\text{bwd. pred.}\\ \text{transformers}\end{array}\right) \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] & & \left(\begin{array}{c}\text{fwd. pred.}\\ \text{transformers}\end{array}\right) \arrow[ll, bend left, "\mathbb{P}^{\mathcal{EM}}"] \\
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- Here predicates are
  - $d \in \mathcal{D}X$ "uncertain whereabouts" (subj. to normalization $\sum_x d(x) \leq 1$)
  - unlike $p\colon X \to \mathcal{D}1 \cong [0,1]$ (that can be everywhere 1)
- No modality yet

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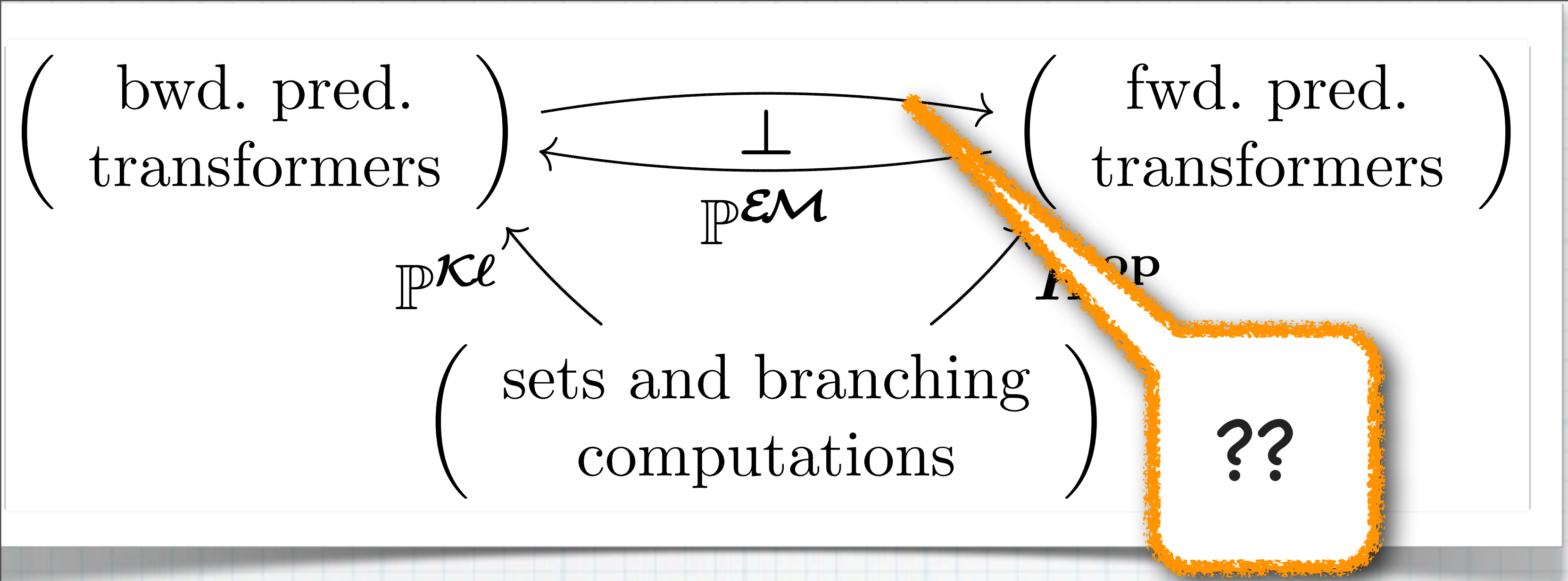

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richer structure?
(e.g. effect modules)

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PT Situation: Examples

Lem. $\begin{array}{c} T(T\Omega) \\ \downarrow \mu^T_\Omega \\ T\Omega \end{array}$ is monotone, hence a modality.

PT Situation: Examples

$T = \mathcal{P}, \Omega = 1$

*

$$\begin{tikzcd}\mathcal{P}(\mathcal{P}1) \arrow[d, "\mu^{\mathcal{P}} = \bigcup =: \tau_{\Diamond}"] \\ \mathcal{P}1\end{tikzcd}$$

→ "existential"
predicate transformer

$$\mathbf{wp}_{\Diamond}(f)(Q) = \{x \mid f(x) \cap Q \neq \emptyset\}$$

PT Situation: Examples

$$\boxed{T = \mathcal{P}, \Omega = 1}$$

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  → “existential” predicate transformer

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- $$\begin{tikzcd} \mathcal{P}(\mathcal{P}1) \arrow[r, "\mathcal{P}\sigma"] \arrow[d, "\tau_\square"'] & \mathcal{P}(\mathcal{P}1) \arrow[d, "\tau_\diamond"] \\ \mathcal{P}1 \arrow[r, "\cong", "\sigma\ (\mathrm{swap})"'] & \mathcal{P}1 \end{tikzcd}$$

  → “universal” predicate transformer

  $$\mathbf{wp}_\square(f)(Q) = \{x \mid f(x) \subseteq Q\}$$

PT Situation: Ex

$T = \mathcal{P}, \Omega = 1$

These are the only modalities

- $$\begin{tikzcd} \mathcal{P}(\mathcal{P}1) \arrow[d, "\mu^{\mathcal{P}} = \bigcup =: \tau_{\Diamond}"] \\ \mathcal{P}1 \end{tikzcd}$$

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PT Situation: Examples

$$T = \mathcal{D}, \Omega = 1$$

- $$\begin{tikzcd} \mathcal{D}(\mathcal{D}1) \arrow[d, "\mu^{\mathcal{D}}"] \\ \mathcal{D}1 \end{tikzcd} =: \tau_{\mathrm{total}}$$

➔ probab. pred. transformer, "as expected"

$$\mathbf{wp}_{\mathrm{total}}(X \xrightarrow{f} \mathcal{D}Y)(Y \xrightarrow{q} [0,1])$$

$$= \left[\begin{array}{ccl} x & \mapsto & \sum_y f(x)(y) \cdot q(y) \\ & = & \sum_y \mathbf{Pr}[x \to y] \cdot q(y) \end{array} \right]$$

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- $$\begin{tikzcd} \mathcal{D}(\mathcal{D}1) \arrow[r, "\mathcal{D}\sigma"] \arrow[d, "\tau_{\mathrm{partial}}"'] & \mathcal{D}(\mathcal{D}1) \arrow[d, "\tau_{\mathrm{total}}"] \\ \mathcal{D}1 \arrow[r, "\cong", "\sigma\ (\mathrm{swap})"'] & \mathcal{D}1 \end{tikzcd}$$

➔ "partial" pred. transf.

$$\mathbf{wp}_{\mathrm{partial}}(X \xrightarrow{f} \mathcal{D}Y)(Y \xrightarrow{q} [0,1]) = \left[\begin{array}{ccl} x & \mapsto & \sum_y f(x)(y) \cdot q(y) + \left(1 - \sum_y f(x)(y)\right) \\ & = & \mathbf{wp}_{\mathrm{total}}(f)(q)(x) + \mathbf{Pr}[\mathrm{deadlock}] \end{array} \right]$$

1-Player Setting: Summary

$$\left(T, \Omega, \begin{tikzcd}T(T\Omega) \arrow[d, "\tau"] \\ T\Omega\end{tikzcd}\right)$$

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order-enriched monad
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- modality:
  "monotone" EM-alg.
  = monotone pred lifting
  compat. w/ monad str.
- induces

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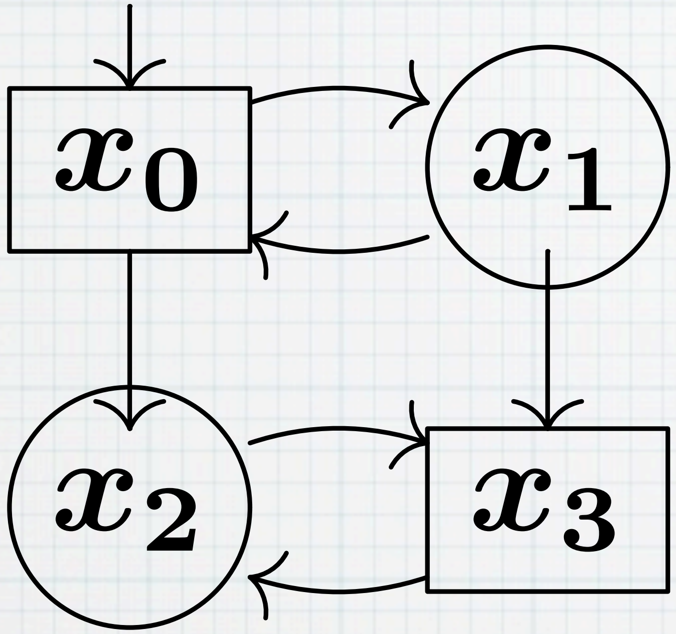
- Various modalities
  that can be "enumerated"

MONADIC FOUNDATION OF WEAKEST PRECONDITION SEMANTICS

1-player setting

➡ 2-player setting

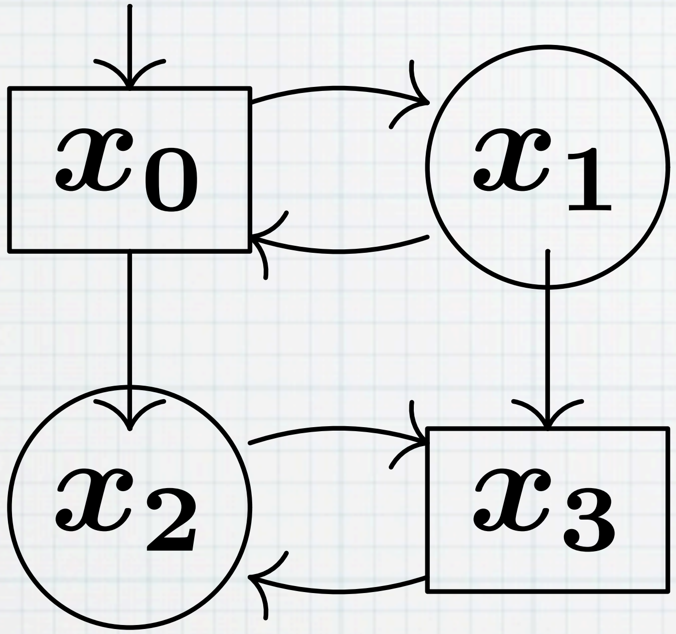
2-Player Setting: Games



□: Player's state

○: Opponent's state

2-Player Setting: Games



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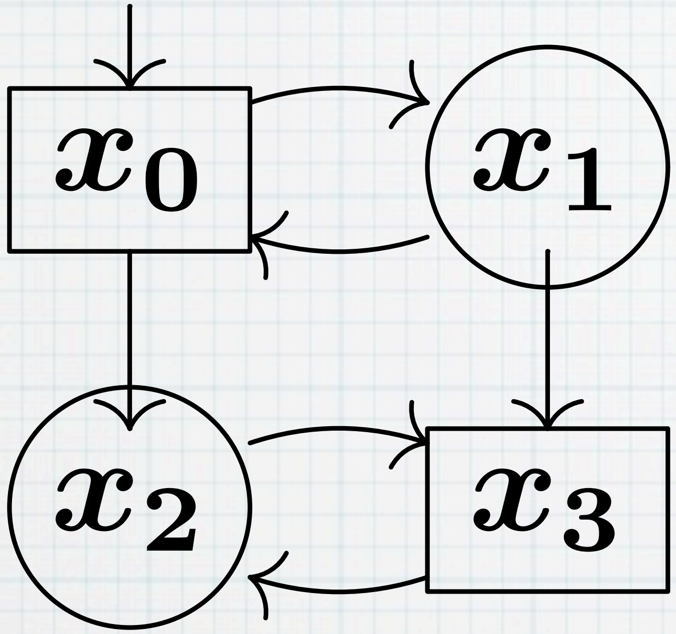

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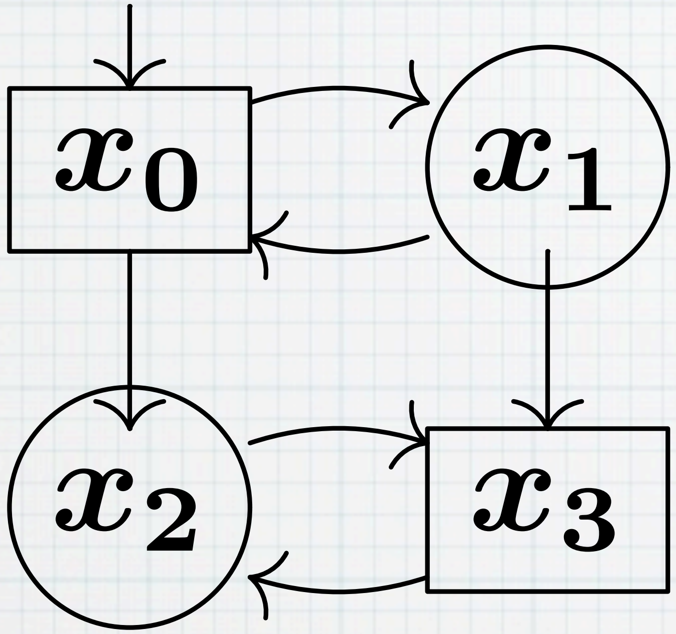

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- cf. nonnormal modal logic [Hansen, Kupke, ...]

$$\Box\varphi \wedge \Box\psi \not\Rightarrow \Box(\varphi \wedge \psi)$$

2-Player Setting: Probability & Nondeterminism

- Many prob. sys. are modeled as
  [Segala PhD'95, Vardi FOCS'85, ...]

$$\boldsymbol{X \longrightarrow \mathcal{P}DFX}$$

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environment's
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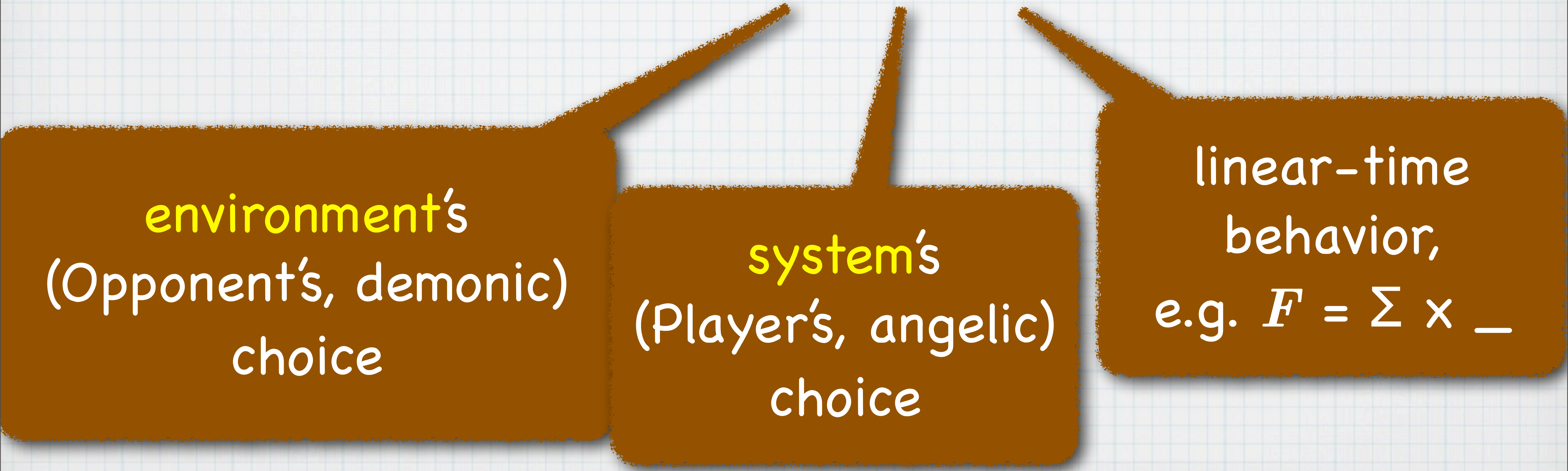

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linear-time
behavior,
e.g. $F = \Sigma \times \_$

Technical Challenges

$\mathcal{PD}$

$\mathcal{PP}$

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- No distr. law $\mathcal{DP} \Rightarrow \mathcal{PD}$ [Plotkin]

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- $\mathcal{P} \mapsto$ convex subset constr. [Tix, Keimel & Plotkin]

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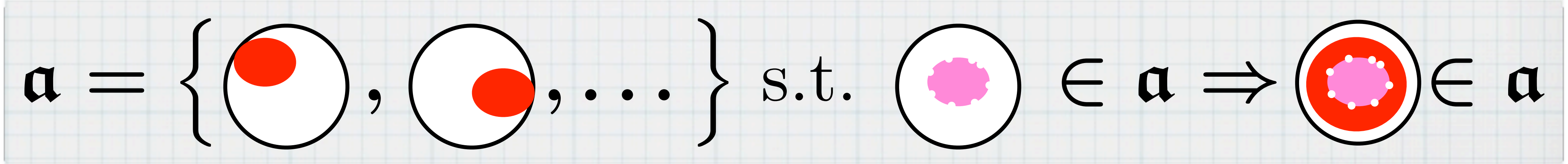

$\mathfrak{a} = \{ \ , \ , \ldots \}$ s.t. $\in \mathfrak{a} \Rightarrow \in \mathfrak{a}$

➔ def. by direct/inverse img. coincide

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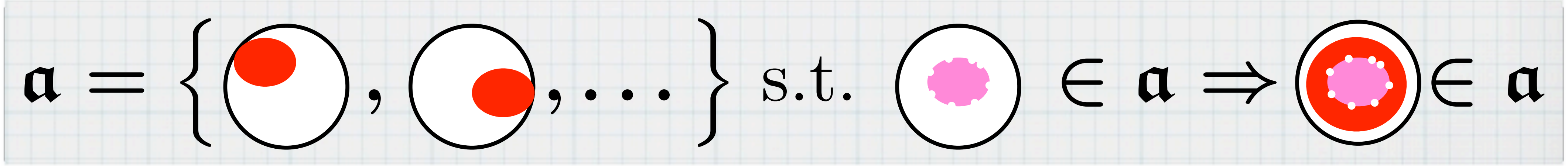

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Contributions

In the paper in the proceedings...

- **Modular presentation** of 2-player settings
  - Not
$$\begin{tikzcd}\mathbb{C} \arrow[loop above, "T"] \arrow[loop below, "R"]\end{tikzcd}$$
but
$$\begin{tikzcd}\mathbb{C} \arrow[loop left, "T"] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{EM}(T) \arrow[l, bend left] \arrow[loop below, "R"]\end{tikzcd}$$
- **Backward pred. transf. semantics**, e.g.

$$\frac{X \xrightarrow{f} \mathcal{UP}Y \qquad Y \xrightarrow{q} 2}{X \xrightarrow{\mathsf{wp}(f,q)} 2}$$

Categorical Axiomatics

Defn. A *2-player PT situation* on $\mathbb{C}$ is

$$\left(\boldsymbol{T}, \boldsymbol{\Omega}, \begin{array}{c} \boldsymbol{T}(\boldsymbol{T\Omega}) \\ \downarrow \boldsymbol{\tau} \\ \boldsymbol{T\Omega} \end{array}, \boldsymbol{R}, \begin{array}{c} \boldsymbol{R\tau} \\ \downarrow \boldsymbol{\rho} \\ \boldsymbol{\tau} \end{array} \right)$$

in

$$\begin{tikzcd}
\mathbb{C} \arrow[loop above, "\boldsymbol{T}"] \arrow[r, bend left, "\boldsymbol{F^T}"] \arrow[r, phantom, "\perp"] & \overset{\boldsymbol{\tau}\,\in}{\mathcal{EM}(\boldsymbol{T})} \arrow[l, bend left, "\boldsymbol{U^T}"] \arrow[loop above, "\boldsymbol{R}"] \arrow[r, bend left, "\boldsymbol{F^R}"] \arrow[r, phantom, "\perp"] & \overset{\boldsymbol{\rho}\,\in}{\mathcal{EM}(\boldsymbol{R})} \arrow[l, bend left, "\boldsymbol{U^R}"]
\end{tikzcd}$$

s.t. $\boldsymbol{\tau}, \boldsymbol{\rho}$ are “monotone”

Categorical Axiomatics

Defn. A *2-player PT situation* on $\mathbb{C}$ is

$$\left(\boldsymbol{T},\boldsymbol{\Omega},\ \begin{array}{c}\boldsymbol{T(T\Omega)}\\ \downarrow\boldsymbol{\tau}\\ \boldsymbol{T\Omega}\end{array},\ \boldsymbol{R},\ \begin{array}{c}\boldsymbol{R\tau}\\ \downarrow\boldsymbol{\rho}\\ \boldsymbol{\tau}\end{array}\ \right)$$

in

$$\begin{tikzcd}
 & \boldsymbol{\tau} \arrow[d, phantom, sloped, "\in"] & \boldsymbol{\rho} \arrow[d, phantom, sloped, "\in"] \\
\mathbb{C} \arrow[loop above, "\boldsymbol{T}"] \arrow[r, bend left, "\boldsymbol{F^T}"] \arrow[r, phantom, "\perp"] & \boldsymbol{\mathcal{EM}(T)} \arrow[loop above, "\boldsymbol{R}"] \arrow[l, bend left, "\boldsymbol{U^T}"] \arrow[r, bend left, "\boldsymbol{F^R}"] \arrow[r, phantom, "\perp"] & \boldsymbol{\mathcal{EM}(R)} \arrow[l, bend left, "\boldsymbol{U^R}"]
\end{tikzcd}$$

s.t. $\boldsymbol{\tau},\boldsymbol{\rho}$ are “monotone”

Categorical Axiomatics

Defn. A *2-player PT situation* on $\mathbb{C}$ is

$$\left(\boldsymbol{T}, \boldsymbol{\Omega}, \begin{array}{c} \boldsymbol{T}(\boldsymbol{T}\boldsymbol{\Omega}) \\ \downarrow \boldsymbol{\tau} \\ \boldsymbol{T}\boldsymbol{\Omega} \end{array}, \boldsymbol{R}, \begin{array}{c} \boldsymbol{R}\boldsymbol{\tau} \\ \downarrow \boldsymbol{\rho} \\ \boldsymbol{\tau} \end{array} \right)$$

in

$\tau \in$ $\qquad$ $\rho \in$

$$\begin{tikzcd}
\mathbb{C} \arrow[loop above, "\boldsymbol{T}"] \arrow[r, bend left, "\boldsymbol{F}^{\boldsymbol{T}}"] \arrow[r, phantom, "\perp"] & \boldsymbol{\mathcal{EM}}(\boldsymbol{T}) \arrow[l, bend left, "\boldsymbol{U}^{\boldsymbol{T}}"] \arrow[loop above, "\boldsymbol{R}"] \arrow[r, bend left, "\boldsymbol{F}^{\boldsymbol{R}}"] \arrow[r, phantom, "\perp"] & \boldsymbol{\mathcal{EM}}(\boldsymbol{R}) \arrow[l, bend left, "\boldsymbol{U}^{\boldsymbol{R}}"]
\end{tikzcd}$$

s.t. $\boldsymbol{\tau}, \boldsymbol{\rho}$ are "monotone"

Categorical Axiomatics

Defn. A *2-player PT situation* on $\mathbb{C}$ is

$$\left(T, \Omega, \begin{array}{c} T(T\Omega) \\ \downarrow \tau \\ T\Omega \end{array}, R, \begin{array}{c} R\tau \\ \downarrow \rho \\ \tau \end{array} \right)$$

in

$\tau \in$ $\rho \in$

$$\begin{tikzcd}
\mathbb{C} \arrow[loop above, "T"] \arrow[loop left, "U^T R F^T"] \arrow[r, bend left, "F^T"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(T) \arrow[l, bend left, "U^T"] \arrow[loop above, "R"] \arrow[r, bend left, "F^R"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(R) \arrow[l, bend left, "U^R"]
\end{tikzcd}$$

s.t. τ, ρ are "monotone"

Categorical Axiomatics

Defn. A *2-player PT situation* on $\mathbb{C}$ is

$$\Bigl(T, \Omega, \begin{array}{c} T(T\Omega) \\ \downarrow\tau \\ T\Omega \end{array}, R, \begin{array}{c} R\tau \\ \downarrow\rho \\ \tau \end{array}\Bigr)$$

in

$$\begin{tikzcd}
& \tau \arrow[d, phantom, "\in"] & \rho \arrow[d, phantom, "\in"] \\
\mathbb{C} \arrow[loop above, "T"] \arrow[loop left, "U^TRF^T"] \arrow[r, bend left, "F^T"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(T) \arrow[l, bend left, "U^T"] \arrow[loop above, "R"] \arrow[r, bend left, "F^R"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(R) \arrow[l, bend left, "U^R"]
\end{tikzcd}$$

s.t. τ, ρ are "monotone"

Prop.

$$\mathcal{K}\ell(U^TRF^T)^{\mathrm{op}} \xrightarrow{\mathsf{wp}} \mathbf{Posets}$$

arises, realizing

$$\frac{X \xrightarrow{f} U^TRF^TY \qquad Y \xrightarrow{q} T\Omega}{X \xrightarrow{\mathsf{wp}(f,q)} T\Omega}$$

Categorical Axiomatics

Defn. A *2-player PT situation* on $\mathbb{C}$ is

$$\Bigl(T, \Omega, \begin{array}{c} T(T1) \\ \downarrow\tau \\ T1 \end{array}, R, \begin{array}{c} R\tau \\ \downarrow\rho \\ \tau \end{array}\Bigr)$$

in

$$\begin{tikzcd}
T & & \tau \arrow[d, phantom, sloped, "\in"] & R & & \rho \arrow[d, phantom, sloped, "\in"] \\
\mathbb{C} \arrow[loop above, "T"] \arrow[loop left, "U^T R F^T"] \arrow[rr, bend left, "F^T"] \arrow[rr, phantom, "\perp"] & & \mathcal{EM}(T) \arrow[ll, bend left, "U^T"] \arrow[loop above, "R"] \arrow[rrr, bend left, "F^R"] \arrow[rrr, phantom, "\perp"] & & & \mathcal{EM}(R) \arrow[lll, bend left, "U^R"]
\end{tikzcd}$$

s.t. τ, ρ are "monotone"

Prop.

$$\begin{tikzcd}
\textbf{Posets} & & \mathcal{EM}(R)^{\mathrm{op}} \arrow[ll, bend right, "\mathbb{P}^{\mathcal{EM}}"'] \\
& \mathcal{K}\ell(U^T R F^T)^{\mathrm{op}} \arrow[ul, "\mathbb{P}^{\mathcal{K}\ell}"] \arrow[ur, "K^{\mathrm{op}}"'] &
\end{tikzcd}$$

Categorical Axiomatics

Defn. A *2-player PT situation*

$$(T, \Omega, \begin{array}{c} T(T1) \\ \downarrow\tau \\ T1 \end{array}, R, \begin{array}{c} R\tau \\ \downarrow\rho \end{array}$$

in

$$\begin{tikzcd}
T \ni \tau & R \ni \rho & \\
\mathbb{C} \arrow[loop above, "T"] \arrow[loop left, "U^T R F^T"] \arrow[r, bend left, "F^T"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(T) \arrow[l, bend left, "U^T"] \arrow[loop above, "R"] \arrow[r, bend left, "F^R"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(R) \arrow[l, bend left, "U^R"]
\end{tikzcd}$$

s.t. τ, ρ are "monotone"

$$\begin{tikzcd}
\mathbb{C} \arrow[loop above, "T"] \arrow[loop left, "U^T R F^T"] \arrow[r, bend left, "F^T"] \arrow[r, phantom, "\perp"] \arrow[dr, bend right, "\top"'] & \mathcal{EM}(T) \arrow[l, bend left, "U^T"] \arrow[loop above, "R"] \arrow[r, bend left, "F^R"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(R) \arrow[l, bend left, "U^R"] \\
& & \mathcal{K}\ell(U^T R F^T) \arrow[u, dashed, "K"'] \arrow[ull, bend left]
\end{tikzcd}$$

Prop.

$$\begin{tikzcd}
\mathbf{Posets} & & \mathcal{EM}(R)^{\mathrm{op}} \arrow[ll, "\mathbb{P}^{\mathcal{EM}}"'] \\
& \mathcal{K}\ell(U^T R F^T)^{\mathrm{op}} \arrow[ul, "\mathbb{P}^{\mathcal{K}\ell}"] \arrow[ur, "K^{\mathrm{op}}"'] &
\end{tikzcd}$$

Examples

first move by...	second move by...
nondet. Player	nondet. Opponent
nondet. Opponent	nondet. Player
nondet. Opponent	probabilistic Player

Example:
Nondet. P → Nondet. O

$$\begin{tikzcd}
\mathbf{Sets} \arrow[loop above, "\mathcal{P}"] \arrow[r, bend left=15, "F^{\mathcal{P}}"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathcal{P}) \arrow[l, bend left=15, "U^{\mathcal{P}}"] \arrow[r, "\cong"] & \mathbf{CL}_{\wedge} \arrow[loop above, "\mathsf{Dw}"] \arrow[r, bend left=15] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathsf{Dw}) \arrow[l, bend left=15]
\end{tikzcd}$$

$\tau_{\Box} \in$ $\qquad$ $\rho_{\mathsf{P}} \in$

- modalities:

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}1) \arrow[d, "\tau_{\Box}"] & \{\,\} \arrow[d, mapsto] & \{\mathrm{ff}\} \arrow[d, mapsto] & \{\mathrm{tt}\} \arrow[d, mapsto] & \{\mathrm{tt}, \mathrm{ff}\} \arrow[d, mapsto] \\
\mathcal{P}1 & \mathrm{tt} & \mathrm{ff} & \mathrm{tt} & \mathrm{ff}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Dw}(\tau_{\Box}) \arrow[d, "\rho_{\mathsf{P}}"] & \{\,\} \arrow[d, mapsto] & \{\mathrm{ff}\} \arrow[d, mapsto] & \{\mathrm{tt}, \mathrm{ff}\} \arrow[d, mapsto] \\
\tau_{\Box} & \mathrm{ff} & \mathrm{ff} & \mathrm{tt}
\end{tikzcd}$$

Example:

Nondet. P → Nondet. O

Downward closed subsets

$$\begin{tikzcd}
 & & \tau_\Box \arrow[d, phantom, sloped, "\in"] & \rho_{\mathsf{P}} \arrow[d, phantom, sloped, "\in"] \\
\mathbf{Sets} \arrow[loop above, "\mathcal{P}"] \arrow[r, bend left, "F^{\mathcal{P}}"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathcal{P}) \arrow[l, bend left, "U^{\mathcal{P}}"] \arrow[r, "\cong"] & \mathbf{CL}_\wedge \arrow[loop above, "\mathsf{Dw}"] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathsf{Dw}) \arrow[l, bend left]
\end{tikzcd}$$

* modalities:

$\mathcal{P}(\mathcal{P}1)$	$\{\,\}$	$\{\mathtt{ff}\}$	$\{\mathtt{tt}\}$	$\{\mathtt{tt},\mathtt{ff}\}$
$\downarrow\tau_\Box$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{P}1$	$\mathtt{tt}$	$\mathtt{ff}$	$\mathtt{tt}$	$\mathtt{ff}$

$\mathsf{Dw}(\tau_\Box)$	$\{\,\}$	$\{\mathtt{ff}\}$	$\{\mathtt{tt},\mathtt{ff}\}$
$\downarrow\rho_{\mathsf{P}}$	$\downarrow$	$\downarrow$	$\downarrow$
$\tau_\Box$	$\mathtt{ff}$	$\mathtt{ff}$	$\mathtt{tt}$

Downward closed subsets

Example:

Nondet. P → Nondet. O

$$\begin{tikzcd}
 & & \tau_{\Box} \arrow[d, phantom, sloped, "\in"] & \rho_{\mathsf{P}} \arrow[d, phantom, sloped, "\in"] \\
\mathbf{Sets} \arrow[loop above, "\mathcal{P}"] \arrow[r, bend left, "F^{\mathcal{P}}"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathcal{P}) \arrow[l, bend left, "U^{\mathcal{P}}"] \arrow[r, "\cong"] & \mathbf{CL}_{\wedge} \arrow[loop above, "\mathsf{Dw}"] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathsf{Dw}) \arrow[l, bend left]
\end{tikzcd}$$

* modalities:

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}1) \arrow[d, "\tau_{\Box}"] & \{\,\} \arrow[d, mapsto] & \{\mathrm{ff}\} \arrow[d, mapsto] & \{\mathrm{tt}\} \arrow[d, mapsto] & \{\mathrm{tt}, \mathrm{ff}\} \arrow[d, mapsto] \\
\mathcal{P}1 & \mathrm{tt} & \mathrm{ff} & \mathrm{tt} & \mathrm{ff}
\end{tikzcd}$$

$\wedge$

$$\begin{tikzcd}
\mathsf{Dw}(\tau_{\Box}) \arrow[d, "\rho_{\mathsf{P}}"] & \{\,\} \arrow[d, mapsto] & \{\mathrm{ff}\} \arrow[d, mapsto] & \{\mathrm{tt}, \mathrm{ff}\} \arrow[d, mapsto] \\
\tau_{\Box} & \mathrm{ff} & \mathrm{ff} & \mathrm{tt}
\end{tikzcd}$$

Example:
Nondet. P → Nondet. O

Downward closed subsets

$$\begin{tikzcd}
\mathbf{Sets} \arrow[loop above, "\mathcal{P}"] \arrow[r, bend left, "F^{\mathcal{P}}"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathcal{P}) \arrow[l, bend left, "U^{\mathcal{P}}"] \arrow[r, "\cong"] & \mathbf{CL}_{\wedge} \arrow[loop above, "\mathsf{Dw}"] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathsf{Dw}) \arrow[l, bend left]
\end{tikzcd}$$

$\tau_{\Box} \in \mathbf{CL}_{\wedge}$, $\rho_{\mathsf{P}} \in \mathcal{EM}(\mathsf{Dw})$

* modalities:

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}1) \arrow[d, "\tau_{\Box}"] & \{\,\} \arrow[d, mapsto] & \{\mathtt{ff}\} \arrow[d, mapsto] & \{\mathtt{tt}\} \arrow[d, mapsto] & \{\mathtt{tt}, \mathtt{ff}\} \arrow[d, mapsto] \\
\mathcal{P}1 & \mathtt{tt} & \mathtt{ff} & \mathtt{tt} & \mathtt{ff}
\end{tikzcd}$$

$\wedge$

$$\begin{tikzcd}
\mathsf{Dw}(\tau_{\Box}) \arrow[d, "\rho_{\mathsf{P}}"] & \{\,\} \arrow[d, mapsto] & \{\mathtt{ff}\} \arrow[d, mapsto] & \{\mathtt{tt}, \mathtt{ff}\} \arrow[d, mapsto] \\
\tau_{\Box} & \mathtt{ff} & \mathtt{ff} & \mathtt{tt}
\end{tikzcd}$$

$\vee$

Example:

Nondet. P → Nondet. O

- From this we obtain...

$$\begin{tikzcd}
 & & \tau_{\Box} \in & \rho_{\mathsf{P}} \in \\
\mathbf{Sets} \arrow[loop above, "\mathcal{P}"] \arrow[r, bend left, "F^{\mathcal{P}}"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathcal{P}) \arrow[l, bend left, "U^{\mathcal{P}}"] \arrow[r, "\cong"] & \mathbf{CL}_{\wedge} \arrow[loop above, "\mathsf{Dw}"] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathsf{Dw}) \arrow[l, bend left]
\end{tikzcd}$$

$$\mathcal{K}\ell(U^{\mathcal{P}}\mathsf{Dw}F^{\mathcal{P}})^{\mathrm{op}} \cong \mathcal{K}\ell(\mathcal{UP})^{\mathrm{op}} \longrightarrow \mathbf{Posets}$$

Technical Challenges

$\mathcal{P}\mathcal{D}$

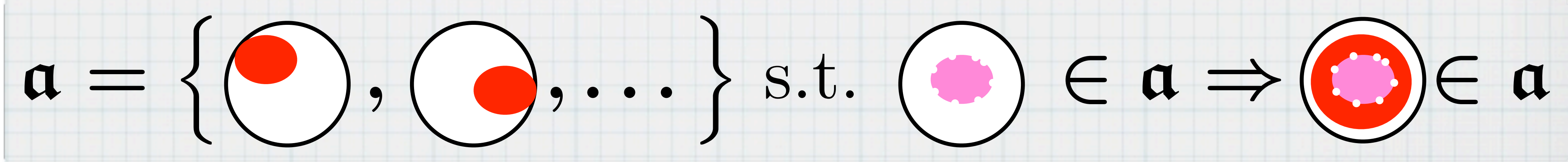
- No distr. law $\mathcal{D}\mathcal{P} \Rightarrow \mathcal{P}\mathcal{D}$ [Plotk
  ➔ not easily a monad!

$$[* \mapsto 1] \neq \left[\begin{array}{l} * \mapsto \frac{1}{2} \\ * \mapsto \frac{1}{2} \end{array}\right]$$

- $\mathcal{D}$ $\mapsto$ indexed valuation monad [Varacca]
- $\mathcal{P}$ $\mapsto$ convex subset constr. [Tix, Keimel & Plotkin]

$\mathcal{P}\mathcal{P}$

- On arrows. By **direct** or **inverse** images?
- $\mathcal{UP}X = \{\mathfrak{a} \subseteq \mathcal{P}X \mid \mathfrak{a}\colon \text{ up-closed}\}$



➔ def. by direct/inverse img. coincide

Technical Challenges

$\mathcal{PD}$

- No distr. law $\mathcal{DP} \Rightarrow \mathcal{PD}$ [Plotkin]

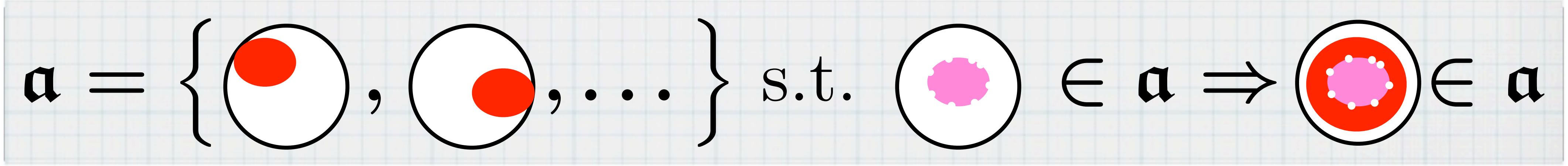
  → not easily a monad!

$[* \mapsto 1] \neq \left[\begin{array}{l} * \mapsto \frac{1}{2} \\ * \mapsto \frac{1}{2} \end{array}\right]$

- $\mathcal{D} \mapsto$ indexed valuation monad [Varacca]
- $\mathcal{P} \mapsto$ convex subset constr. [Tix, Keimel & Plotkin]

$\mathcal{PP}$

- On arrows. By **direct** or **inverse** images?
- $\mathcal{UP}X = \{\mathfrak{a} \subseteq \mathcal{P}X \mid \mathfrak{a}\text{: up-closed}\}$



$\mathfrak{a} = \{ \;, \;, \dots \}$ s.t. $\in \mathfrak{a} \Rightarrow \in \mathfrak{a}$

→ def. by direct/inverse img. coincide

Example:

Nondet. P → Nondet. O

- From this we obtain...

$$\begin{tikzcd}
 & & & \tau_{\Box} \in & & \rho_{\mathsf{P}} \in \\
\mathbf{Sets} \arrow[loop above, "\mathcal{P}"] \arrow[r, bend left, "F^{\mathcal{P}}"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathcal{P}) \arrow[l, bend left, "U^{\mathcal{P}}"] \arrow[rr, "\cong"] & & \mathbf{CL}_{\wedge} \arrow[loop above, "\mathsf{Dw}"] \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] & & \mathcal{EM}(\mathsf{Dw}) \arrow[ll, bend left]
\end{tikzcd}$$

$$\mathcal{K}\ell(U^{\mathcal{P}}\mathsf{Dw}F^{\mathcal{P}})^{\mathrm{op}} \cong \mathcal{K}\ell(\mathcal{U}\mathcal{P})^{\mathrm{op}} \longrightarrow \mathbf{Posets}$$

Example:

Nondet. P → Nondet. O

- From this we obtain...

$$\begin{tikzcd}
 & & \tau_\Box \arrow[d, phantom, sloped, "\in"] & \rho_{\mathsf{P}} \arrow[d, phantom, sloped, "\in"] \\
\mathbf{Sets} \arrow[loop above, "\mathcal{P}"] \arrow[r, bend left, "F^{\mathcal{P}}"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathcal{P}) \arrow[l, bend left, "U^{\mathcal{P}}"] \arrow[r, "\cong"] & \mathbf{CL}_\wedge \arrow[loop above, "\mathsf{Dw}"] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathsf{Dw}) \arrow[l, bend left]
\end{tikzcd}$$

$$\mathcal{K}\ell(U^{\mathcal{P}}\mathsf{Dw}F^{\mathcal{P}})^{\mathrm{op}} \cong \mathcal{K}\ell(\mathcal{UP})^{\mathrm{op}} \longrightarrow \mathbf{Posets}$$

- i.e.

$$\frac{X \xrightarrow{f} \mathcal{UP}Y \qquad Y \xrightarrow{q} \mathcal{P}1}{X \xrightarrow{\mathsf{wp}(f,q)} \mathcal{P}1}$$

s.t.

$$\begin{aligned}\mathsf{wp}(f,q)(x) = \mathtt{tt} &\iff \exists S \in f(x).\, \forall y \in S.\, q(y) = \mathtt{tt} \\ &\iff \text{“}x \text{ can force } q \text{ via } f\text{”}\end{aligned}$$

Example:
Nondet. P → Nondet. O

$$\begin{tikzcd}
 & & \tau_{\Box} \arrow[d, phantom, sloped, "\in"] & \rho_{\mathsf{P}} \arrow[d, phantom, sloped, "\in"] \\
\mathbf{Sets} \arrow[loop above, "\mathcal{P}"] \arrow[r, bend left, "F^{\mathcal{P}}"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathcal{P}) \arrow[l, bend left, "U^{\mathcal{P}}"] \arrow[r, "\cong"] & \mathbf{CL}_{\wedge} \arrow[loop above, "\mathsf{Dw}"] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathsf{Dw}) \arrow[l, bend left]
\end{tikzcd}$$

- modalities:

$$\begin{array}{ccccc}
\mathcal{P}(\mathcal{P}1) & \{\,\} & \{\mathrm{ff}\} & \{\mathrm{tt}\} & \{\mathrm{tt},\mathrm{ff}\} \\
\downarrow\tau_{\Box} & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathcal{P}1 & \mathrm{tt} & \mathrm{ff} & \mathrm{tt} & \mathrm{ff}
\end{array}$$

$$\begin{array}{cccc}
\mathsf{Dw}(\tau_{\Box}) & \{\,\} & \{\mathrm{ff}\} & \{\mathrm{tt},\mathrm{ff}\} \\
\downarrow\rho_{\mathsf{P}} & \downarrow & \downarrow & \downarrow \\
\tau_{\Box} & \mathrm{ff} & \mathrm{ff} & \mathrm{tt}
\end{array}$$

Example:
Nondet. O → Nondet. P

$$\begin{tikzcd}
 & & \tau_{\Diamond} \arrow[d, phantom, sloped, "\in"] & \rho_{\mathsf{O}} \arrow[d, phantom, sloped, "\in"] \\
\mathbf{Sets} \arrow[loop above, "\mathcal{P}"] \arrow[r, bend left, "F^{\mathcal{P}}"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathcal{P}) \arrow[l, bend left, "U^{\mathcal{P}}"] \arrow[r, "\cong"] & \mathbf{CL}_{\vee} \arrow[loop above, "\mathsf{Up}"] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathsf{Up}) \arrow[l, bend left]
\end{tikzcd}$$

- modalities:

$$\begin{array}{ccccc}
\mathcal{P}(\mathcal{P}1) & \{\,\} & \{\mathrm{ff}\} & \{\mathrm{tt}\} & \{\mathrm{tt},\mathrm{ff}\} \\
\downarrow\tau_{\Diamond} & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathcal{P}1 & \mathrm{ff} & \mathrm{ff} & \mathrm{tt} & \mathrm{tt}
\end{array}$$

$$\begin{array}{cccc}
\mathsf{Up}(\tau_{\Diamond}) & \{\,\} & \{\mathrm{tt}\} & \{\mathrm{tt},\mathrm{ff}\} \\
\downarrow\rho_{\mathsf{O}} & \downarrow & \downarrow & \downarrow \\
\tau_{\Diamond} & \mathrm{tt} & \mathrm{tt} & \mathrm{ff}
\end{array}$$

Example:
Nondet. P → Nondet. O

$$\begin{tikzcd}
 & & \tau_{\Box} \in & \rho_{\mathsf{P}} \in \\
\mathbf{Sets} \arrow[loop above, "\mathcal{P}"] \arrow[r, bend left, "F^{\mathcal{P}}"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathcal{P}) \arrow[l, bend left, "U^{\mathcal{P}}"] \arrow[r, "\cong"] & \mathbf{CL}_{\wedge} \arrow[loop above, "\mathsf{Dw}"] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathsf{Dw}) \arrow[l, bend left]
\end{tikzcd}$$

- m $\wedge$ lities:

$\mathcal{P}(\mathcal{P}1)$	$\{\}$	$\{\mathrm{ff}\}$	$\{\mathrm{tt}\}$	$\{\mathrm{tt},\mathrm{ff}\}$
$\downarrow\tau_{\Box}$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{P}1$	tt	ff	tt	ff

$\mathsf{Dw}(\tau_{\Box})$	$\{\}$	$\{\mathrm{ff}\}$	$\{\mathrm{tt},\mathrm{ff}\}$
$\downarrow\rho_{\mathsf{P}}$	$\downarrow$	$\downarrow$	$\downarrow$
$\tau_{\Box}$	ff	ff	tt

$\vee$

Example:
Nondet. O → Nondet. P

$$\begin{tikzcd}
 & & \tau_{\Diamond} \in & \rho_{\mathsf{O}} \in \\
\mathbf{Sets} \arrow[loop above, "\mathcal{P}"] \arrow[r, bend left, "F^{\mathcal{P}}"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathcal{P}) \arrow[l, bend left, "U^{\mathcal{P}}"] \arrow[r, "\cong"] & \mathbf{CL}_{\vee} \arrow[loop above, "\mathsf{Up}"] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathsf{Up}) \arrow[l, bend left]
\end{tikzcd}$$

- m $\vee$ lities:

$\mathcal{P}(\mathcal{P}1)$	$\{\}$	$\{\mathrm{ff}\}$	$\{\mathrm{tt}\}$	$\{\mathrm{tt},\mathrm{ff}\}$
$\downarrow\tau_{\Diamond}$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\mathcal{P}1$	ff	ff	tt	tt

$\mathsf{Up}(\tau_{\Diamond})$	$\{\}$	$\{\mathrm{tt}\}$	$\{\mathrm{tt},\mathrm{ff}\}$
$\downarrow\rho_{\mathsf{O}}$	$\downarrow$	$\downarrow$	$\downarrow$
$\tau_{\Diamond}$	tt	tt	ff

$\wedge$

Example:
Nondet. P → Nondet. O

$$\begin{tikzcd}
 & & \tau_{\Box} \in & \rho_{\mathsf{P}} \in \\
\mathbf{Sets} \arrow[loop above, "\mathcal{P}"] \arrow[r, bend left, "F^{\mathcal{P}}"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathcal{P}) \arrow[l, bend left, "U^{\mathcal{P}}"] \arrow[r, "\cong"] & \mathbf{CL}_{\wedge} \arrow[loop above, "\mathsf{Dw}"] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathsf{Dw}) \arrow[l, bend left]
\end{tikzcd}$$

- pred. trans.:

$$\mathbf{wp}(f, q)(x) = \mathtt{tt}$$

$$\Longleftrightarrow$$

$$\exists S \in f(x).\, \forall y \in S.\, q(y) = \mathtt{tt}$$

$$\Longleftrightarrow$$

“x can force q via f”

Example:
Nondet. O → Nondet. P

$$\begin{tikzcd}
 & & \tau_{\Diamond} \in & \rho_{\mathsf{O}} \in \\
\mathbf{Sets} \arrow[loop above, "\mathcal{P}"] \arrow[r, bend left, "F^{\mathcal{P}}"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathcal{P}) \arrow[l, bend left, "U^{\mathcal{P}}"] \arrow[r, "\cong"] & \mathbf{CL}_{\vee} \arrow[loop above, "\mathsf{Up}"] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathsf{Up}) \arrow[l, bend left]
\end{tikzcd}$$

- pred. trans.:

$$\mathbf{wp}(f, q)(x) = \mathtt{tt}$$

$$\Longleftrightarrow$$

$$\forall S \in f(x).\, \exists y \in S.\, q(y) = \mathtt{tt}$$

$$\Longleftrightarrow$$

“x can force q via f”

Examples

first move by...	second move by...
nondet. Player	nondet. Opponent
nondet. Opponent	nondet. Player
nondet. Opponent	probabilistic Player

Example:
Nondet. O → Probab. P

$$\begin{tikzcd}
 & & \tau_{\text{total}} \arrow[dr, phantom, "\in" sloped] & & \rho_{\text{inf}} \arrow[dr, phantom, "\in" sloped] & \\
\mathbf{Sets} \arrow[loop above, "\mathcal{D}"] \arrow[rr, bend left, "F^{\mathcal{D}}"] \arrow[rr, phantom, "\perp"] & & \mathcal{EM}(\mathcal{D}) \arrow[ll, bend left, "U^{\mathcal{D}}"] \arrow[r, "\cong"] & \mathbf{CCone} \arrow[loop above, "\mathsf{Cv}"] \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] & & \mathcal{EM}(\mathsf{Cv}) \arrow[ll, bend left]
\end{tikzcd}$$

- modalities:

$$\begin{tikzcd}
\mathcal{D}(\mathcal{D}1) \arrow[d, "\tau_{\text{total}}"] & {[s_i \mapsto r_i]_i} \arrow[d, mapsto] \\
\mathcal{D}1 & \sum_i s_i r_i
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Cv}(\tau_{\text{total}}) \arrow[dd, "\rho_{\text{inf}}"] & I \in \mathsf{Cv}[0,1] \arrow[d, mapsto] \\
 & \inf I \\
\tau_{\text{total}} &
\end{tikzcd}$$

Example:
Nondet. O → Probab. P

- From this we obtain:

$$\begin{tikzcd}
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\end{tikzcd}$$

$$\frac{X \xrightarrow{f} \mathsf{Cv}\mathcal{D}Y \qquad Y \xrightarrow{q} \mathcal{D}1 \cong [0,1]}{X \xrightarrow{\mathsf{wp}(f,q)} \mathcal{D}1}$$

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Nondet. O → Probab. P

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\end{tikzcd}$$

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s.t. $\mathbf{wp}(q, f)(x) = \inf_{d \in f(x)} \sum_{y} d(y) \cdot q(y)$

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\end{tikzcd}$$

- From this we obtain:

like $f(x) = \left\{ \begin{bmatrix} y_1 \mapsto \frac{1}{4} \\ y_1 \mapsto \frac{3}{4} \end{bmatrix}, \begin{bmatrix} y_1 \mapsto \frac{3}{4} \\ y_1 \mapsto \frac{1}{4} \end{bmatrix}, \text{(hence)} \begin{bmatrix} y_1 \mapsto \frac{1}{2} \\ y_1 \mapsto \frac{1}{2} \end{bmatrix}, \dots \right\}$

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"prob. pred. transformer" in [Morgan, McIver, Seidel, TOPLAS'96]

2-Player Setting: Summary

- 2-player branching
  - Not $$\begin{tikzcd} \mathbb{C} \arrow[loop above, "T"] \arrow[loop below, "R"] \end{tikzcd}$$ but

$$\begin{tikzcd}
 & \tau \in & & \rho \in \\
\mathbb{C} \arrow[loop above, "T"] \arrow[rr, bend left, "F^T"] \arrow[rr, phantom, "\perp"] & & \mathcal{EM}(T) \arrow[ll, bend left, "U^T"] \arrow[loop above, "R"] \arrow[rr, bend left, "F^R"] \arrow[rr, phantom, "\perp"] & & \mathcal{EM}(R) \arrow[ll, bend left, "U^R"]
\end{tikzcd}$$

- From which we obtain

$$\begin{tikzcd}
\mathbf{Posets} & & \mathcal{EM}(R)^{\mathrm{op}} \arrow[ll, bend right, "\mathbb{P}^{\mathcal{EM}}"'] \\
 & \mathcal{K}\ell(U^T R F^T)^{\mathrm{op}} \arrow[ul, "\mathbb{P}^{\mathcal{K}\ell}"] \arrow[ur, "K^{\mathrm{op}}"'] &
\end{tikzcd}$$

- Examples: game logic ("what is forced"), probab. pred. transf. [Morgan et al.]

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\end{tikzcd}$$

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- Examples: game logic ("what is forced"), probab. pred. transf. [Morgan et al.]

Conclusions

- **Monadic foundation** of weakest precondition semantics
  - "Branching" as an order-enriched monad
  - $T(1)$ = {truth values}
  - "Modality" as an EM-alg.
  - Extends to a **2-player setting**

Epilogue:
Alternating Branching

- (Automata-theoretic) **model checking**

  See e.g. [LNCS 2500]

$$\mathcal{S} \models \psi$$

$$\iff \mathcal{S} \text{ is accepted by } \mathcal{A}_\psi$$

$$\iff \text{Player has a winning str. in } \mathcal{G}(\mathcal{S}, \mathcal{A}_\psi)$$

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Kripke frame

modal μ-fml.

$$\mathcal{S} \models \psi$$

alternating parity tree automaton

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parity game for acceptance

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Algorithms like [Jurdzinski '98]

parity game for acceptance

Prologue:
Alternating Branching

$\mathcal{S} \models \psi$

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Alternating Branching

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$$\psi :\equiv$$

$$\nu X.\, \mu Y.$$

$$\bigl(\, (p \wedge \Diamond X)$$

$$\vee\ (q \wedge \Diamond Y)\, \bigr)$$

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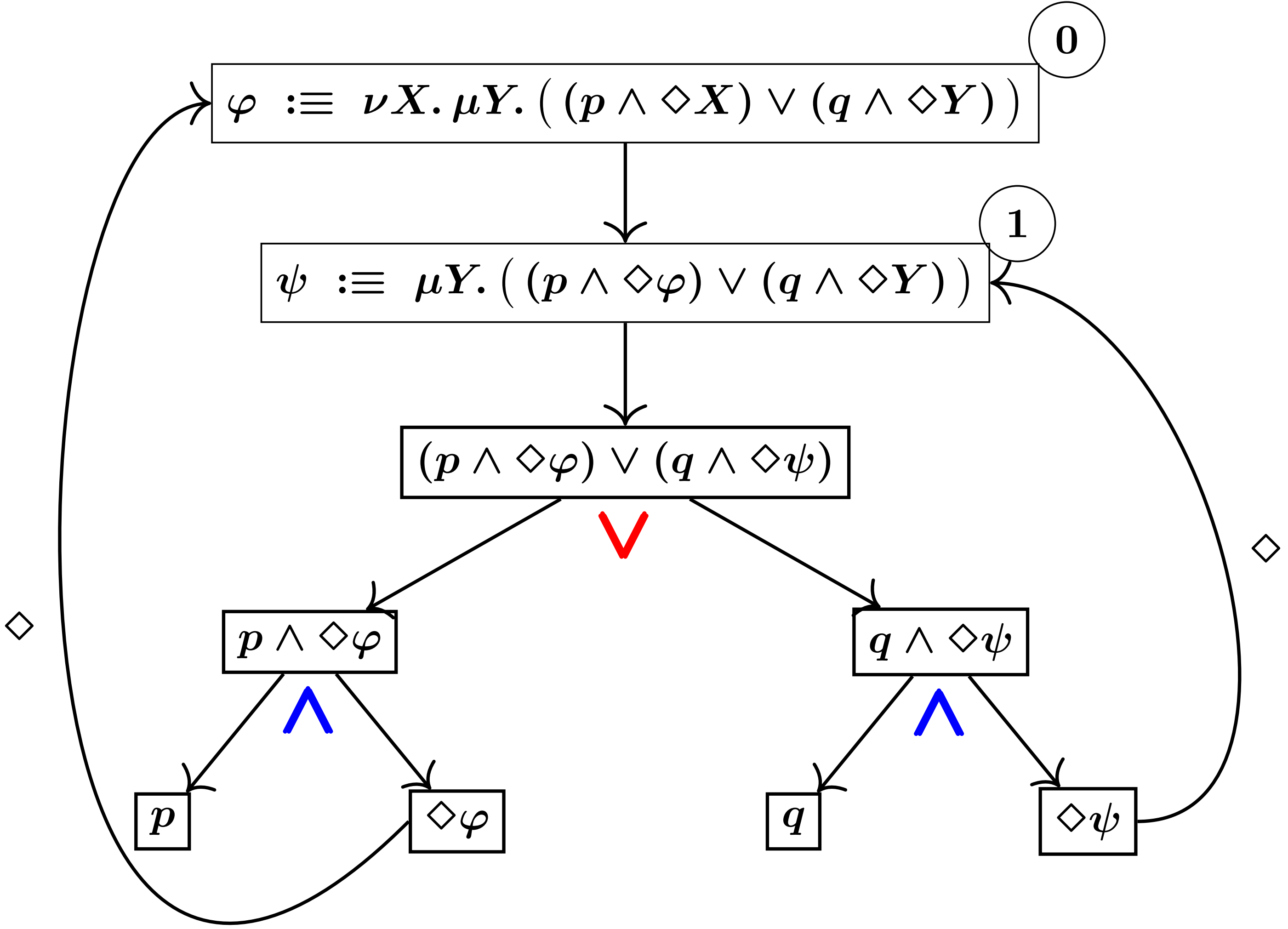
alternating parity tree automaton



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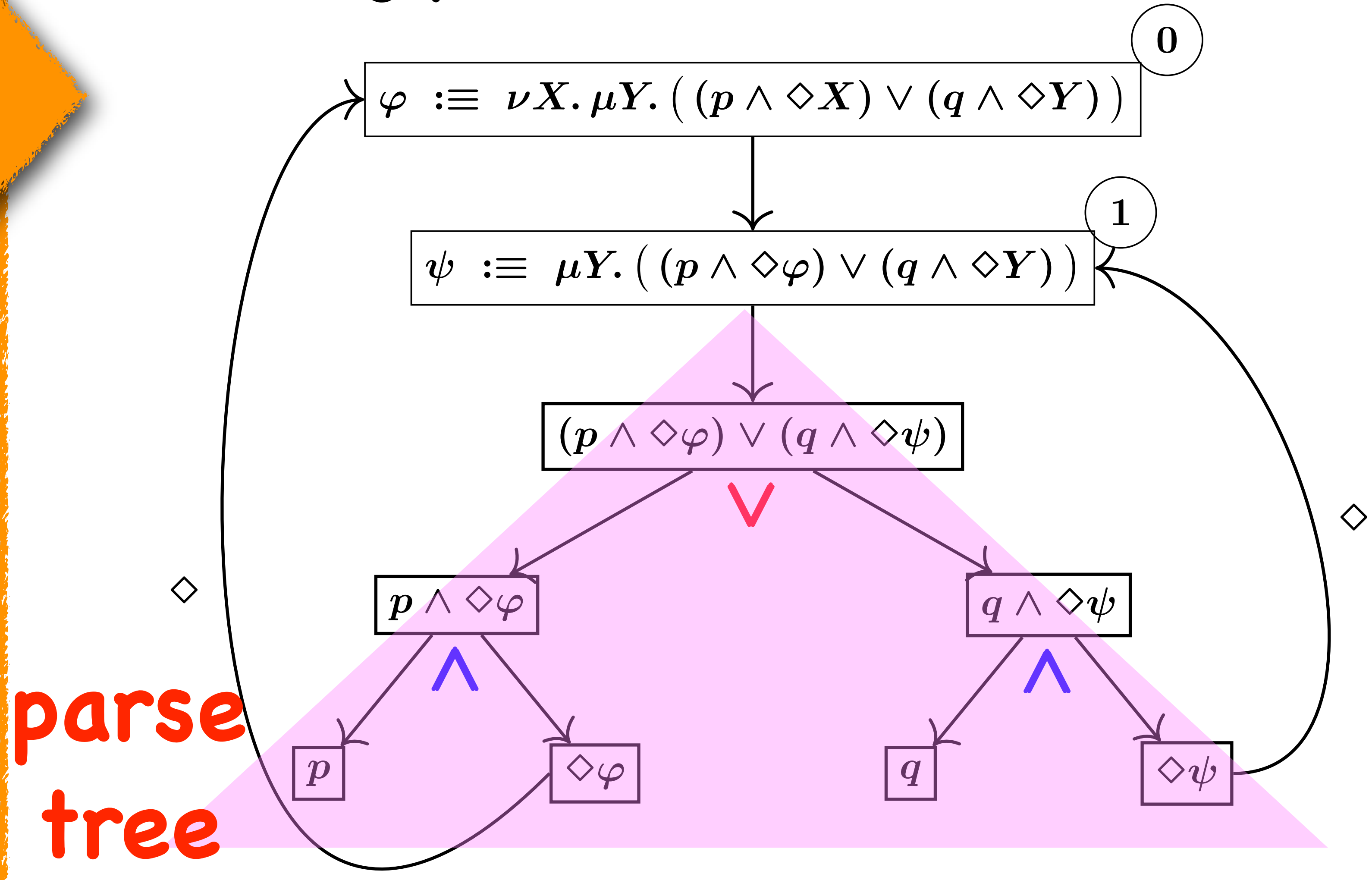

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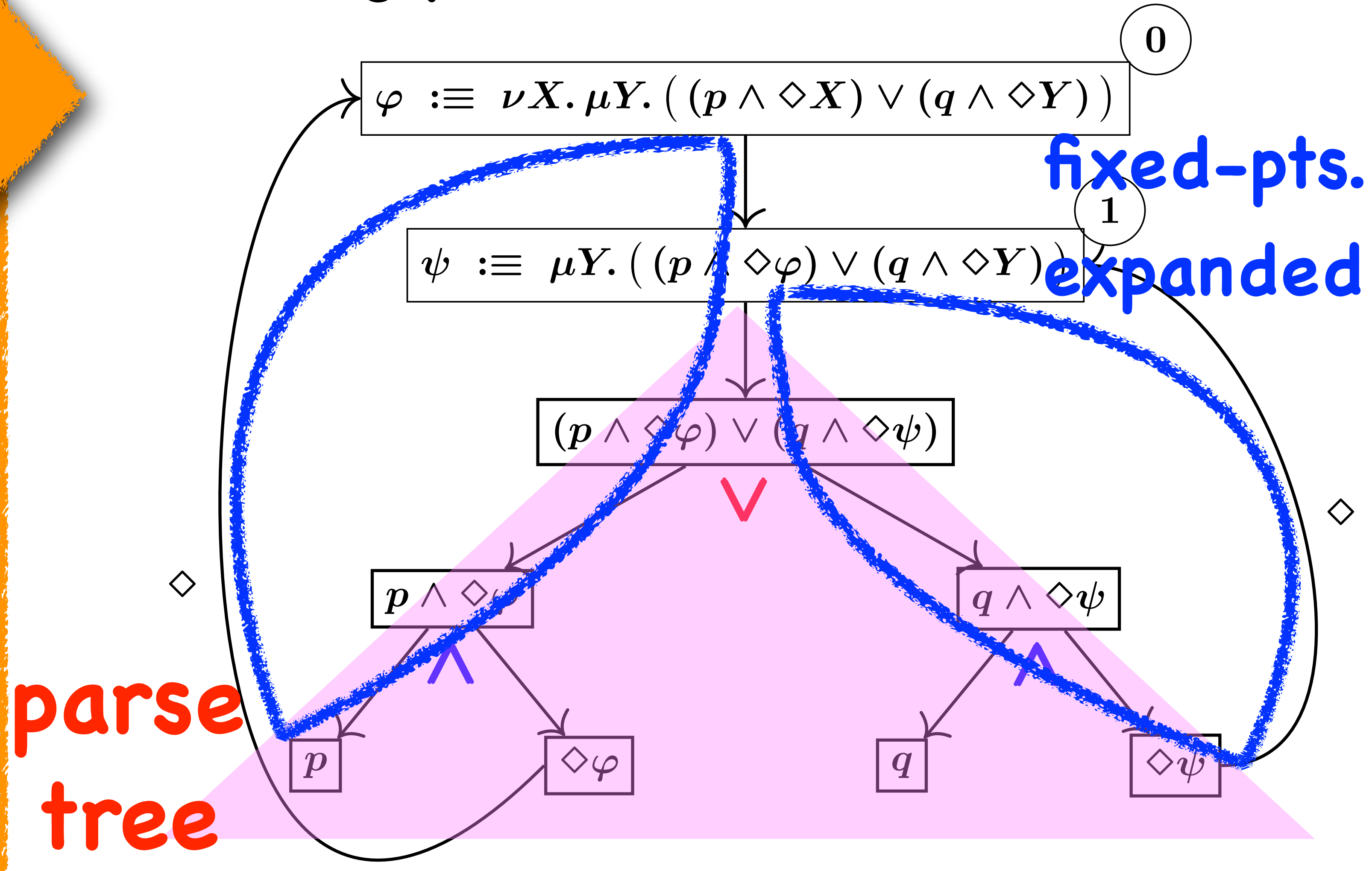

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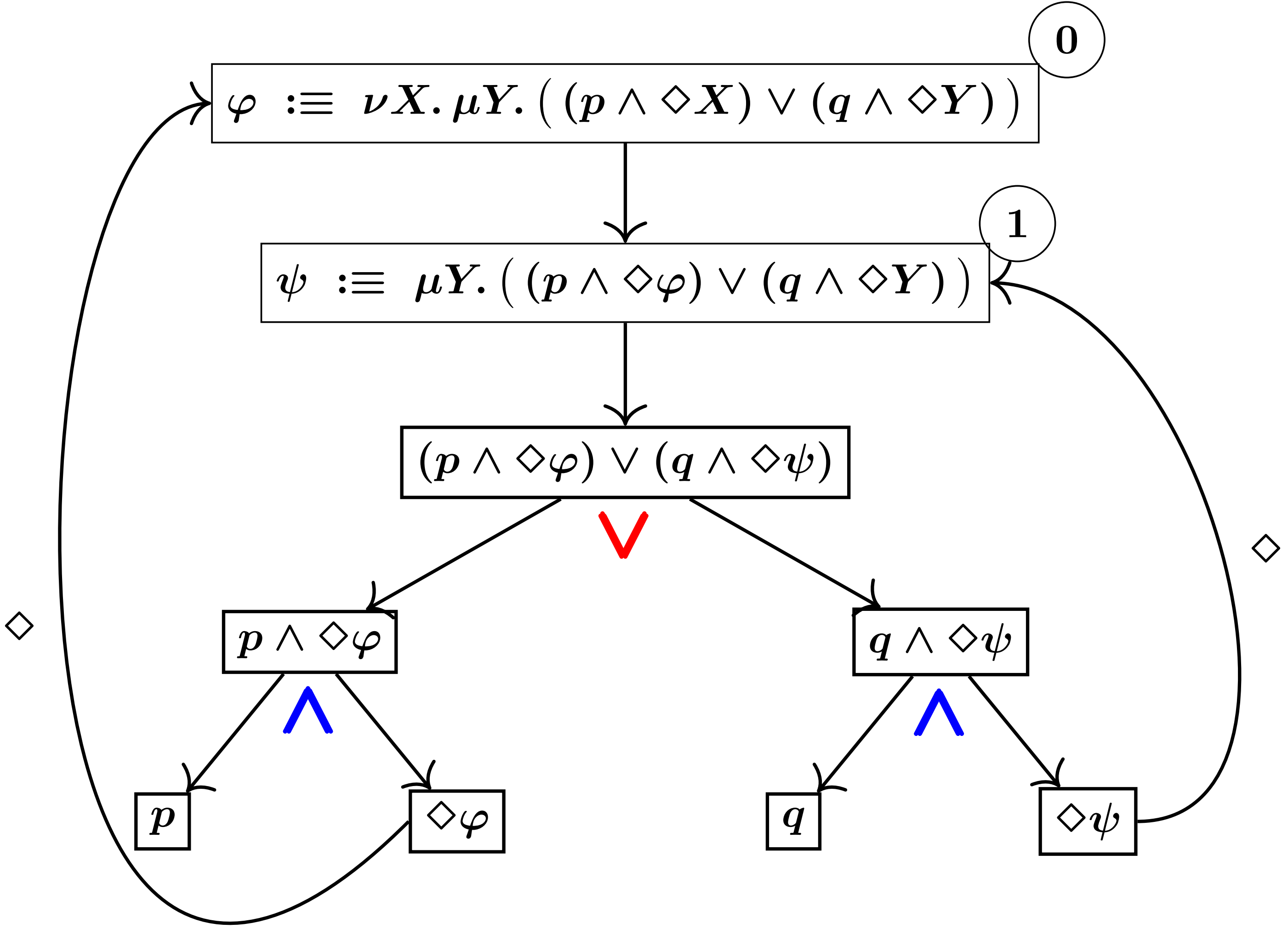

Epilogue:
Alternating Branching

- Monadic modeling?

$$\begin{tikzcd}
X \arrow[r] & \mathcal{P}\mathcal{P}X \\
\{x\} \arrow[r, mapsto] & \{\, \{x_1, x_2\}, \{x_3\} \,\}
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intention: positive DNF

$$(x_1 \wedge x_2) \vee x_3$$

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Q1. How?

Epilogue: Alternating Branching

- Monadic modeling?

$$\begin{tikzcd}
X \arrow[r] & \overset{U^{\mathcal{P}}\mathrm{Dw}F^{\mathcal{P}}}{\cancel{\mathcal{P}\mathcal{P}}}X \\
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\end{tikzcd}$$

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Example:

Nondet. P → Nondet. O

- From this we obtain...

$$\begin{tikzcd}
 & & \tau_{\Box} \arrow[d, phantom, sloped, "\in"] & \rho_{\mathsf{P}} \arrow[d, phantom, sloped, "\in"] \\
\mathbf{Sets} \arrow[loop above, "\mathcal{P}"] \arrow[r, bend left, "F^{\mathcal{P}}"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathcal{P}) \arrow[l, bend left, "U^{\mathcal{P}}"] \arrow[r, "\cong"] & \mathbf{CL}_{\wedge} \arrow[loop above, "\mathsf{Dw}"] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathsf{Dw}) \arrow[l, bend left]
\end{tikzcd}$$

$$\mathcal{K}\ell(U^{\mathcal{P}}\mathsf{Dw}F^{\mathcal{P}})^{\mathrm{op}} \cong \mathcal{K}\ell(\mathcal{UP})^{\mathrm{op}} \longrightarrow \mathbf{Posets}$$

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- i.e.

$$\frac{X \xrightarrow{f} \mathcal{UP}Y \qquad Y \xrightarrow{q} \mathcal{P}1}{X \xrightarrow{\mathsf{wp}(f,q)} \mathcal{P}1}$$

s.t.

$$\begin{aligned}\mathsf{wp}(f,q)(x) = \mathrm{tt} &\iff \exists S \in f(x).\, \forall y \in S.\, q(y) = \mathrm{tt} \\ &\iff \text{“}x \text{ can force } q \text{ via } f\text{”}\end{aligned}$$

Epilogue: Alternating Branching

- Monadic modeling?

$$X \longrightarrow \overset{U^{\mathcal{P}}\mathbf{Dw}F^{\mathcal{P}}}{\cancel{\mathcal{P}\mathcal{P}}}X$$

$$\{x\} \longmapsto \{\,\{x_1, x_2\}, \{x_3\}\,\}$$

intention: positive DNF

$$(x_1 \wedge x_2) \vee x_3$$

Q1. How?

Epilogue: Alternating Branching

- Monad multiplication

$$\begin{tikzcd}
 & & \mathcal{P}\mathcal{P}(\mathcal{P}\mathcal{P}X) \arrow[d, "\mu^{\mathcal{P}\mathcal{P}}"] \\
 & \mathcal{P}\mathcal{P}X \arrow[ur] & \mathcal{P}\mathcal{P}X \\
X \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \begin{array}{l} \left(\begin{array}{l} \bigl((y_1 \wedge y_2) \vee y_3 \bigr) \\ \wedge (z_1 \wedge z_2) \end{array} \right) \\ \vee \bigl(u_1 \wedge (u_2 \wedge u_3)\bigr) \end{array} \arrow[d, mapsto] \\
 & (y \wedge z) \vee u \arrow[ur, mapsto] & \left[\begin{array}{c} \text{some positive DNF} \\ \text{that is logically equiv.} \end{array} \right] \\
x \arrow[ur, mapsto] & &
\end{tikzcd}$$

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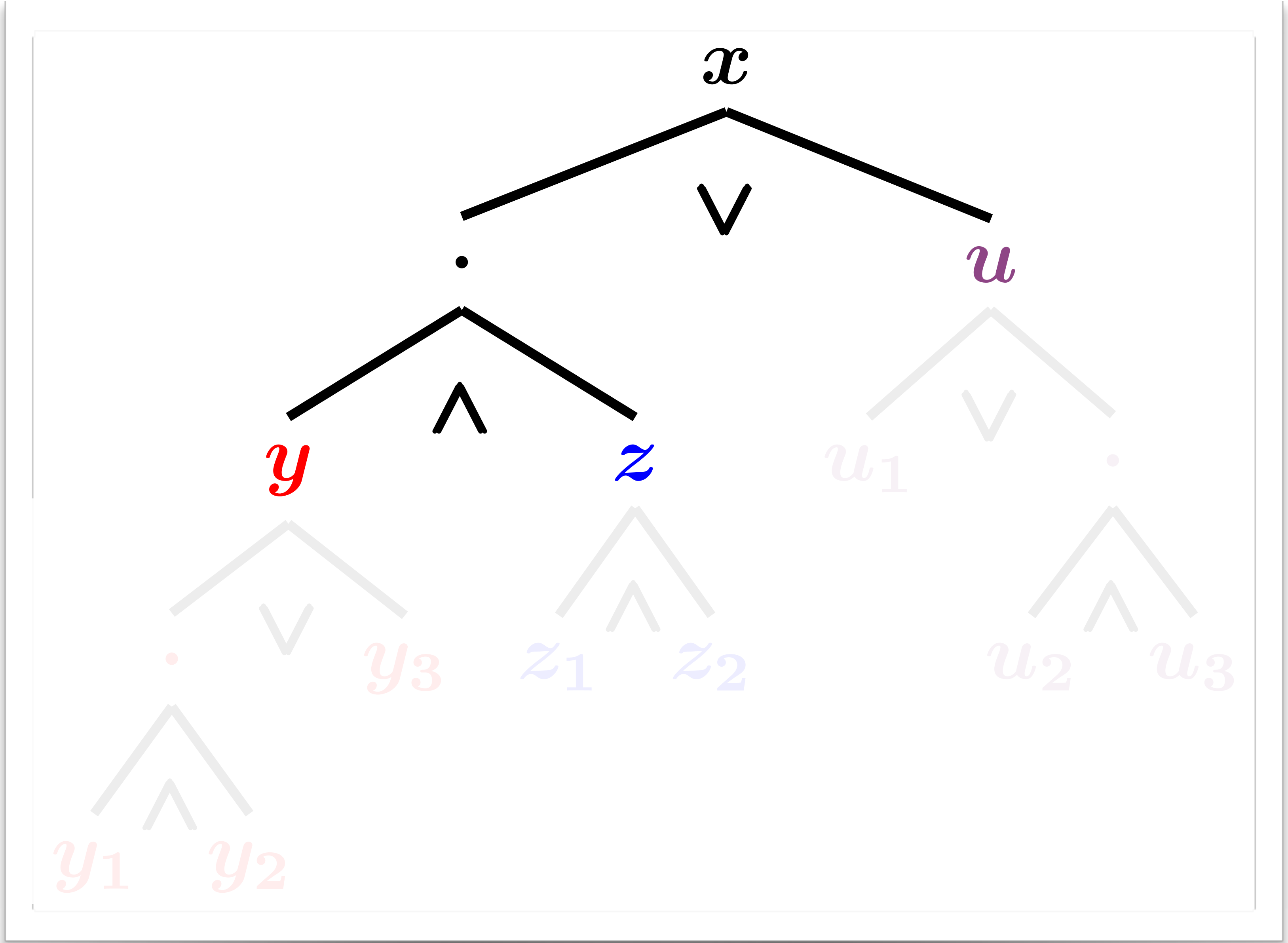

$$\begin{tikzcd}
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\end{tikzcd}$$

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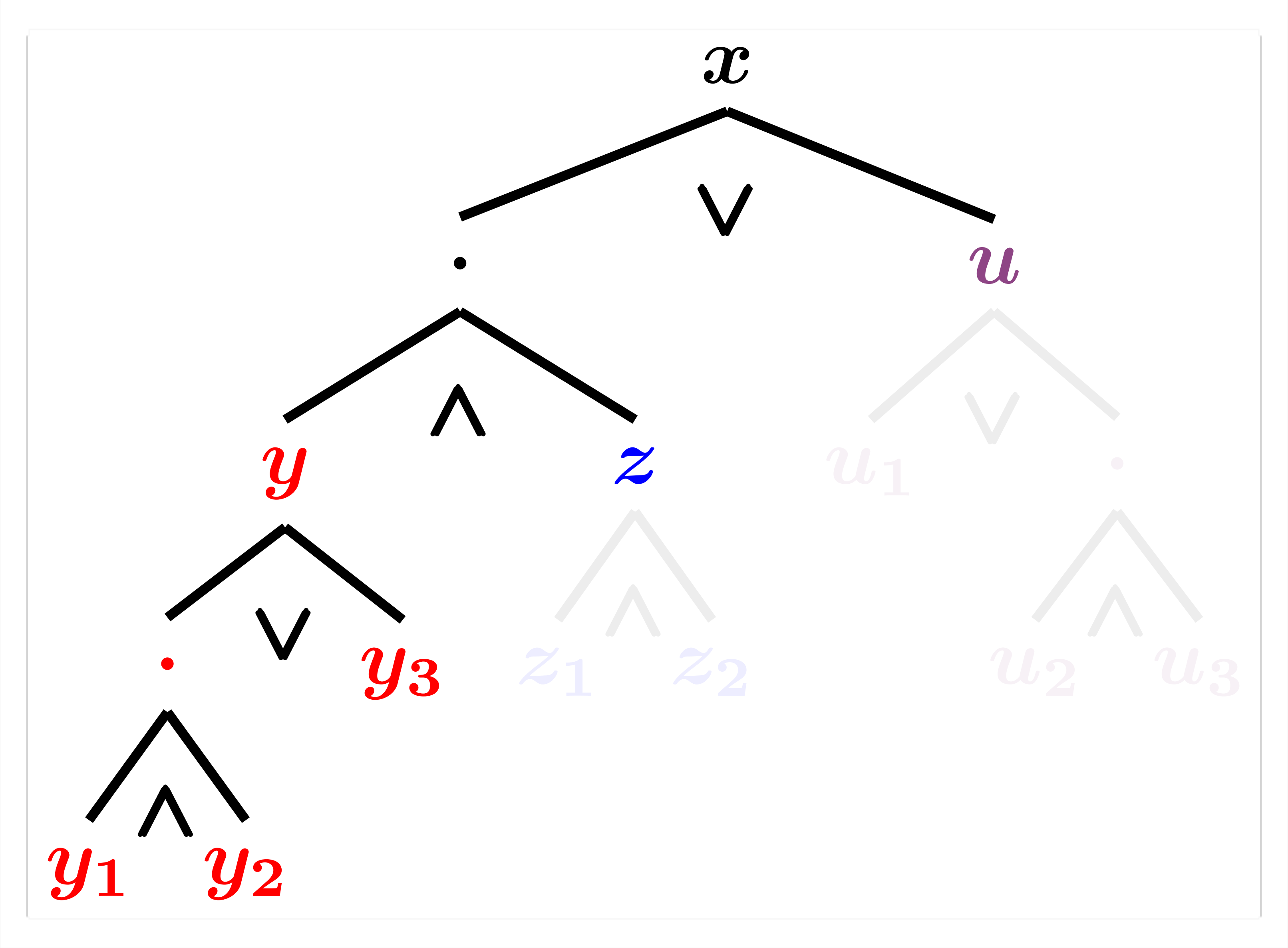

$$\begin{tikzcd}
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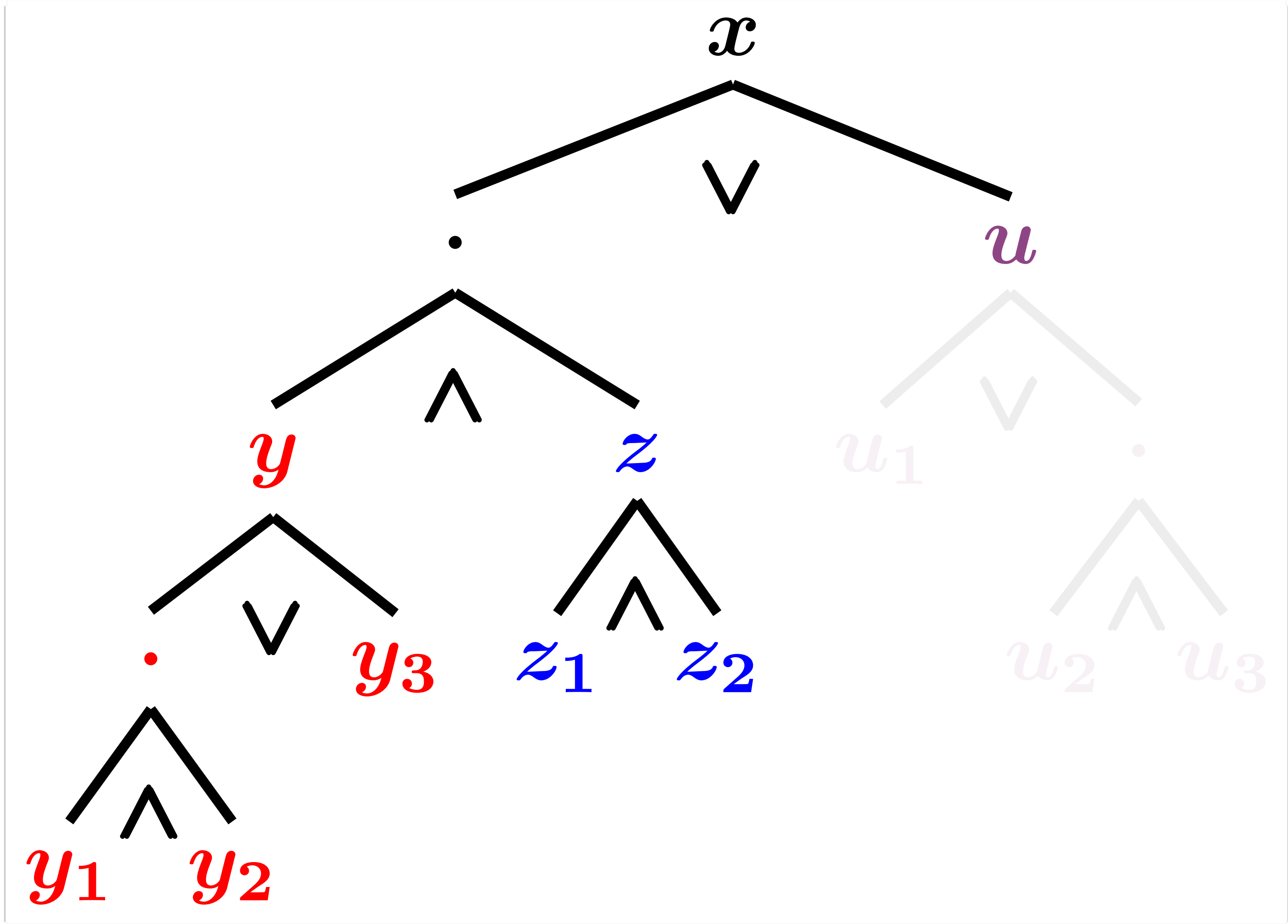

$$\begin{tikzcd}
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X \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \left(\begin{array}{l}\big(\,(y_1 \wedge y_2) \vee y_3\,\big) \\ \wedge(z_1 \wedge z_2)\end{array}\right) \vee\big(u_1 \wedge (u_2 \wedge u_3)\big) \arrow[d, mapsto] \\
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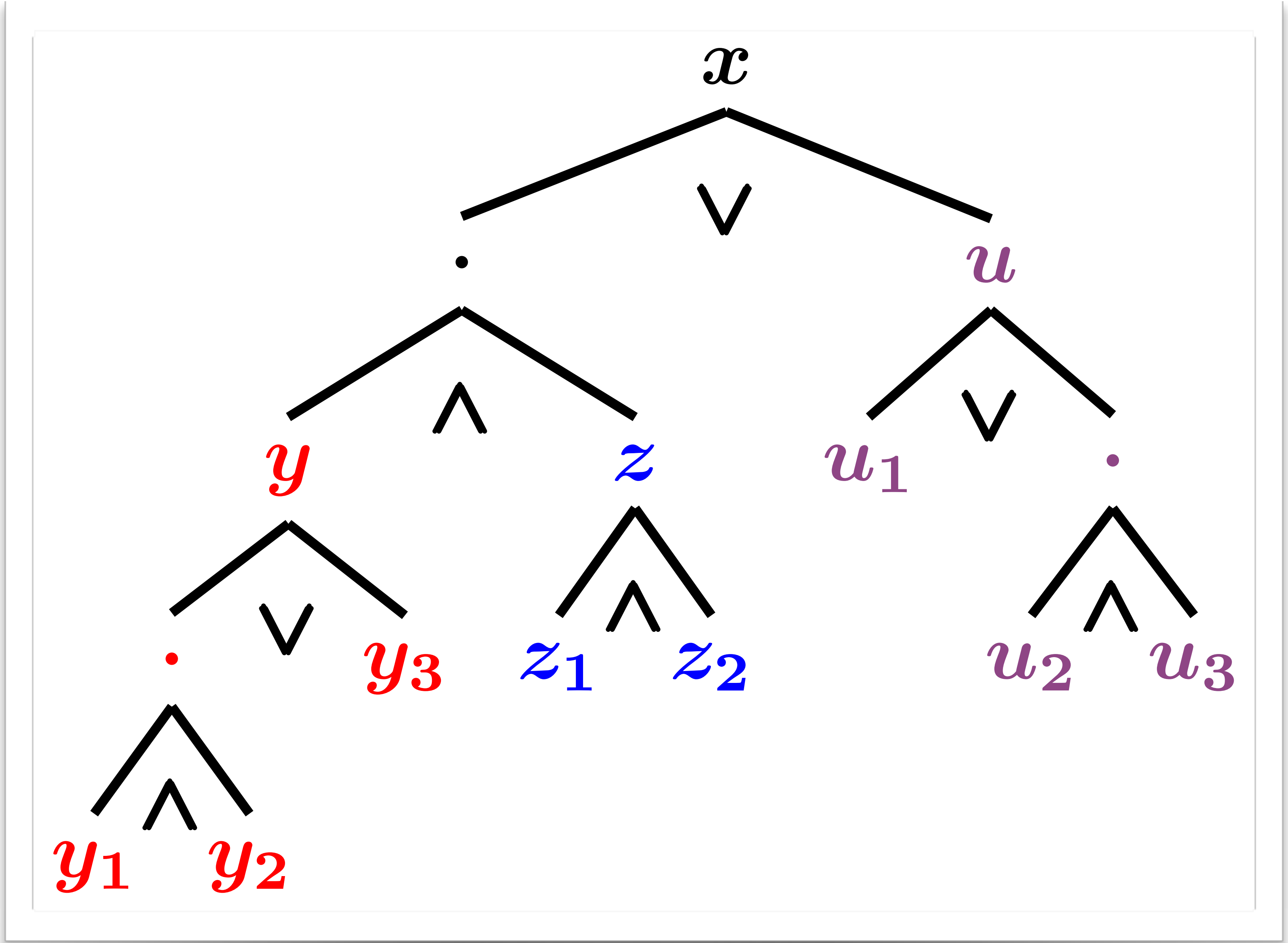
- Monad multiplica



Epilogue:
Alternating Branching

- Monad multiplication

$$\begin{tikzcd}
 & & \mathcal{P}\mathcal{P}(\mathcal{P}\mathcal{P}X) \arrow[d, "\mu^{\mathcal{P}\mathcal{P}}"] \\
 & \mathcal{P}\mathcal{P}X \arrow[ur] & \mathcal{P}\mathcal{P}X \\
X \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \begin{array}{l} \left(\begin{array}{l} \big((y_1 \wedge y_2) \vee y_3 \big) \\ \wedge(z_1 \wedge z_2) \end{array} \right) \\ \vee\big(u_1 \wedge (u_2 \wedge u_3)\big) \end{array} \arrow[d, mapsto] \\
 & (y \wedge z) \vee u \arrow[ur, mapsto] & \left[\begin{array}{c} \text{some positive DNF} \\ \text{that is logically equiv.} \end{array} \right] \\
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\end{tikzcd}$$

Epilogue: Alternating Branching

- Monad multiplication

$$\begin{tikzcd}
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Q2. How?

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- Monad multiplication

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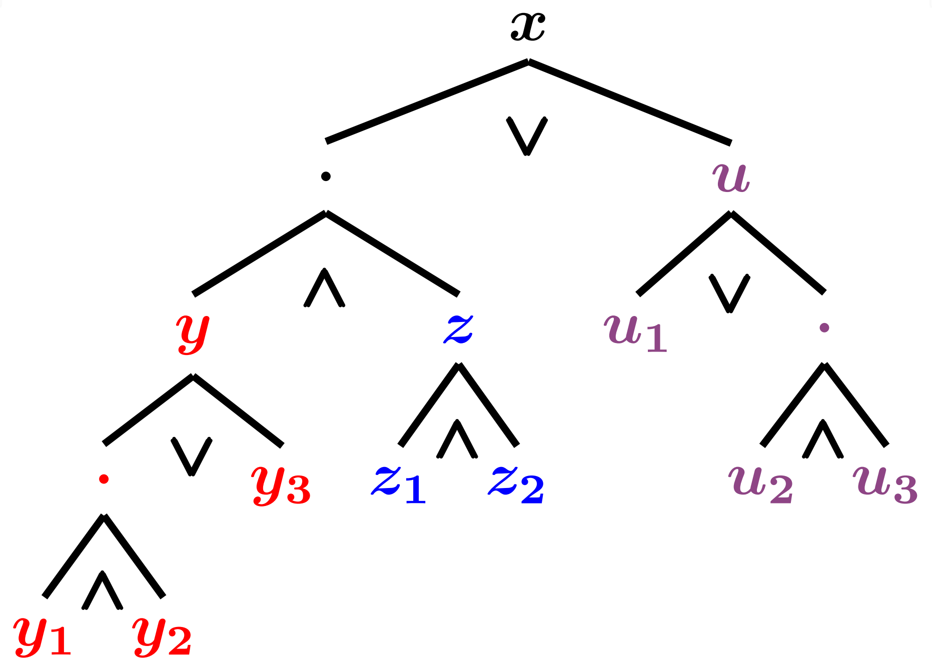

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Q2. How?

Epilogue:
Alternating Branching

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X \arrow[ur] & &
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- arises from

- Logical equivalence: from **functoriality/compositionality**

Epilogue: Alternating Branching

- Monad multiplication

- arises from

$U^{\mathcal{P}}\mathbf{Dw}F^{\mathcal{P}}$

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Epilogue: Alternating Branching

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X \arrow[ur] & &
\end{tikzcd}$$

- arises from

$$\begin{tikzcd}
 & & \tau_{\Box} \arrow[d, phantom, "\in" sloped] & \rho_{\mathsf{P}} \arrow[d, phantom, "\in" sloped] \\
\mathrm{Sets} \arrow[loop above, "\mathcal{P}"] \arrow[r, bend left, "F^{\mathcal{P}}"] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathcal{P}) \arrow[l, bend left, "U^{\mathcal{P}}"] \arrow[r, "\cong"] & \mathbf{CL}_{\wedge} \arrow[loop above, "\mathsf{Dw}"] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{EM}(\mathsf{Dw}) \arrow[l, bend left]
\end{tikzcd}$$

Epilogue: Alternating Branching

- Monad multiplication

$U^{\mathcal{P}}\mathsf{Dw}F^{\mathcal{P}}$

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$\tau_{\Box} \in$ $\rho_{\mathsf{P}} \in$

- Logical equivalence: from **functoriality/compositionality**

$$\mathbf{wp}(f, \mathbf{wp}(g, Q)) = \mathbf{wp}(g \circ f, Q)$$

Special Thanks

Bart Jacobs

Kenta Cho

Kazuyuki Asada

Corina Cirstea

Tetsuri Moriya

Future Work

- Complete
  Bart's picture:

$$\begin{tikzcd}
\left(\begin{array}{c}\text{bwd. pred.}\\\text{transformers}\end{array}\right) \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] & & \left(\begin{array}{c}\text{fwd. pred.}\\\text{transformers}\end{array}\right) \arrow[ll, bend left, "\mathbb{P}^{\mathcal{EM}}"] \\
& \left(\begin{array}{c}\text{sets and branching}\\\text{computations}\end{array}\right) \arrow[ul, "\mathbb{P}^{\mathcal{K}\ell}"] \arrow[ur, "\boldsymbol{K}^{\mathrm{op}}"'] &
\end{tikzcd}$$

- Syntactic calculus
- "Healthiness" [Dijkstra]

Characterize those

$$\mathbb{C}(\boldsymbol{Y},\boldsymbol{T\Omega}) \rightarrow \mathbb{C}(\boldsymbol{X},\boldsymbol{T\Omega})$$

in the form of $\mathbf{wp}(\boldsymbol{f})$

- Game bisimulation [Kissig, Venema]
- Trace semantics & simulation
- LTL model checking [Cirstea, FoSSaCS'14]
- Higher-order extension
- Three (or more) players
  [Cirstea, CMCS'14]

Future Work

- Complete Bart's picture:

$$\begin{tikzcd}
\left(\begin{array}{c}\text{bwd. pred.}\\ \text{transformers}\end{array}\right) \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] & & \left(\begin{array}{c}\text{fwd. pred.}\\ \text{transformers}\end{array}\right) \arrow[ll, bend left, "\mathbb{P}^{\mathcal{EM}}"] \\
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Thank you for your attention!

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