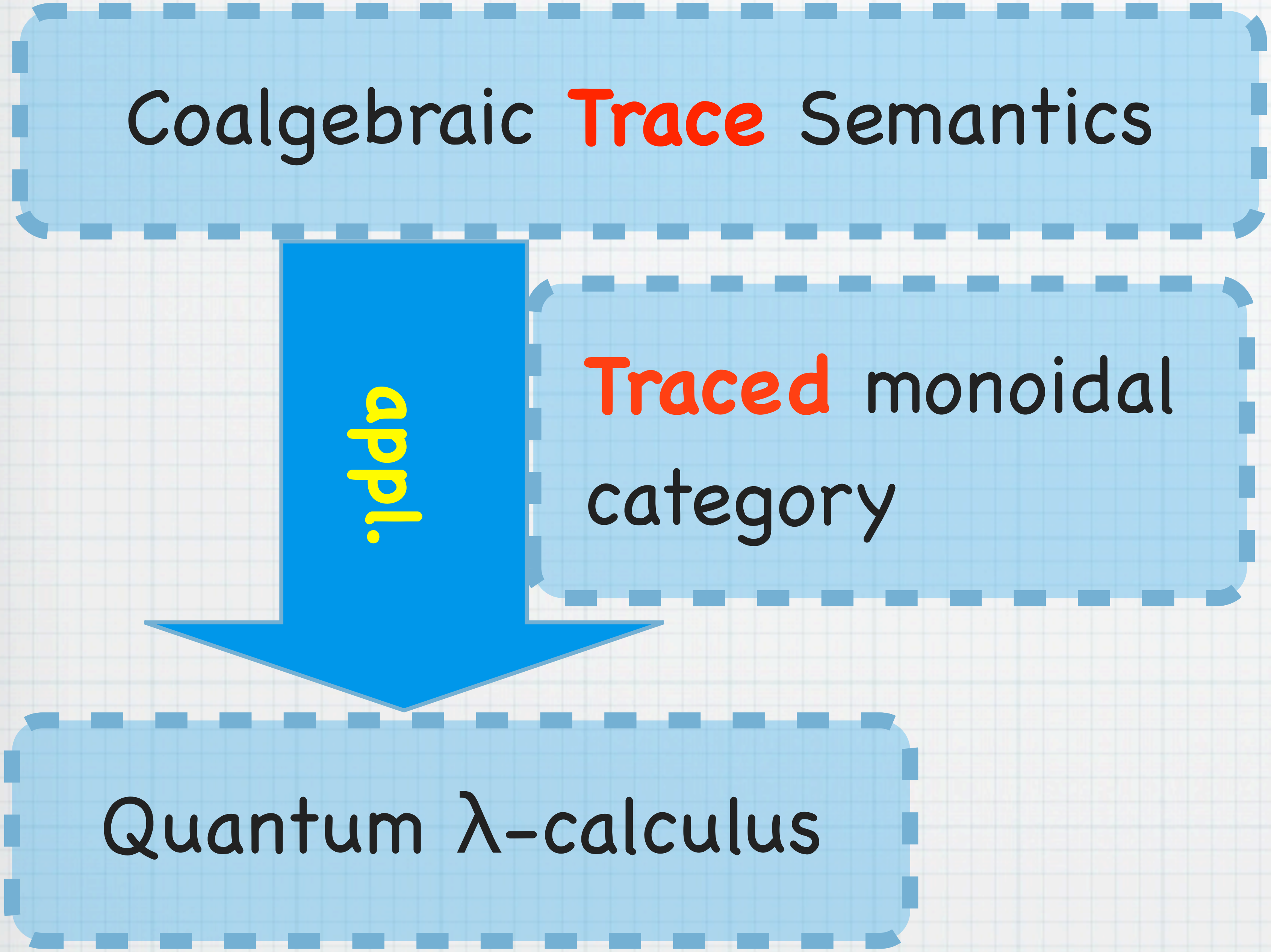
Trace Everywhere

Based on: IH & N. Hoshino, **Semantics of Higher-Order Quantum Computation via Geometry of Interaction,** Proc. LICS 2011

Ichiro Hasuo
University of Tokyo (JP)

東京大学
THE UNIVERSITY OF TOKYO

Three "Traces"



Coalgebraic **Trace** Semantics

appl.

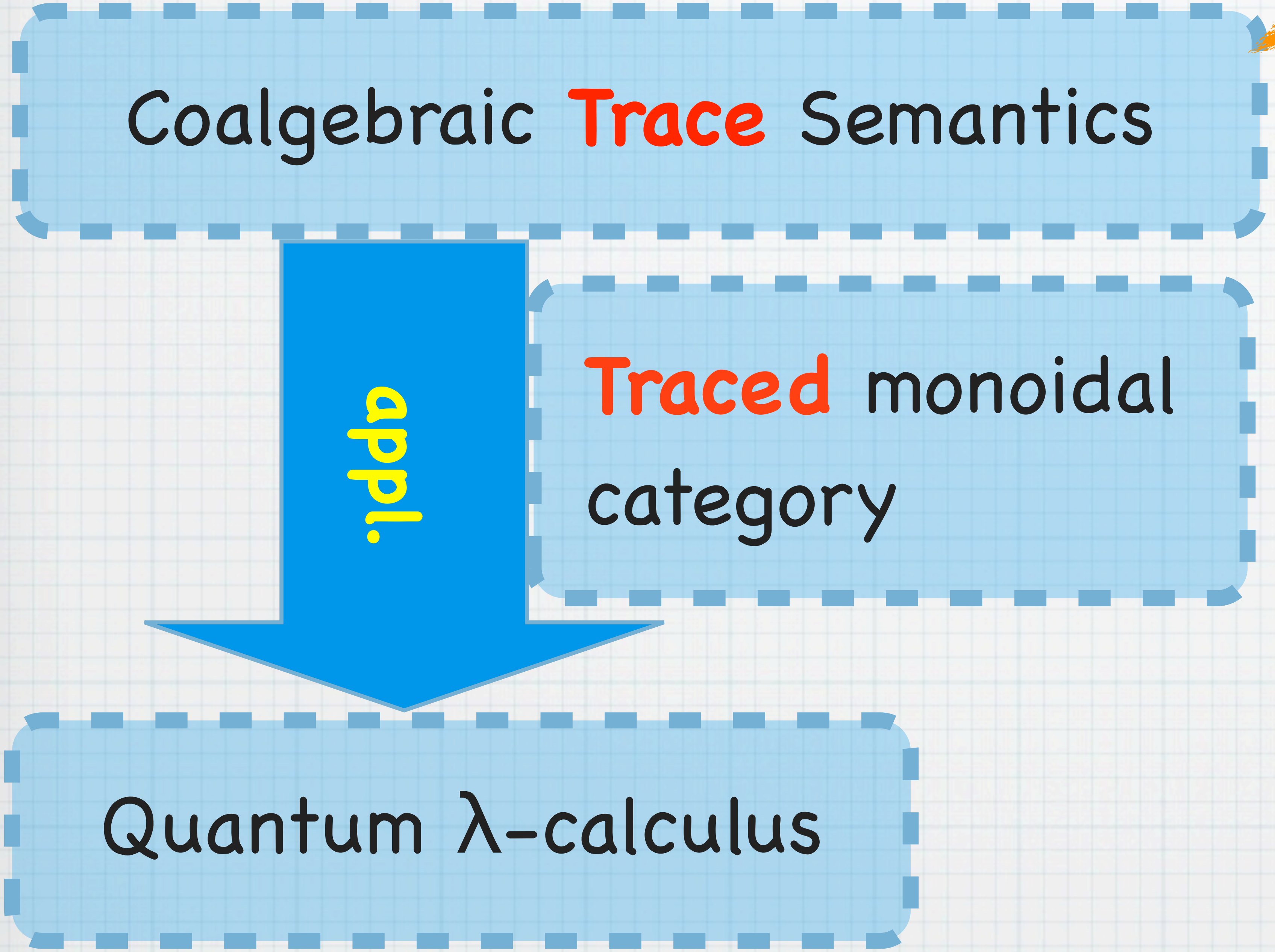
Traced monoidal category

Quantum λ-calculus

Three "Traces"

$$\begin{tikzcd}
\boldsymbol{FX} \arrow[r, dashed, "\boldsymbol{F}\mathbf{beh}(\boldsymbol{c})"] & \boldsymbol{FZ} \\
\boldsymbol{X} \arrow[u, "\boldsymbol{c}"] \arrow[r, dashed, "\mathbf{beh}(\boldsymbol{c})"'] & \boldsymbol{Y} \arrow[u, "\mathrm{final}"']
\end{tikzcd}$$

Coinduction in $\boldsymbol{Kl}(\mathbf{B})$



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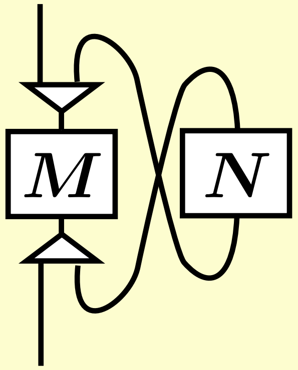
[Abramsky, Haghverdi, Scott]



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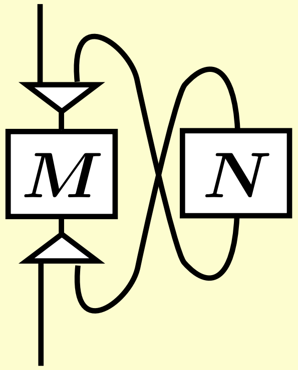

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Measurements by **tracing out** matrices

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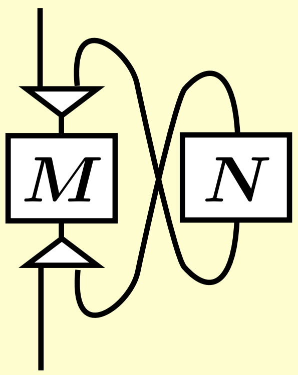

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- Goal: Denotational model of a quantum λ-calculus

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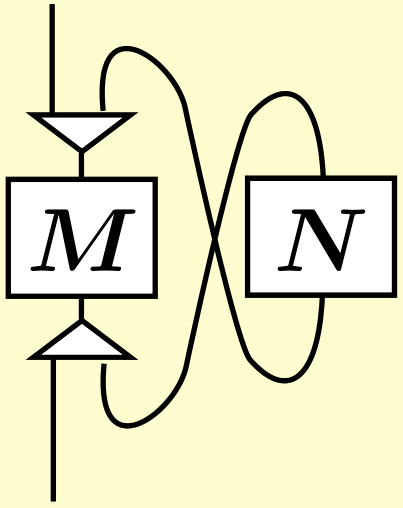

[Abramsky, Haghverdi, Scott]

Quantum λ-calculus

Mea

trac

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GoI:
Geometry of Interaction

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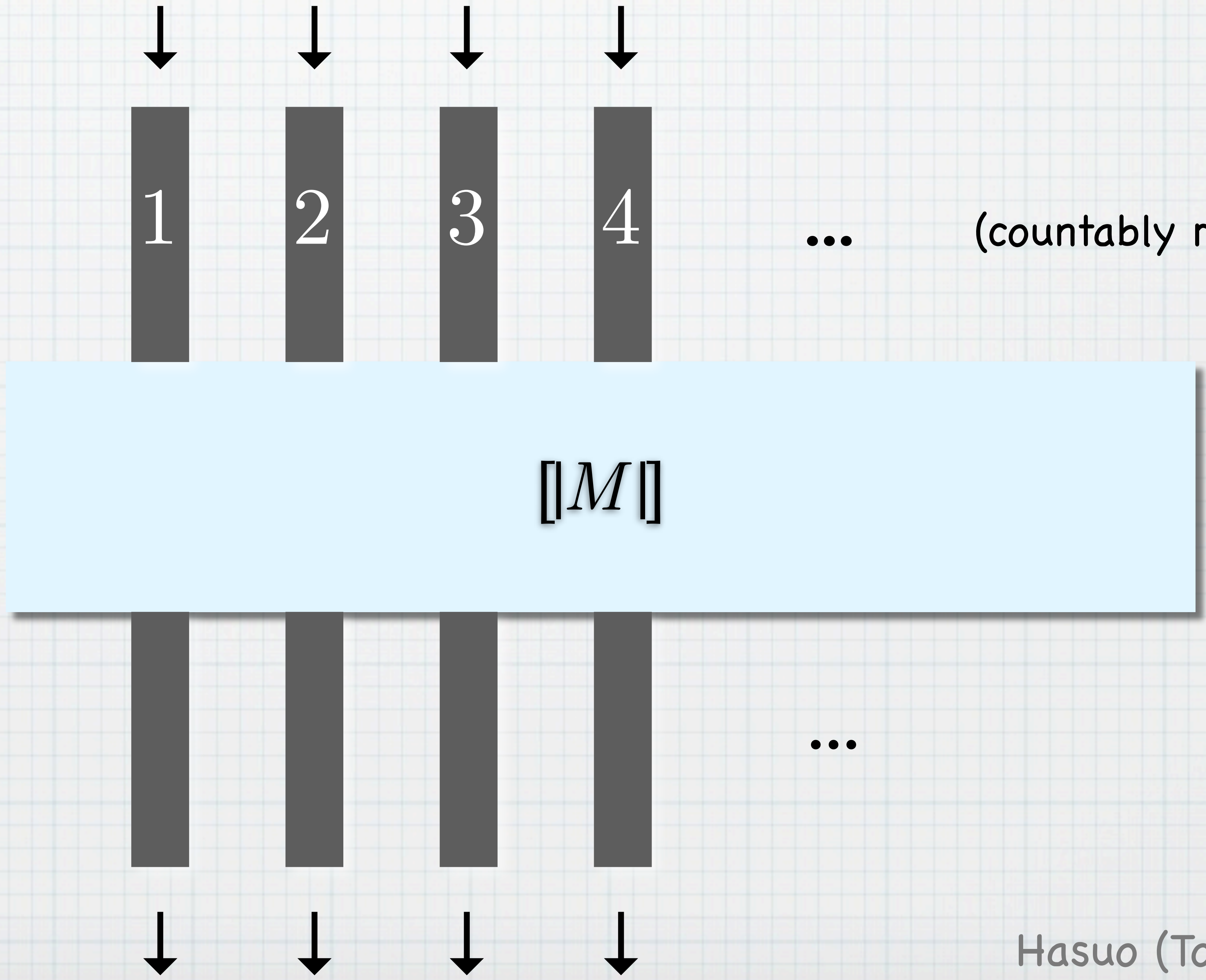
- In this talk:
  - Its categorical formulation [Abramsky, Haghverdi, Scott '02]
  - "The GoI Animation"

The GoI Animation

$$[\![M]\!] \;=\; (\,\mathbb{N} \rightharpoonup \mathbb{N},\;\text{a partial function}\,)$$

$$=\;\text{“piping”}$$



(countably many)

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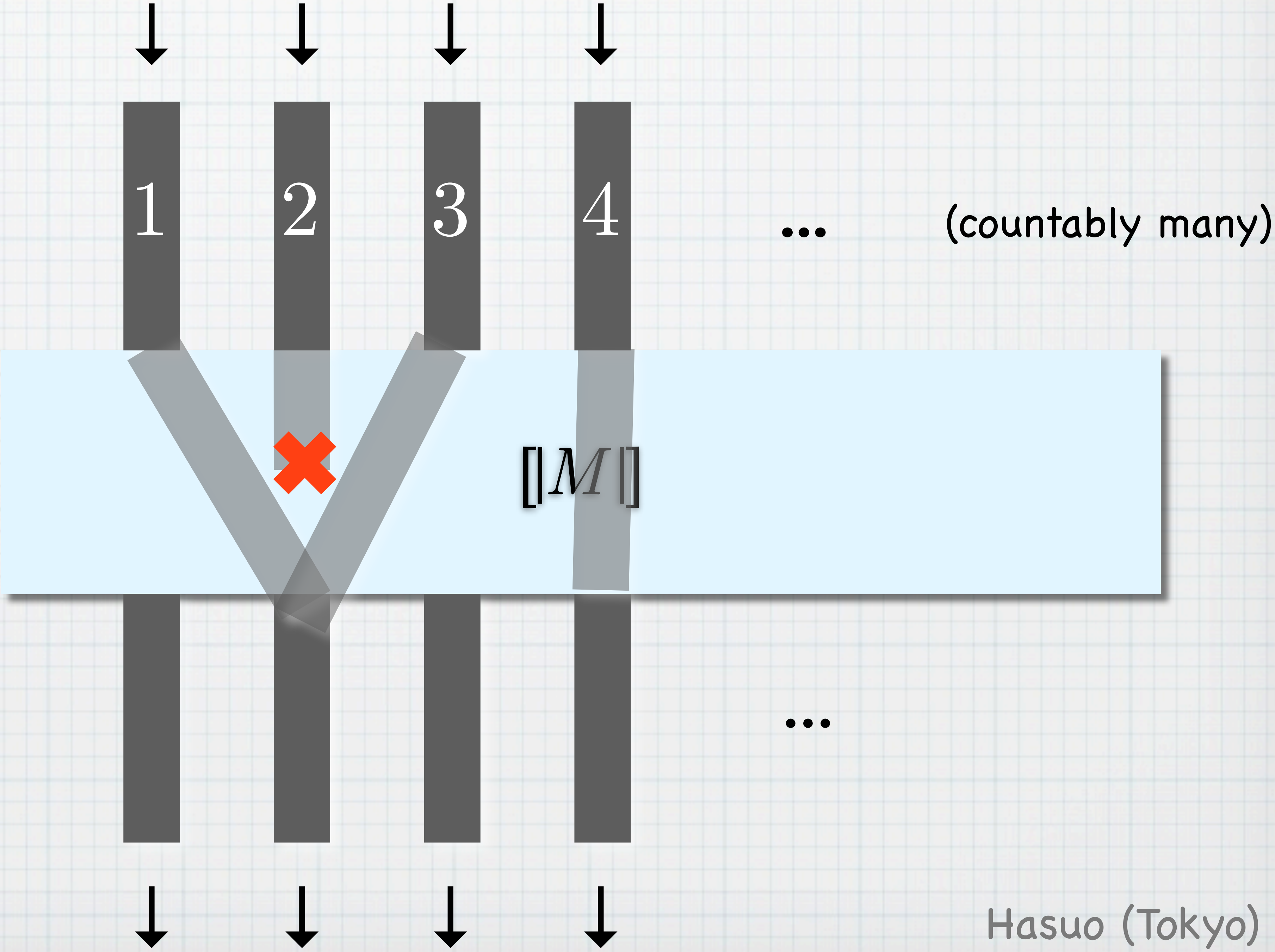

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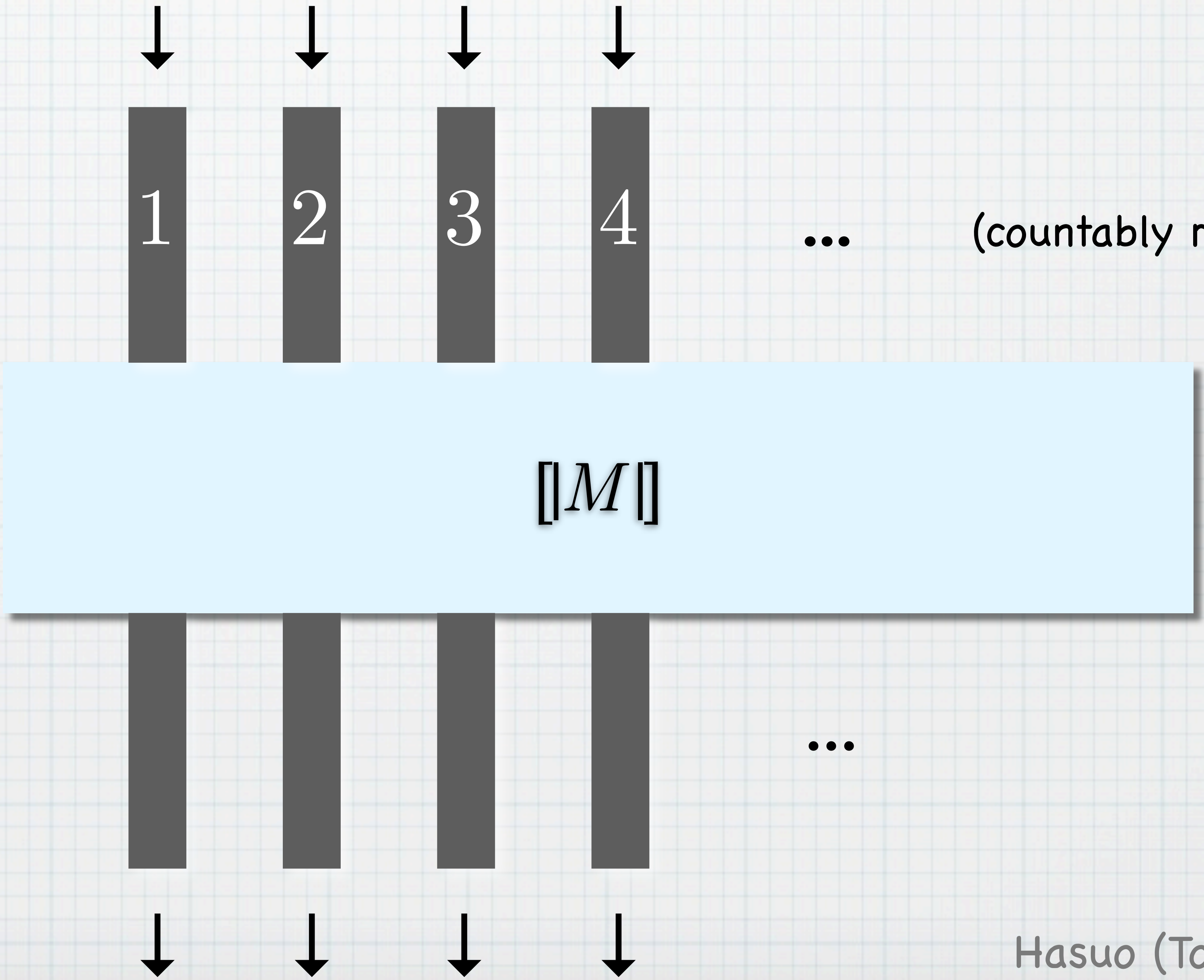

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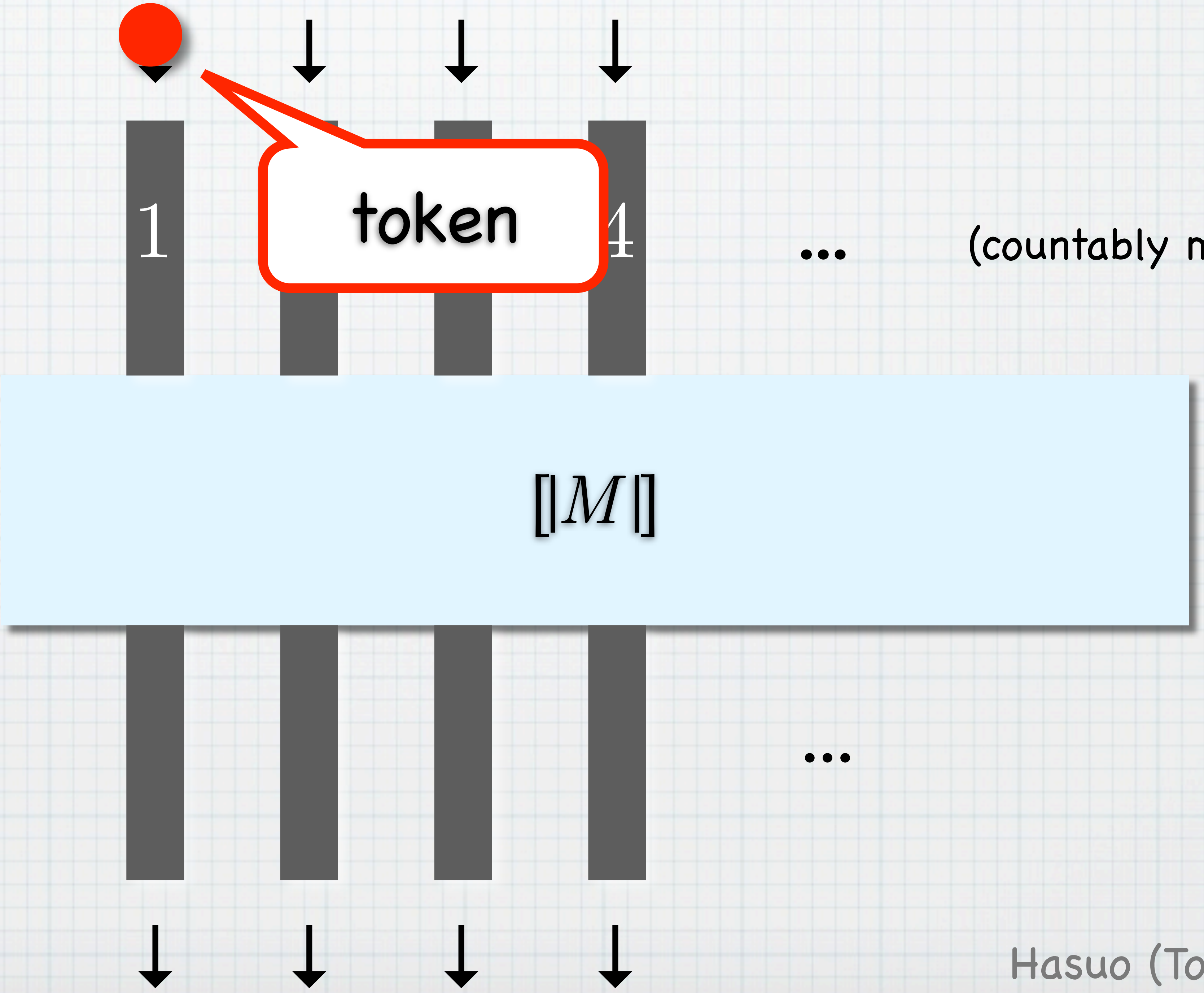

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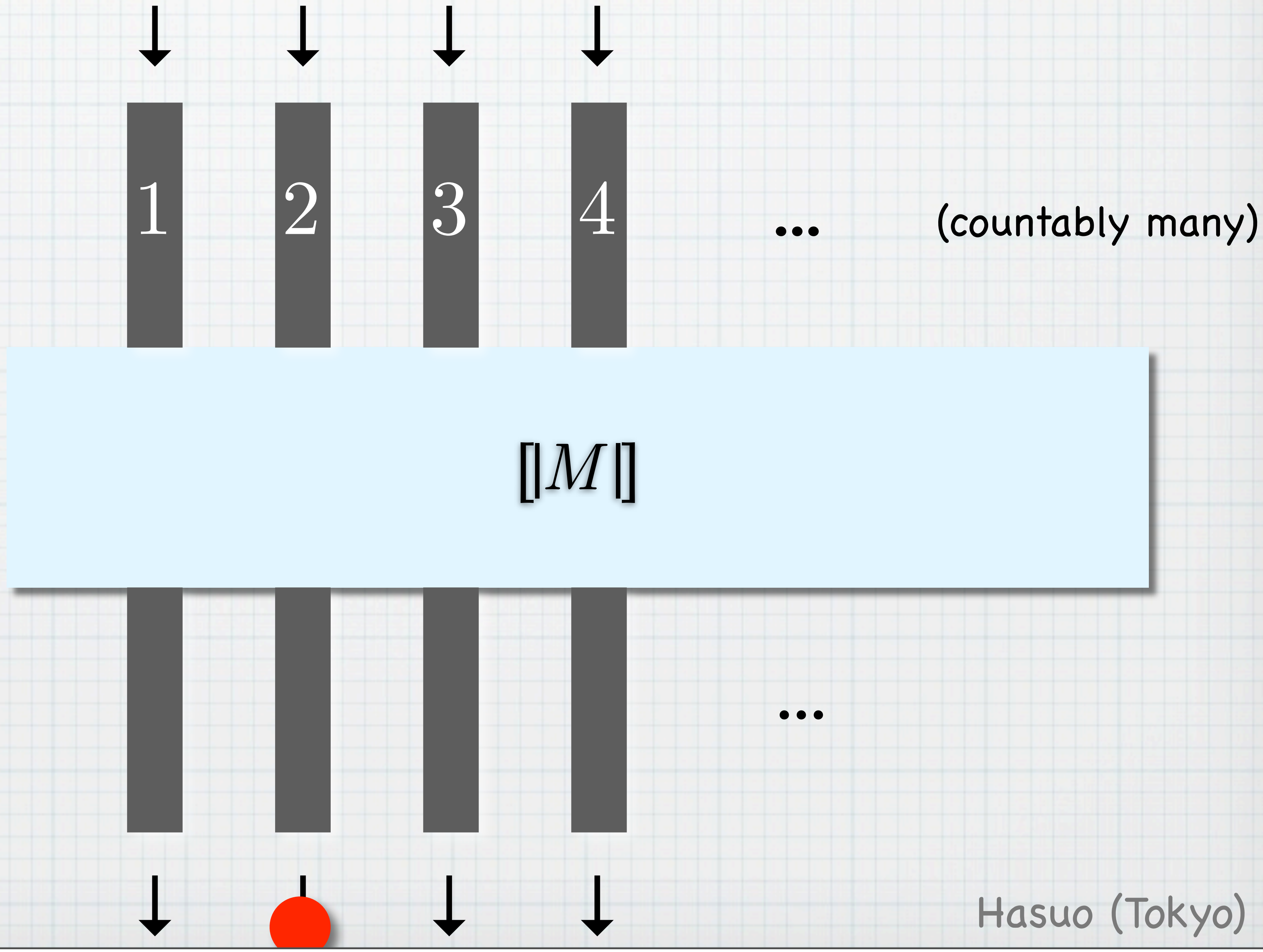

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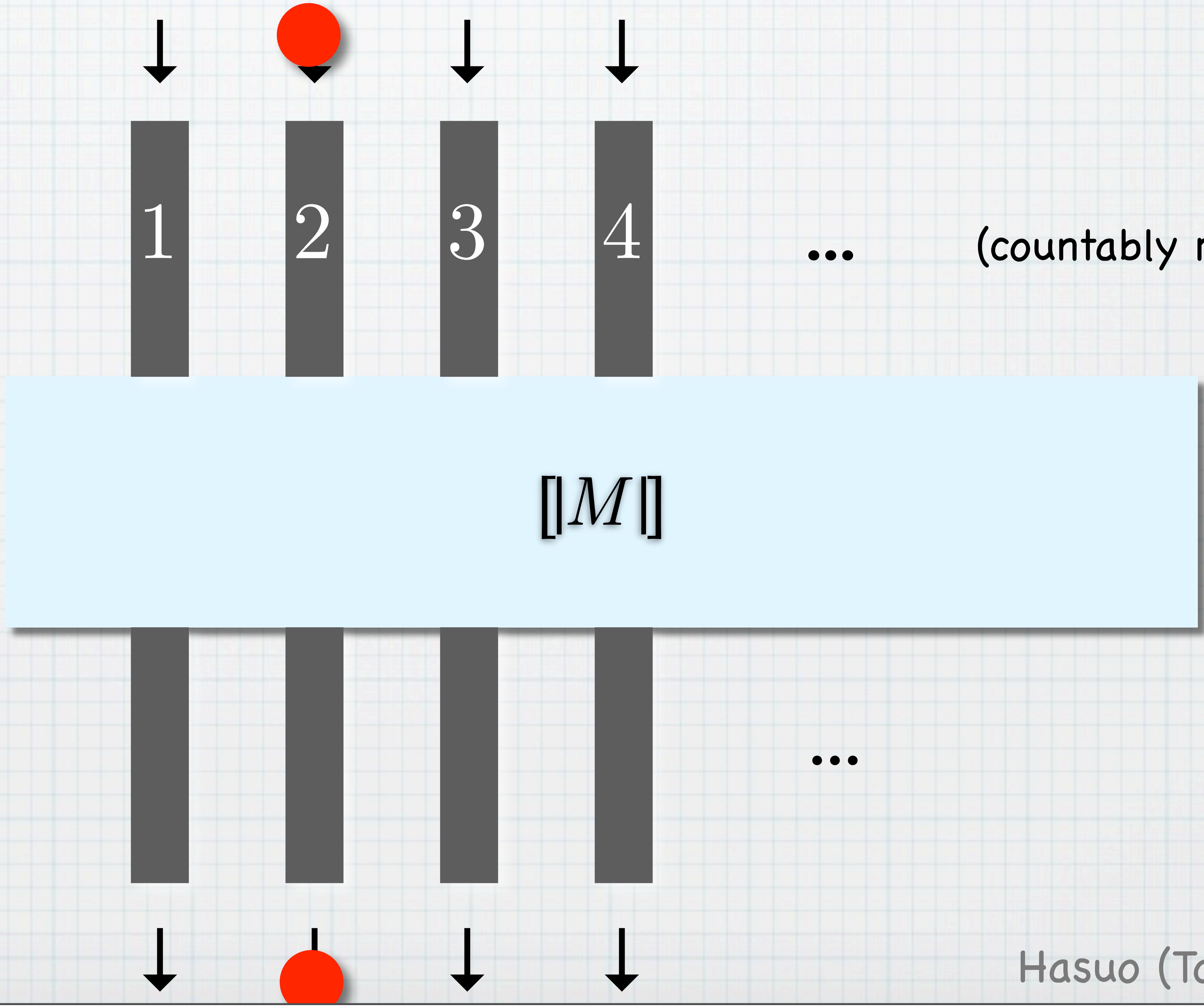

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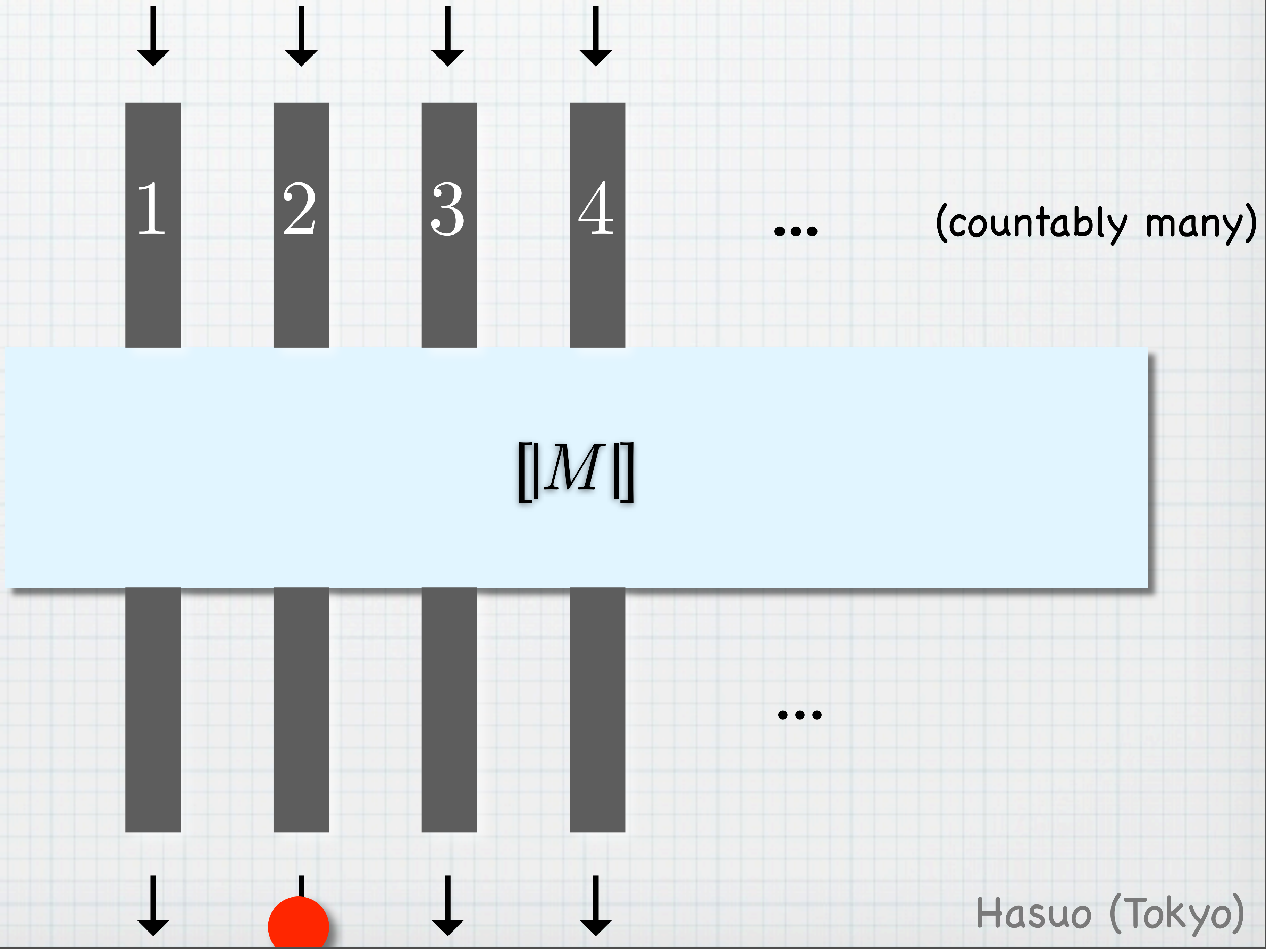

The GoI Animation

- Function application $[\![MN]\!]$
  - by “parallel composition + hiding”

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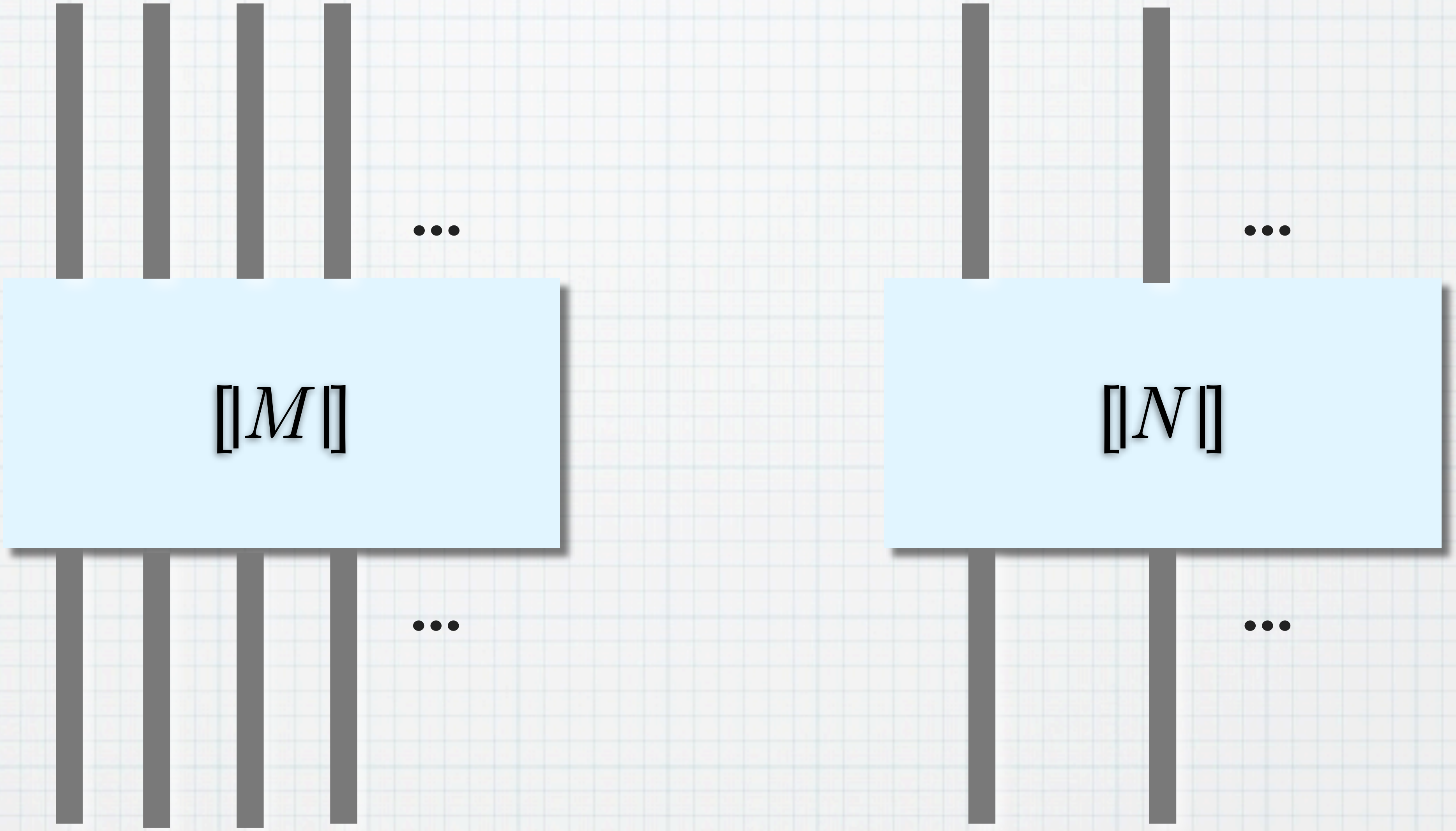

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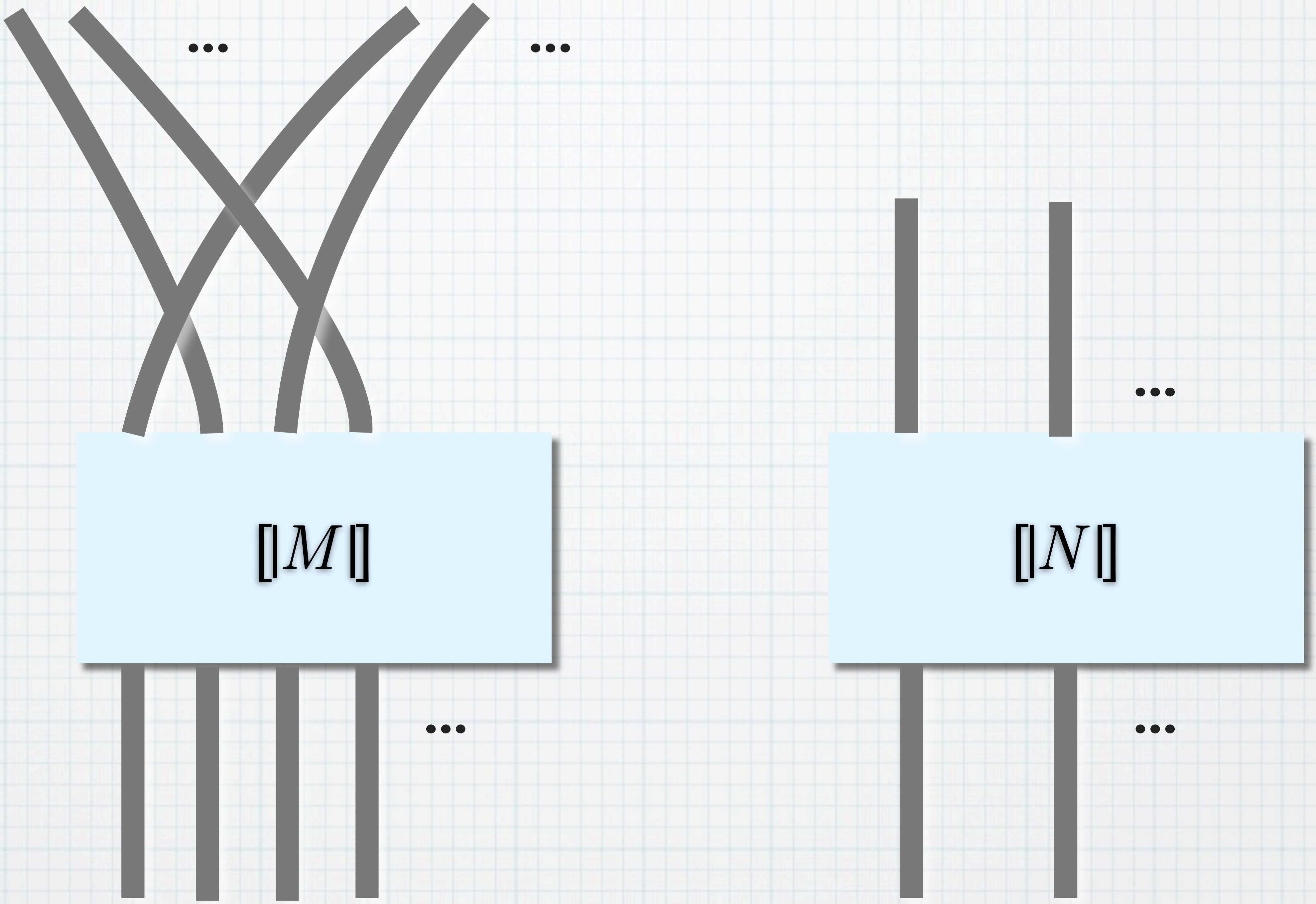

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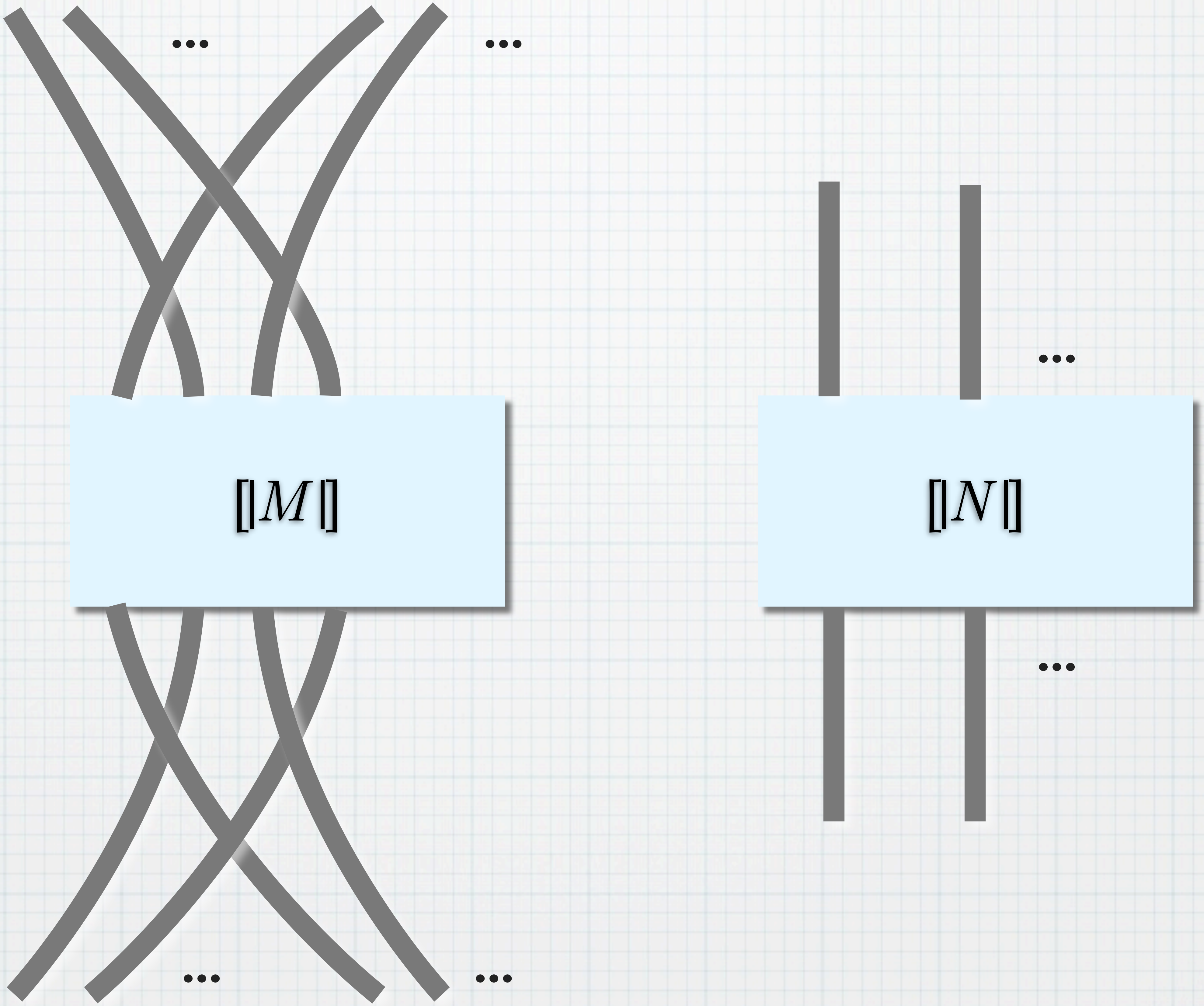

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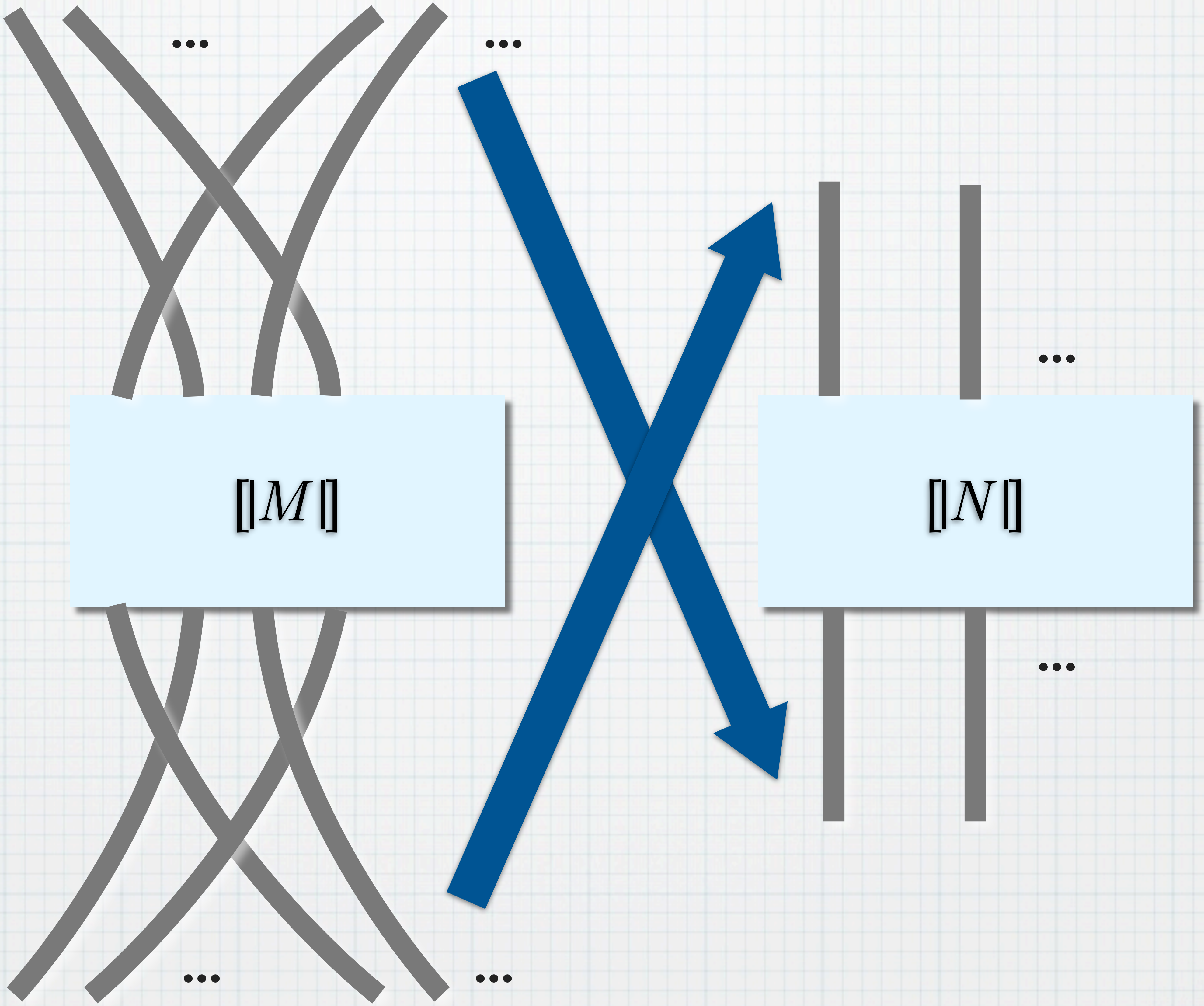

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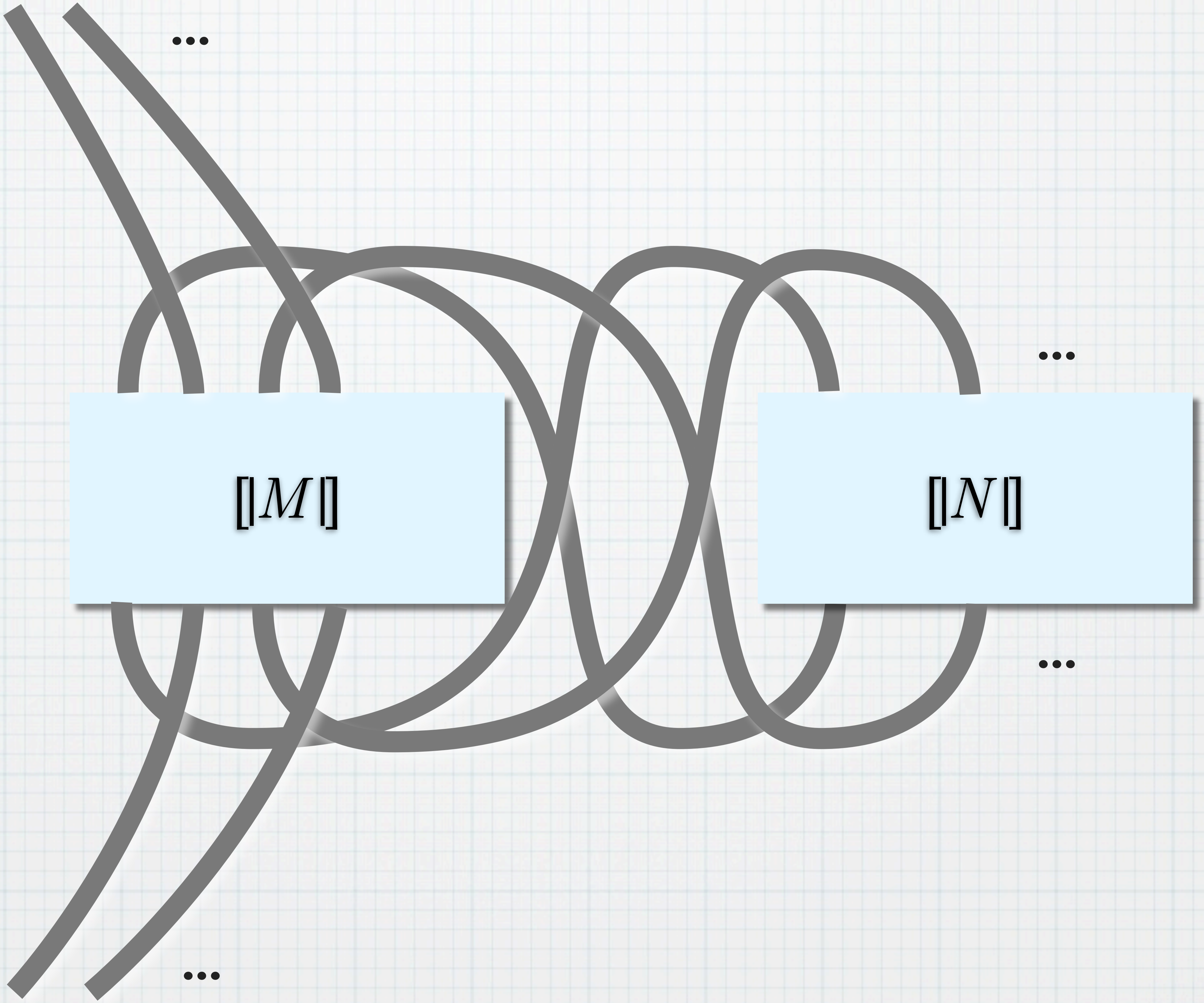

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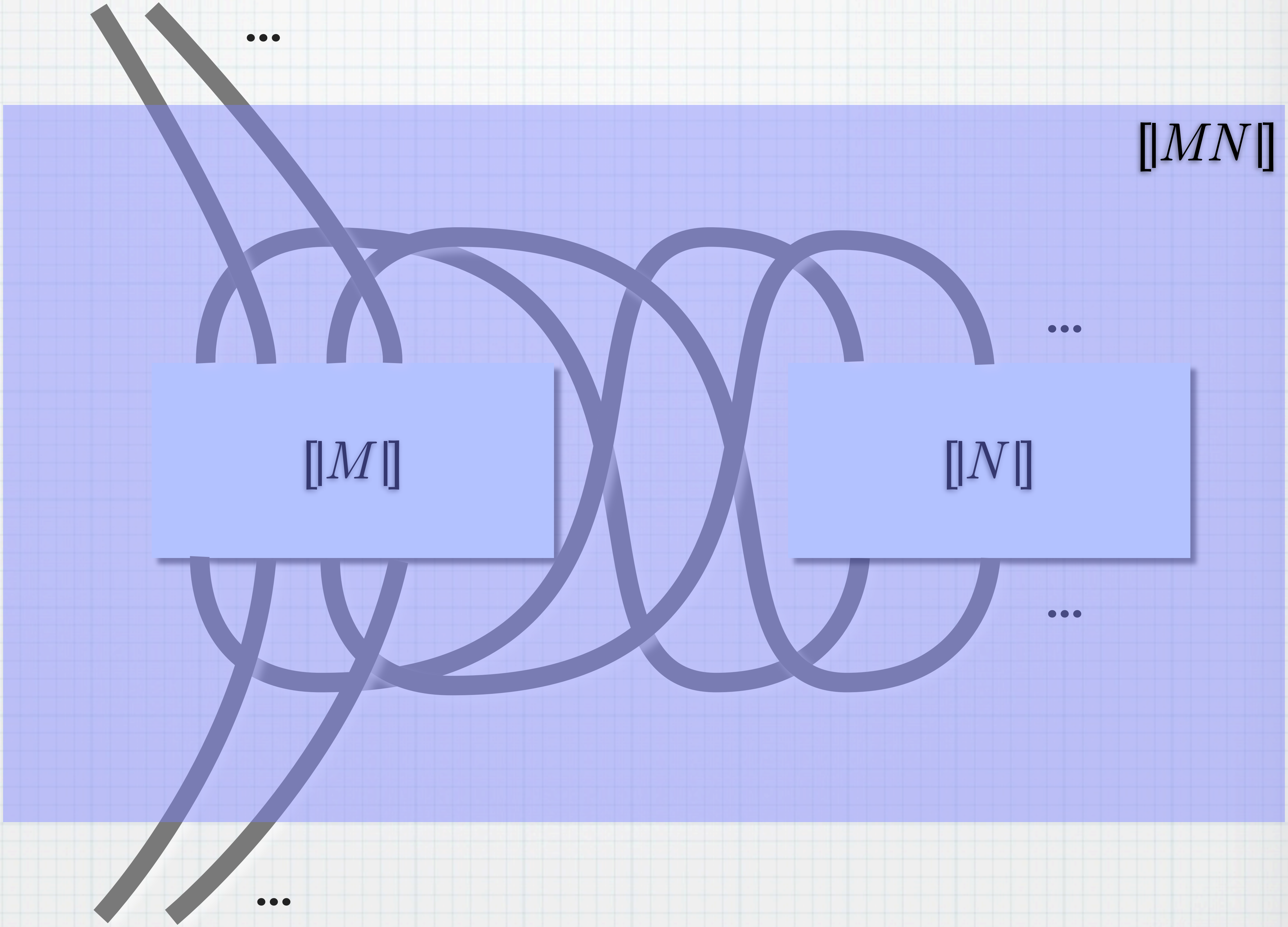

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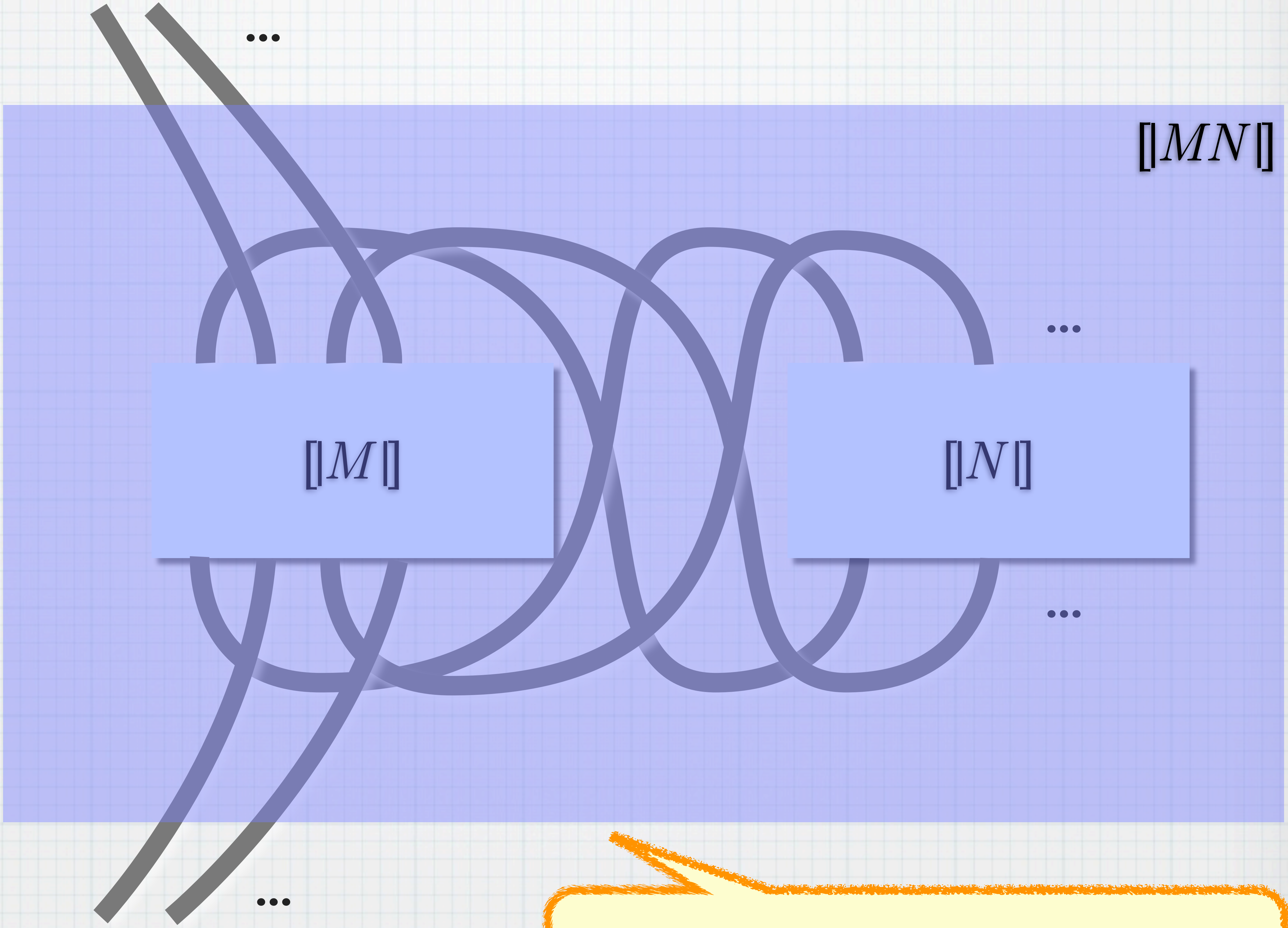

"parallel composition + hiding"
(cf. AJM games)

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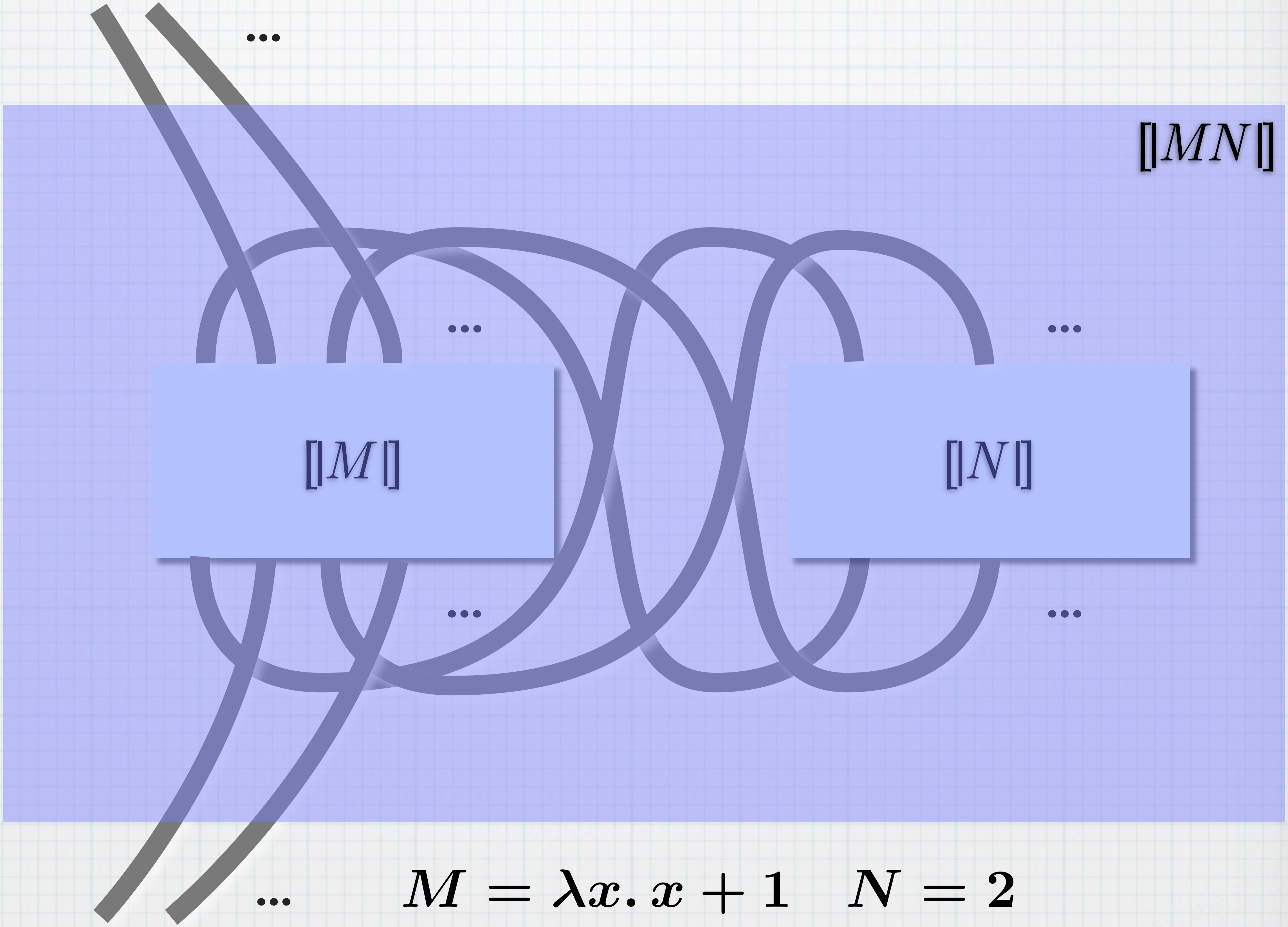

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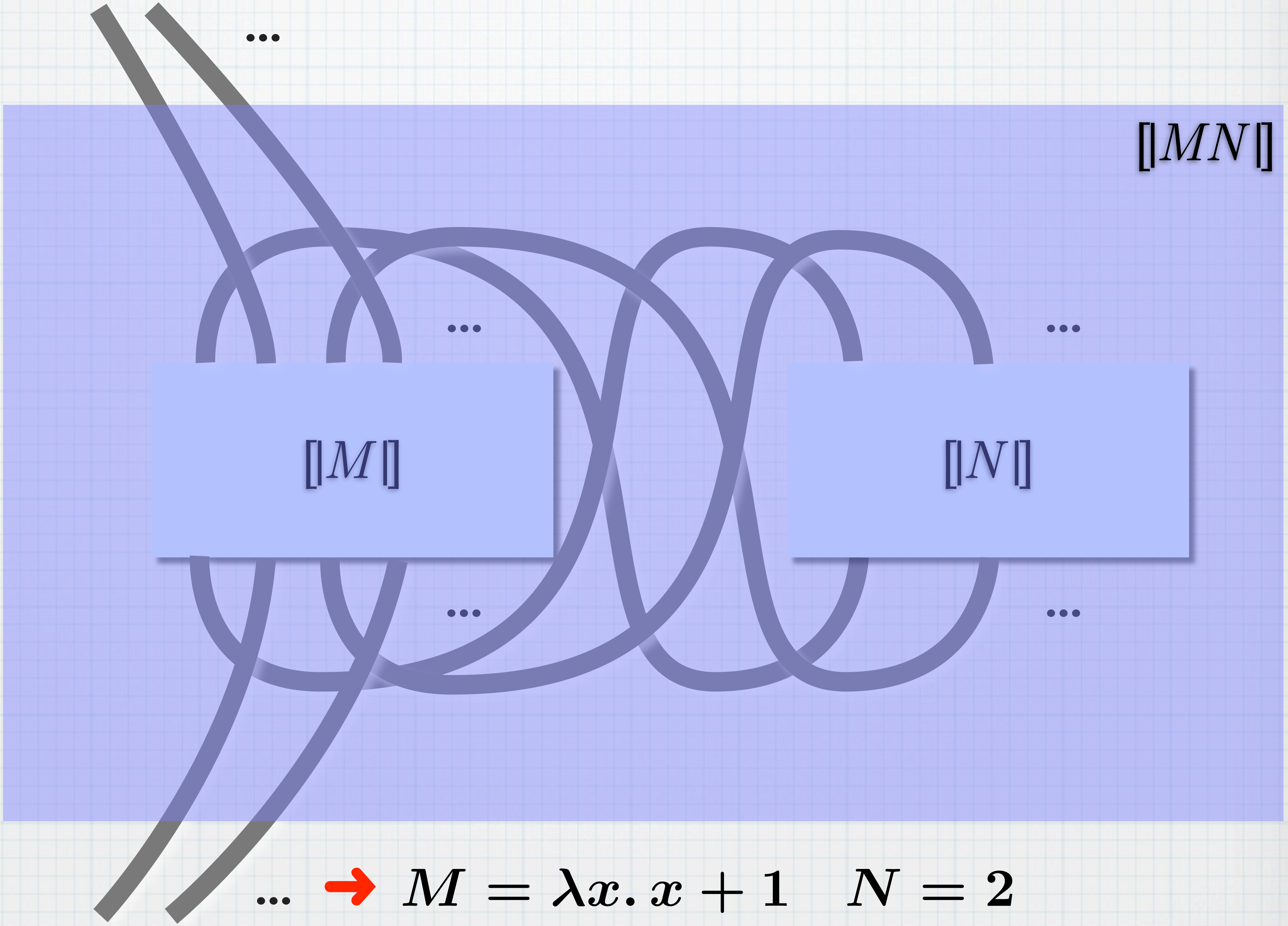
$[\![MN]\!]$

$=$



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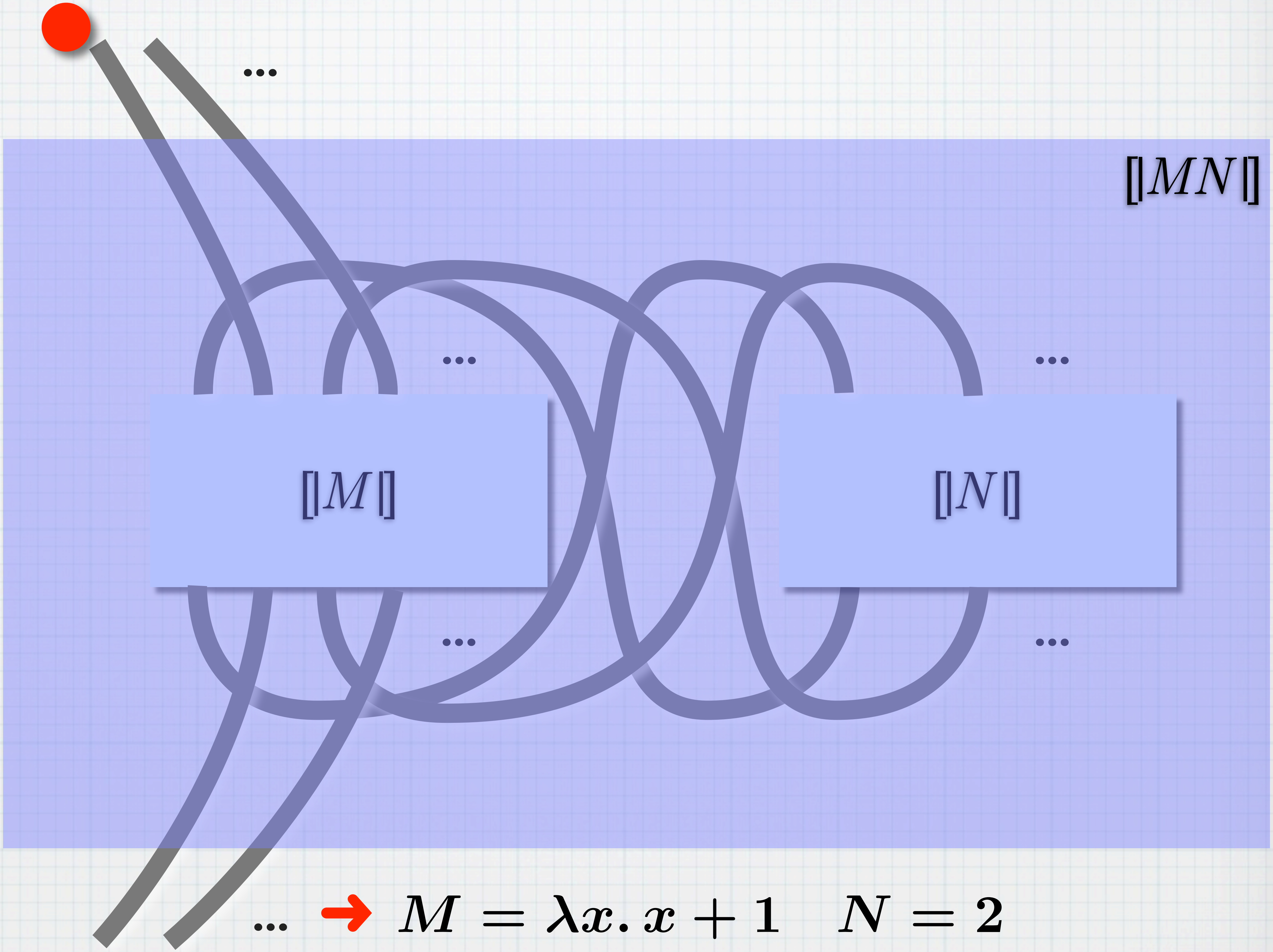

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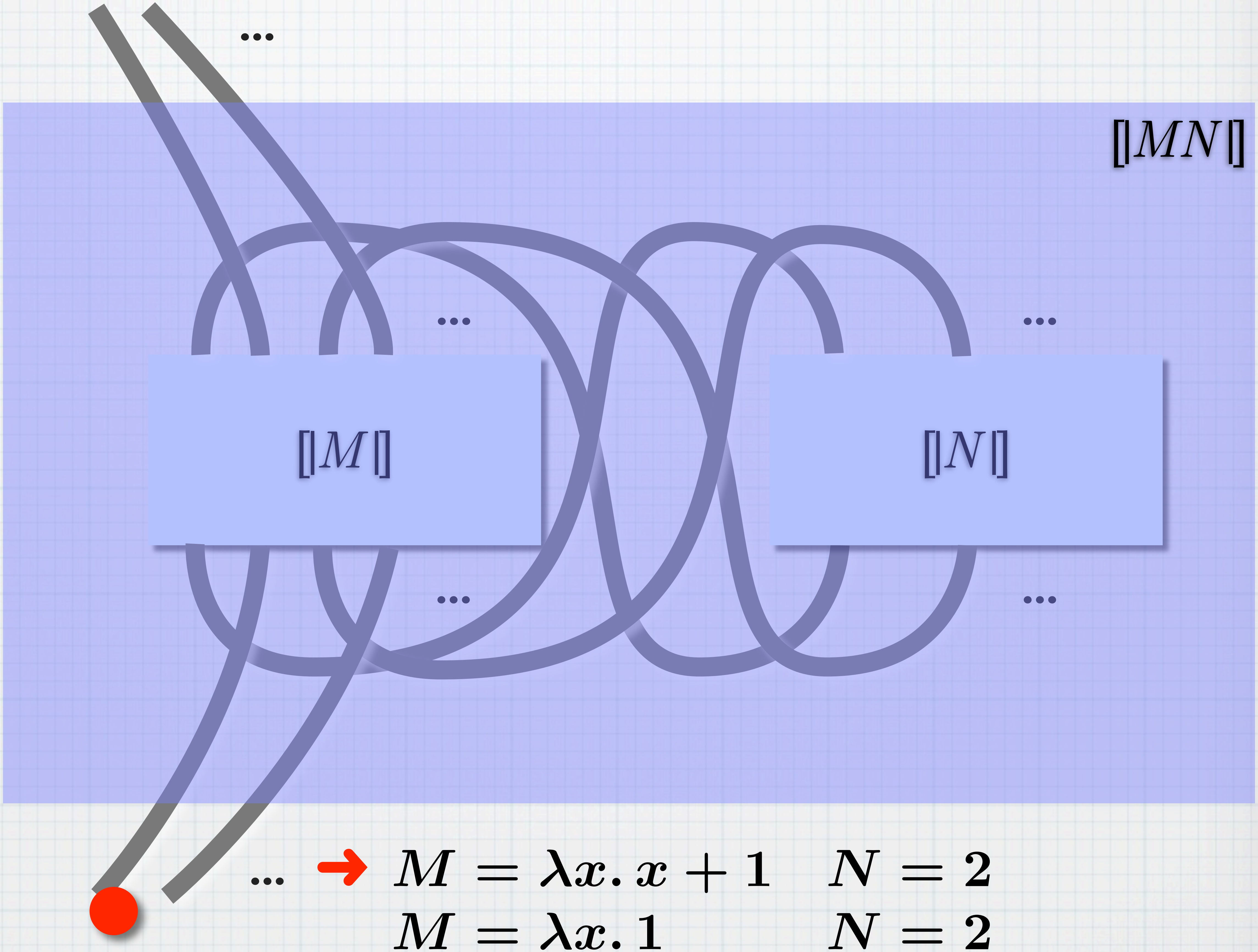

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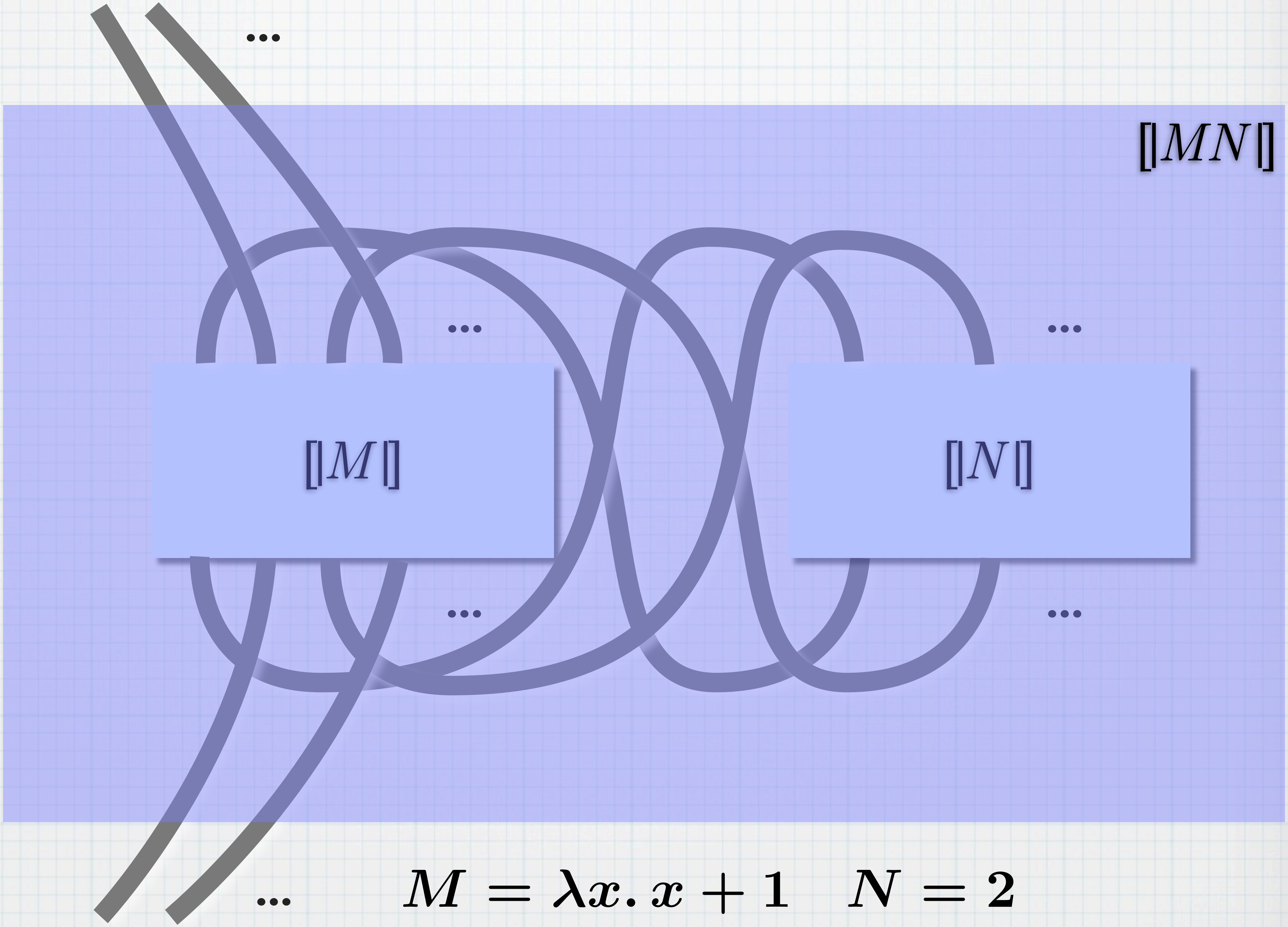

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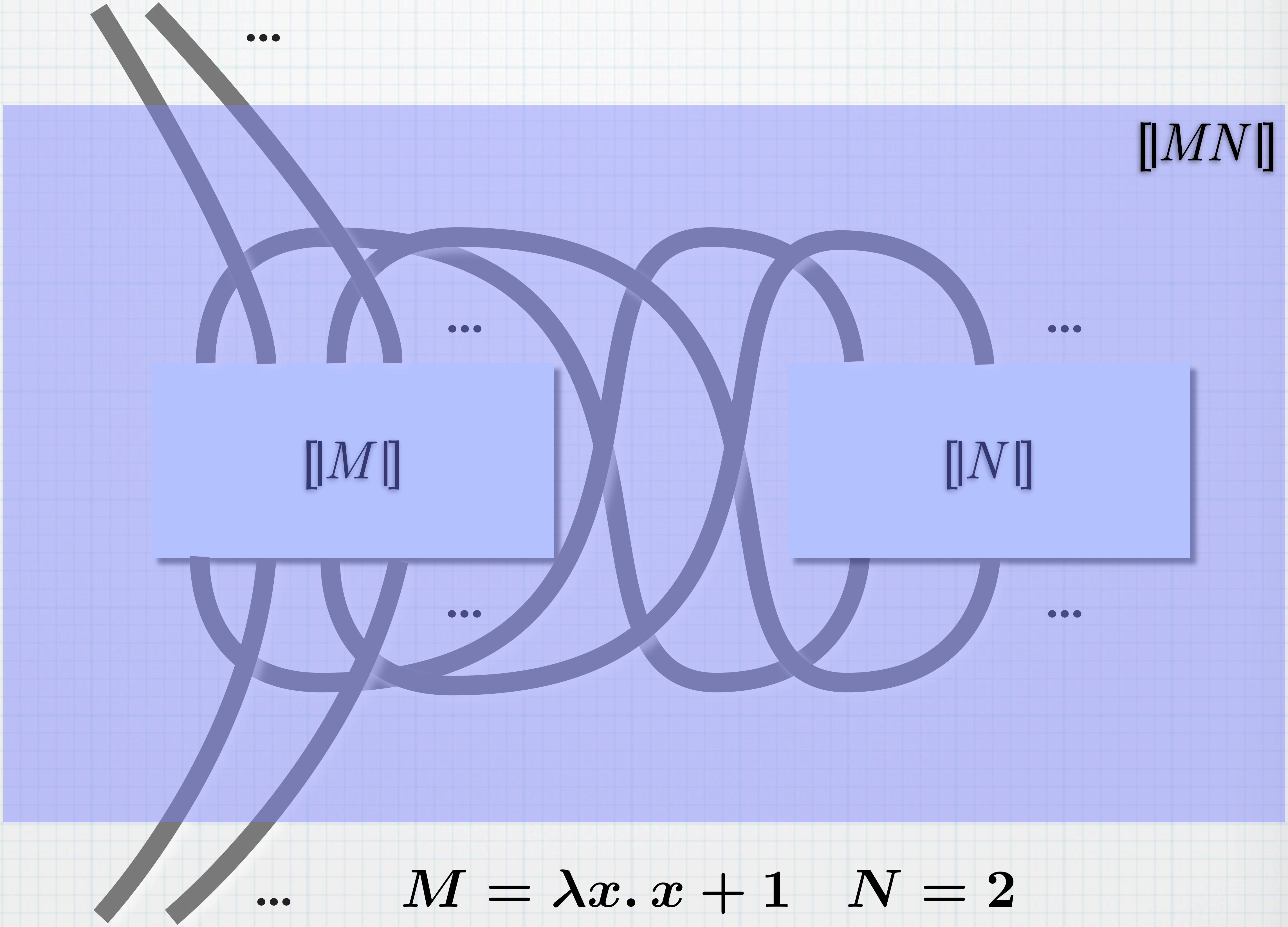

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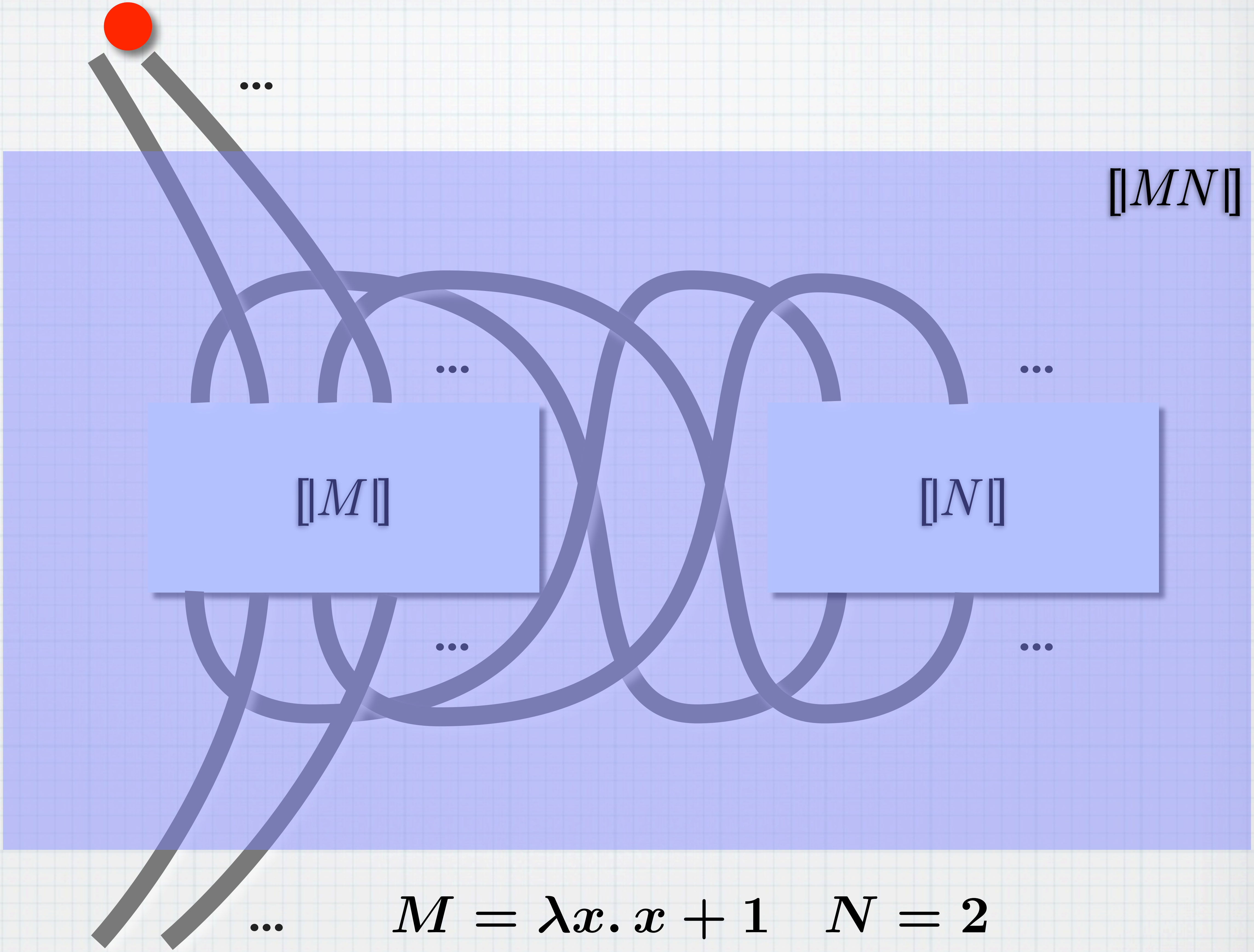

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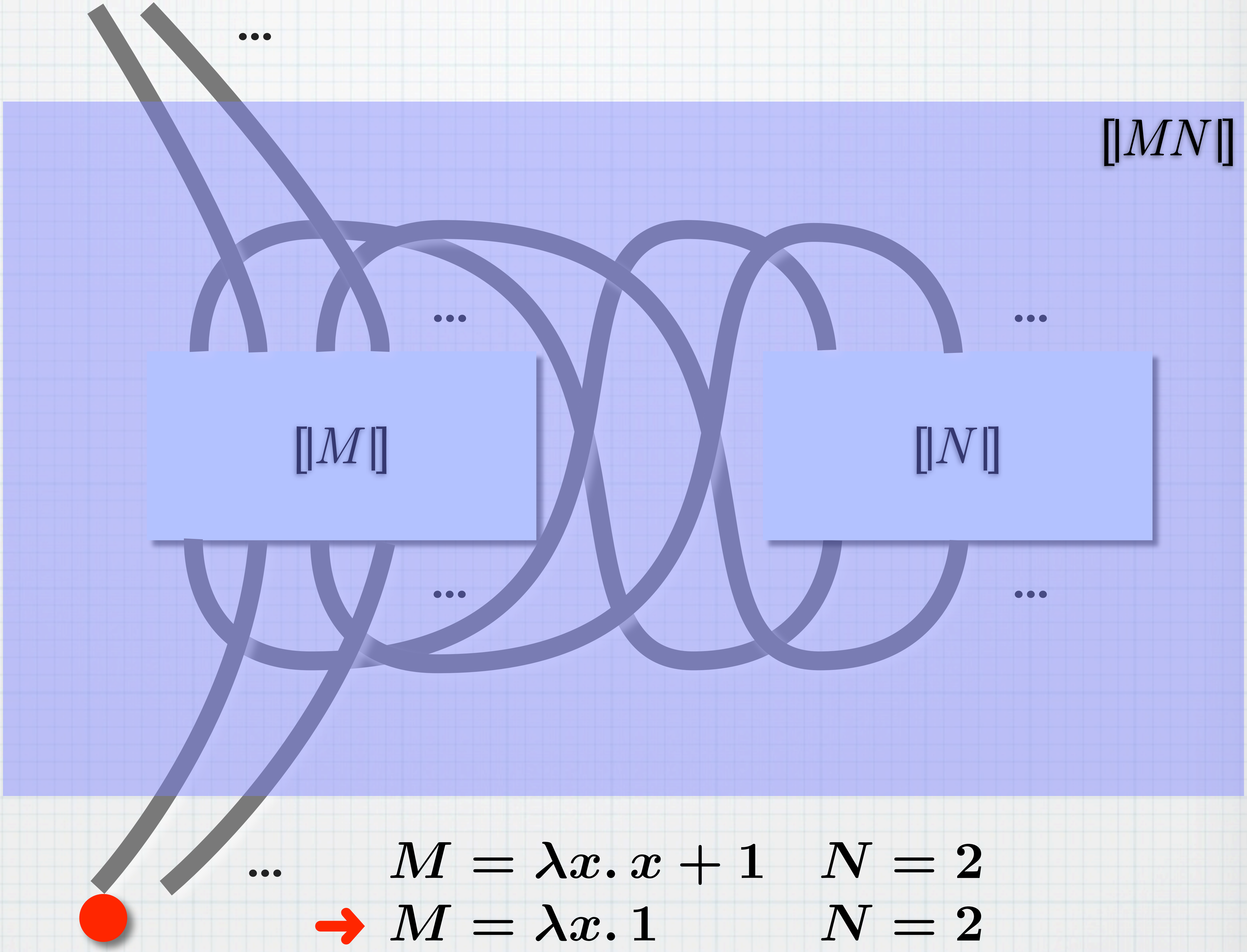

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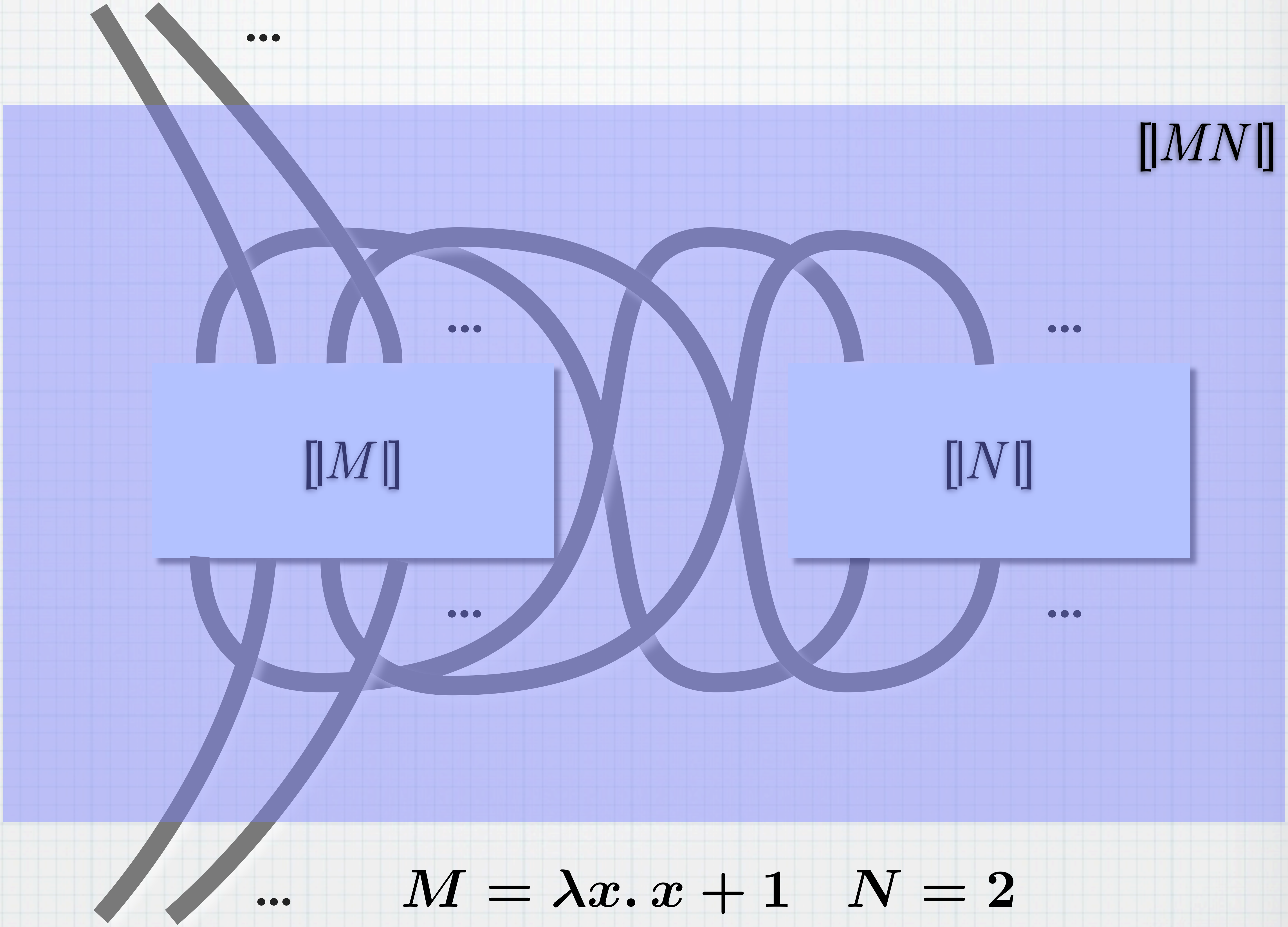

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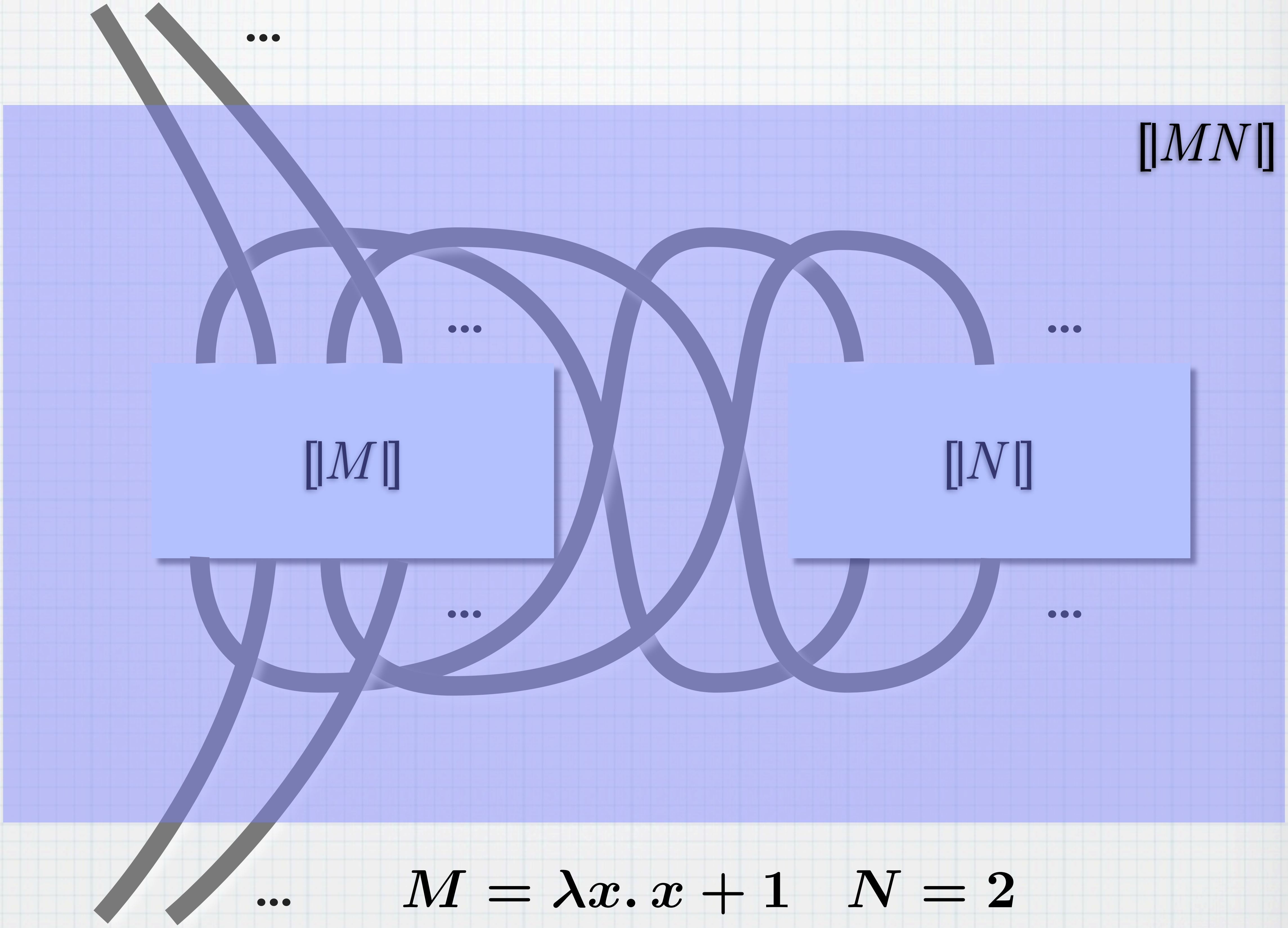

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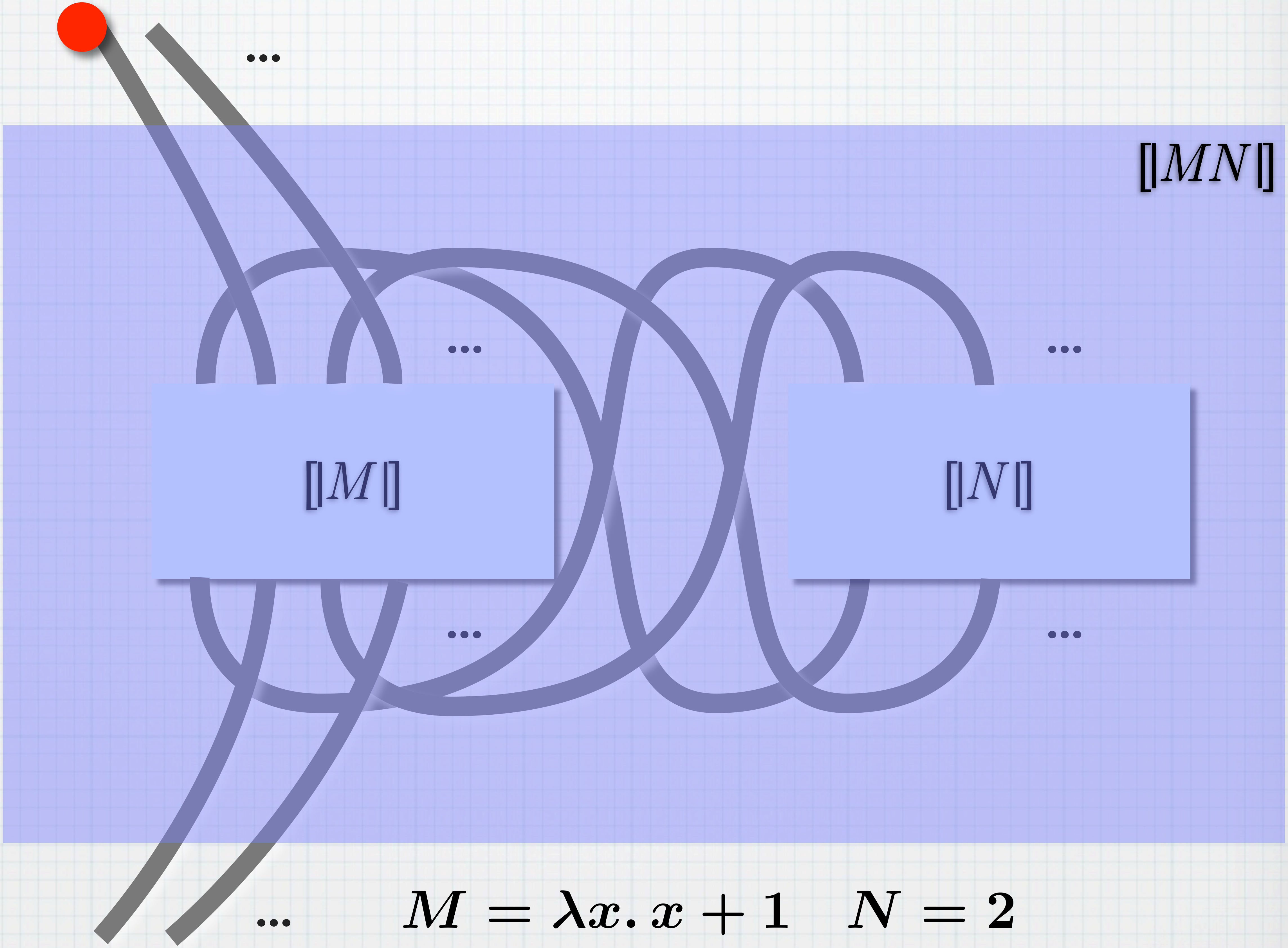

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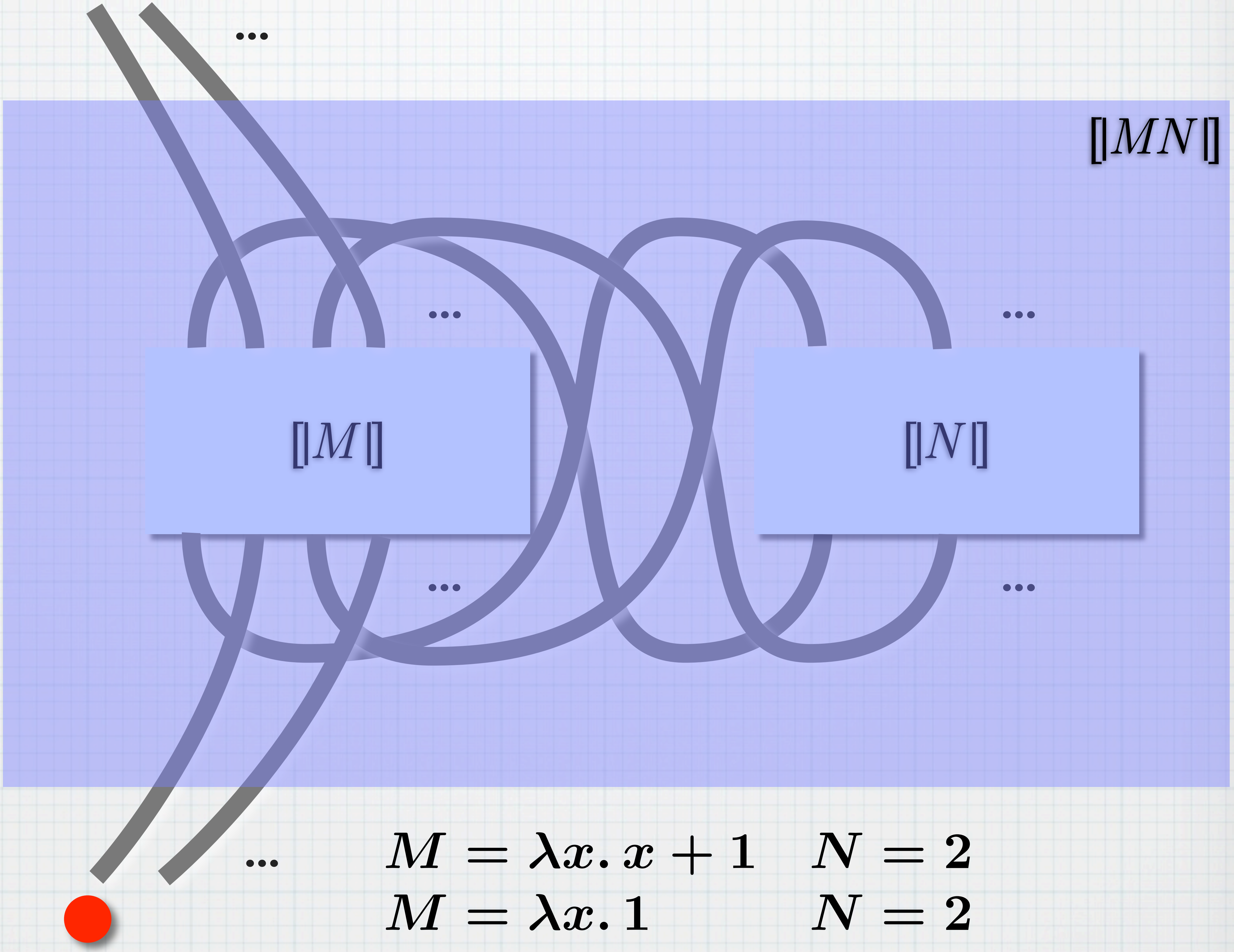

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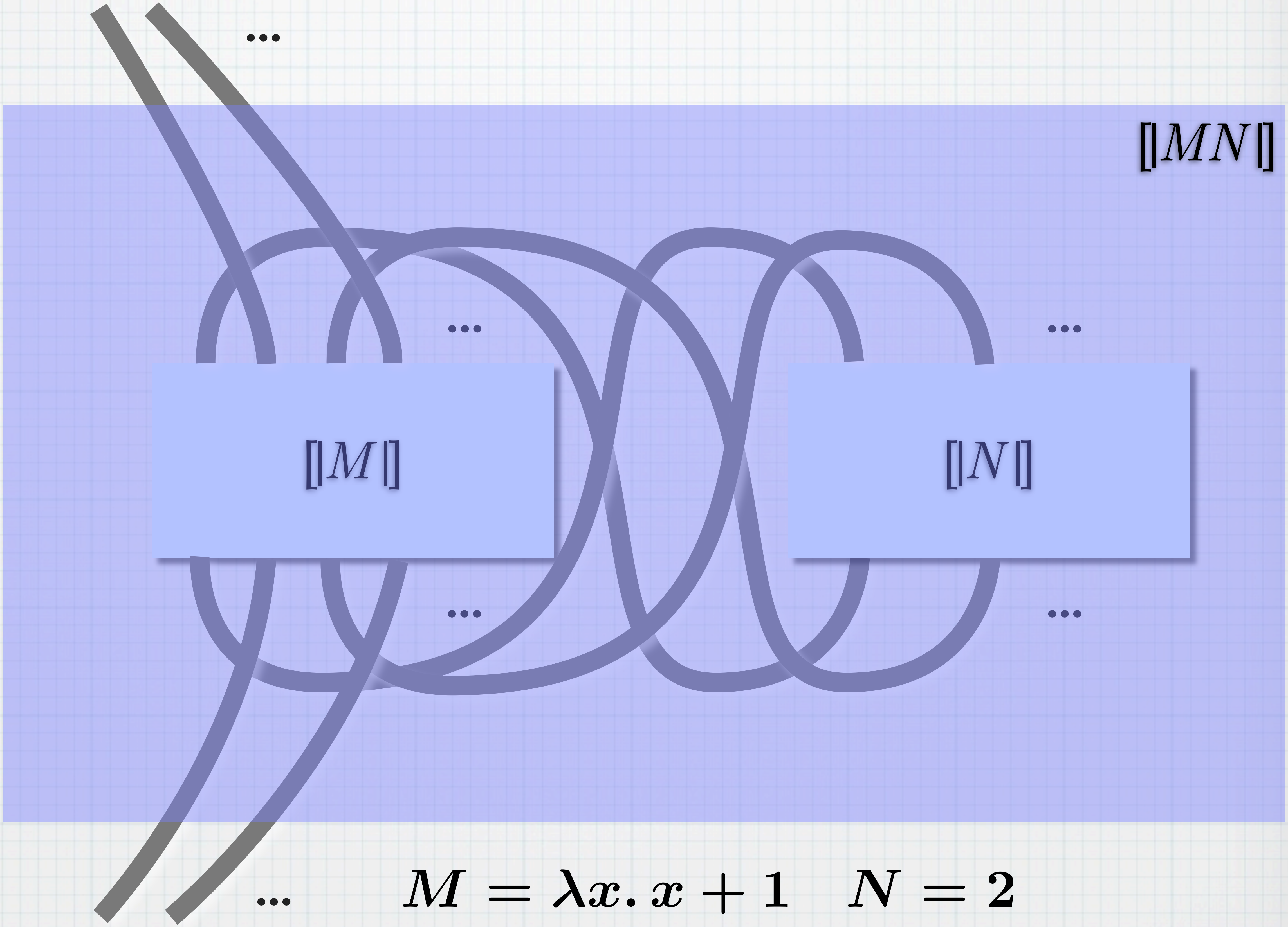

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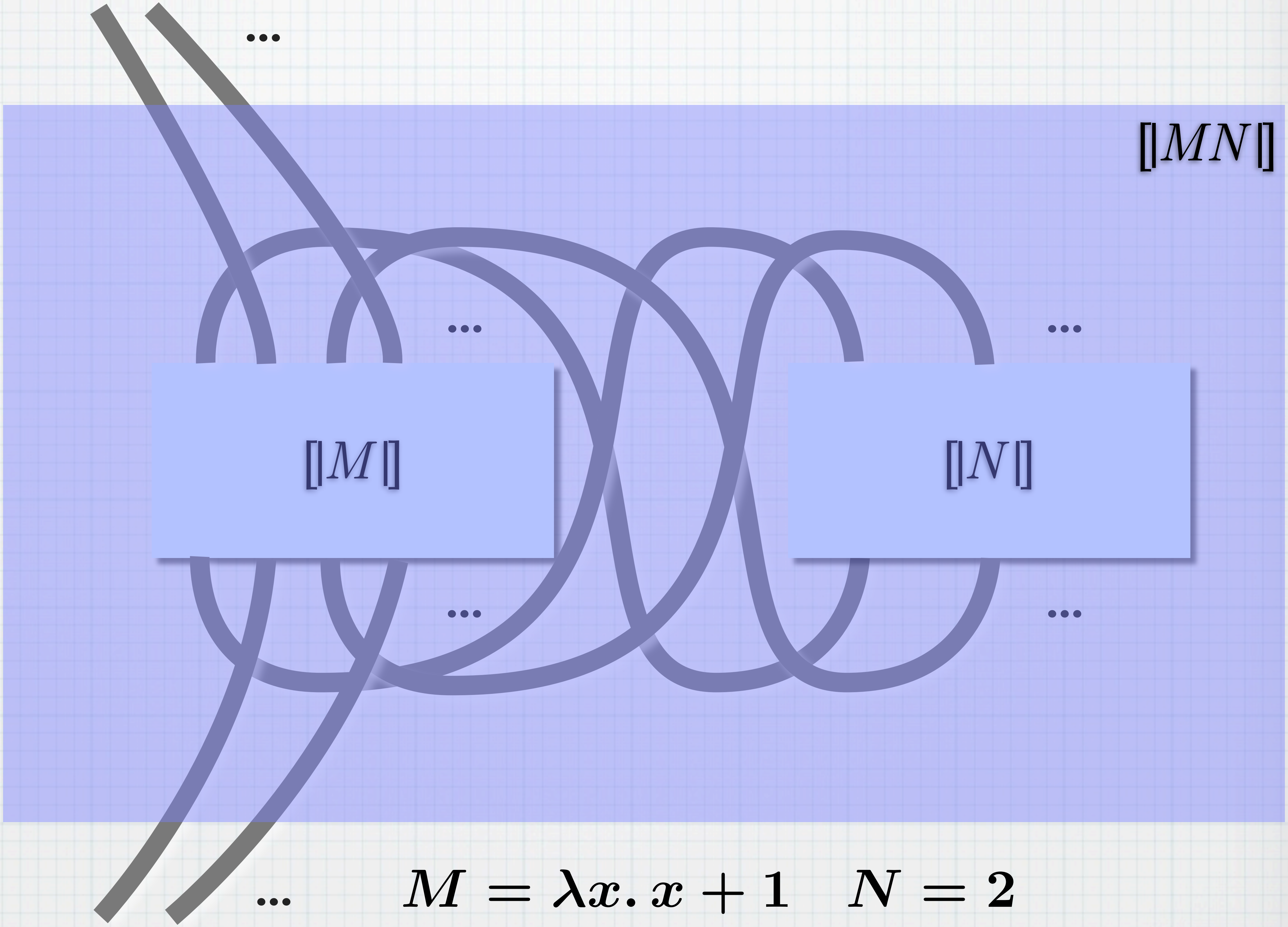

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$A \to B$

as $!A \multimap B$

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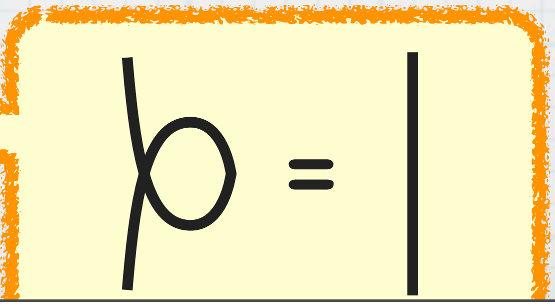
- "Geometry": invariant under β-reductions



(Tokyo)

Categorical GoI

- Axiomatics of GoI in the categorical language
- Our main reference:
  - [AHS02] S. Abramsky, E. Haghverdi, and P. Scott, "Geometry of interaction and linear combinatory algebras," MSCS 2002
  - Especially its technical report version (Oxford CL), since it's a bit more detailed

The Categorical GoI Workflow

Traced monoidal category $\mathbb{C}$

+ other constructs ➜ "GoI situation" [AHS02]

Categorical GoI [AHS02]

Linear combinatory algebra

Realizability

Linear category

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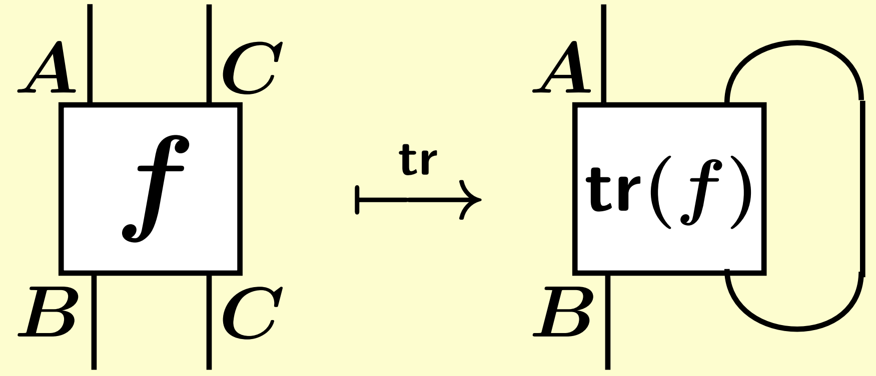

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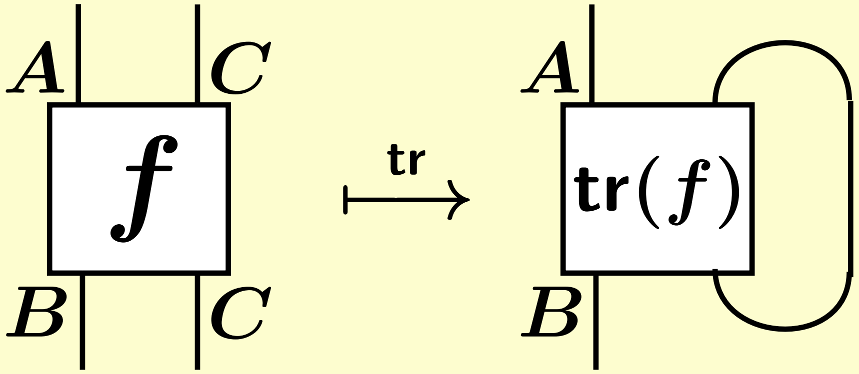

Categorical GoI [AHS02]

Linear combinatory algebra

- ✱ Applicative str. + combinators
- ✱ Model of untyped calculus

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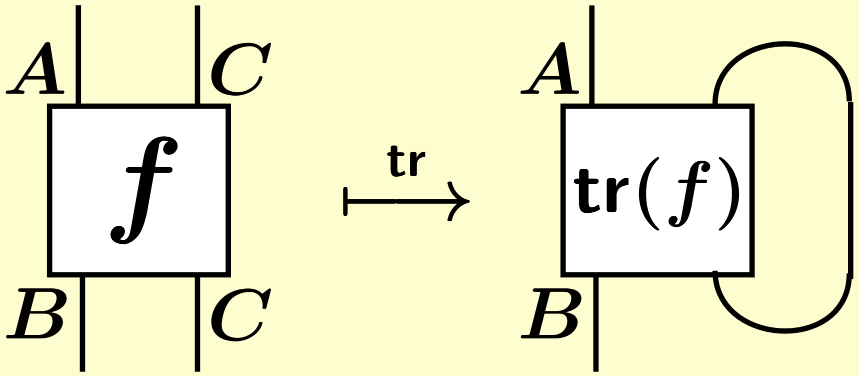

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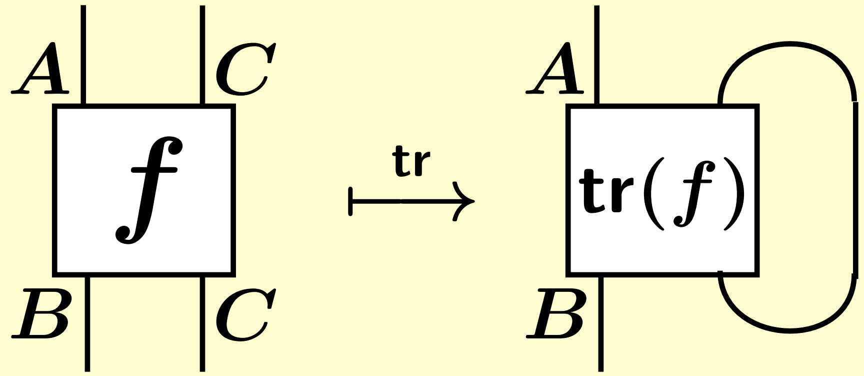

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Linear combinatory algebra

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- PER, ω-set, assembly, ...
- "Programming in untyped λ"

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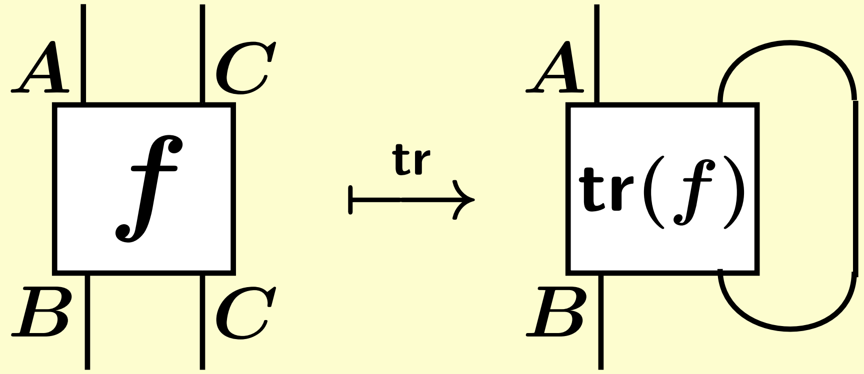

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Linear Combinatory Algebra (LCA)

Defn. (LCA)
A *linear combinatory algebra (LCA)* is a set $\boldsymbol{A}$ equipped with

- a binary operator (called an *applicative structure*)

$$\cdot \;:\; \boldsymbol{A}^2 \longrightarrow \boldsymbol{A}$$

- a unary operator

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- (*combinators*) distinguished elements $\mathsf{B}, \mathsf{C}, \mathsf{I}, \mathsf{K}, \mathsf{W}, \mathsf{D}, \delta, \mathsf{F}$ satisfying

$\mathsf{B}xyz = x(yz)$	Composition, Cut
$\mathsf{C}xyz = (xz)y$	Exchange
$\mathsf{I}x = x$	Identity
$\mathsf{K}x\,!\,y = x$	Weakening
$\mathsf{W}x\,!\,y = x\,!\,y\,!\,y$	Contraction
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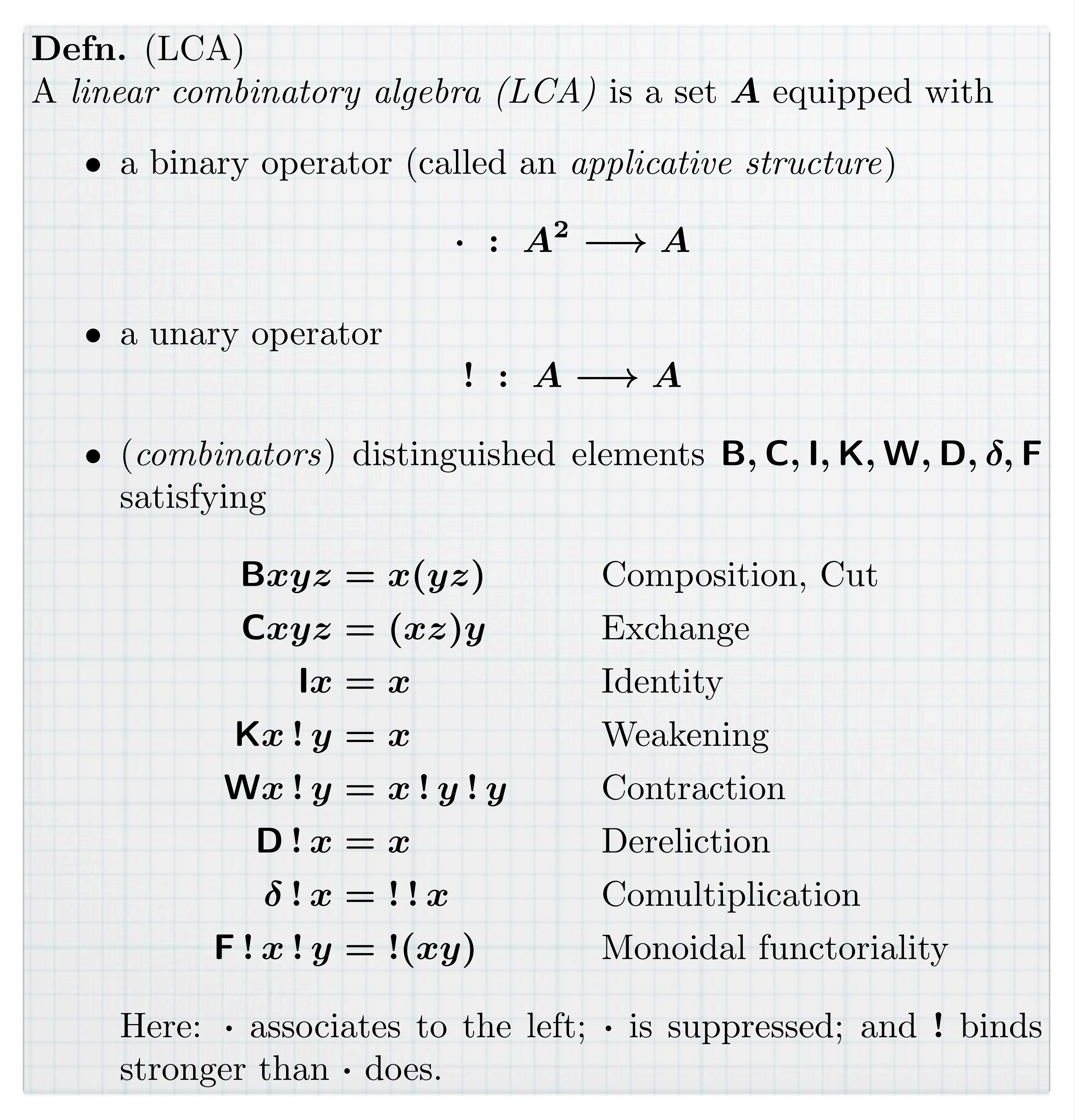
Linear Combinatory Algebra (LCA)

What
we want (outcome)



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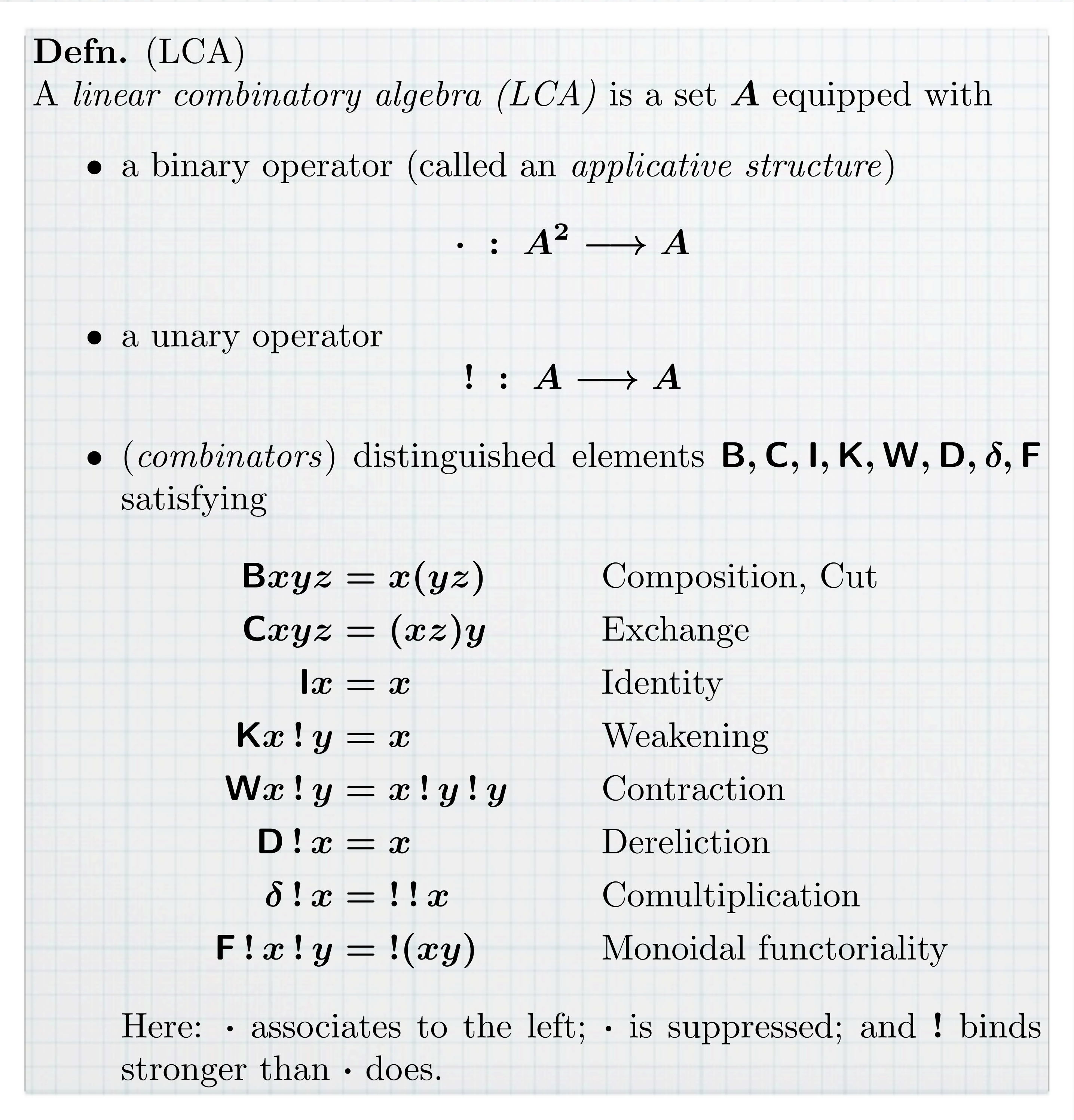

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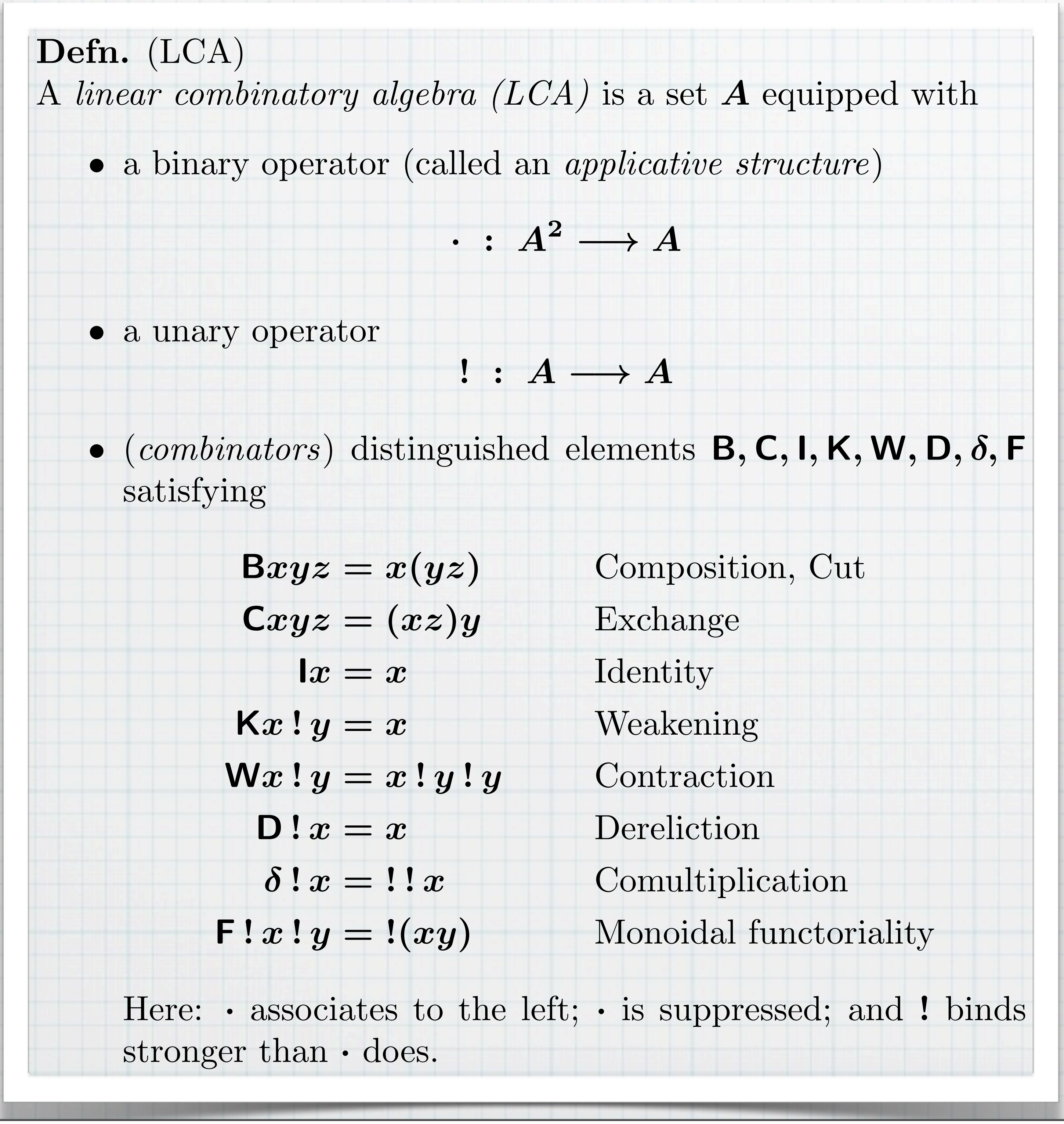
- Model of untyped linear λ

Linear Combinatory Algebra (LCA)



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Linear Combinatory Algebra (LCA)

What
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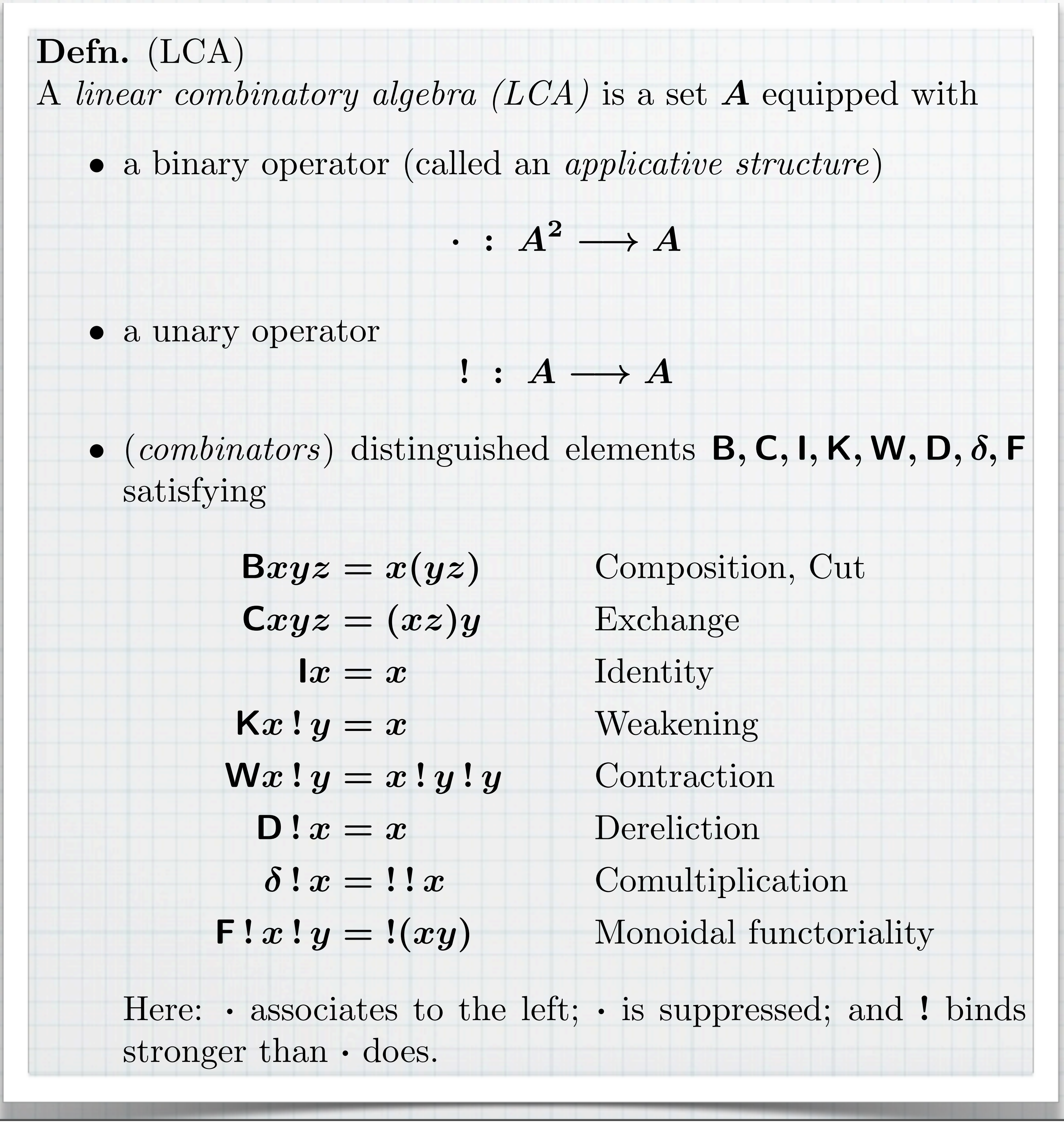

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- Model of untyped linear λ
- a ∈ A ≈ closed linear λ-term
- No **S** or **K** (linear!)
- Combinatory completeness: e.g.

$$\lambda xyz.\, zxy$$

designates an elem. of A

What we use (ingredient)

GoI situation

Defn. (GoI situation [AHS02])
A *GoI situation* is a triple $(\mathbb{C}, F, U)$ where

- $\mathbb{C} = (\mathbb{C}, \otimes, I)$ is a traced symmetric monoidal category (TSMC);
- $F : \mathbb{C} \to \mathbb{C}$ is a traced symmetric monoidal functor, equipped with the following retractions (which are monoidal natural transformations).

$$\begin{array}{rcl l}
e : & FF \lhd F & : e' & \text{Comultiplication} \\
d : & \mathrm{id} \lhd F & : d' & \text{Dereliction} \\
c : & F \otimes F \lhd F & : c' & \text{Contraction} \\
w : & K_I \lhd F & : w' & \text{Weakening}
\end{array}$$

Here K_I is the constant functor into the monoidal unit I;

- $U \in \mathbb{C}$ is an object (called *reflexive object*), equipped with the following retractions.

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 & I \lhd U & \\
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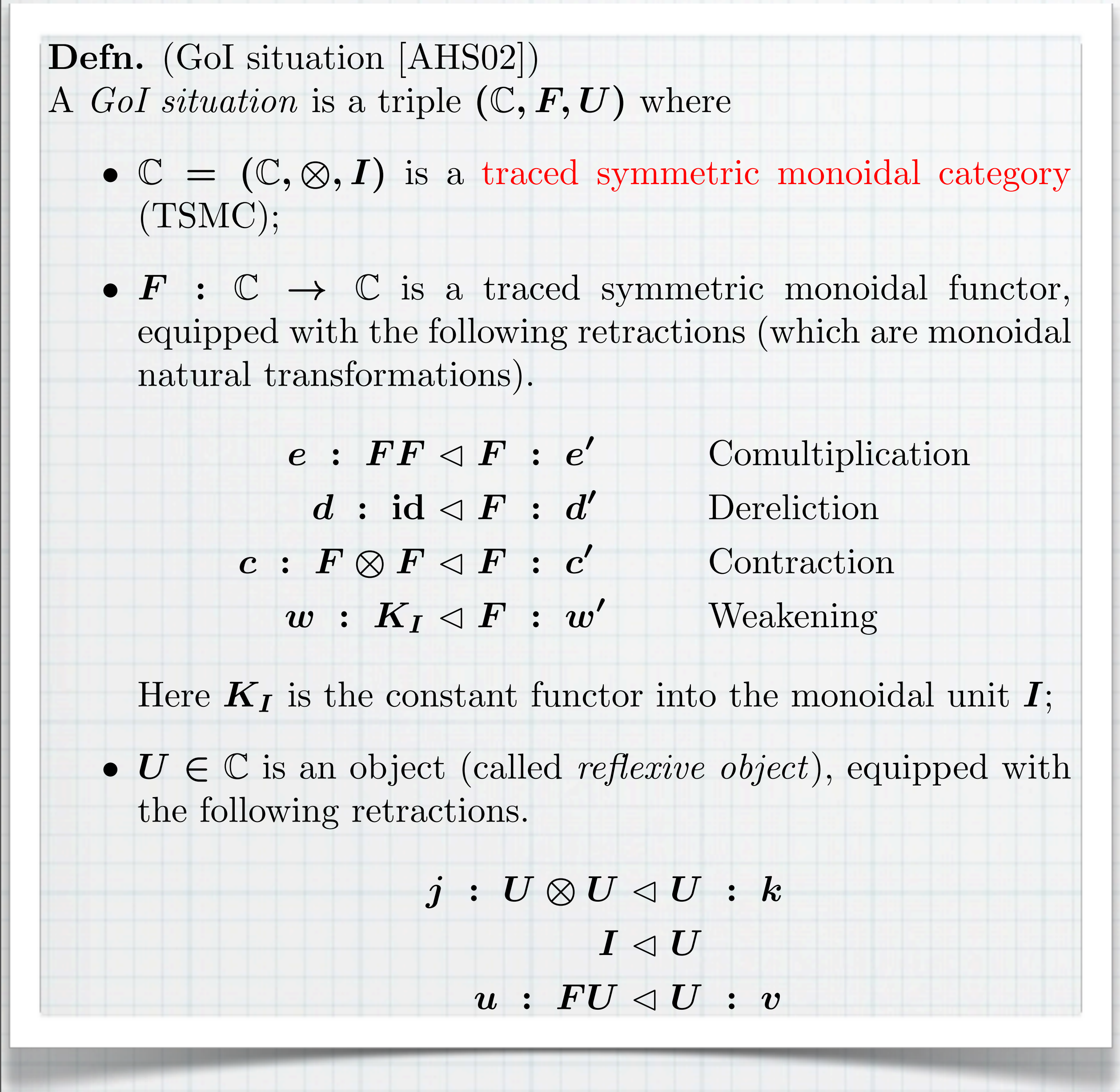
GoI situation



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\boldsymbol{I} \lhd \boldsymbol{U} \\
\boldsymbol{u} : \boldsymbol{F}\boldsymbol{U} \lhd \boldsymbol{U} : \boldsymbol{v}
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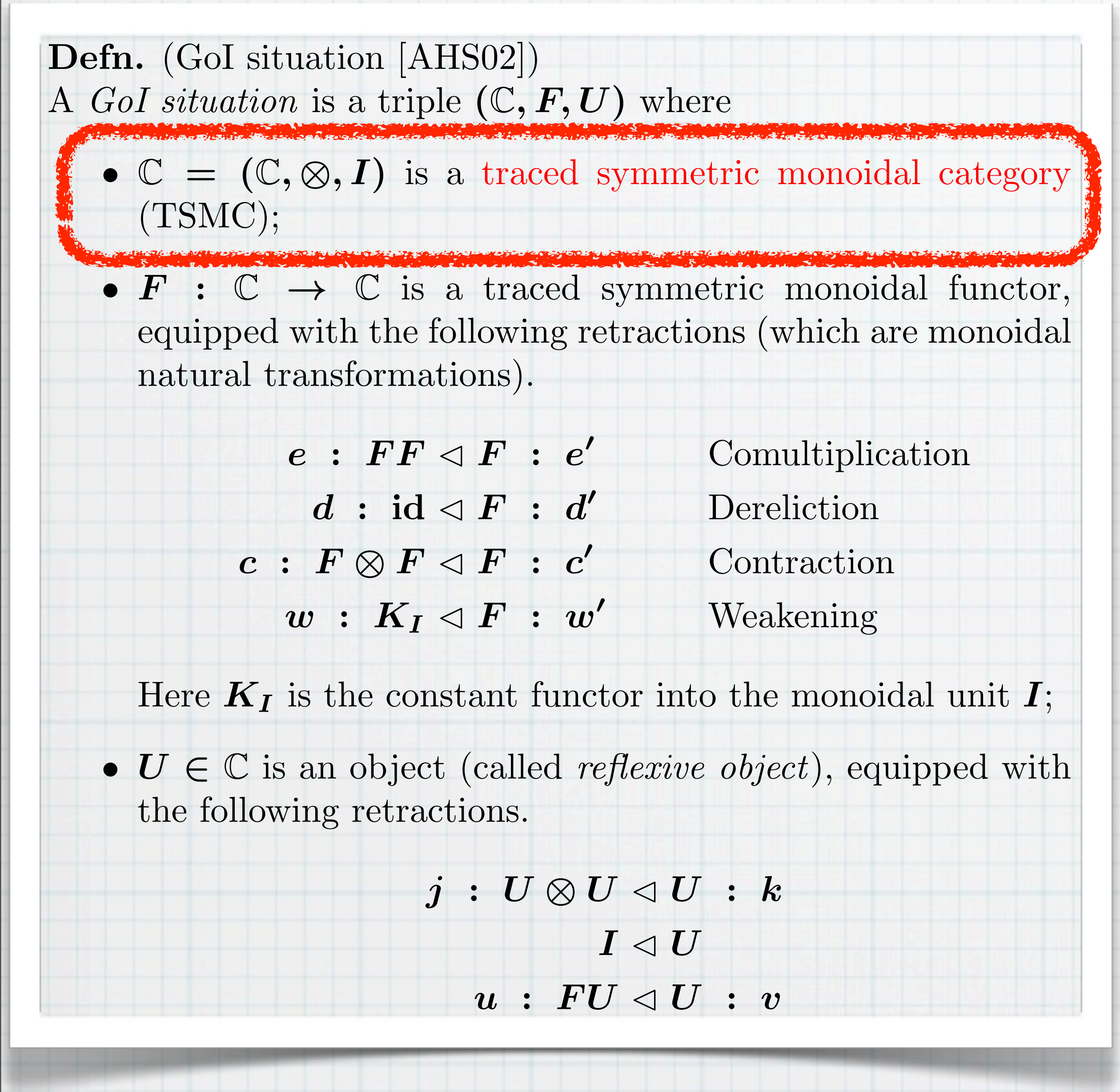
- Monoidal category $(\mathbb{C}, \otimes, I)$
  - String diagrams

GoI situation



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- **Monoidal category** $(\mathbb{C}, \otimes, \boldsymbol{I})$
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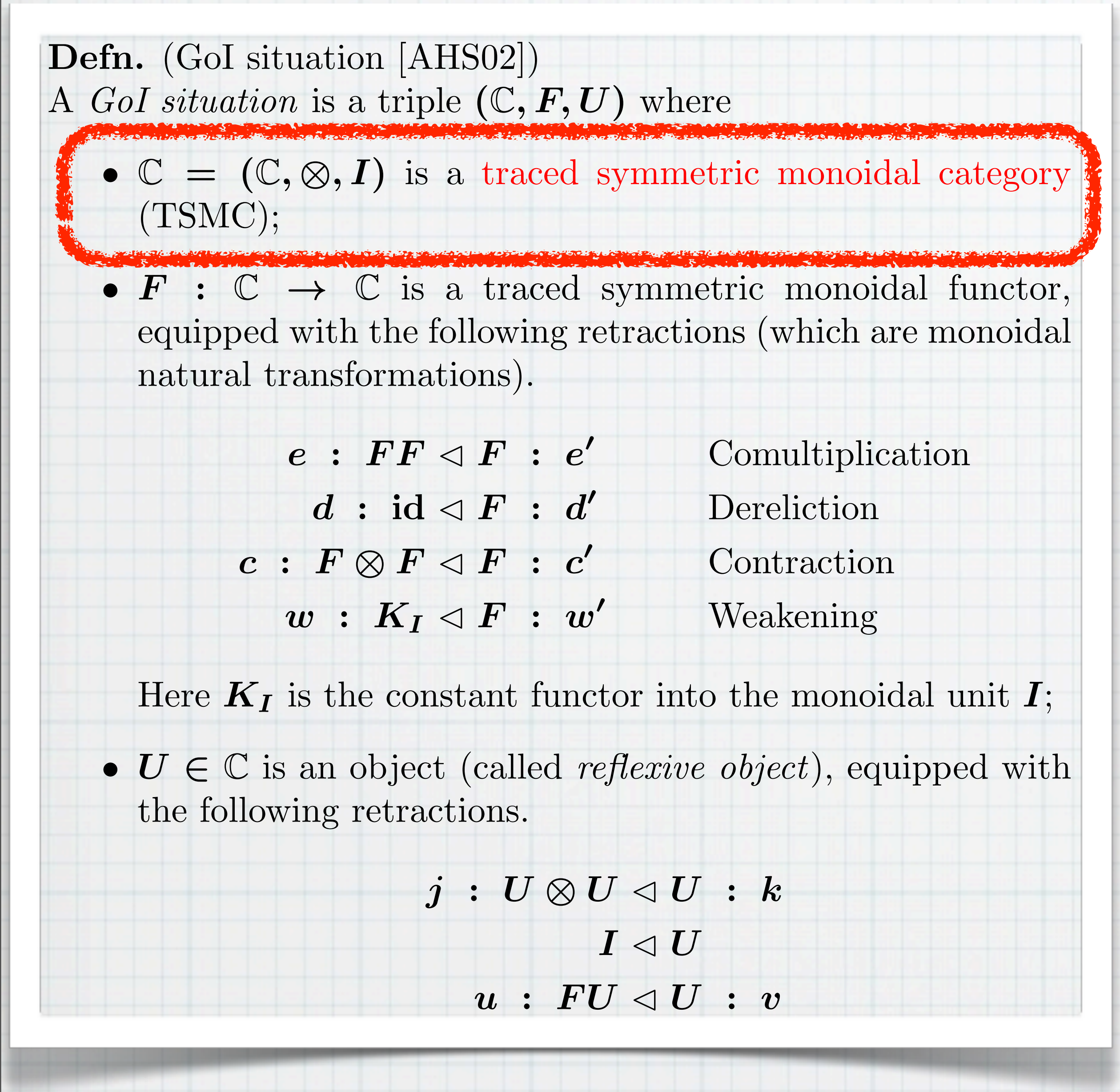
GoI situation



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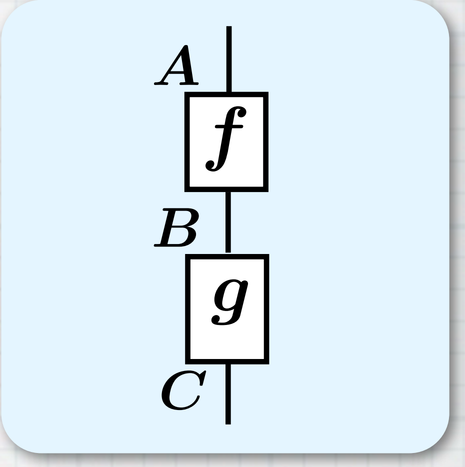
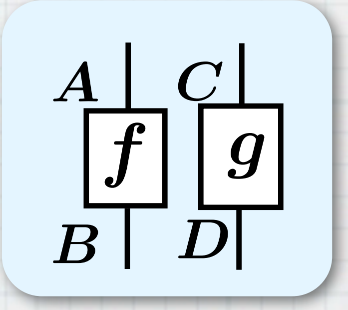
- Monoidal category $(\mathbb{C}, \otimes, I)$

- String diagrams

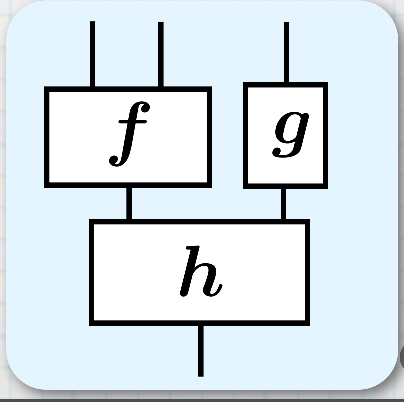
$$\frac{A \xrightarrow{f} B \quad B \xrightarrow{g} C}{A \xrightarrow{g \circ f} C}$$



$$\frac{A \xrightarrow{f} B \quad C \xrightarrow{g} D}{A \otimes C \xrightarrow{f \otimes g} B \otimes D}$$



$$h \circ (f \otimes g)$$



o)

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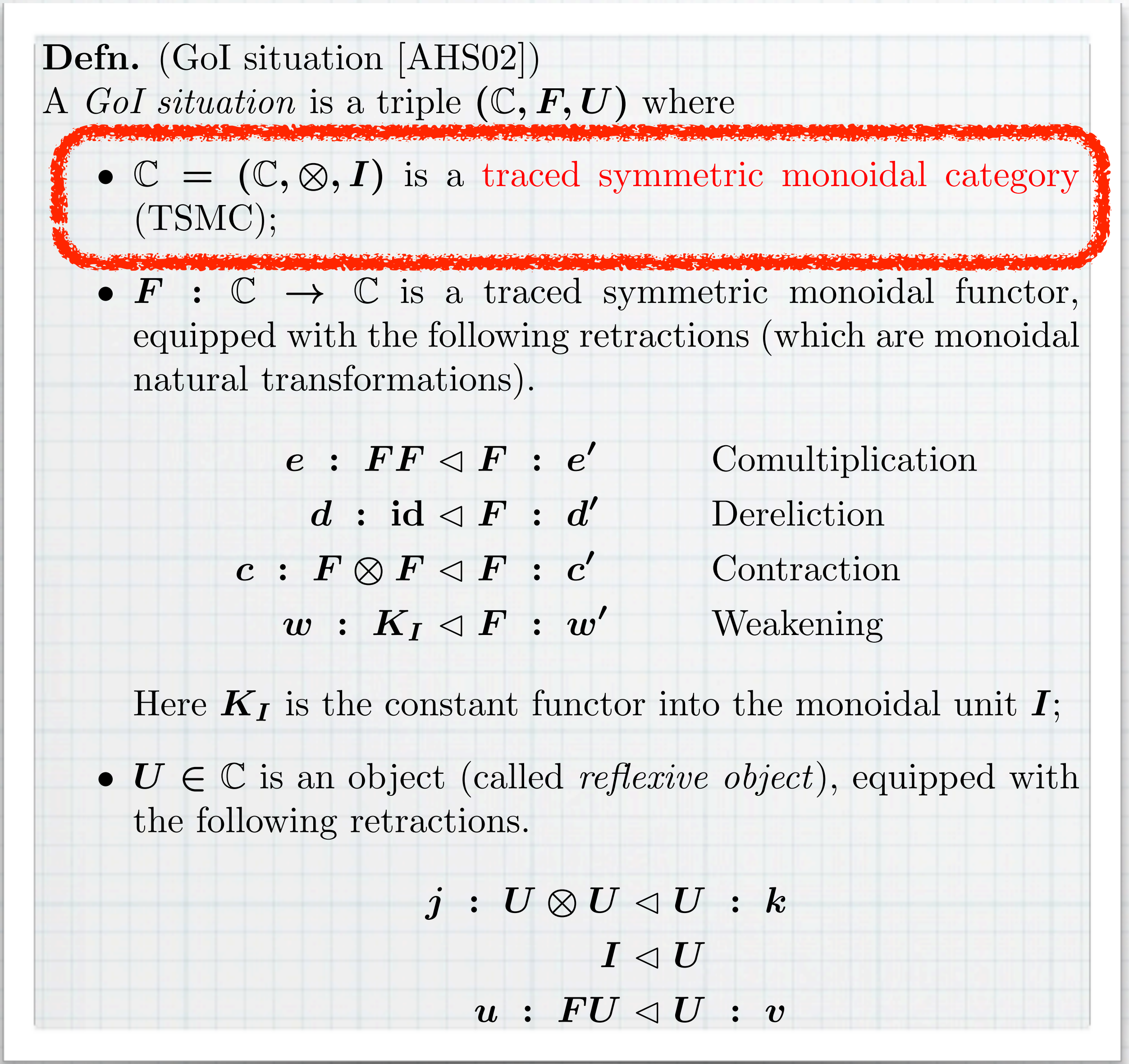

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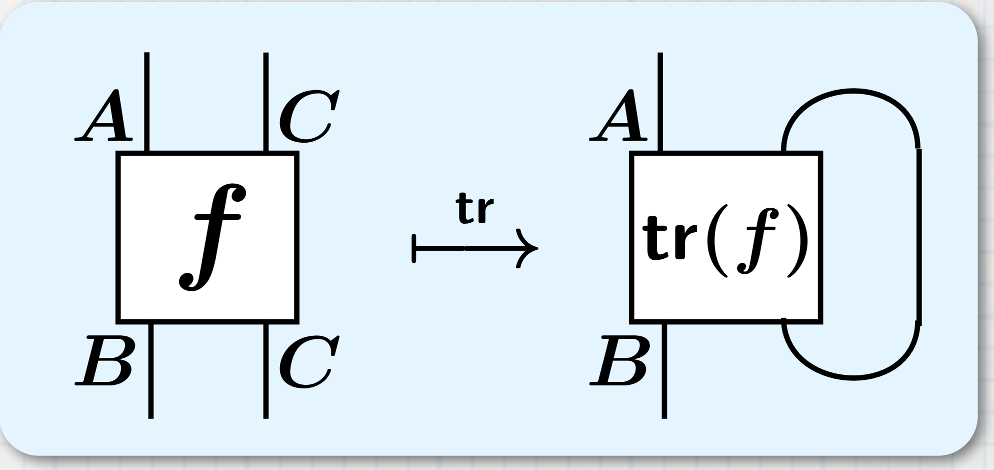
- **Traced** monoidal category
  - “feedback”

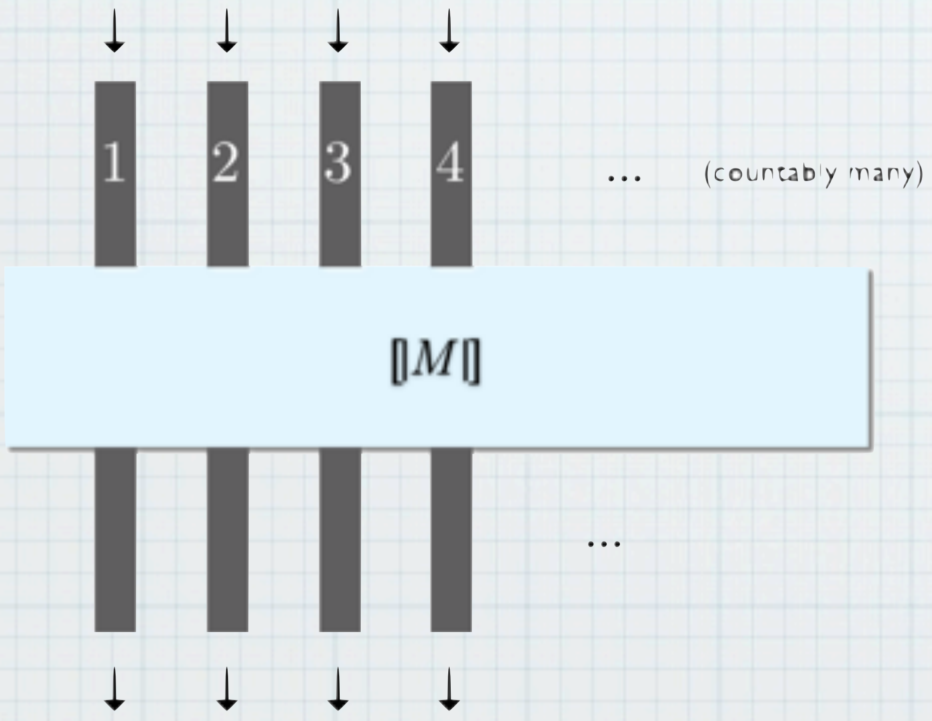
$$\frac{A \otimes C \xrightarrow{f} B \otimes C}{A \xrightarrow{\mathrm{tr}(f)} B}$$

that is

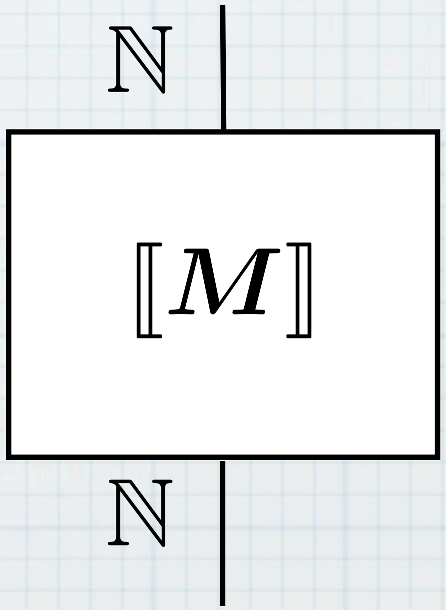


String Diagram vs. "Pipe Diagram"

- I use two ways of depicting partial functions $\mathbb{N} \rightharpoonup \mathbb{N}$



Pipe diagram



String diagram

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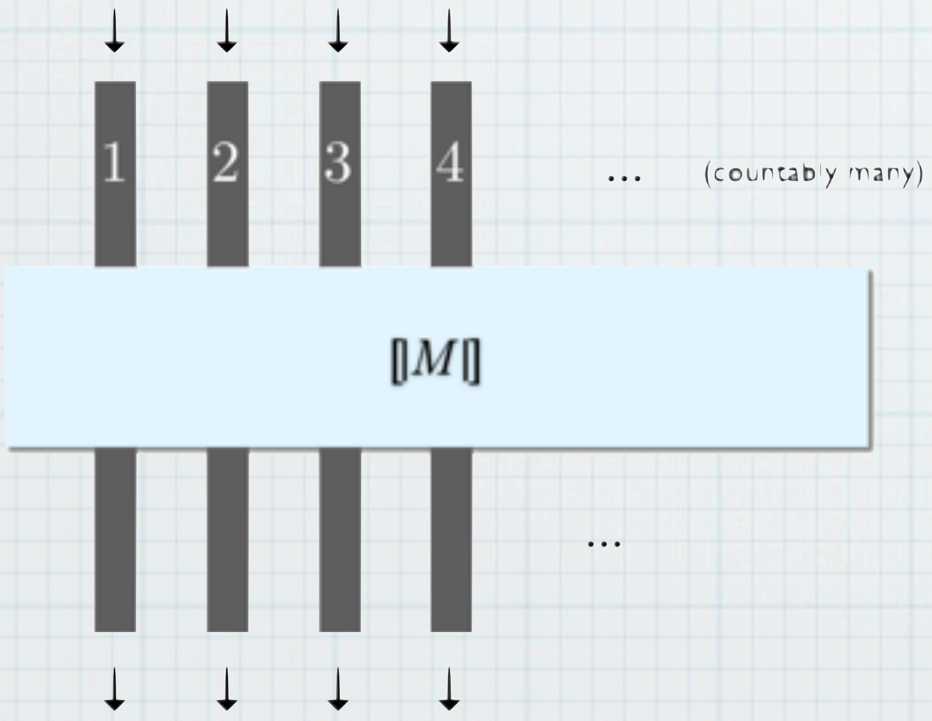
In the monoidal category $(\mathbf{Pfn}, +, 0)$



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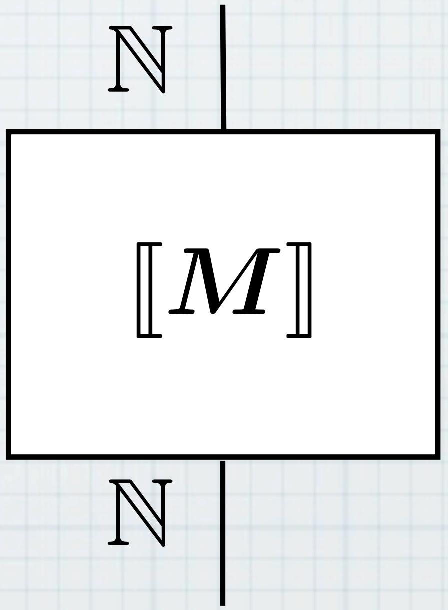


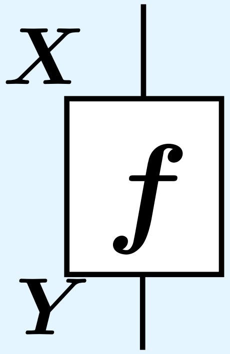
String diagram

Traced Sym. Monoidal Category

$(\mathbf{Pfn}, +, 0)$

- Category **Pfn** of partial functions
  - **Obj.** A set X
  - **Arr.** A partial function

$$\begin{array}{c} X \to Y \ \text{ in } \mathbf{Pfn} \\ \hline\hline X \rightharpoonup Y, \text{ partial function} \end{array}$$



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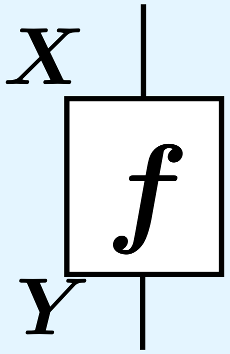

- is traced symmetric monoidal

Traced Sym. Monoidal Category

$(\mathbf{Pfn}, +, 0)$

$$\frac{X + Z \xrightarrow{f} Y + Z \qquad \text{in } \mathbf{Pfn}}{X \xrightarrow{\mathbf{tr}(f)} Y \qquad \text{in } \mathbf{Pfn}}$$

How?

Traced Sym. Monoidal Category

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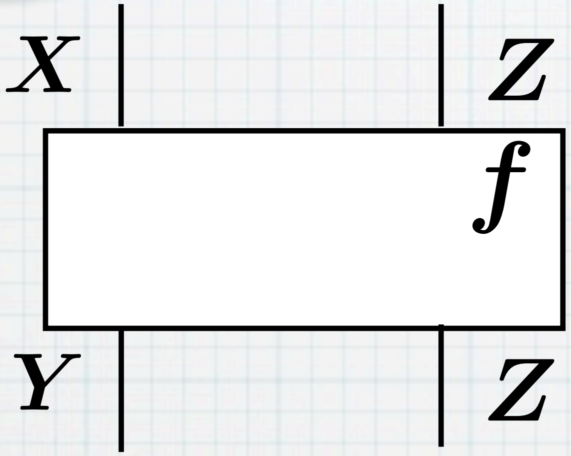
- 

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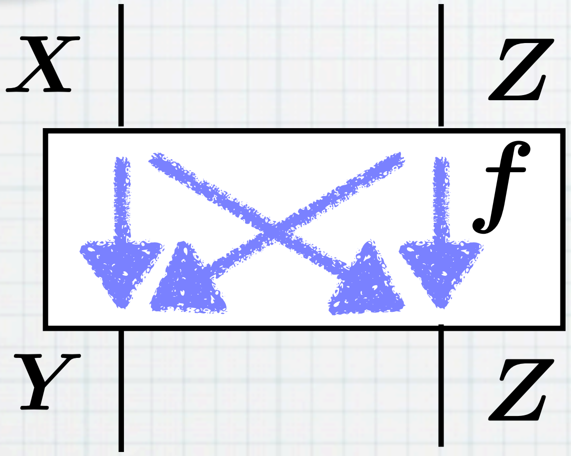
- 

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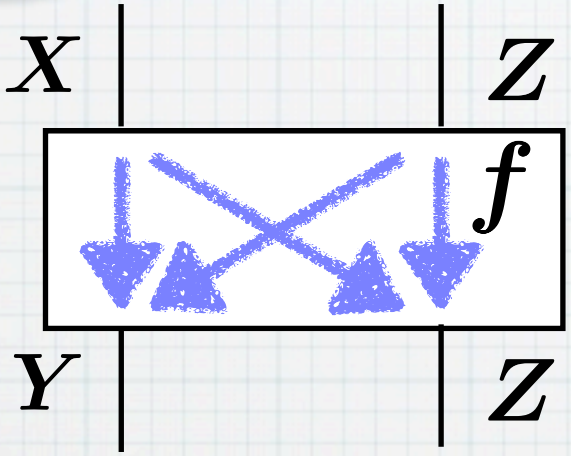
- 

$$f_{XY}(x) := \begin{cases} f(x) & \text{if } f(x) \in Y \\ \perp & \text{o.w.} \end{cases}$$

Similar for f_{XZ}, f_{ZY}, f_{ZZ}

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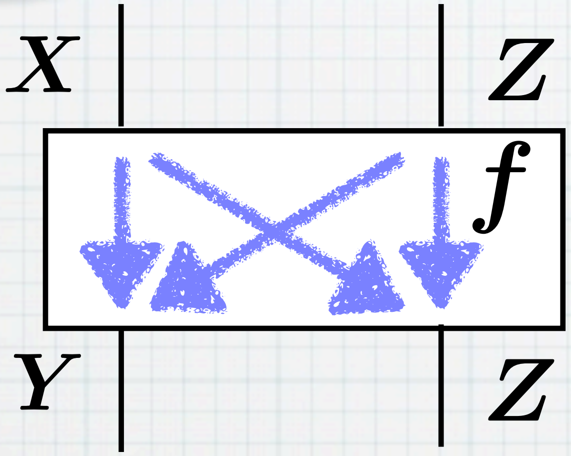
- 

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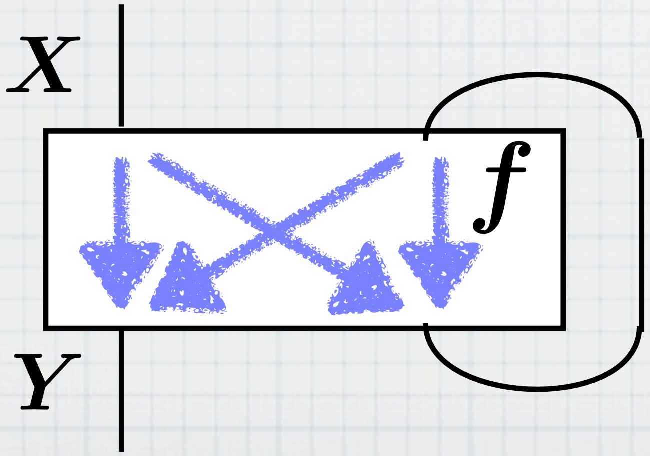
- Trace operator:



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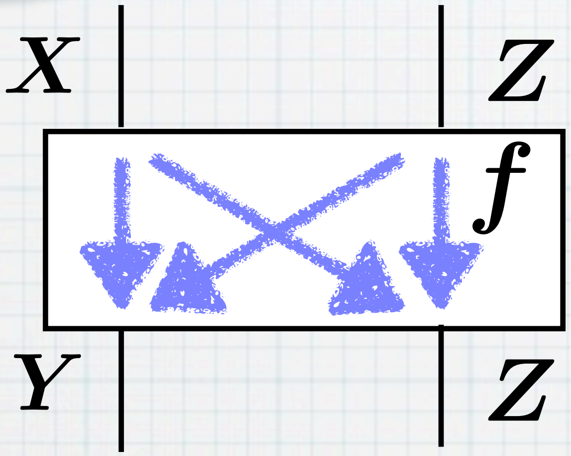
- 

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Similar for f_{XZ}, f_{ZY}, f_{ZZ}

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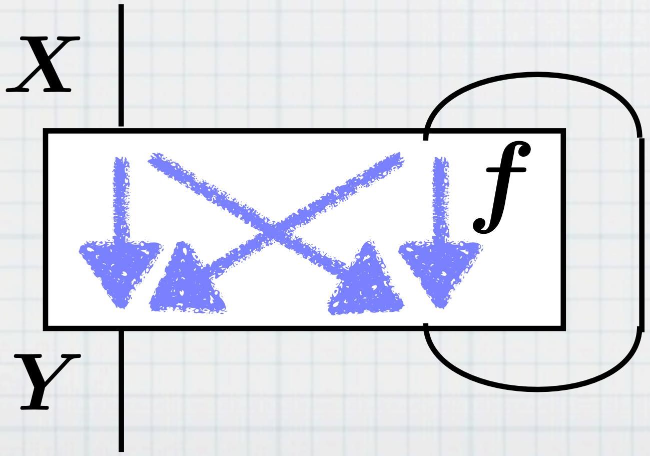

$$\mathsf{tr}(f) =$$

$$f_{XY} \sqcup \left(\coprod_{n \in \mathbb{N}} f_{ZY} \circ (f_{ZZ})^n \circ f_{XZ} \right)$$

Traced Sym. Monoidal Category

$(\mathbf{Pfn}, +, 0)$

- $$\frac{\boldsymbol{X} + \boldsymbol{Z} \xrightarrow{f} \boldsymbol{Y} + \boldsymbol{Z} \qquad \text{in } \mathbf{Pfn}}{\boldsymbol{X} \xrightarrow{\mathsf{tr}(f)} \boldsymbol{Y} \qquad \text{in } \mathbf{Pfn}}$$

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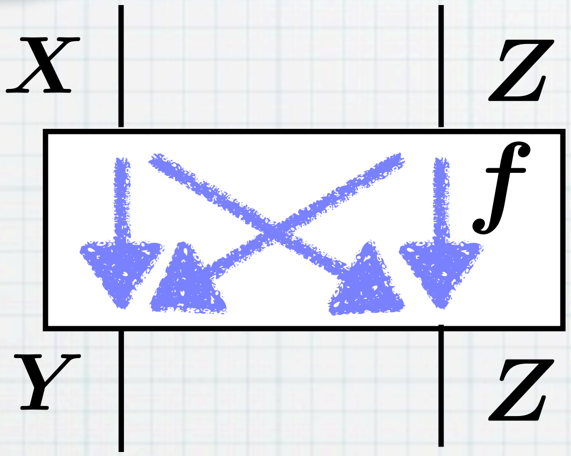
- 

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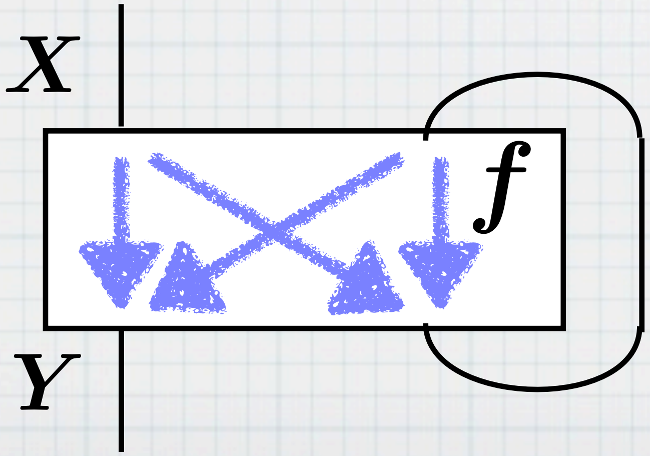
- **Execution formula** (Girard)
- Partiality is essential (infinite loop)

- Trace operator:



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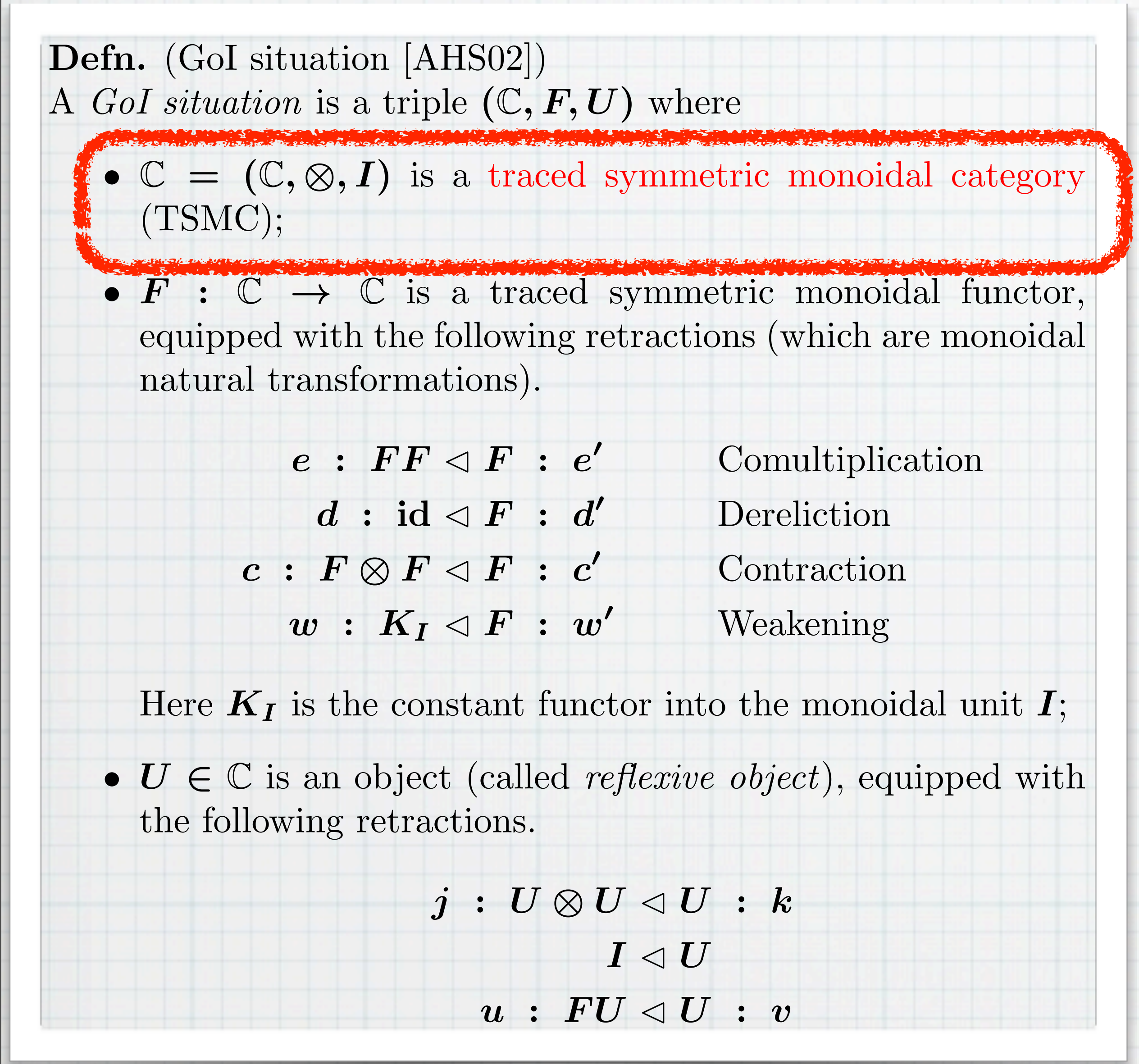
GoI situation



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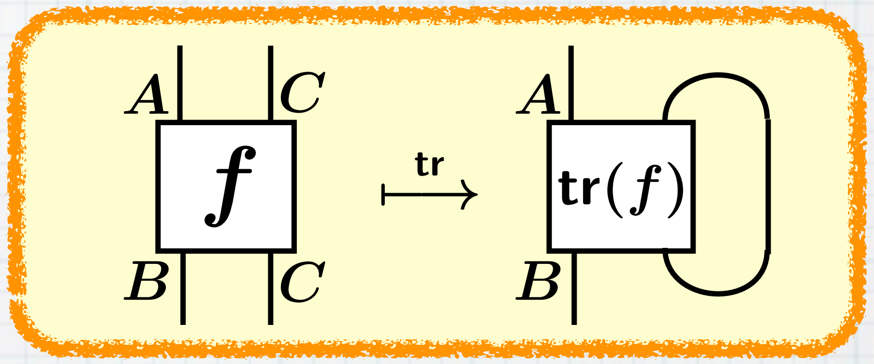
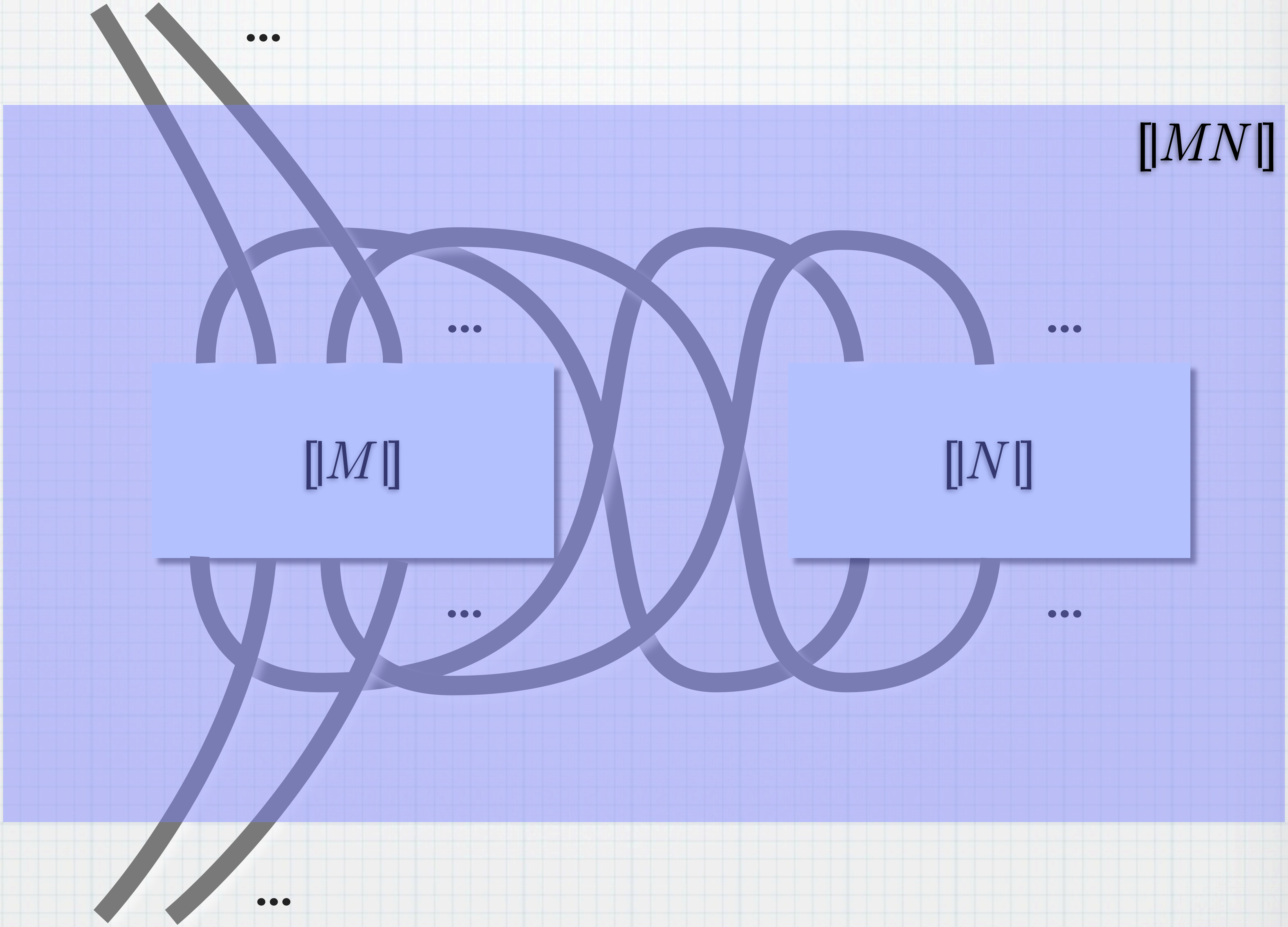
- Traced sym. monoidal cat.
  - Where one can "feedback"



  - Why for GoI?

$$[\![\boldsymbol{M N}]\!] =$$



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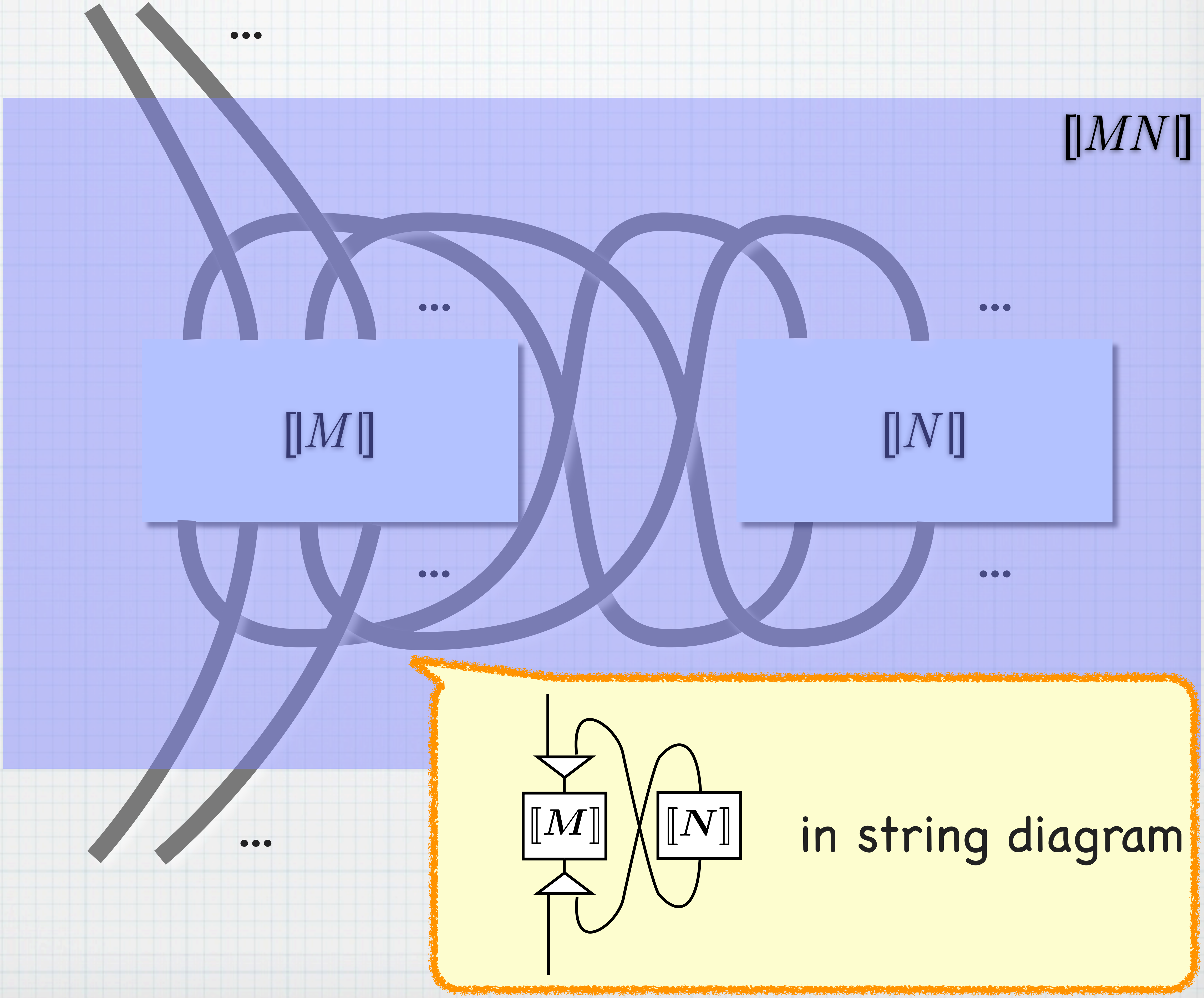
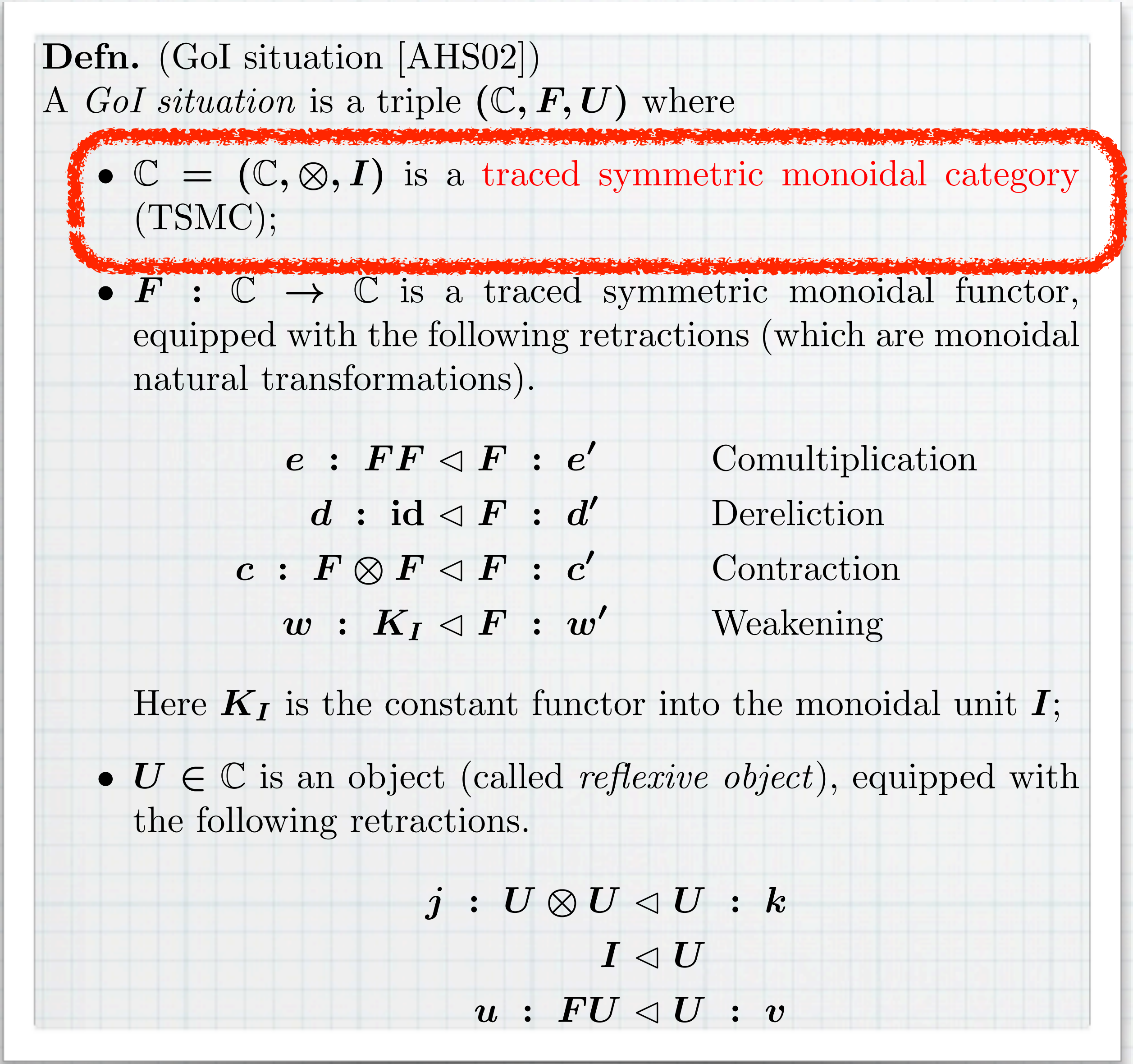


in string diagram

GoI situation



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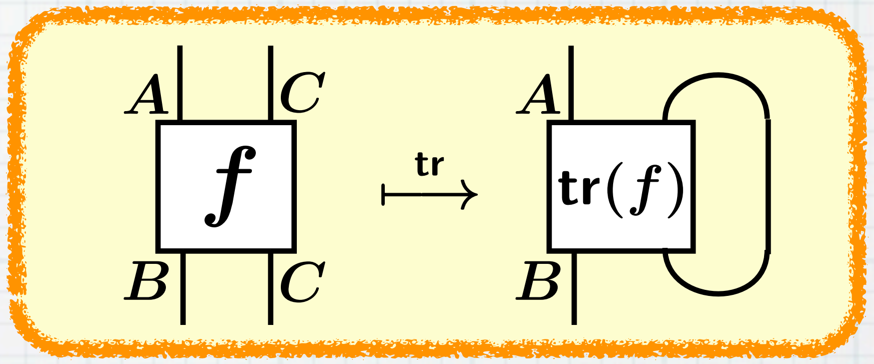

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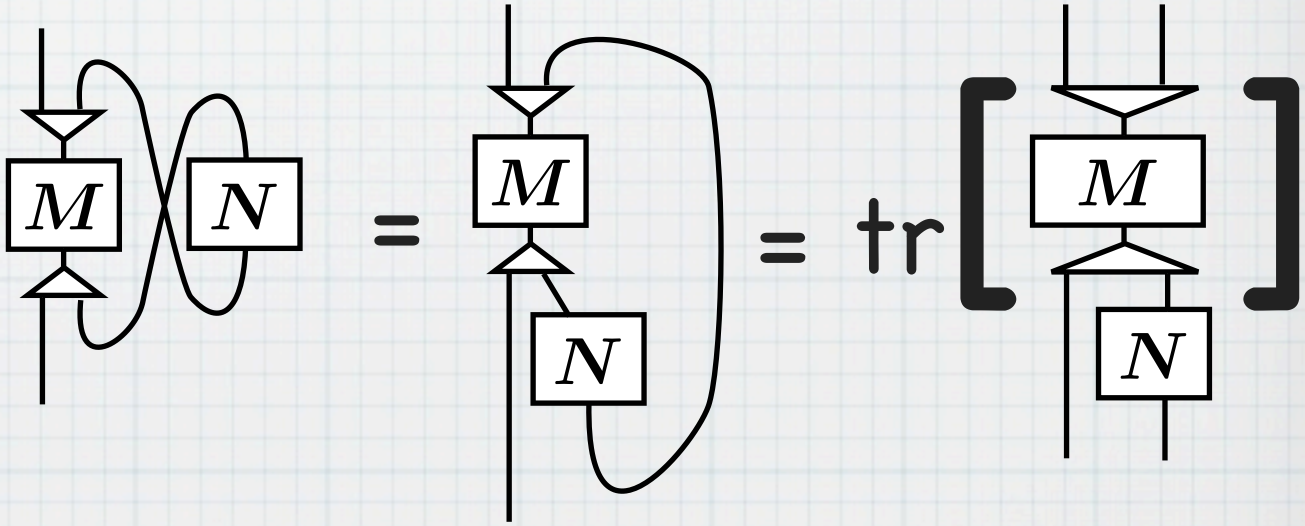

- Leading example: **Pfn**

GoI situation

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Defn. (Retraction)
A *retraction* from X to Y,

$$f : X \lhd Y : g \ ,$$

is a pair of arrows

$$\begin{tikzcd} X \arrow[loop left, "\mathrm{id}"] \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

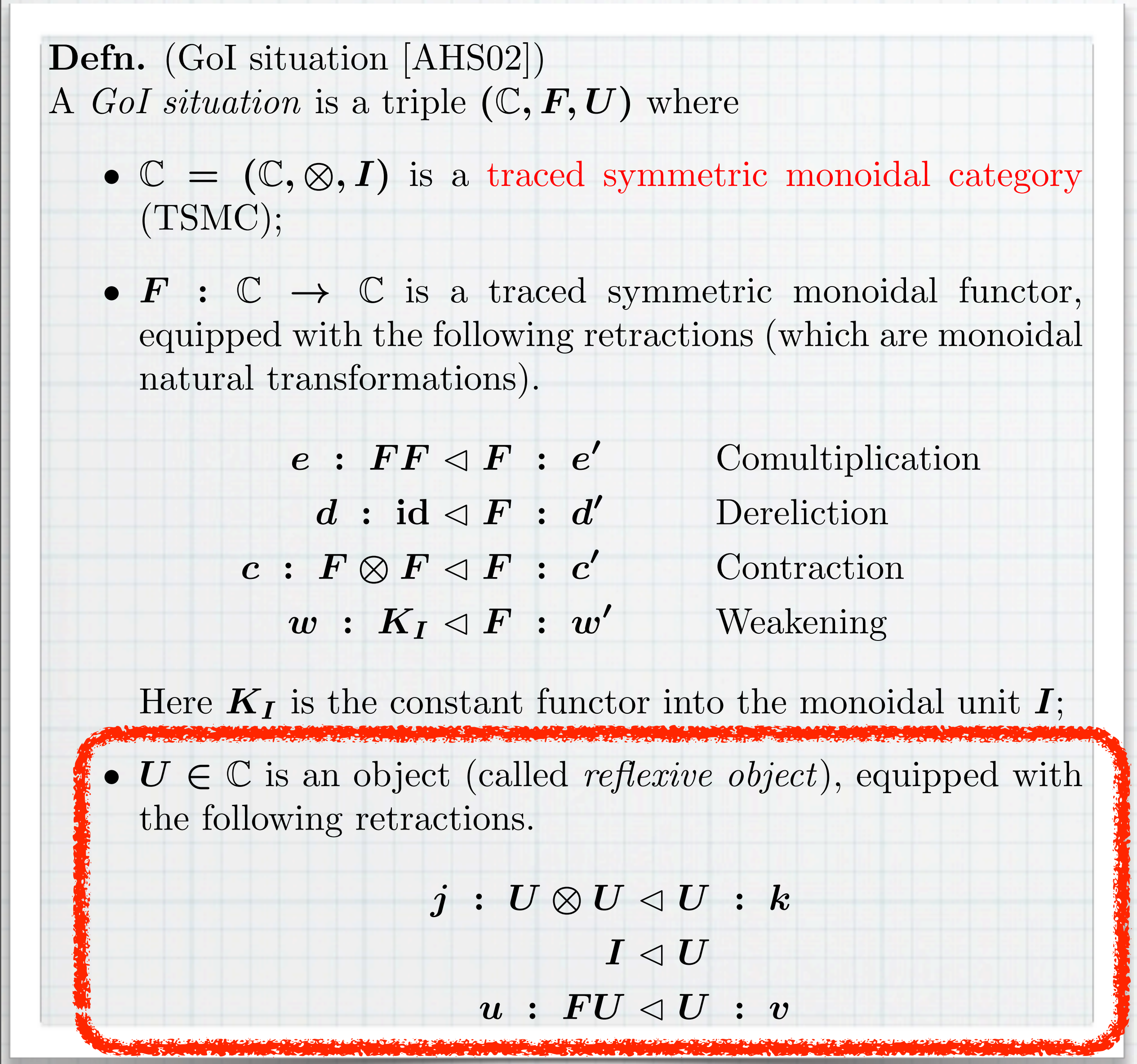
"embedding"

"projection"

such that $g \circ f = \mathrm{id}_X$.

- ✱ Functor F
  - ✱ For obtaining $! : A \to A$

GoI situation



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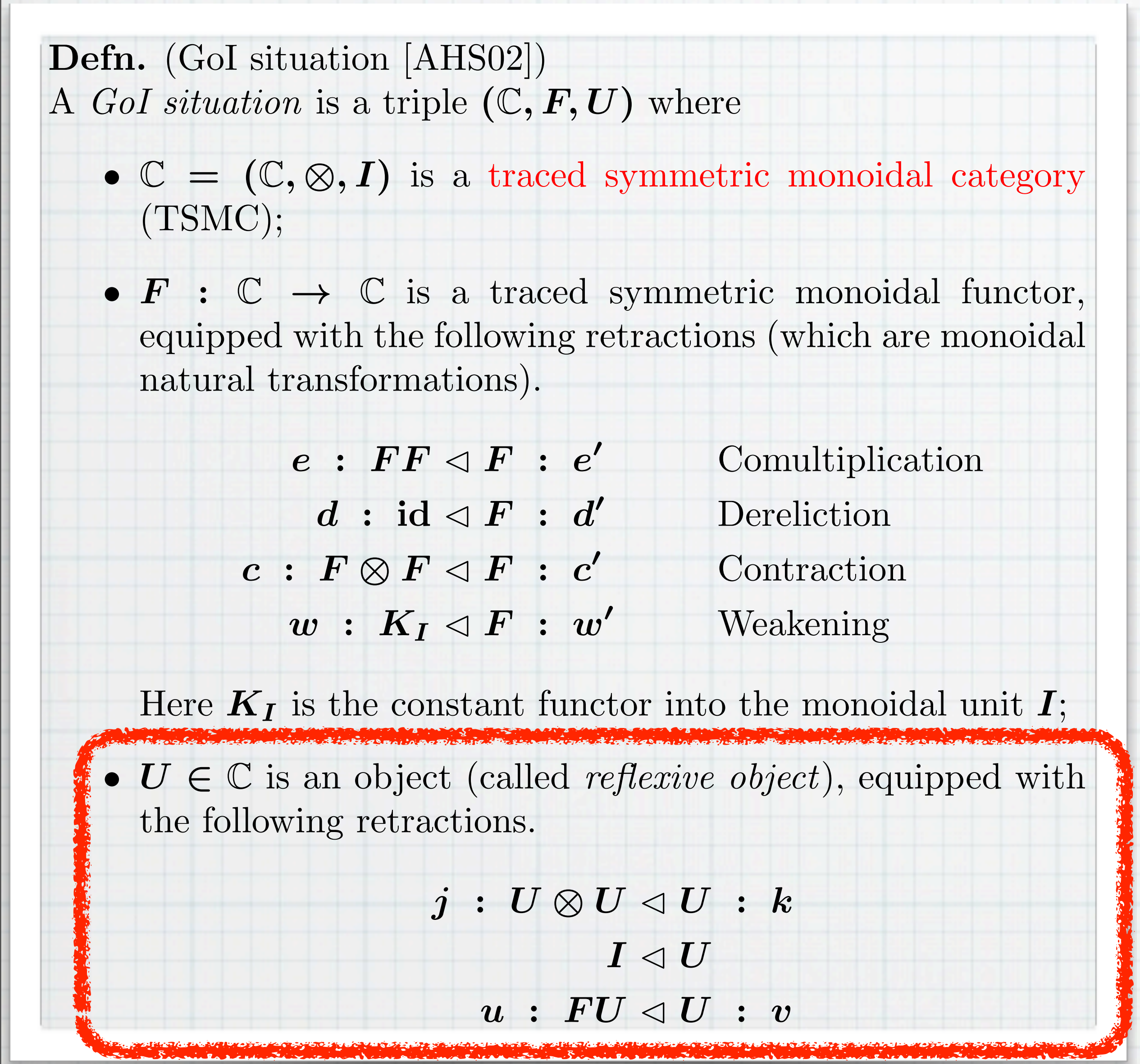
GoI situation



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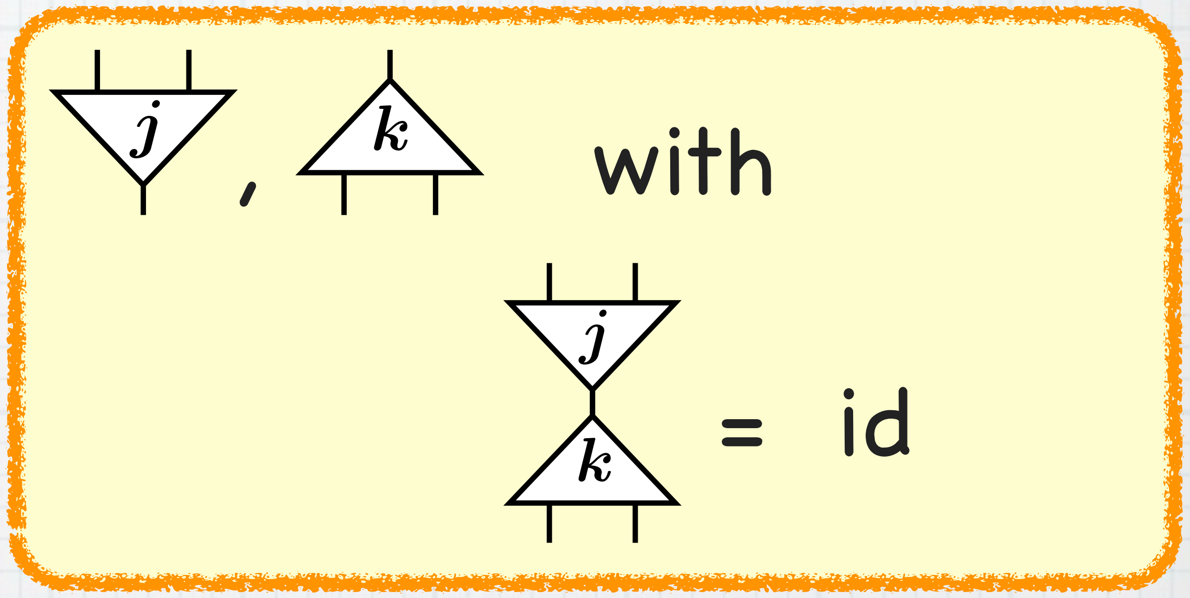

with

= id

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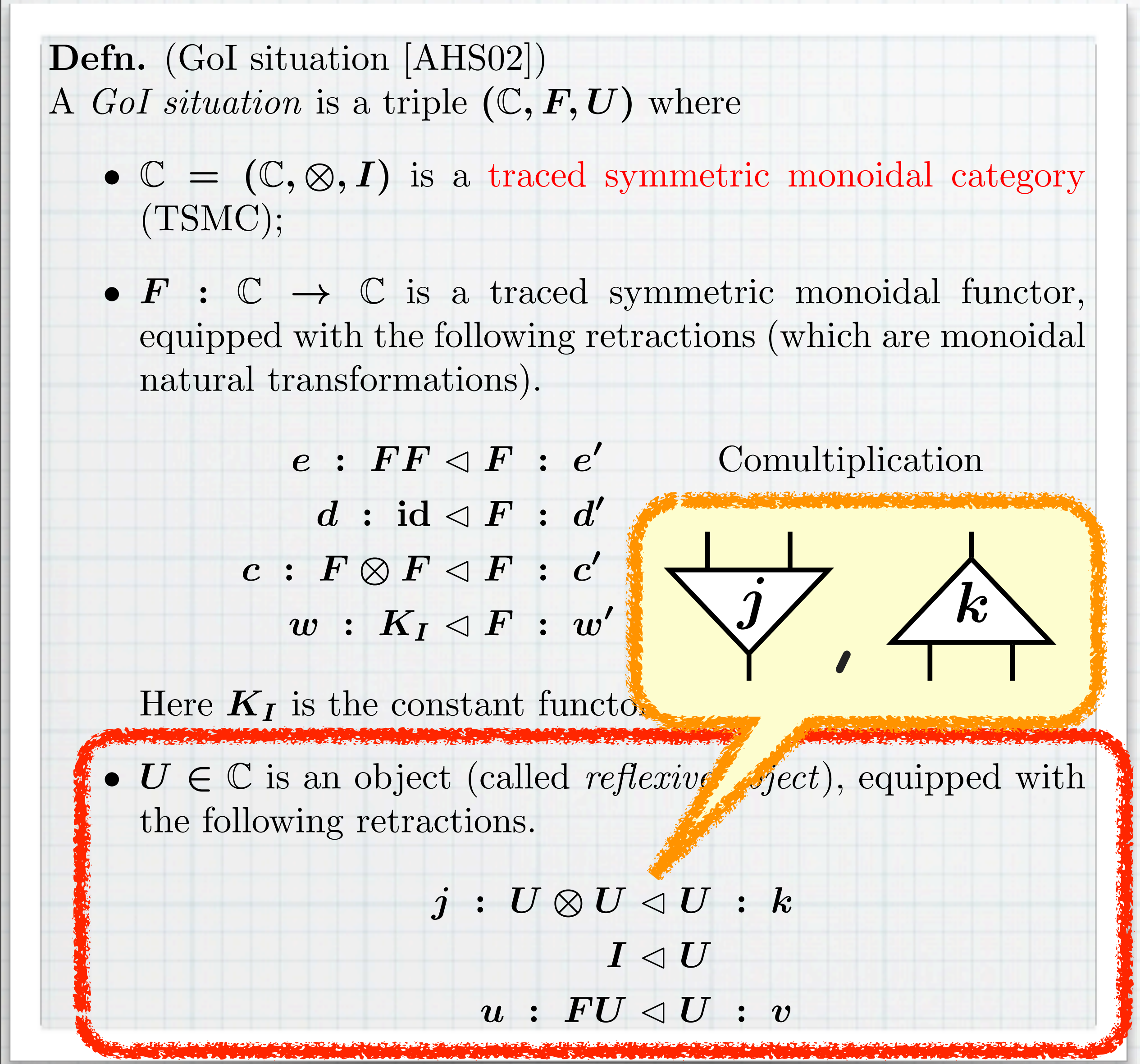

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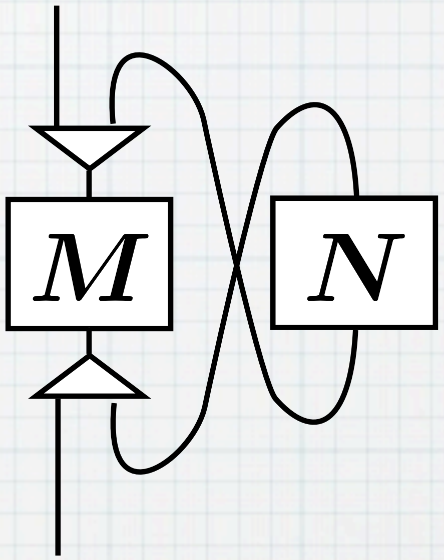
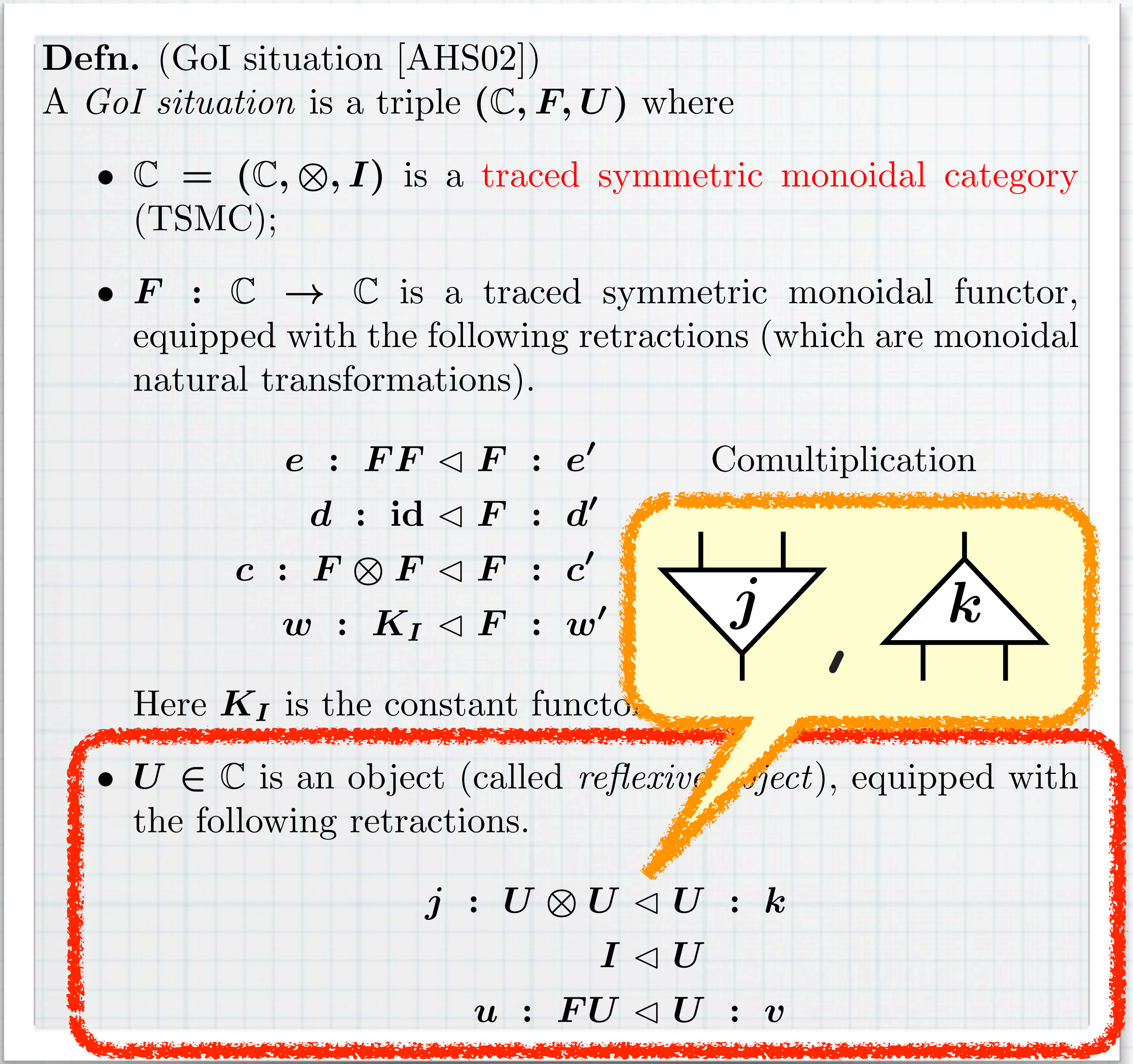


  - Example in **Pfn**:

GoI situation



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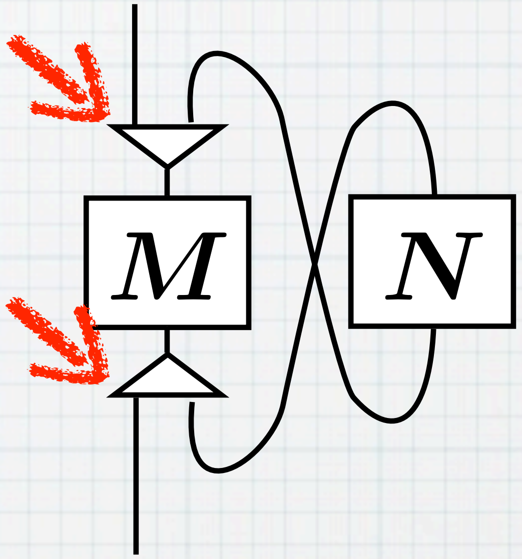

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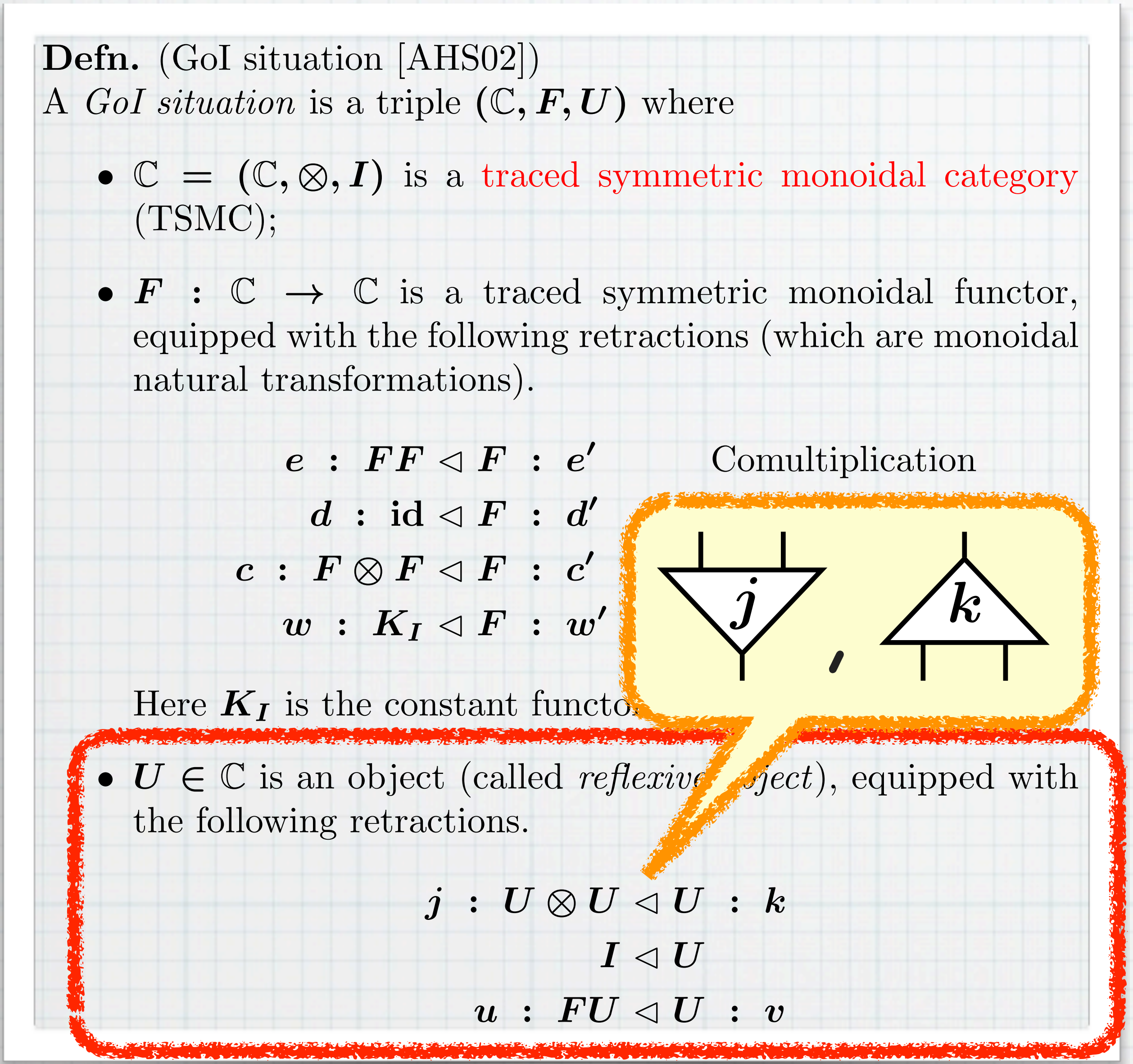

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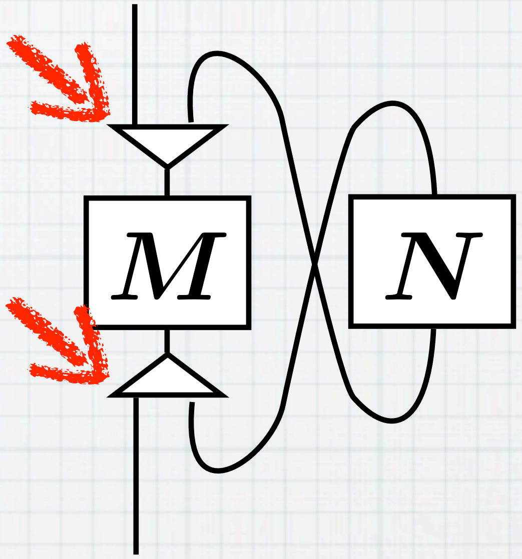

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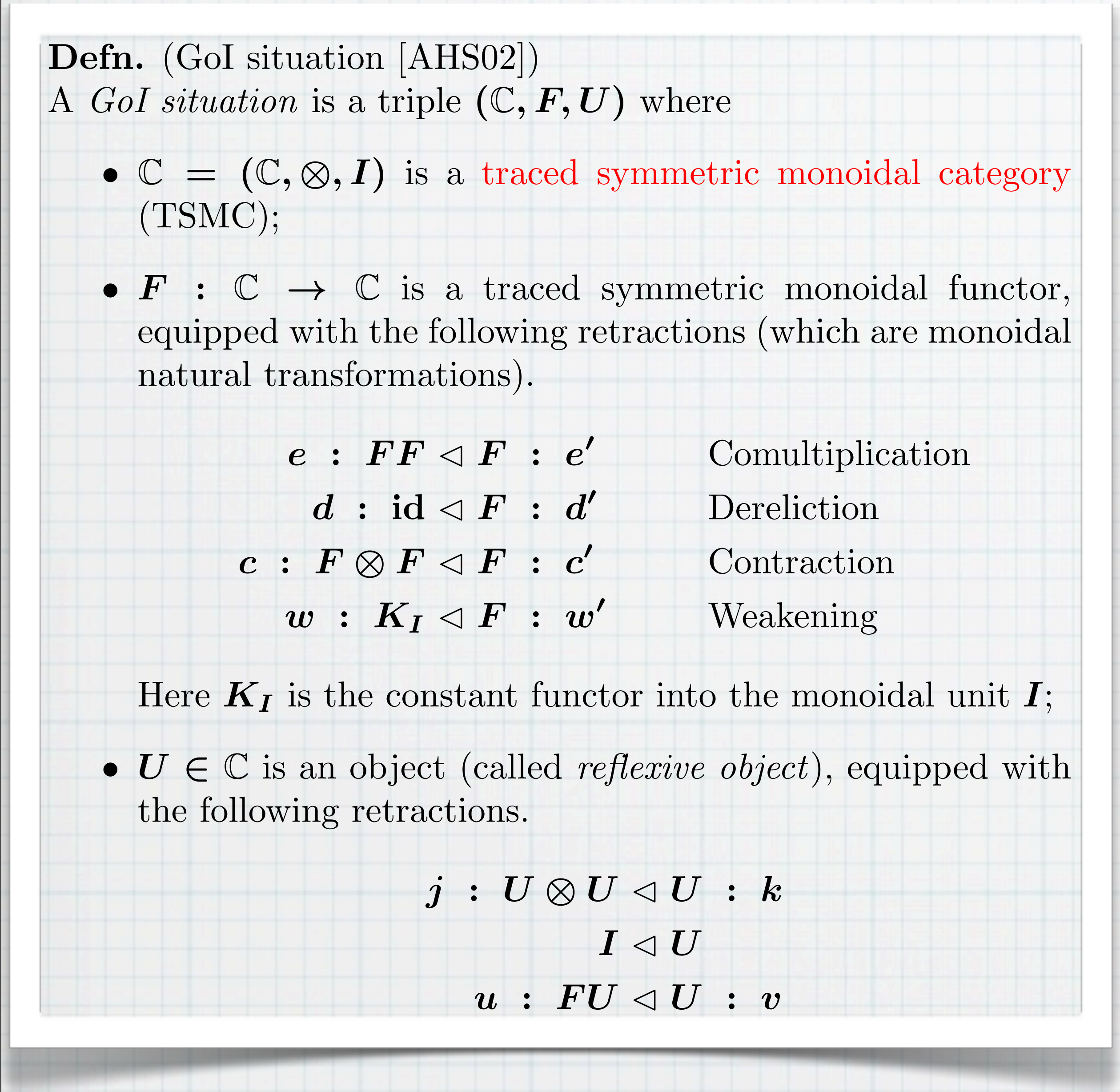
$$\mathbb{N} \in \mathbf{Pfn}, \text{ with}$$
$$\mathbb{N} + \mathbb{N} \cong \mathbb{N},$$
$$\mathbb{N} \cdot \mathbb{N} \cong \mathbb{N}$$

GoI Situation: Summary



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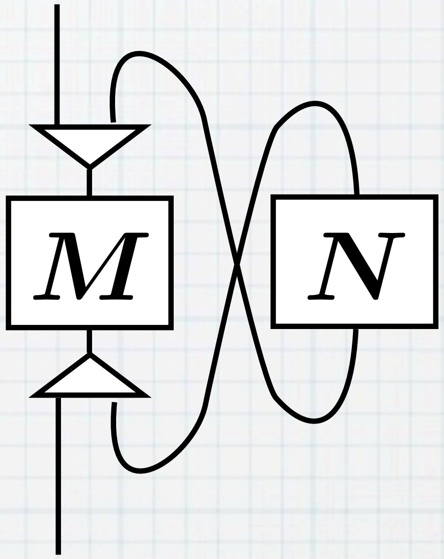
- Categorical axiomatics of the "GoI animation"



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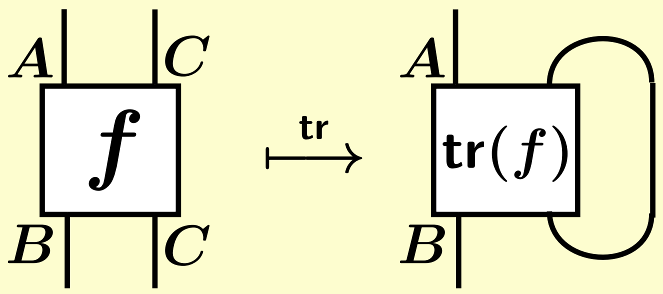

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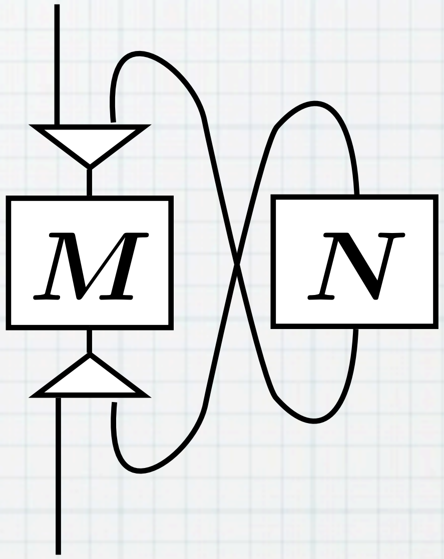

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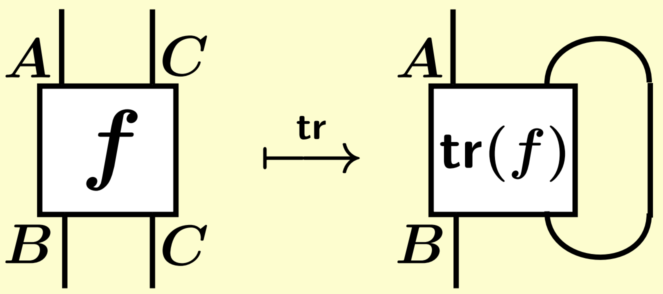

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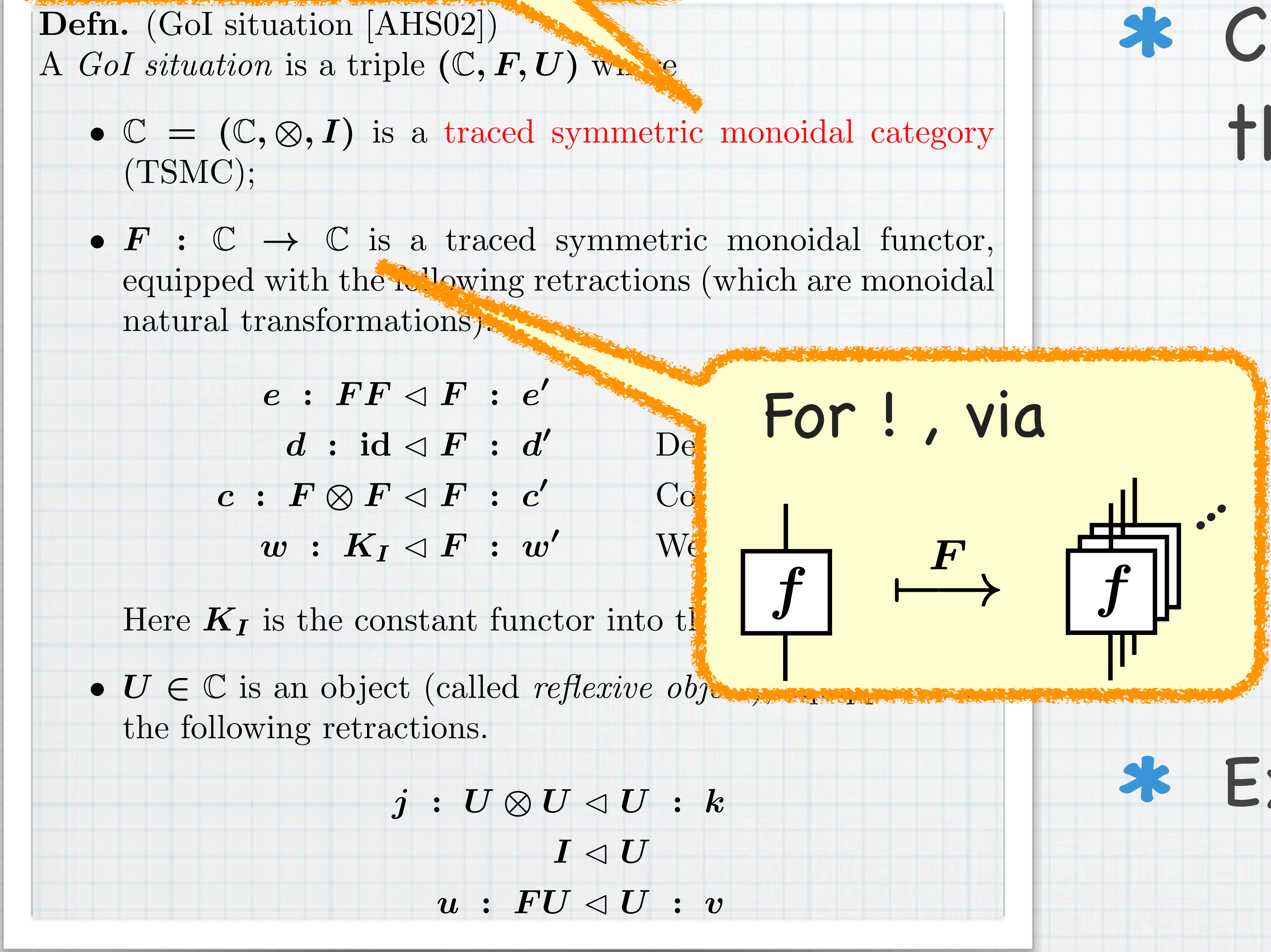

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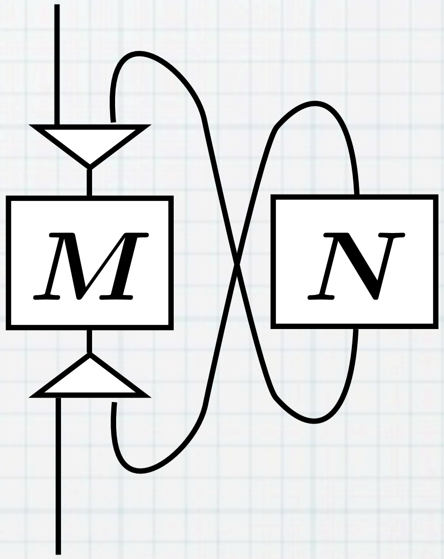

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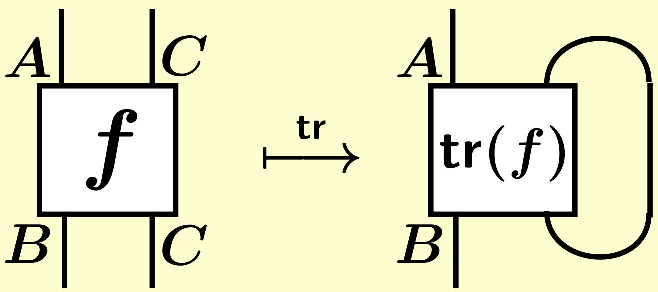

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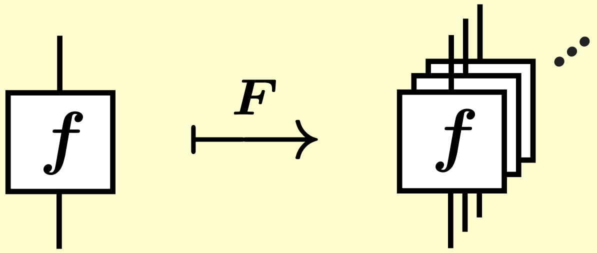

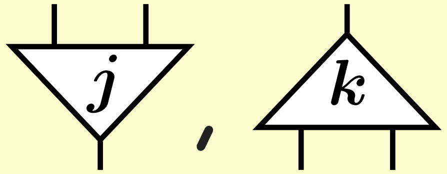

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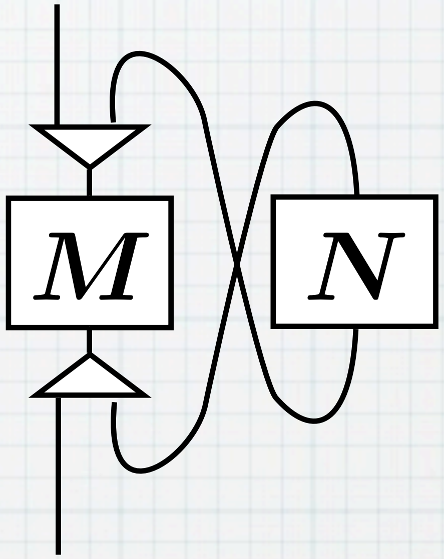

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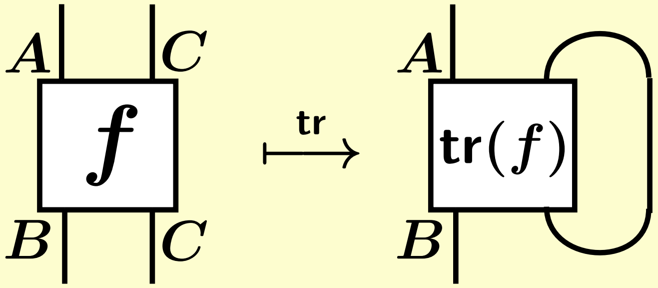

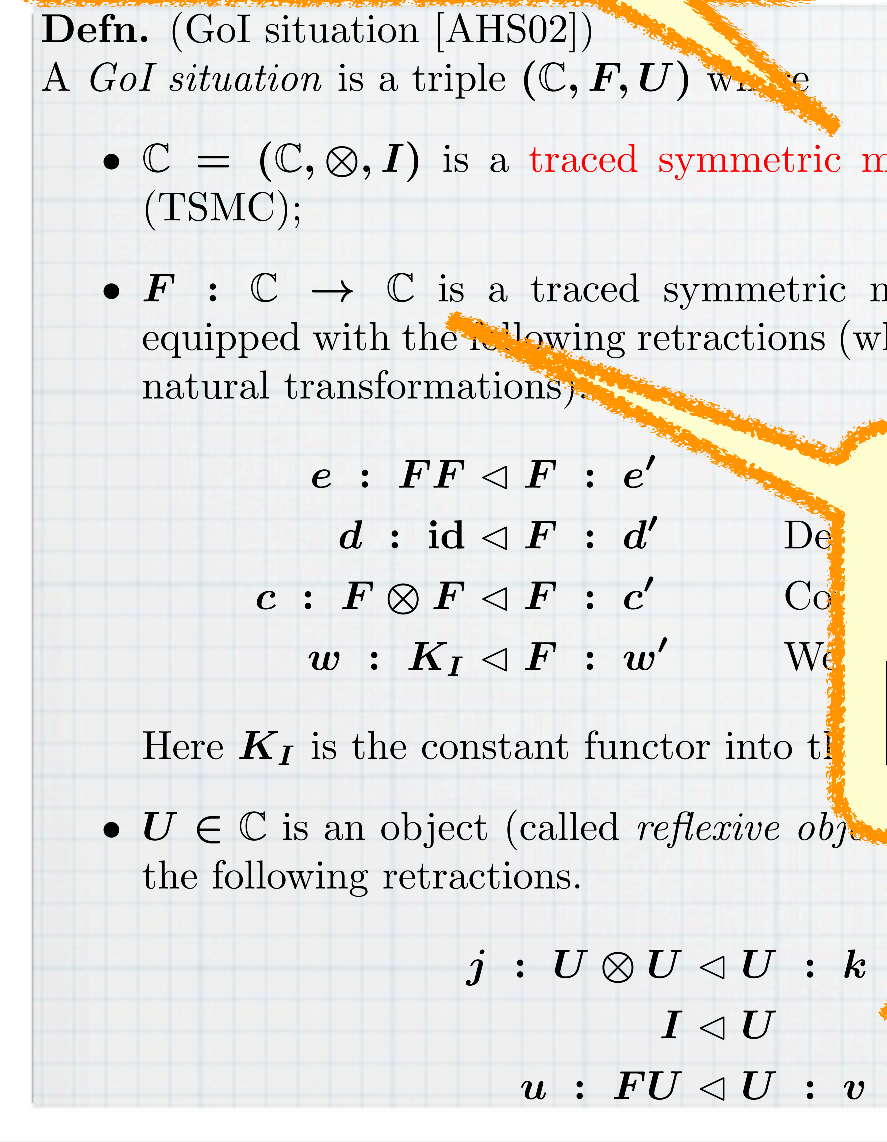

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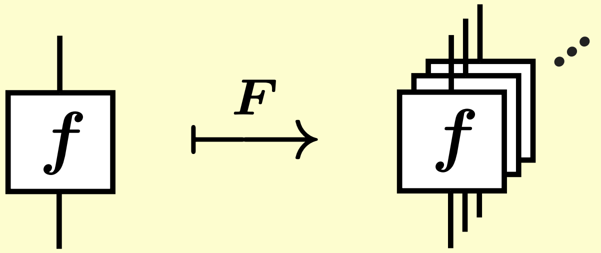
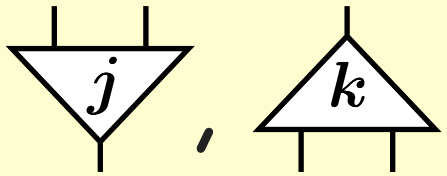
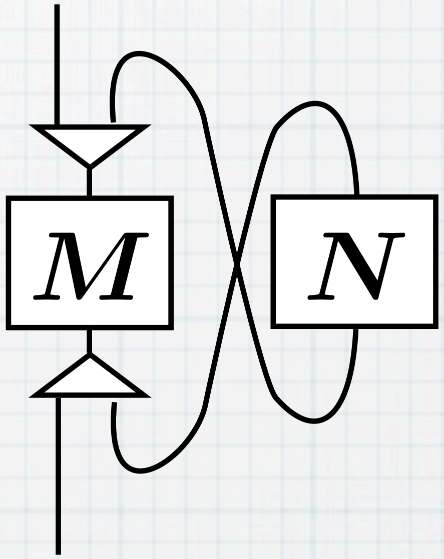


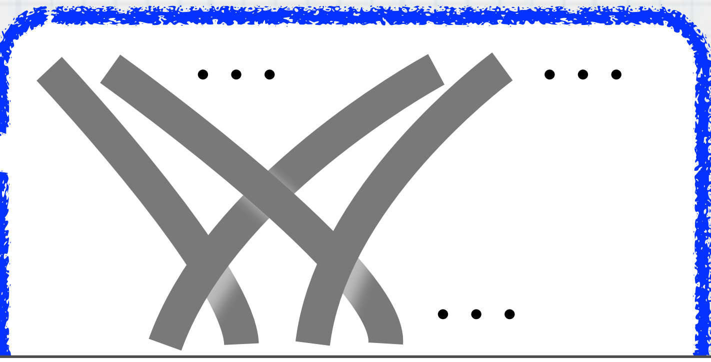


- Categorical axiomatics of the "GoI animation"



- Example:

$(\mathbf{Pfn}, \mathbb{N} \cdot \_\,, \mathbb{N})$



Categorical GoI: Constr. of an LCA

Thm. ([AHS02])
Given a GoI situation $(\mathbb{C}, \boldsymbol{F}, \boldsymbol{U})$, the homset

$$\mathbb{C}(\boldsymbol{U}, \boldsymbol{U})$$

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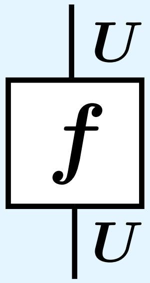

$\in \mathbb{C}(U, U)$

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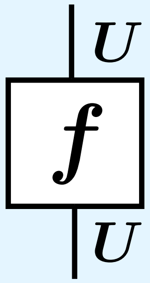
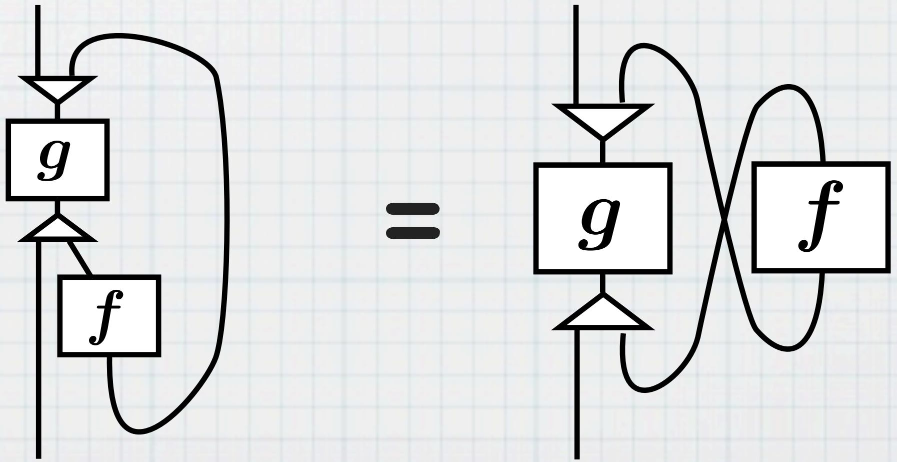


$\in \mathbb{C}(U, U)$

- $g \cdot f$

$$:= \mathbf{tr}\big((U \otimes f) \circ k \circ g \circ j \big)$$

$=$  $=$

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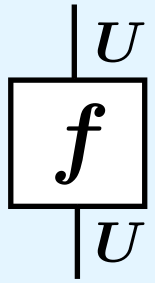
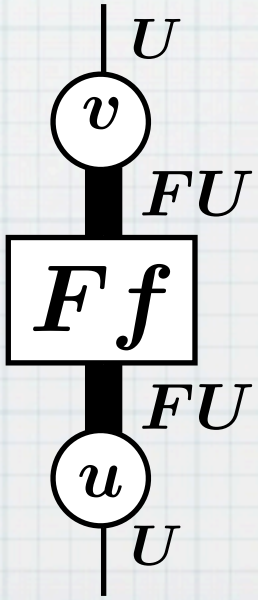


$\in \mathbb{C}(\boldsymbol{U}, \boldsymbol{U})$

- $!\boldsymbol{f} \;:=\; \boldsymbol{u} \circ \boldsymbol{F}\boldsymbol{f} \circ \boldsymbol{v}$

$=$



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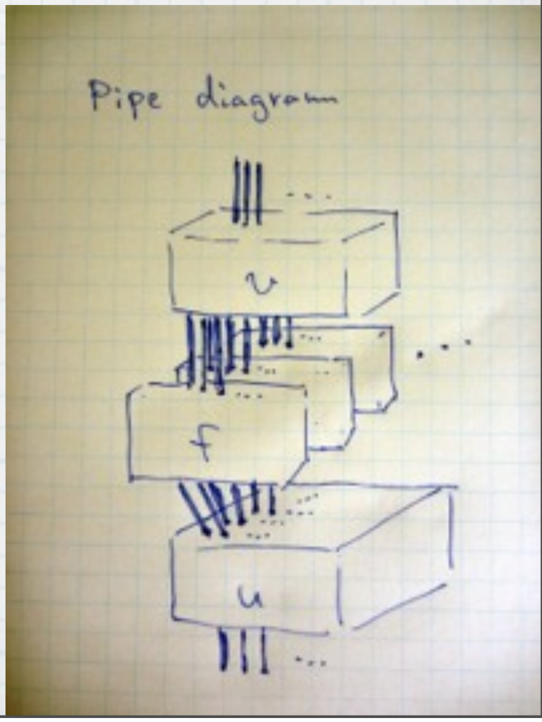

Categorical GoI: Constr. of an LCA

- Combinator $Bxyz = x(yz)$

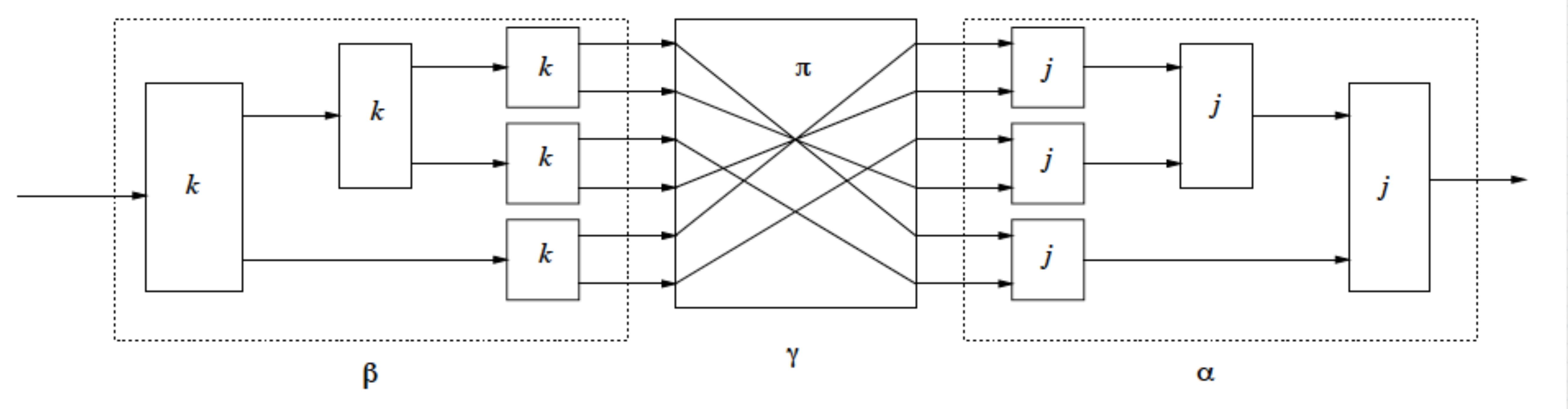

Figure 7: Composition Combinator B

from [AHS02]

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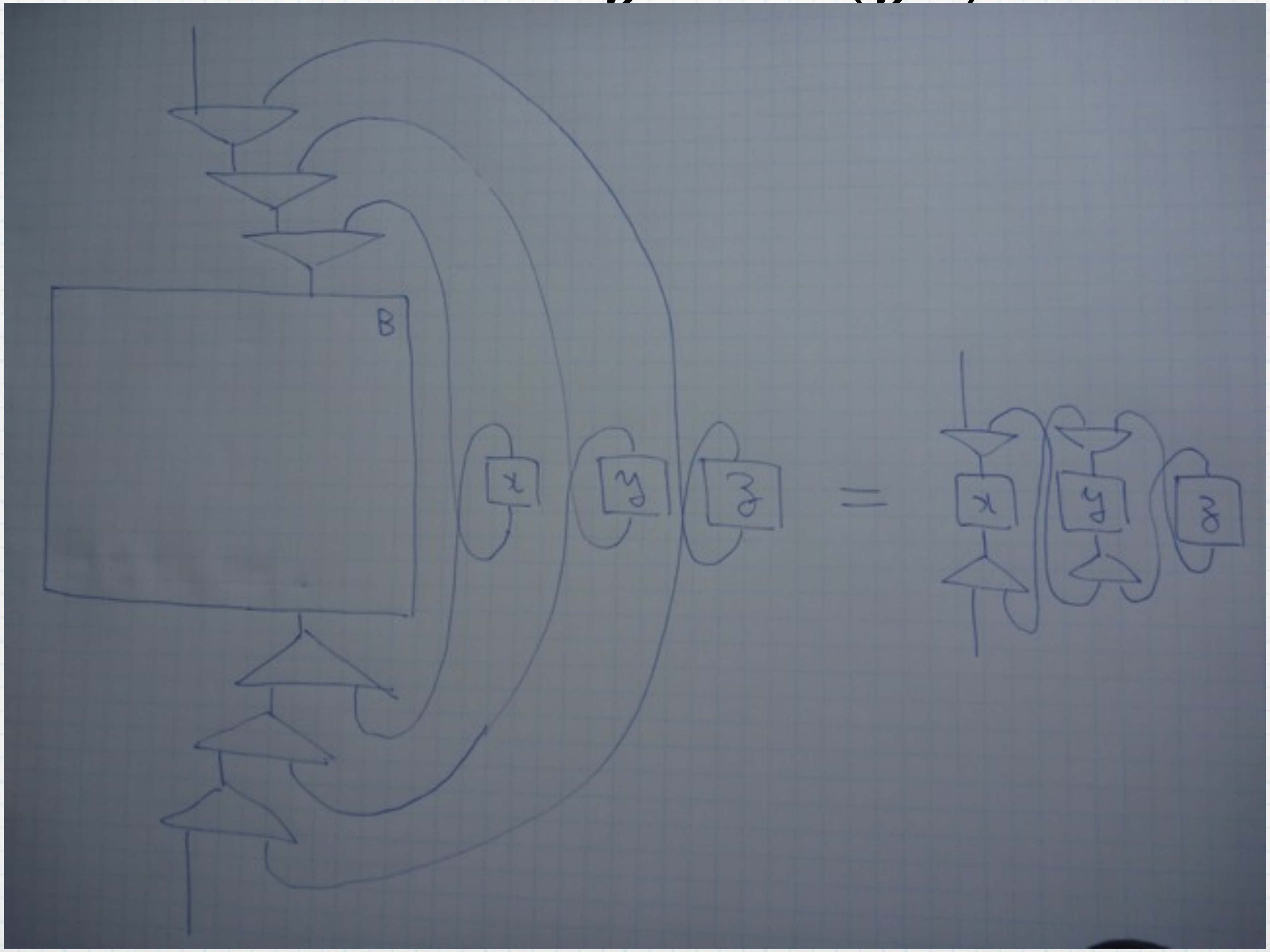
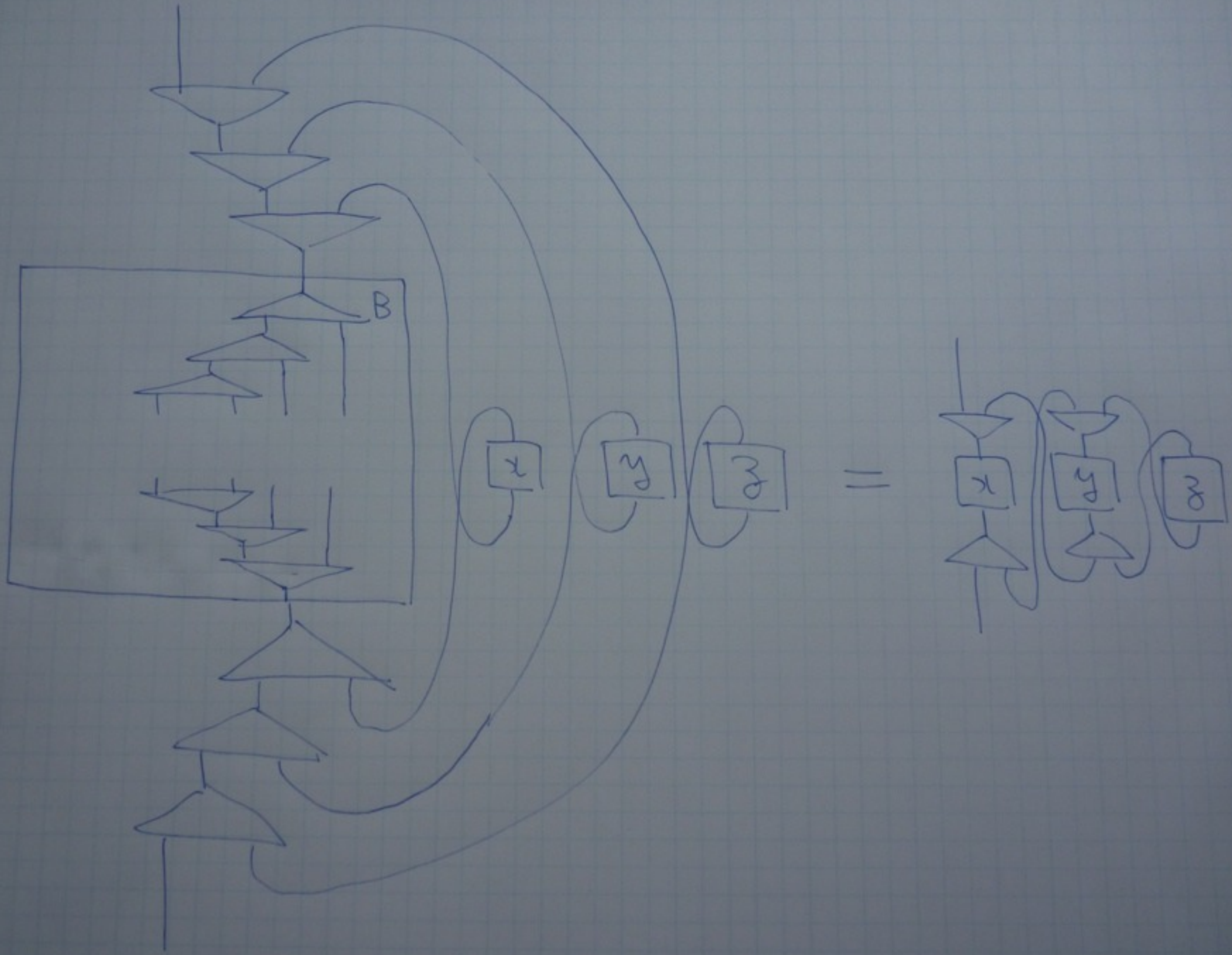
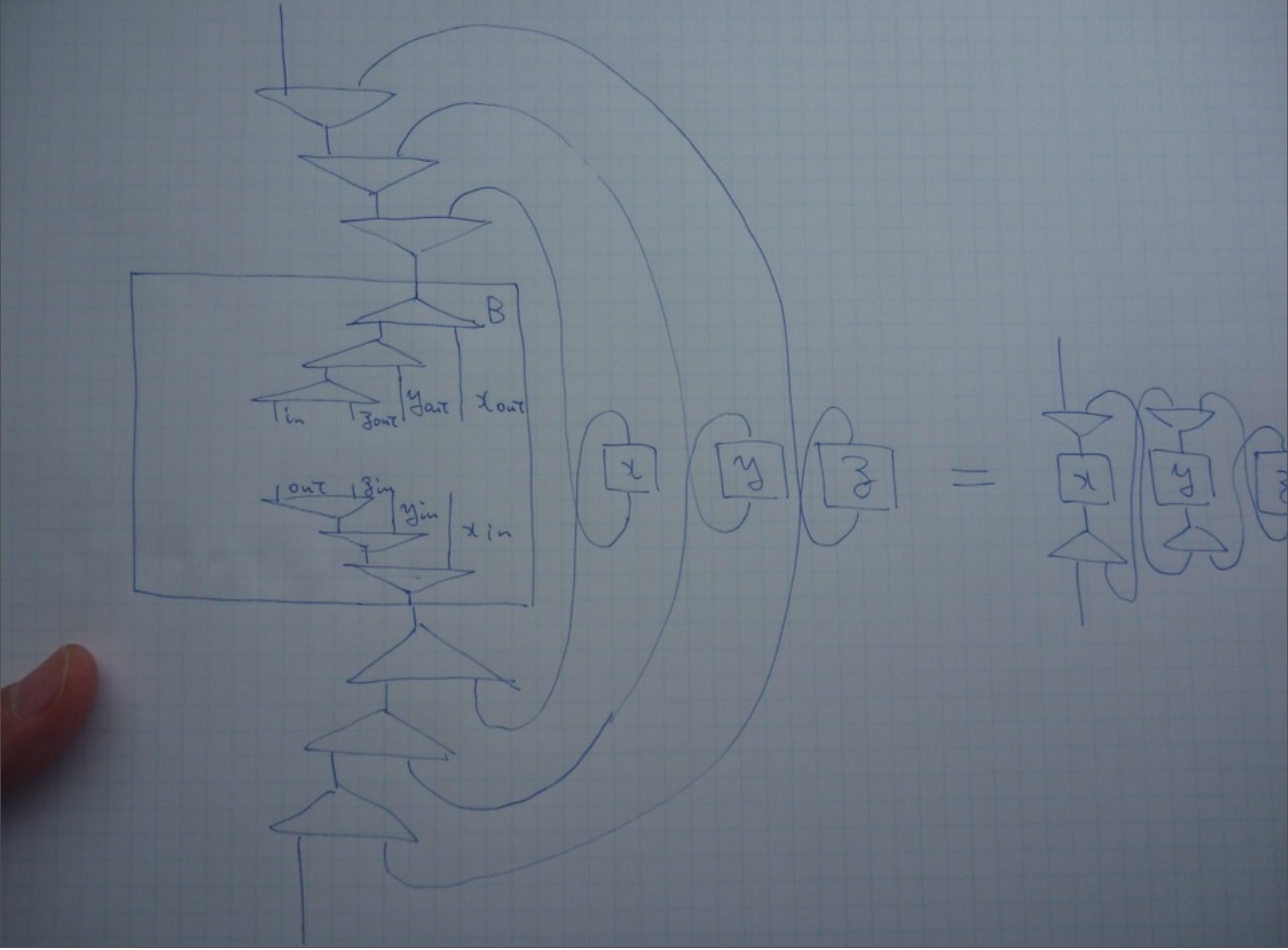
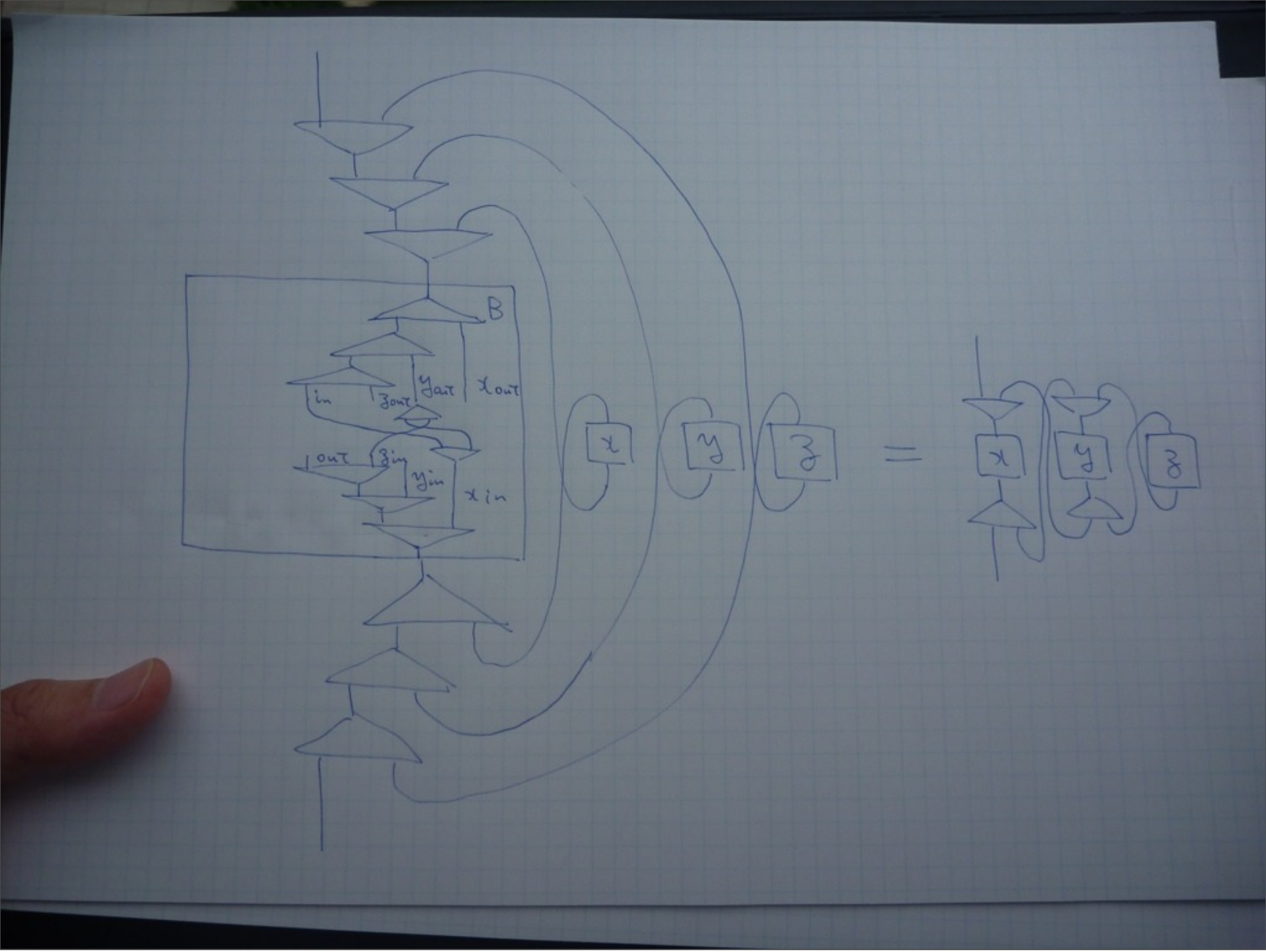
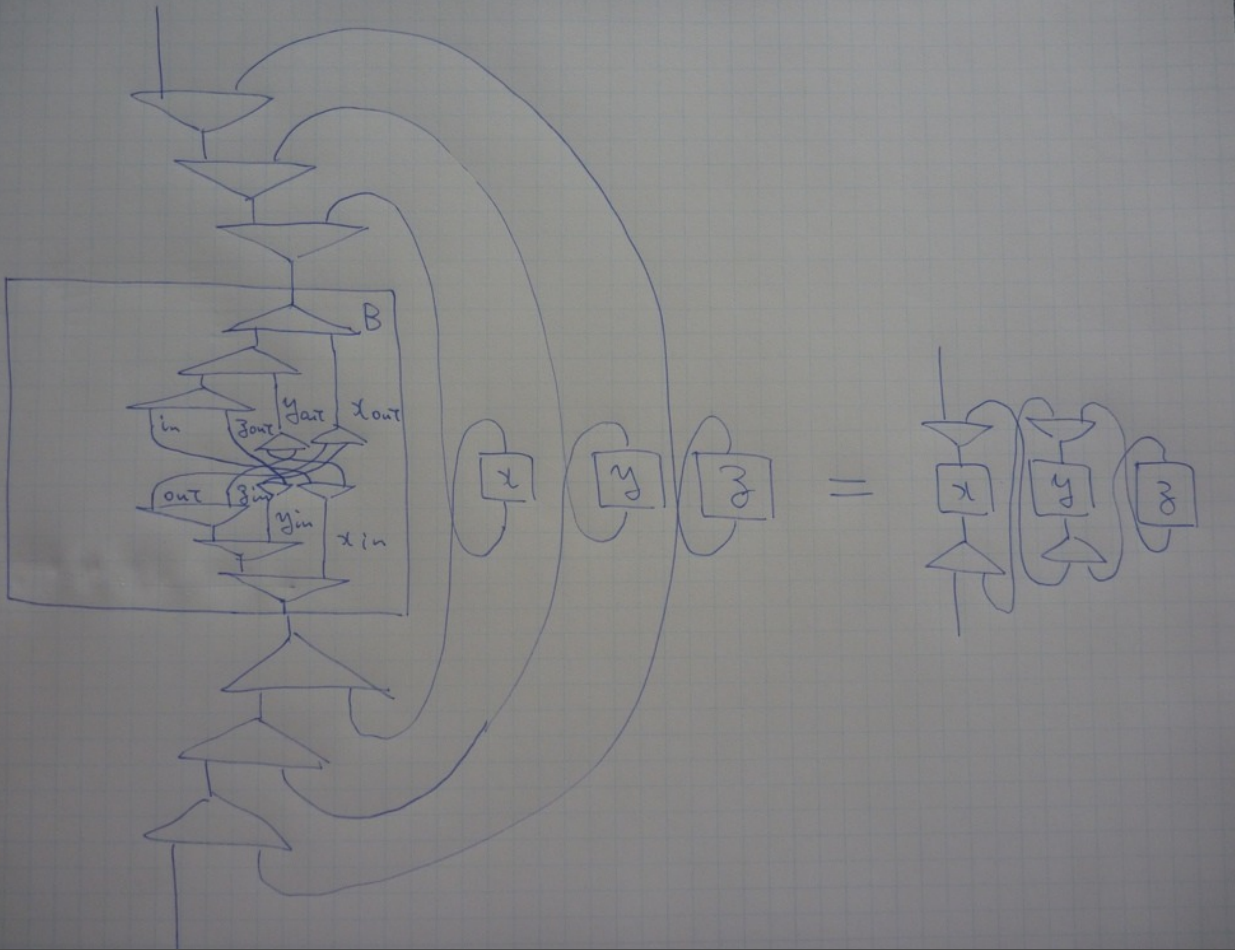



B
x
y
z
=
x
y
z


B
in
z out
y out
x out
out
z in
y in
x in
x
y
z
=
x
y
z


B
in
z out
y out
x out
out
z in
y in
x in
x
y
z
=
x
y
z


B
in
zout
yout
xout
out
zin
yin
xin
x
y
z
=
x
y
z

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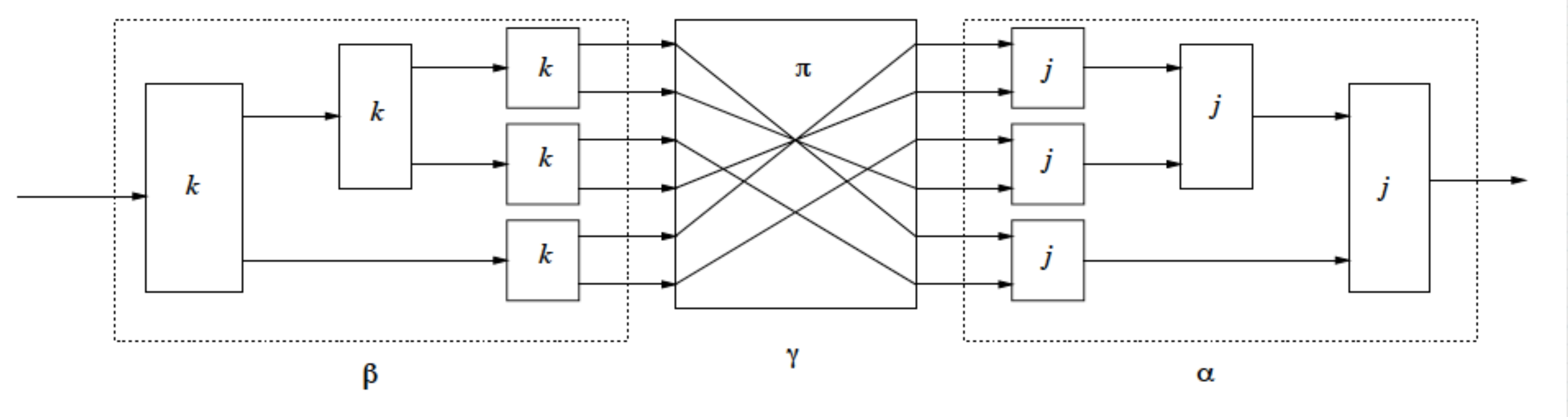

Figure 7: Composition Combinator B

from [AHS02]

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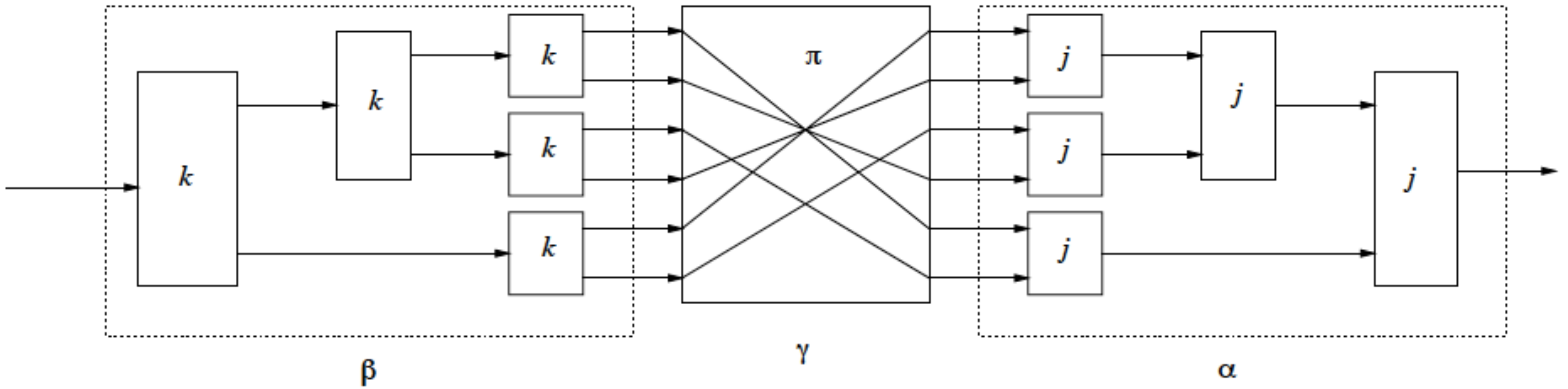

Figure 7: Composition Combinator B

from [AHS02]

Nice dynamic interpretation of (linear) computation!!

Summary: Categorical GoI

Defn. (GoI situation [AHS02])
A *GoI situation* is a triple $(\mathbb{C}, \boldsymbol{F}, \boldsymbol{U})$ where

- $\mathbb{C} = (\mathbb{C}, \otimes, \boldsymbol{I})$ is a traced symmetric monoidal category (TSMC);
- $\boldsymbol{F} : \mathbb{C} \to \mathbb{C}$ is a traced symmetric monoidal functor, equipped with the following retractions (which are monoidal natural transformations).

$$\begin{array}{rcll} \boldsymbol{e} : \boldsymbol{FF} & \lhd & \boldsymbol{F} : \boldsymbol{e'} & \text{Comultiplication} \\ \boldsymbol{d} : \mathbf{id} & \lhd & \boldsymbol{F} : \boldsymbol{d'} & \text{Dereliction} \\ \boldsymbol{c} : \boldsymbol{F} \otimes \boldsymbol{F} & \lhd & \boldsymbol{F} : \boldsymbol{c'} & \text{Contraction} \\ \boldsymbol{w} : \boldsymbol{K_I} & \lhd & \boldsymbol{F} : \boldsymbol{w'} & \text{Weakening} \end{array}$$

  Here $\boldsymbol{K_I}$ is the constant functor into the monoidal unit $\boldsymbol{I}$;
- $\boldsymbol{U} \in \mathbb{C}$ is an object (called *reflexive object*), equipped with the following retractions.

$$\begin{array}{c} \boldsymbol{j} : \boldsymbol{U} \otimes \boldsymbol{U} \lhd \boldsymbol{U} : \boldsymbol{k} \\ \boldsymbol{I} \lhd \boldsymbol{U} \\ \boldsymbol{u} : \boldsymbol{FU} \lhd \boldsymbol{U} : \boldsymbol{v} \end{array}$$

Thm. ([AHS02])
Given a GoI situation $(\mathbb{C}, \boldsymbol{F}, \boldsymbol{U})$, the homset

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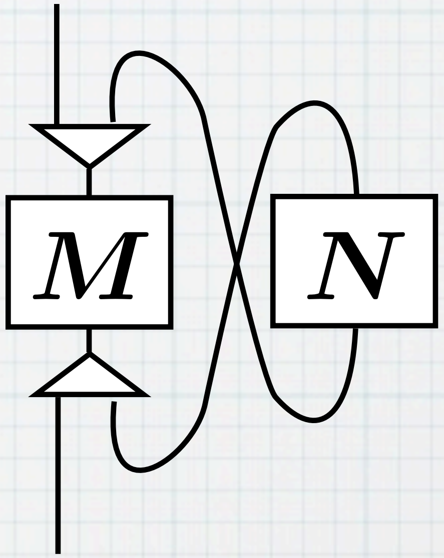
Why Categorical Generalization?: Examples Other Than Pfn [AHS02]

- Strategy: find a TSMC!

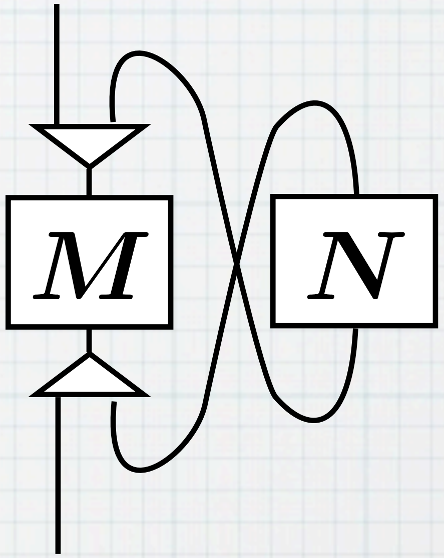


- "Wave-style" examples
  - ⊗ is Cartesian product(-like)
    - in which case,

      trace ≈ fixed point operator [Hasegawa/Hyland]
  - An example: $\big(\,(\omega\text{-}\mathbf{Cpo}, \times, \mathbf{1}),\ (\_)^{\mathbb{N}},\ A^{\mathbb{N}}\,\big)$
  - (... less of a dynamic flavor)

Why Categorical Generalization?: Examples Other Than $\mathbf{Pfn}$ [AHS02]

- "Particle-style" examples
  - Obj. X∈C is set-like; ⊗ is coproduct-like
    - The GoI animation is valid



  - Examples:
    - Partial functions $\big(\,(\mathbf{Pfn},+,\mathbf{0}),\ \mathbb{N}\cdot\_\,,\ \mathbb{N}\,\big)$
    - Binary relations $\big(\,(\mathbf{Rel},+,\mathbf{0}),\ \mathbb{N}\cdot\_\,,\ \mathbb{N}\,\big)$
    - "Discrete stochastic relations" $\big(\,(\mathbf{DSRel},+,\mathbf{0}),\ \mathbb{N}\cdot\_\,,\ \mathbb{N}\,\big)$

Why Categorical Generalization?: Examples Other Than Pfn [AHS02]

- **Pfn** (partial functions)

$$\begin{array}{c} X \to Y \text{ in } \mathbf{Pfn} \\ \hline\hline X \rightharpoonup Y, \text{ partial function} \\ \hline\hline X \to \mathcal{L}Y \text{ in } \mathbf{Sets} \end{array} \quad \text{where } \mathcal{L}Y = \{\bot\} + Y$$

- **Rel** (relations)

$$\begin{array}{c} X \to Y \text{ in } \mathbf{Rel} \\ \hline\hline R \subseteq X \times Y, \text{ relation} \\ \hline\hline X \to \mathcal{P}Y \text{ in } \mathbf{Sets} \end{array} \quad \text{where } \mathcal{P} \text{ is the powerset monad}$$

- **DSRel**

$$\begin{array}{c} X \to Y \text{ in } \mathbf{DSRel} \\ \hline\hline X \to \mathcal{D}Y \text{ in } \mathbf{Sets} \end{array}$$

$$\text{where } \mathcal{D}Y = \{d : Y \to [0,1] \mid \sum_{y} d(y) \leq 1\}$$

Why Catego
Examples Other Than Pfn [AHS02]

Categories of sets and
(functions with different branching/partiality)

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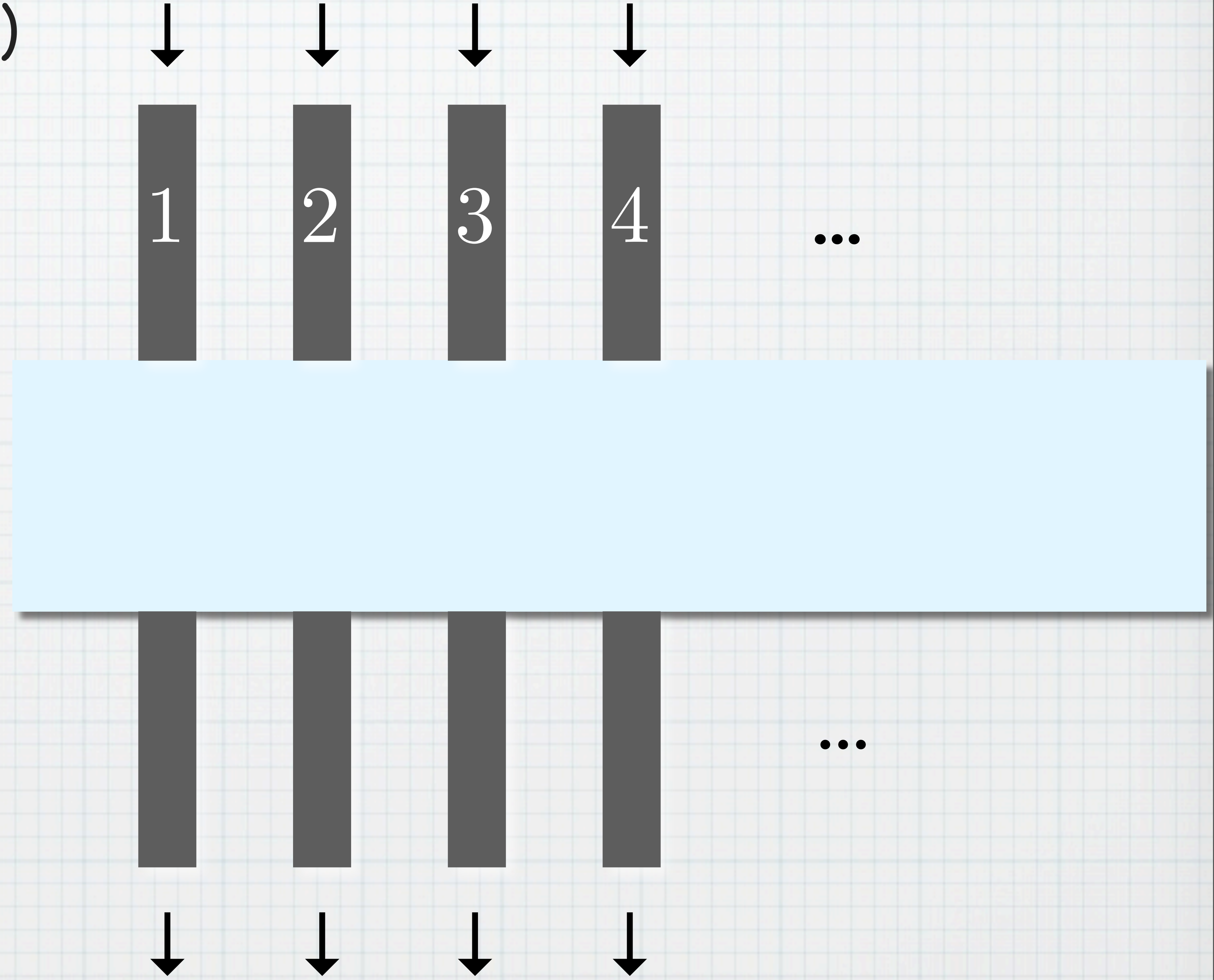
Probabilistic branching

Different Branching in The GoI Animation

- **Pfn** (partial functions)
  - Pipes can be stuck
- **Rel** (relations)
  - Pipes can branch
- **DSRel**
  - Pipes can branch probabilistically



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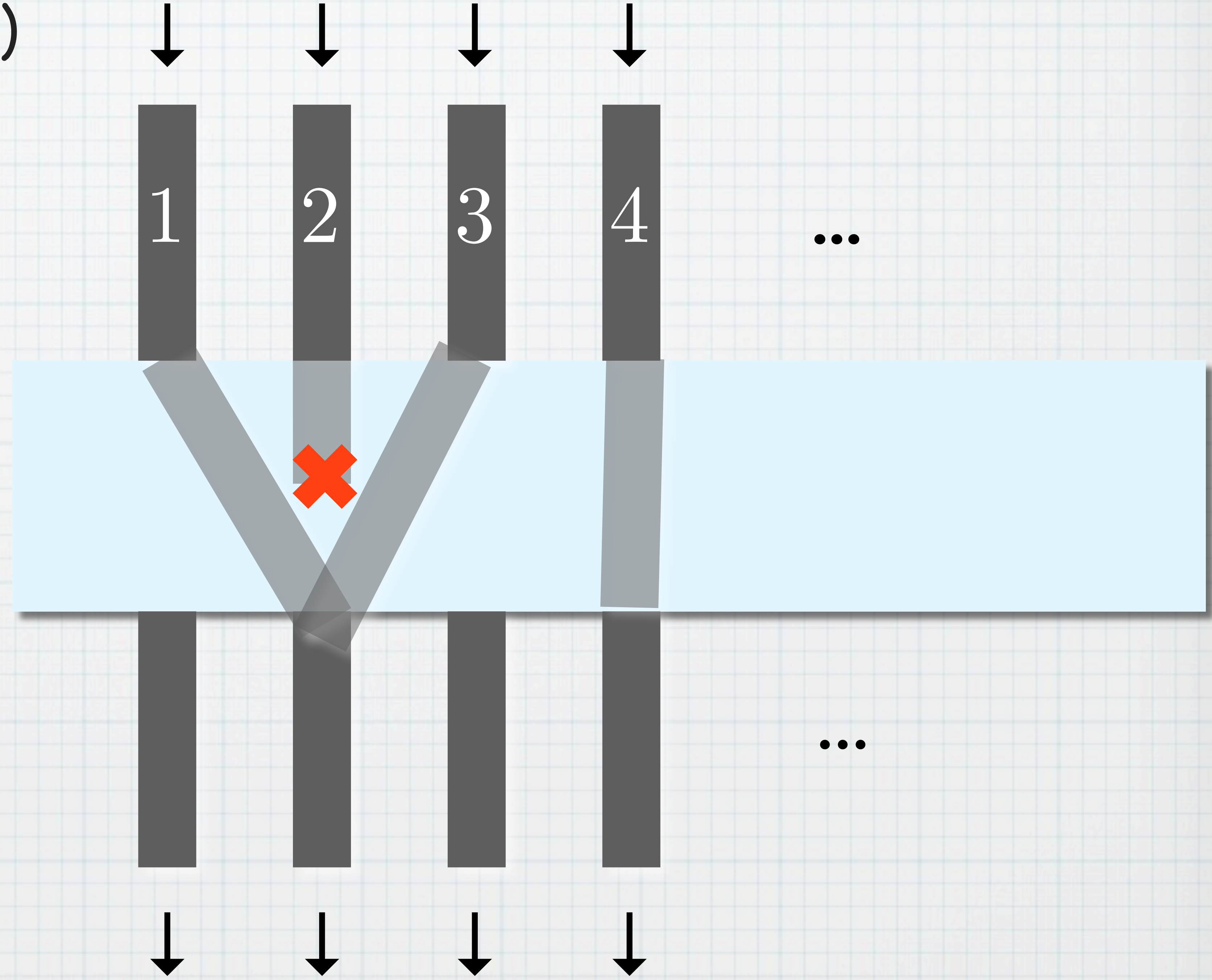

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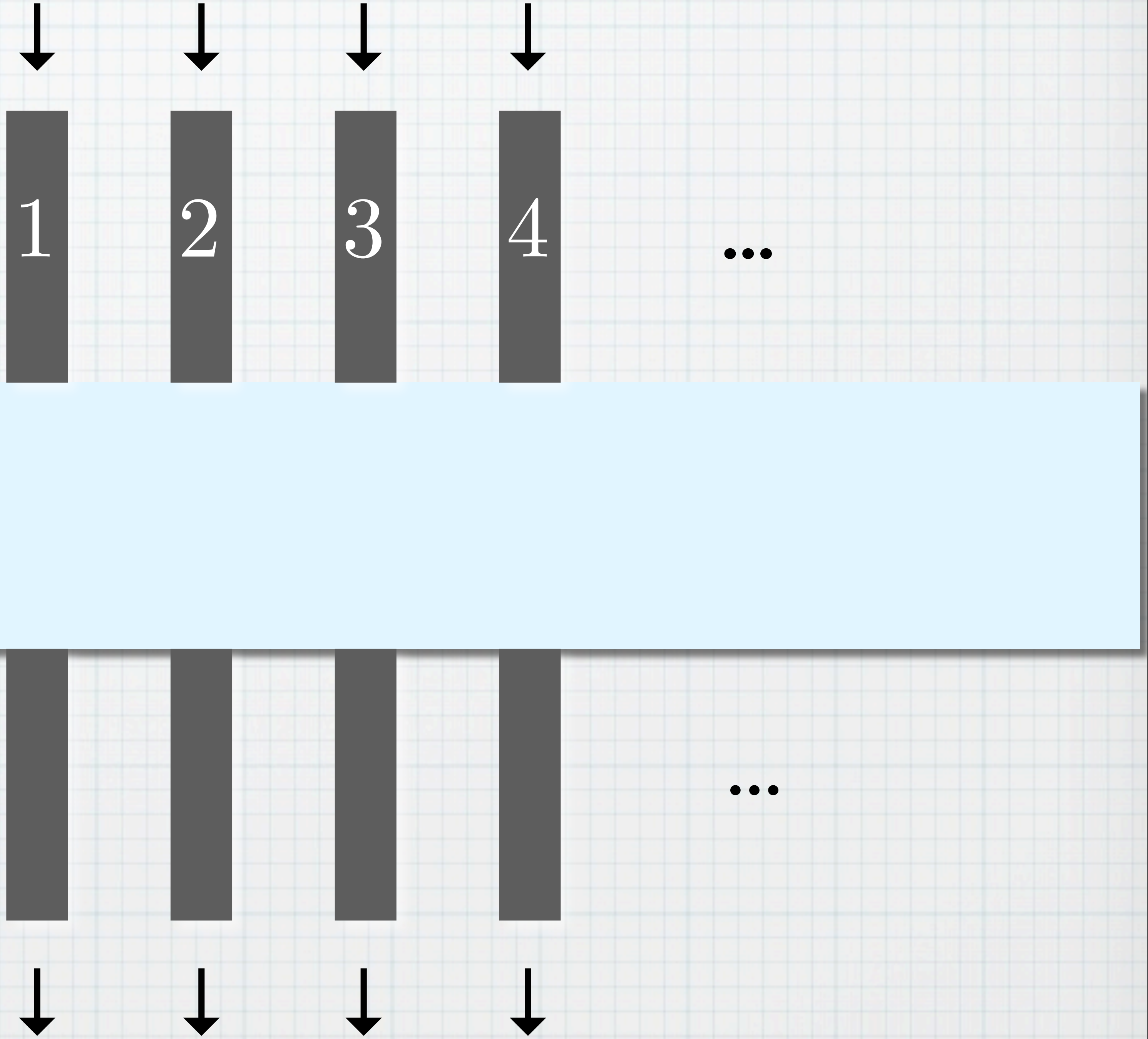

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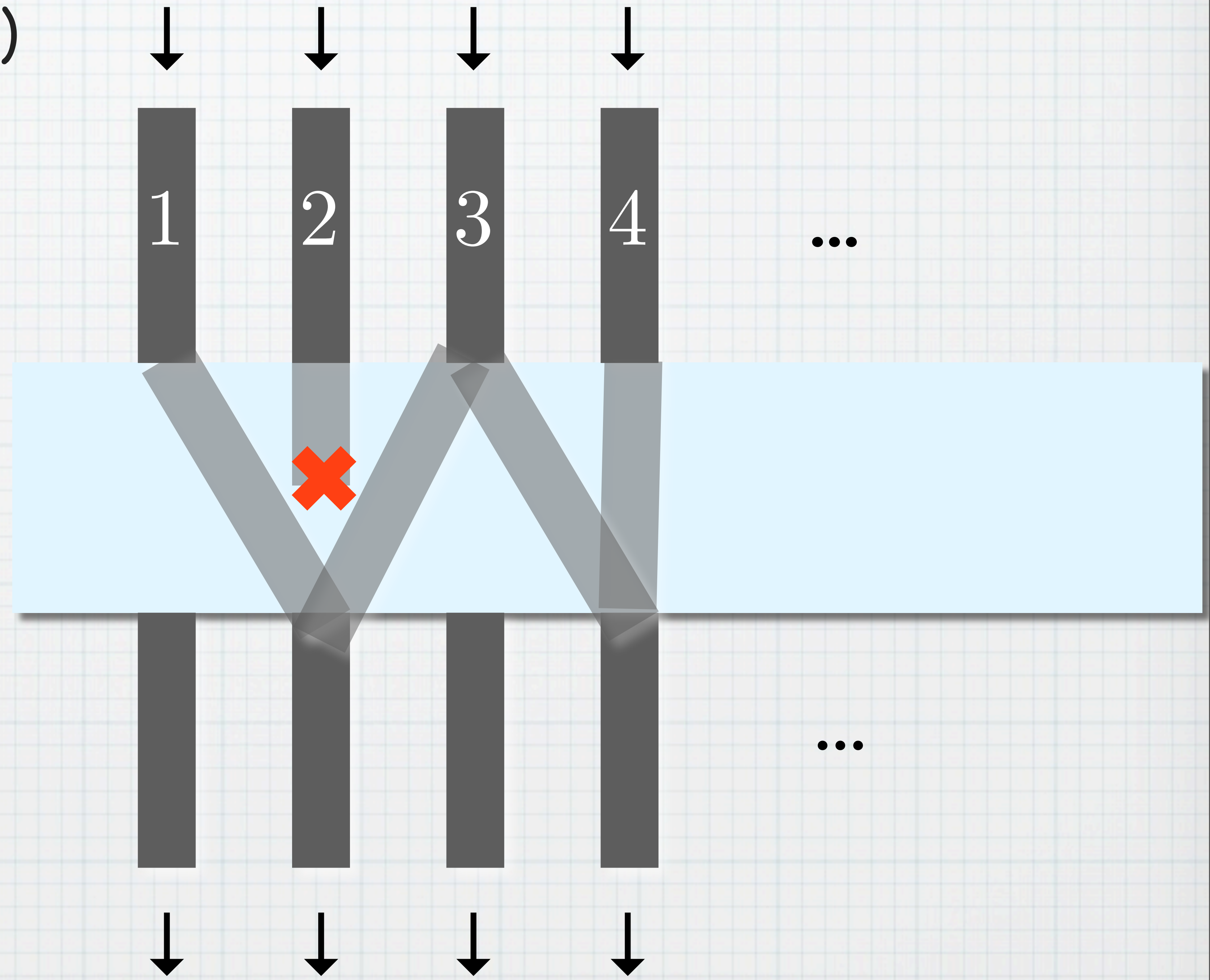

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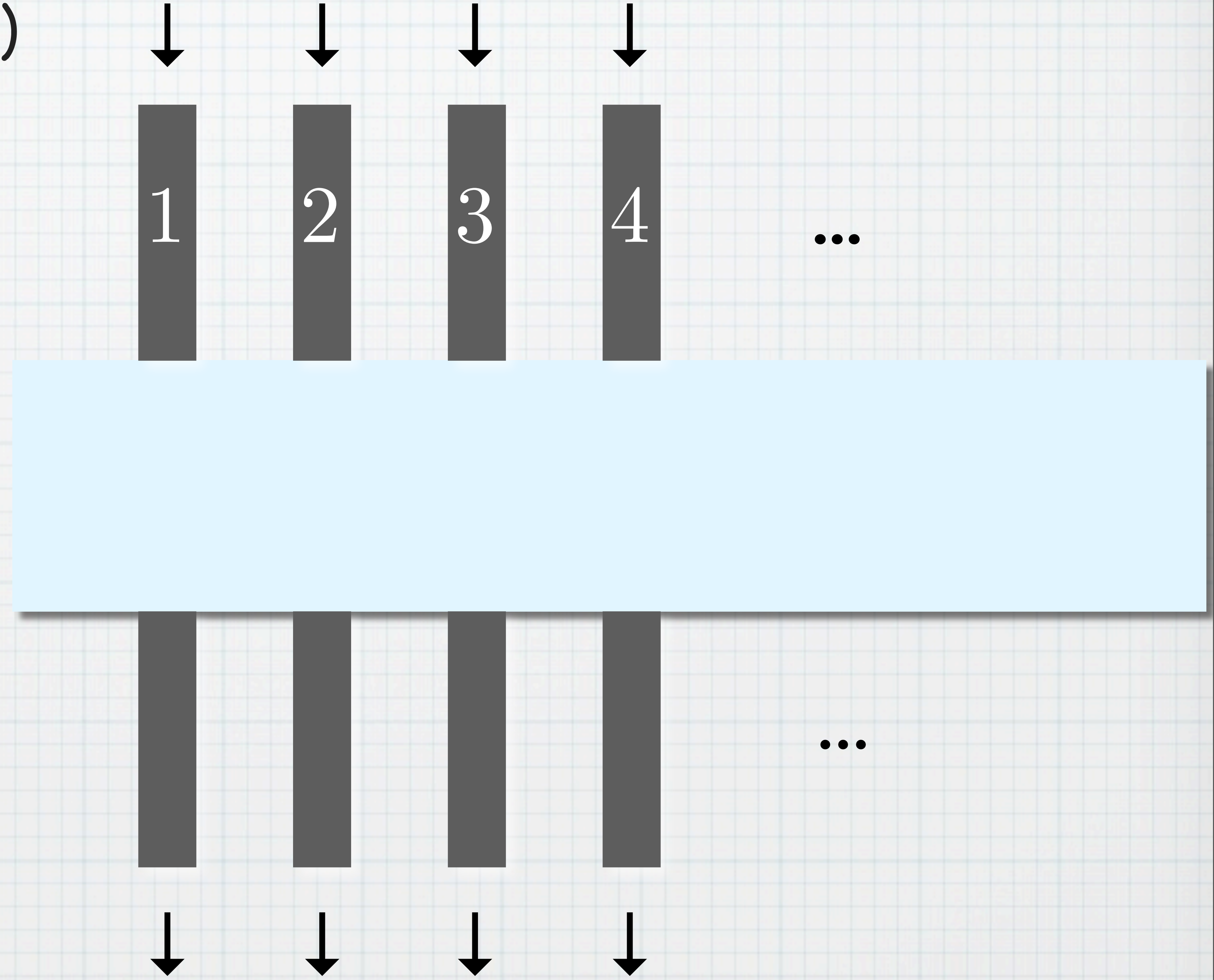

Hasuo (Tokyo)

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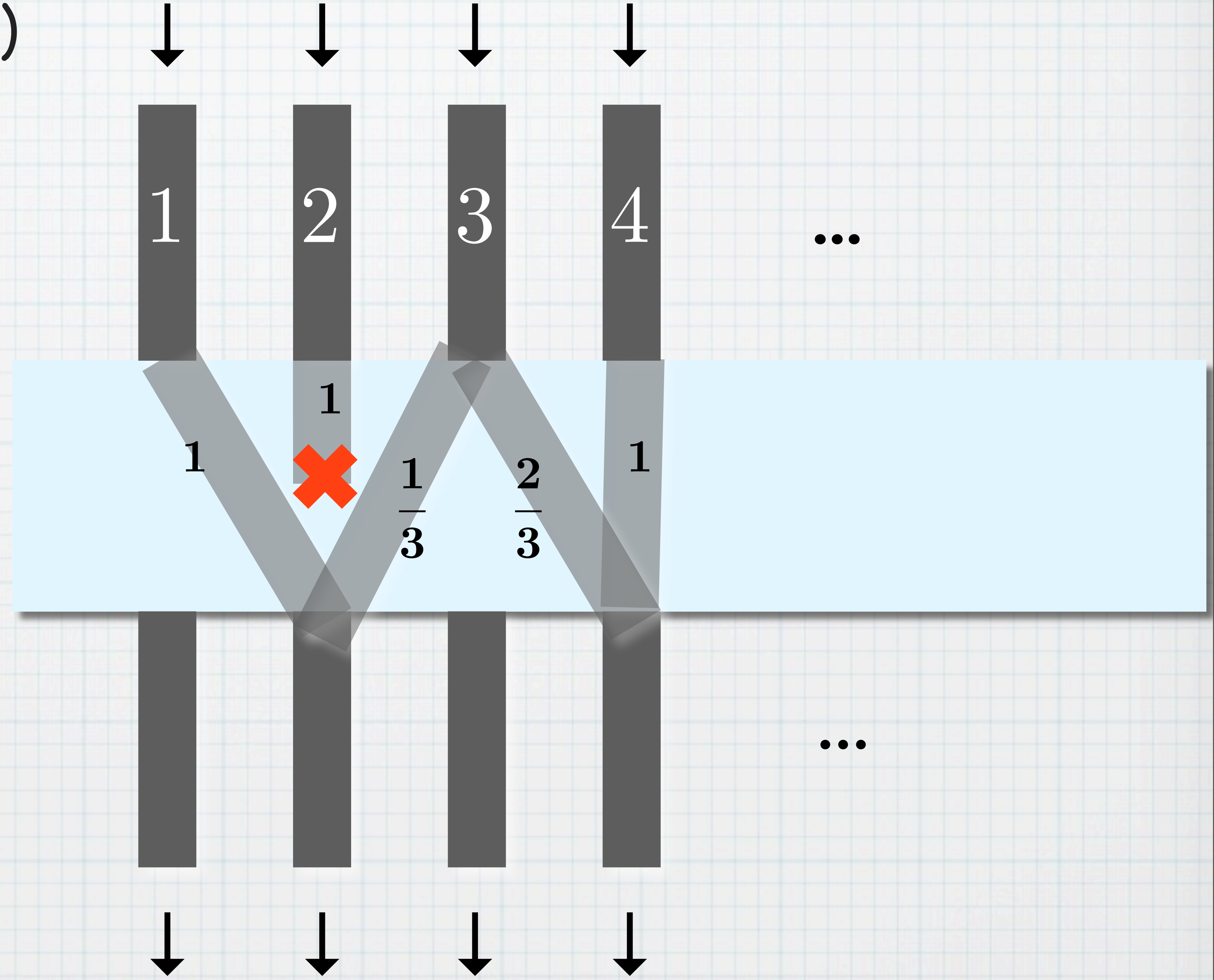

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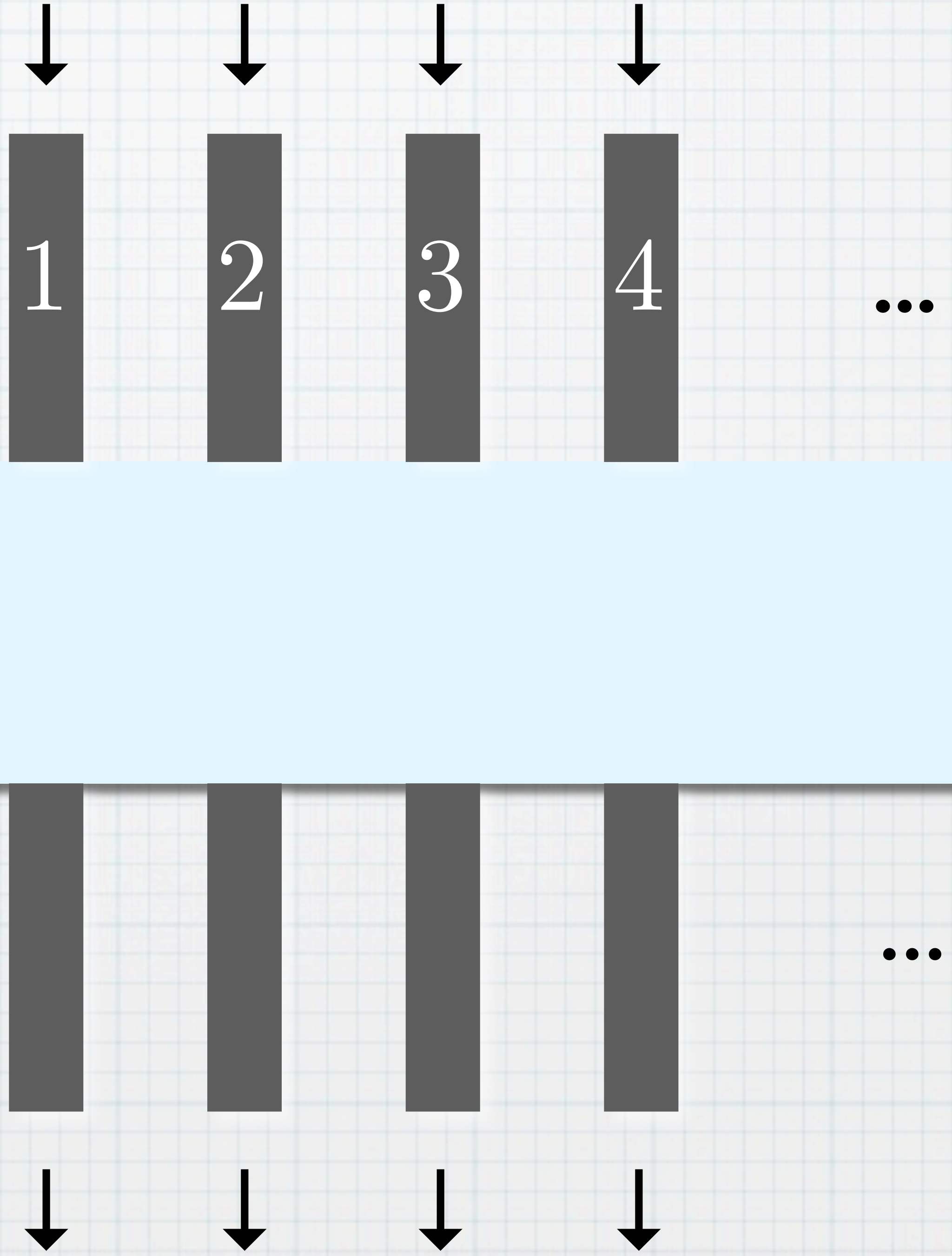

Hasuo (Tokyo)

Why Catego
Examples

Categories of sets and
(functions with different branching/partiality)

(Potential) non-termination

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Probabilistic branching

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Why Catego
Examples

$Kl(B)$ for different branching monads B

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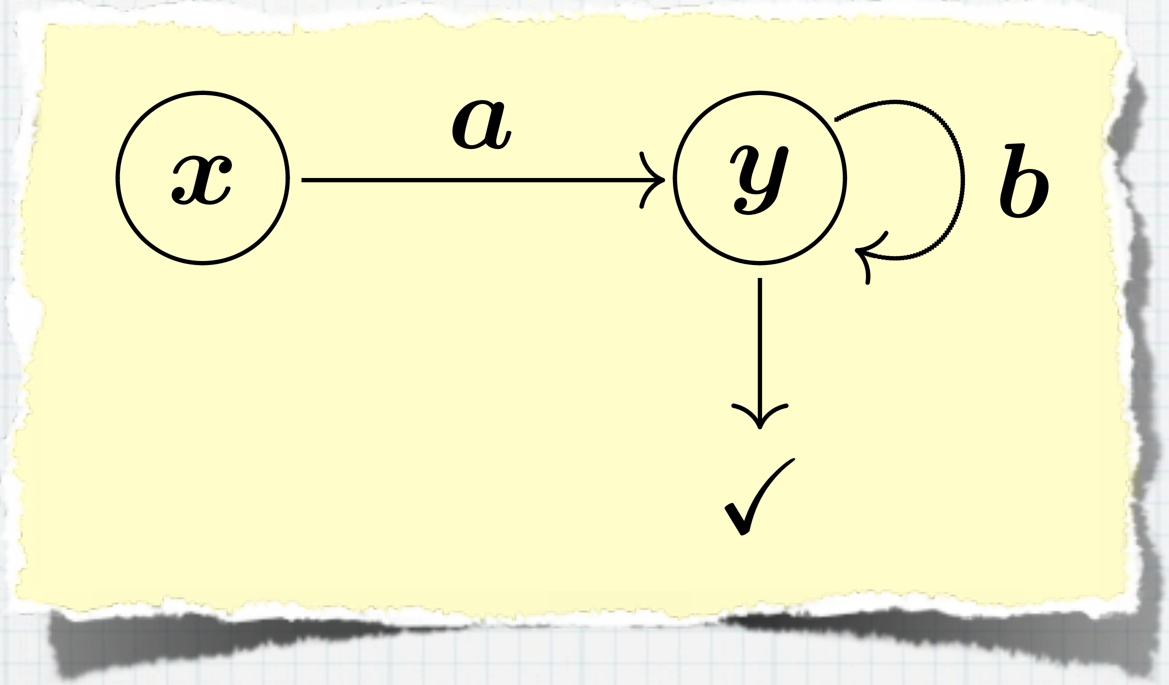
Probabilistic branching

Part 2

Coalgebraic
Trace Semantics

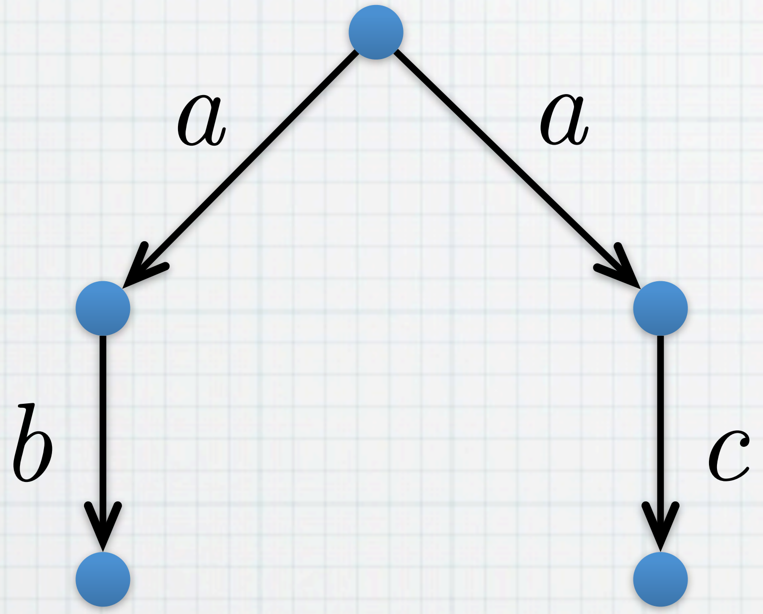
Trace Semantics of Systems

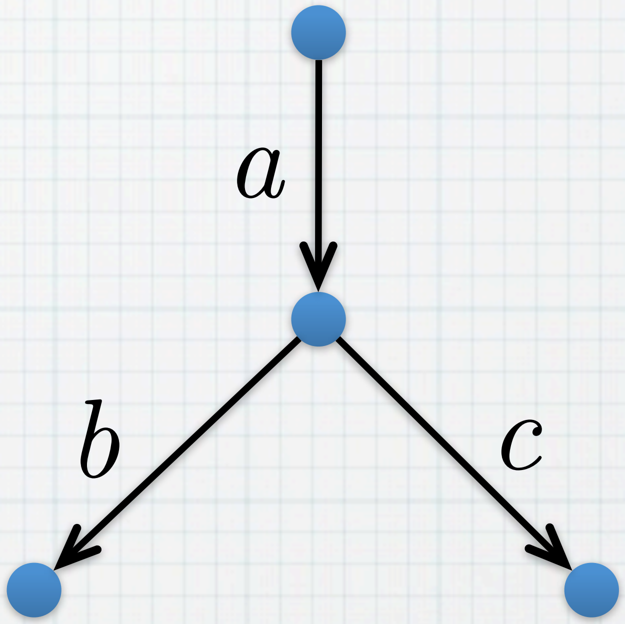


$$\mathbf{tr}(x) = \{a, ab, abb, \ldots\} = ab^*$$

- **Non-deterministic** branching:
  sign. functor is $\mathcal{P}(1 + \Sigma \times \_)$

Bisimilarity vs. Trace Sem.





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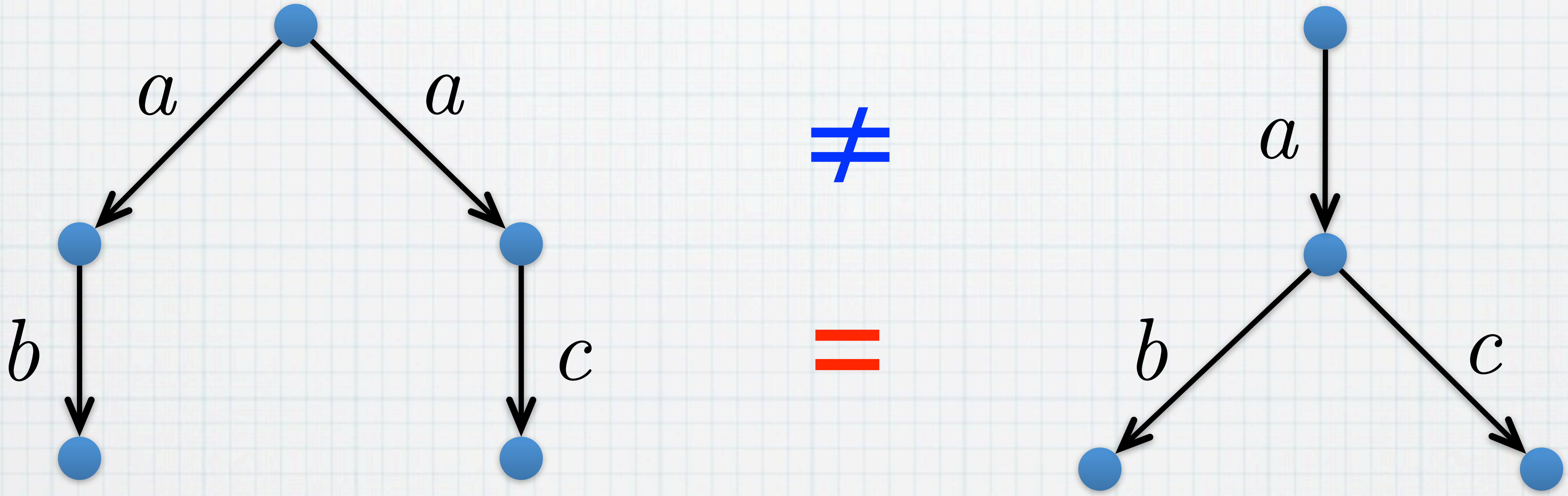

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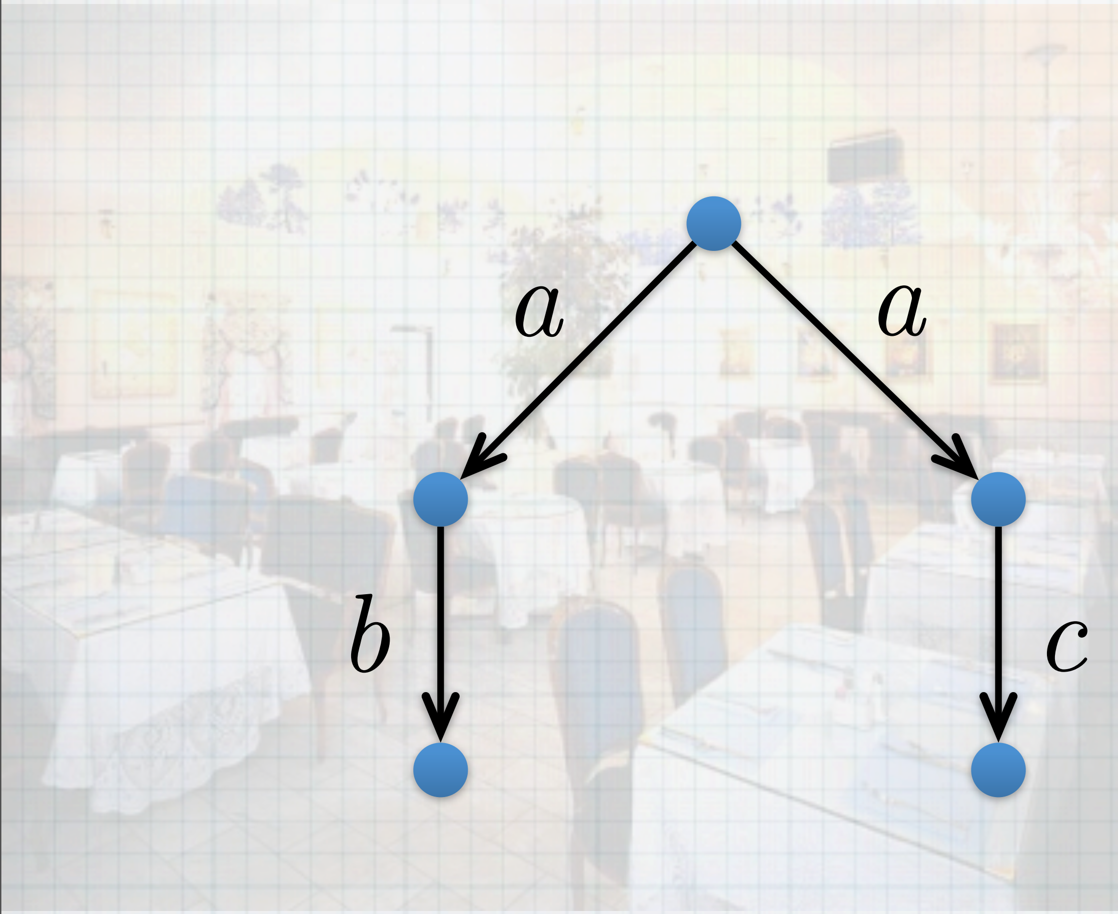

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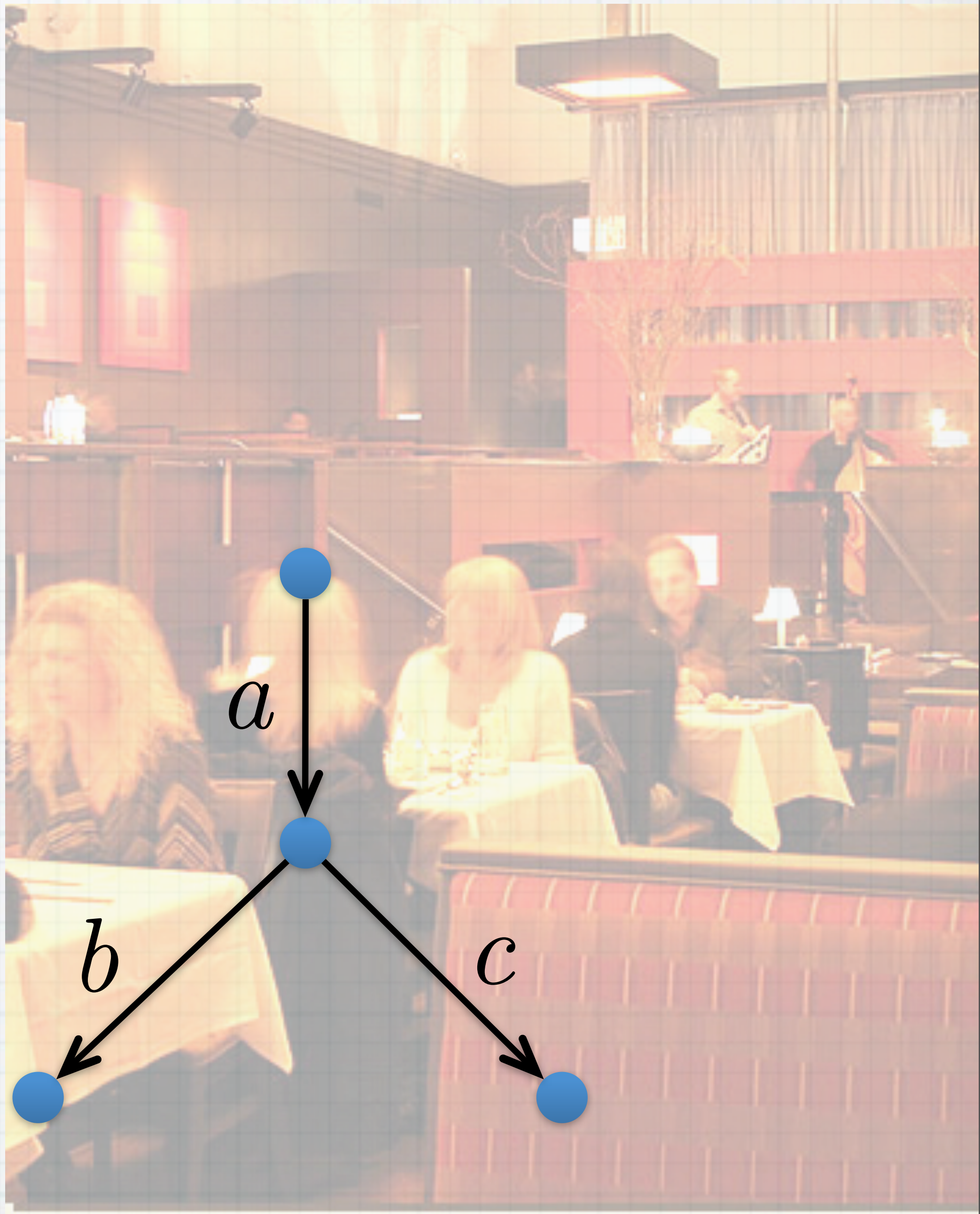

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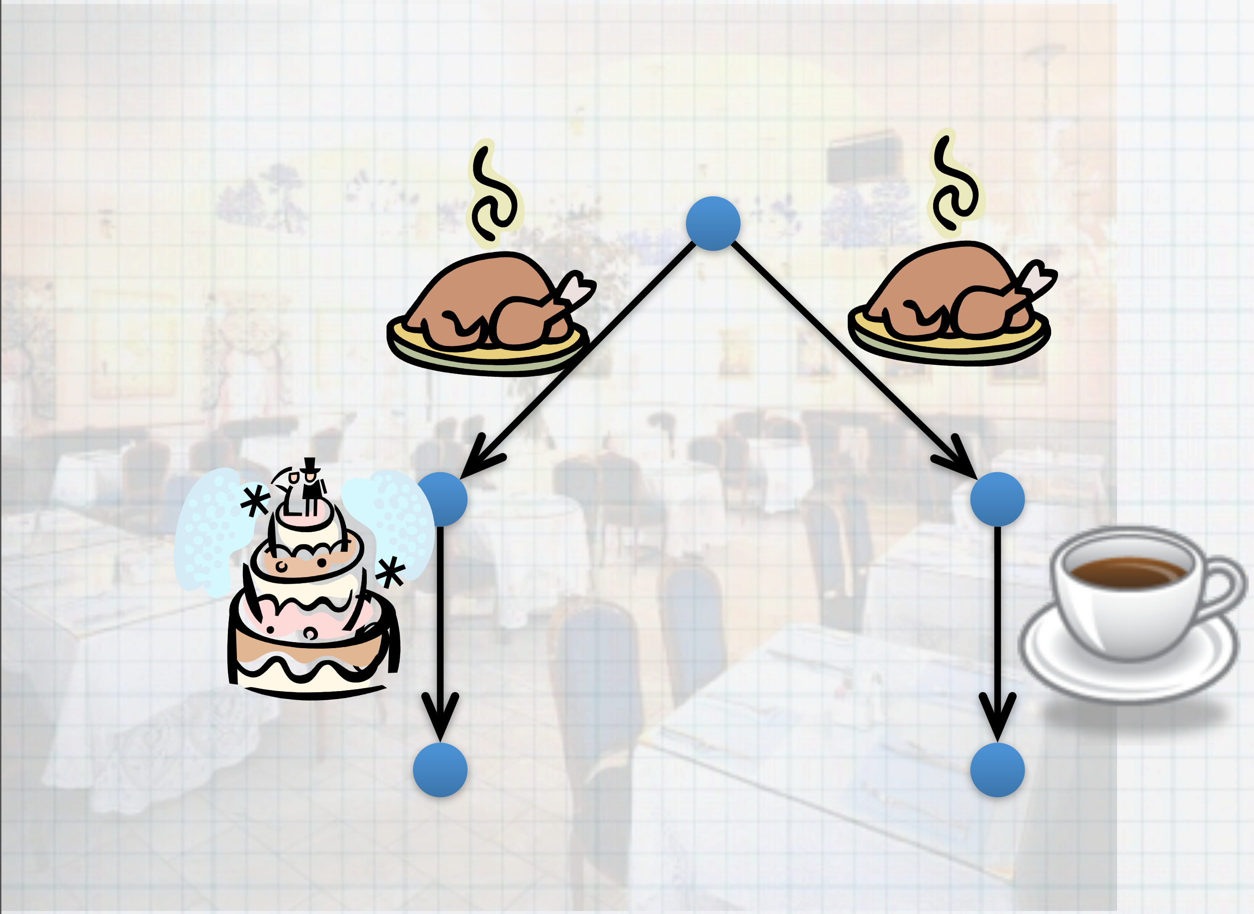

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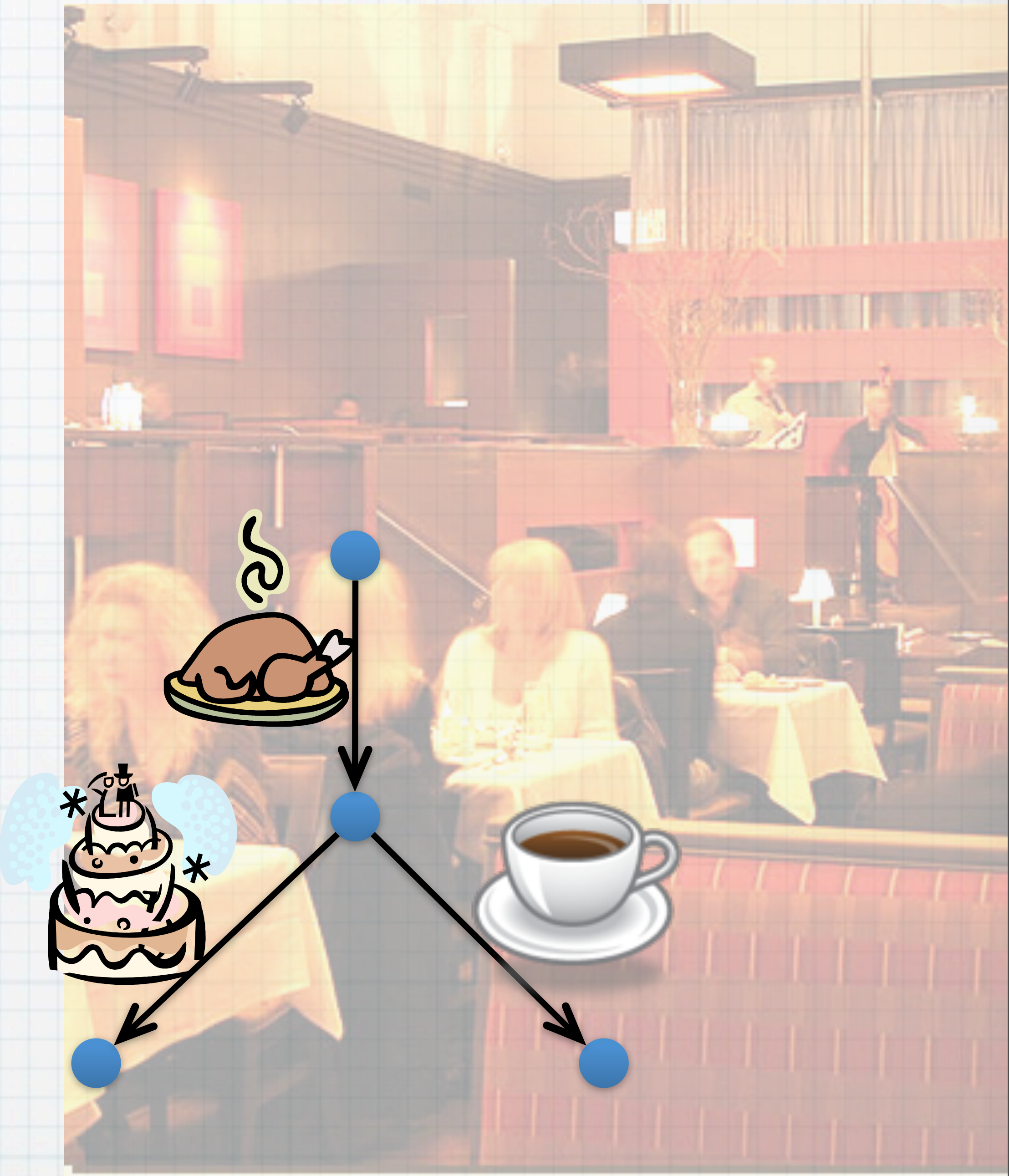


Bisimilarity

Branching structure matters.

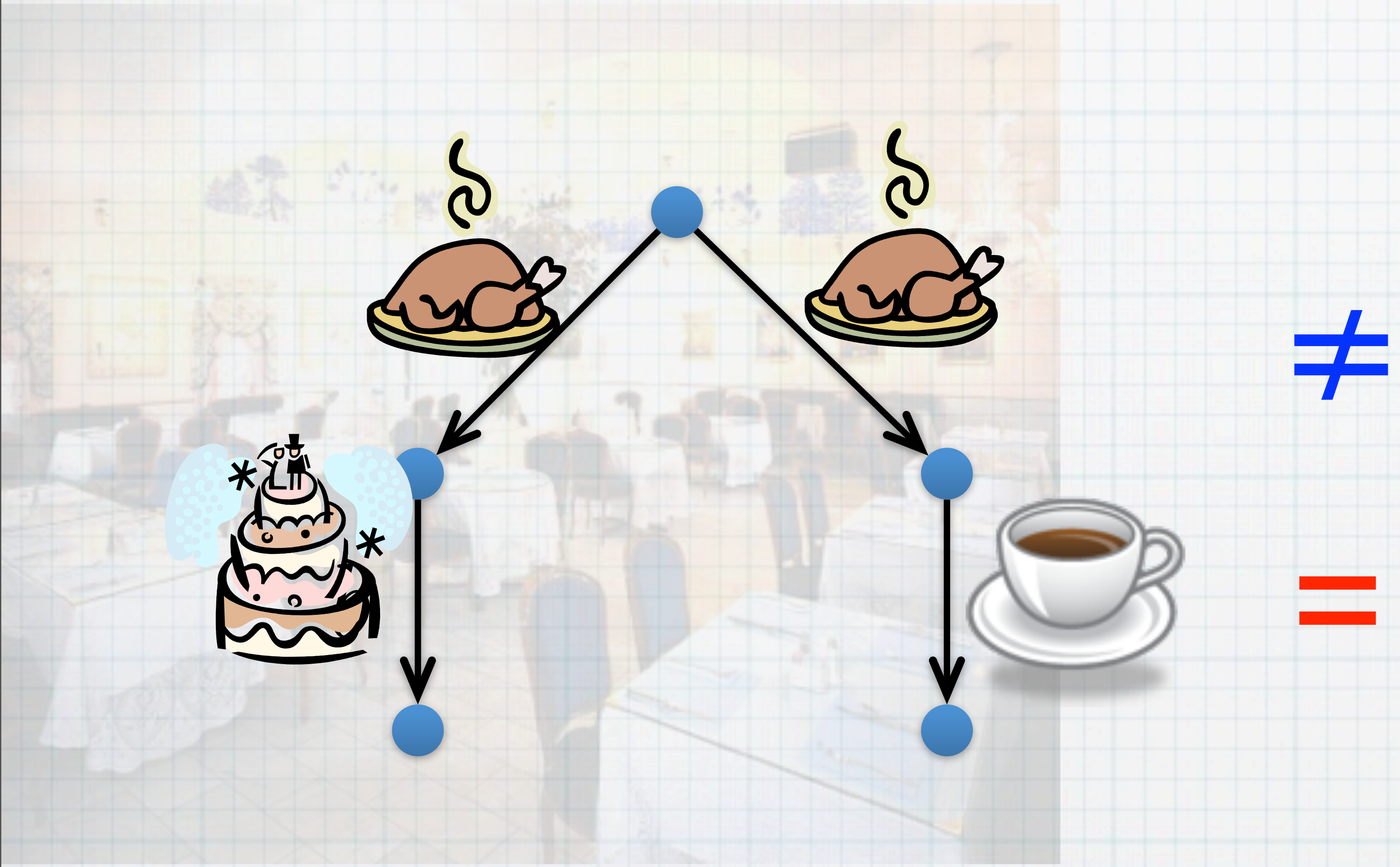
Can I choose later?

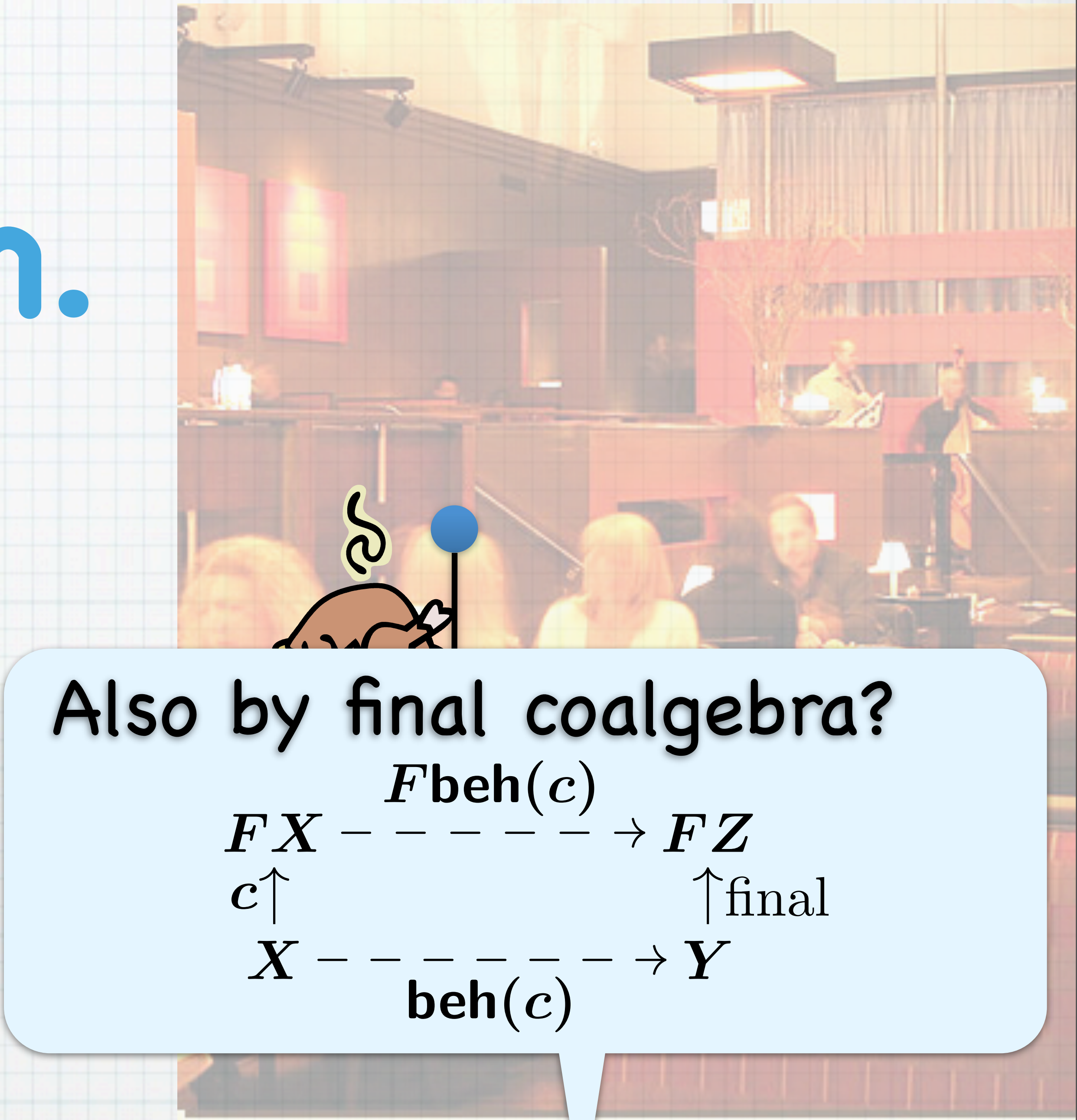
Trace semantics

Branching structure does not matter.

Anyway we'll get the same sets of food.

Bisimilarity vs. Trace Sem.





Also by final coalgebra?

$$\begin{tikzcd}
FX \arrow[r, dashed, "F\mathbf{beh}(c)"] & FZ \\
X \arrow[u, "c"] \arrow[r, dashed, "\mathbf{beh}(c)"'] & Y \arrow[u, "\mathrm{final}"']
\end{tikzcd}$$

Bisimilarity

Branching structure matters.

Can I choose later?

Trace semantics

Branching structure does not matter.

Anyway we'll get the same sets of food.

Coinduction in a Kleisli Category

[IH, Jacobs, Sokolova, '07]

$$\frac{X \longrightarrow\!\!\!\!\!\!\!+\;\; Y \quad \text{in } \mathcal{K}\ell(B)}{X \longrightarrow BY \quad \text{in } \mathbf{Sets}}$$

Thm. Let $\boldsymbol{F}$ be an endofunctor, and $\boldsymbol{B}$ be a monad, both on **Sets**. Assume:

1. We have a distributive law $\boldsymbol{\lambda : FB \Rightarrow BF}$.
2. The functor $\boldsymbol{F}$ preserves $\boldsymbol{\omega}$-colimits, yielding an initial algebra $\begin{tikzcd} \boldsymbol{FA} \arrow[d, "\boldsymbol{\alpha}", "\cong"'] \\ \boldsymbol{A} \end{tikzcd}$.
3. The Kleisli category $\mathcal{K}\ell(\boldsymbol{B})$ is $\mathbf{Cpo}_{\perp}$-enriched and composition in $\mathcal{K}\ell(\boldsymbol{B})$ is left-strict.

Then:

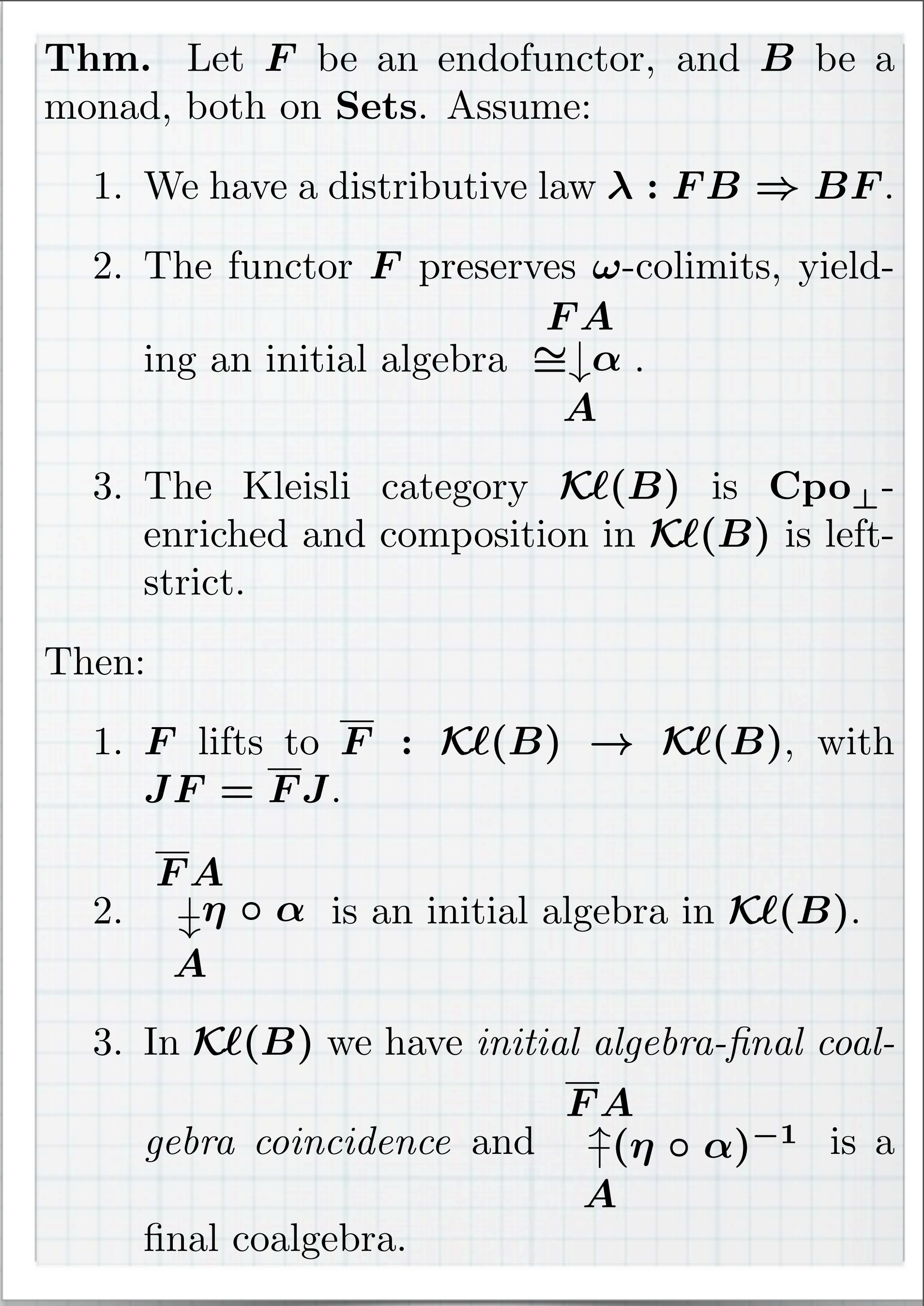
1. $\boldsymbol{F}$ lifts to $\overline{\boldsymbol{F}} : \mathcal{K}\ell(\boldsymbol{B}) \to \mathcal{K}\ell(\boldsymbol{B})$, with $\boldsymbol{JF = \overline{F}J}$.
2. $\begin{tikzcd} \overline{\boldsymbol{F}}\boldsymbol{A} \arrow[d, "\boldsymbol{\eta \circ \alpha}", "+"' marking] \\ \boldsymbol{A} \end{tikzcd}$ is an initial algebra in $\mathcal{K}\ell(\boldsymbol{B})$.
3. In $\mathcal{K}\ell(\boldsymbol{B})$ we have *initial algebra-final coalgebra coincidence* and $\begin{tikzcd} \overline{\boldsymbol{F}}\boldsymbol{A} \\ \boldsymbol{A} \arrow[u, "(\boldsymbol{\eta \circ \alpha})^{-1}", "+"' marking] \end{tikzcd}$ is a final coalgebra.

Coinduction in a Kleisli Category

[IH, Jacobs, Sokolova, '07]

$$\begin{array}{c} X \relbar\joinrel\mapstochar\joinrel\rightarrow Y \quad \text{in } \mathcal{K}\ell(B) \\ \hline\hline X \longrightarrow BY \quad \text{in } \mathbf{Sets} \end{array}$$

- Initial algebra lifts from Sets to $Kl(B)$
  - diagram chasing [Johnstone]



Thm. Let F be an endofunctor, and B be a monad, both on **Sets**. Assume:

1. We have a distributive law $\lambda : FB \Rightarrow BF$.
2. The functor F preserves ω-colimits, yielding an initial algebra $\begin{tikzcd} FA \arrow[d, "\cong"', "\alpha"] \\ A \end{tikzcd}$.
3. The Kleisli category $\mathcal{K}\ell(B)$ is $\mathbf{Cpo}_\perp$-enriched and composition in $\mathcal{K}\ell(B)$ is left-strict.

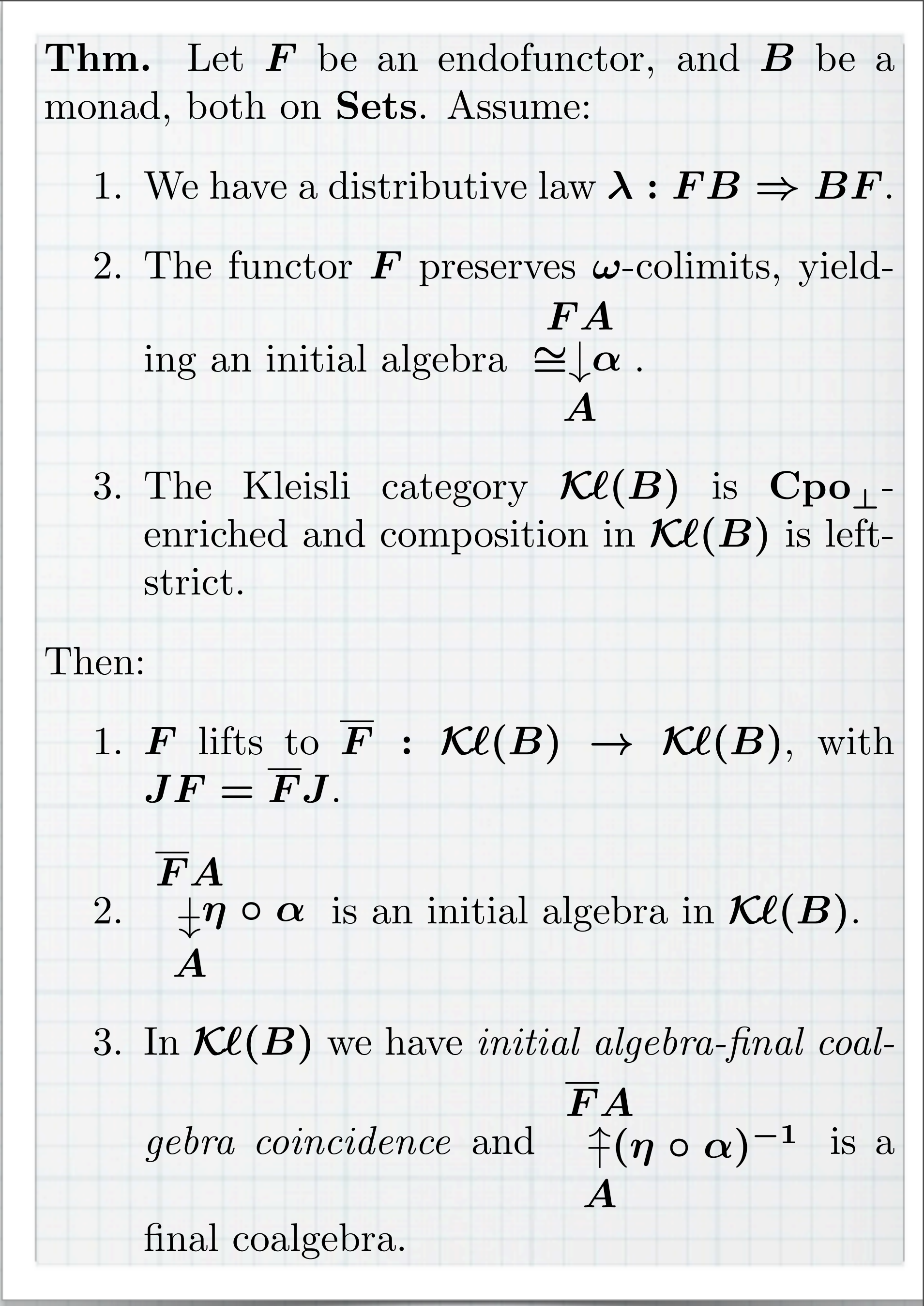
Then:

1. F lifts to $\overline{F} : \mathcal{K}\ell(B) \to \mathcal{K}\ell(B)$, with $JF = \overline{F}J$.
2. $\begin{tikzcd} \overline{F}A \arrow[d, "|" marking, "\eta \circ \alpha"] \\ A \end{tikzcd}$ is an initial algebra in $\mathcal{K}\ell(B)$.
3. In $\mathcal{K}\ell(B)$ we have *initial algebra-final coalgebra coincidence* and $\begin{tikzcd} \overline{F}A \\ A \arrow[u, "|" marking, "(\eta \circ \alpha)^{-1}"'] \end{tikzcd}$ is a final coalgebra.

Coinduction in a Kleisli Category

[IH, Jacobs, Sokolova, '07]

$$\begin{array}{c} X \mathrel{-\!\!\!+\!\!\!\rightarrow} Y \quad \text{in } \mathcal{K}\ell(B) \\ \hline\hline X \longrightarrow BY \quad \text{in } \mathbf{Sets} \end{array}$$



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- Initial algebra lifts from **Sets** to $Kl(B)$
  - diagram chasing [Johnstone]
- In $Kl(B)$ we have **IA-FC coincidence**
  - typical of "domain-theoretic" categories
  - "Algebraically compact" [Freyd]

Coinduction in a Kleisli Category

- E.g. $B = \mathcal{P},\ F = 1 + \Sigma \times (\_\,)$

$$\begin{tikzcd}
1+\Sigma\times X \arrow[r, dashed, "1+\Sigma\times \mathsf{tr}(c)"] & 1+\Sigma\times\Sigma^* & \text{in } \mathcal{K}\ell(\mathcal{P}) \\
X \arrow[u, "c"] \arrow[r, dashed, "\mathsf{tr}(c)"'] & \Sigma^* \arrow[u, "\text{final}"'] &
\end{tikzcd}$$

- Separation between B and F

Coinduction in a Kleisli Category

- E.g. $B = \mathcal{P},\ F = 1 + \Sigma \times (\_\,)$

$$\begin{tikzcd}
1 + \Sigma \times X \arrow[r, dashed, "1 + \Sigma \times \mathsf{tr}(c)", "\shortmid" marking] & 1 + \Sigma \times \Sigma^* \\
X \arrow[u, "c", "\shortmid" marking] \arrow[r, dashed, "\mathsf{tr}(c)"', "\shortmid" marking] & \Sigma^* \arrow[u, "\text{final}"', "\shortmid" marking]
\end{tikzcd} \quad \text{in } \mathcal{K}\ell(\mathcal{P})$$

$$\begin{tikzcd}
\mathcal{P}(1 + \Sigma \times X) \\
X \arrow[u, "c"]
\end{tikzcd} \quad \text{in Sets}$$

- Separation between B and F

Coinduction in a Kleisli Category

- E.g. $B = \mathcal{P}$, $F = 1 + \Sigma \times (\_)$

$$\begin{tikzcd}
1 + \Sigma \times X \arrow[r, dashed, "|", "1 + \Sigma \times \mathrm{tr}(c)"] & 1 + \Sigma \times \Sigma^* \\
X \arrow[u, "|" marking, "c"] \arrow[r, dashed, "|" marking, "\mathrm{tr}(c)"'] & \Sigma^* \arrow[u, "|" marking, "\mathrm{final}"']
\end{tikzcd}$$

in $\mathcal{K}\ell(\mathcal{P})$

induced by

$$\begin{tikzcd}
1 + \Sigma \times \Sigma^* \arrow[d, "\mathrm{initial}"] \\
\Sigma^*
\end{tikzcd}$$

in **Sets**

$$\begin{tikzcd}
\mathcal{P}(1 + \Sigma \times X) \\
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in **Sets**

$$X \xrightarrow{\mathsf{tr}(c)} \mathcal{P}(\Sigma^*)$$

- Separation between B and F

Coinduction in a Kleisli Category

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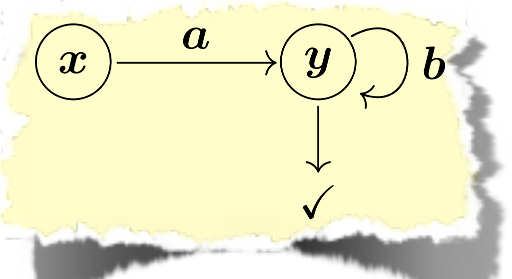

$\mathbf{tr}(x) = \{a, ab, abb, \dots\} = ab^*$

- Separation between B and F

Coinduction in a Kleisli Category

- E.g. $B = \mathcal{P}$, $F = 1 + \Sigma \times (\_)$

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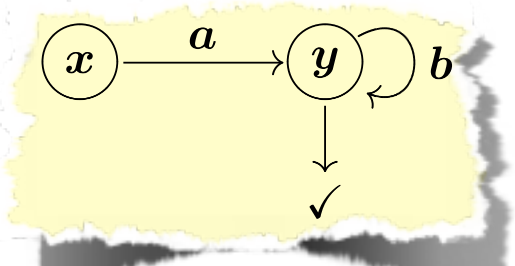
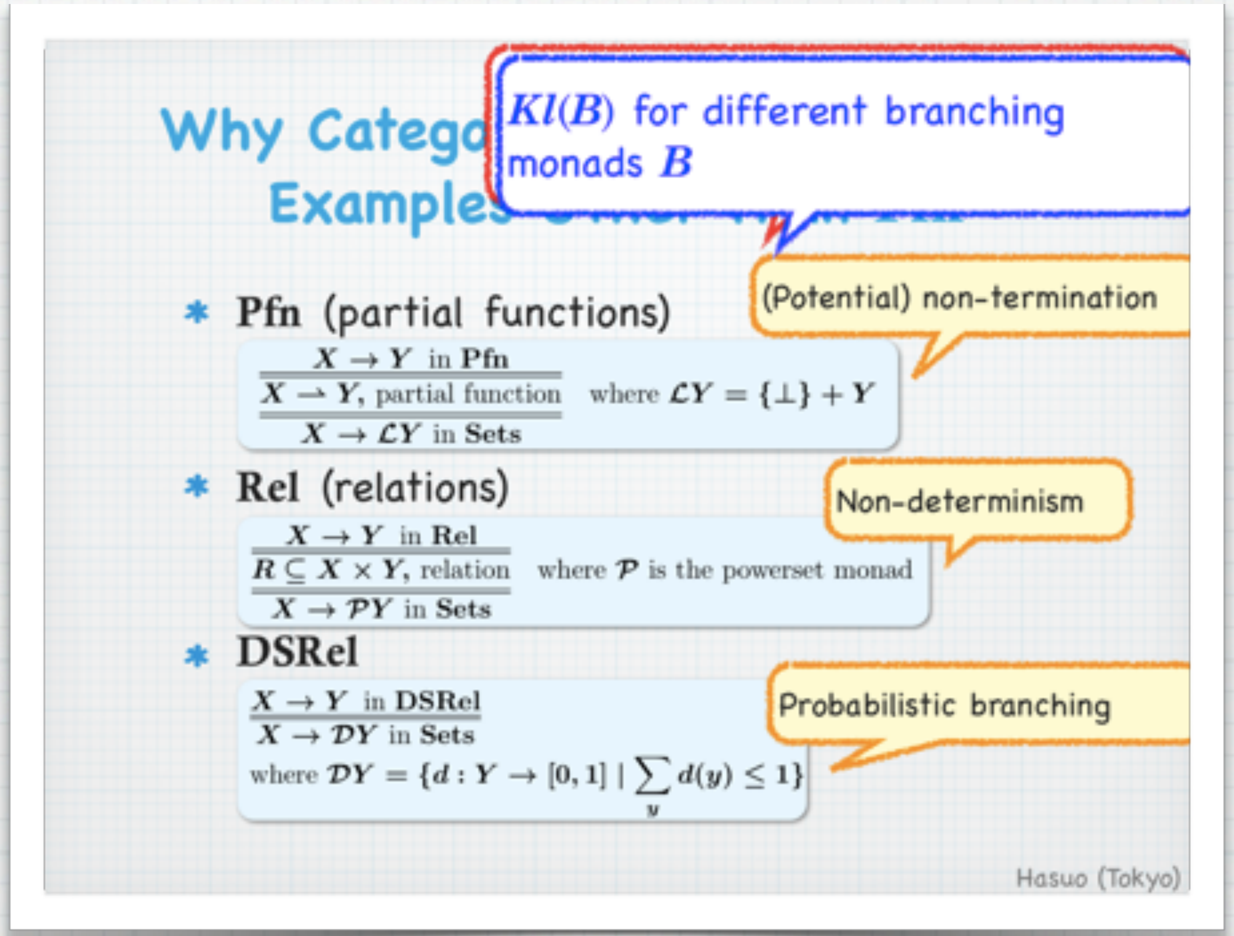


$\mathbf{tr}(x) = \{a, ab, abb, \dots\} = ab^{*}$

- Separation between B and F

Examples

- A branching monad B:
  - **Lift monad** $\mathcal{L} = 1 + (\_)$, **powerset monad** $\mathcal{P}$, **subdistribution monad** $\mathcal{D}$
  - Precisely those in



Why Catego

Examples

$Kl(B)$ for different branching monads B

- **Pfn** (partial functions)

$$\frac{X \to Y \text{ in } \mathbf{Pfn}}{\dfrac{X \rightharpoonup Y, \text{ partial function}}{X \to \mathcal{L}Y \text{ in } \mathbf{Sets}}} \quad \text{where } \mathcal{L}Y = \{\bot\} + Y$$

(Potential) non-termination

- **Rel** (relations)

$$\frac{X \to Y \text{ in } \mathbf{Rel}}{\dfrac{R \subseteq X \times Y, \text{ relation}}{X \to \mathcal{P}Y \text{ in } \mathbf{Sets}}} \quad \text{where } \mathcal{P} \text{ is the powerset monad}$$

Non-determinism

- **DSRel**

$$\frac{X \to Y \text{ in } \mathbf{DSRel}}{X \to \mathcal{D}Y \text{ in } \mathbf{Sets}}$$

$$\text{where } \mathcal{D}Y = \{d : Y \to [0,1] \mid \sum_{y} d(y) \le 1\}$$

Probabilistic branching

Hasuo (Tokyo)

- A functor F: polynomial functors

The Coauthor

- Naohiko Hoshino
  - DSc (Kyoto, 2011)
    - Supervisor: Masahito "Hassei" Hasegawa
  - Currently at RIMS, Kyoto U.
    - `http://www.kurims.kyoto-u.ac.jp/~naophiko/`

From Coalgebraic Trace to Monoidal Trace

Thm. ([Jacobs,CMCS10])
Given a "branching monad" B on **Sets**, the monoidal category

$$(\mathcal{K}\ell(\boldsymbol{B}), +, \mathbf{0})$$

is a traced symmetric monoidal category.

Cor.
$\big(\,(\mathcal{K}\ell(\boldsymbol{B}), +, \mathbf{0}),\ \mathbb{N}\cdot\_\,,\ \mathbb{N}\,\big)$ is a GoI situation.

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Proof. We need

$$\frac{X + Z \overset{f}{\longrightarrow} Y + Z \quad \text{in } \mathcal{K}\ell(T)}{X \overset{\mathsf{tr}(f)}{\longrightarrow} Y \quad \text{in } \mathcal{K}\ell(T)}$$

- $X + Z \overset{f}{\longrightarrow} Y + Z \overset{\kappa}{\longrightarrow} Y + (X + Z)$ is a $Y + (\_)$-coalgebra
- $$\begin{tikzcd} Y + \mathbb{N}\cdot Y \arrow[d, "\cong", "\alpha"'] \\ \mathbb{N}\cdot Y \end{tikzcd}$$ is an initial algebra in **Sets**
- Therefore in $\mathcal{K}\ell(T)$:

$$\begin{tikzcd}
Y + (X + Z) \arrow[r, dashed] & Y + \mathbb{N}\cdot Y \\
X + Z \arrow[u, "\kappa \circ f"] \arrow[r, dashed, "\mathsf{tr}(c)"'] & \mathbb{N}\cdot Y \arrow[u, "\text{final}"'] \arrow[d, "\nabla"'] \\
X \arrow[u, "\kappa_1"] & Y
\end{tikzcd}$$

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The Categorical GoI Workflow

Traced monoidal category $\mathbb{C}$

+ other constructs ➜ "GoI situation" [AHS02]

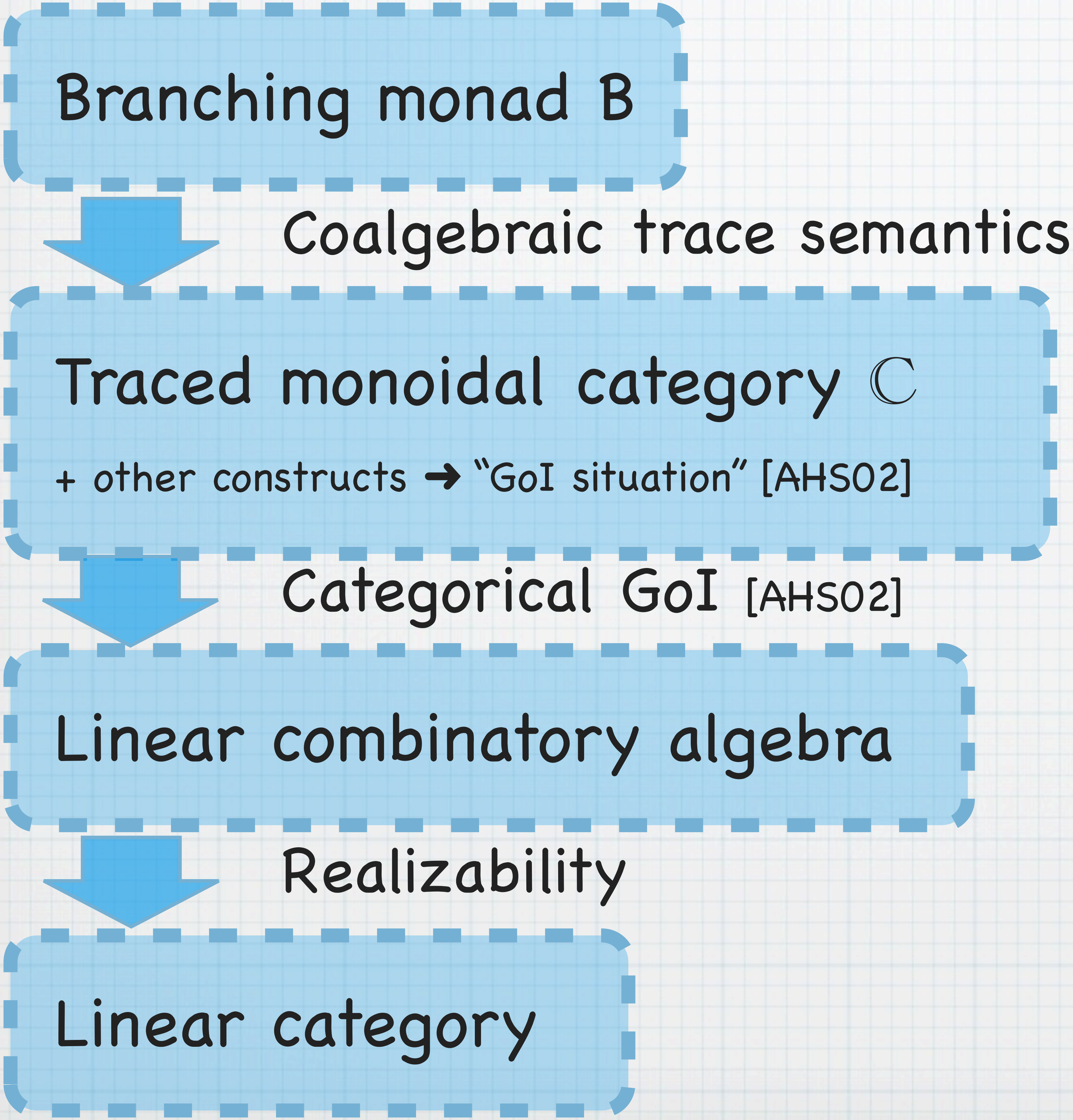
Categorical GoI [AHS02]

Linear combinatory algebra

Realizability

Linear category

The Categorical GoI Workflow



Branching monad B

Coalgebraic trace semantics

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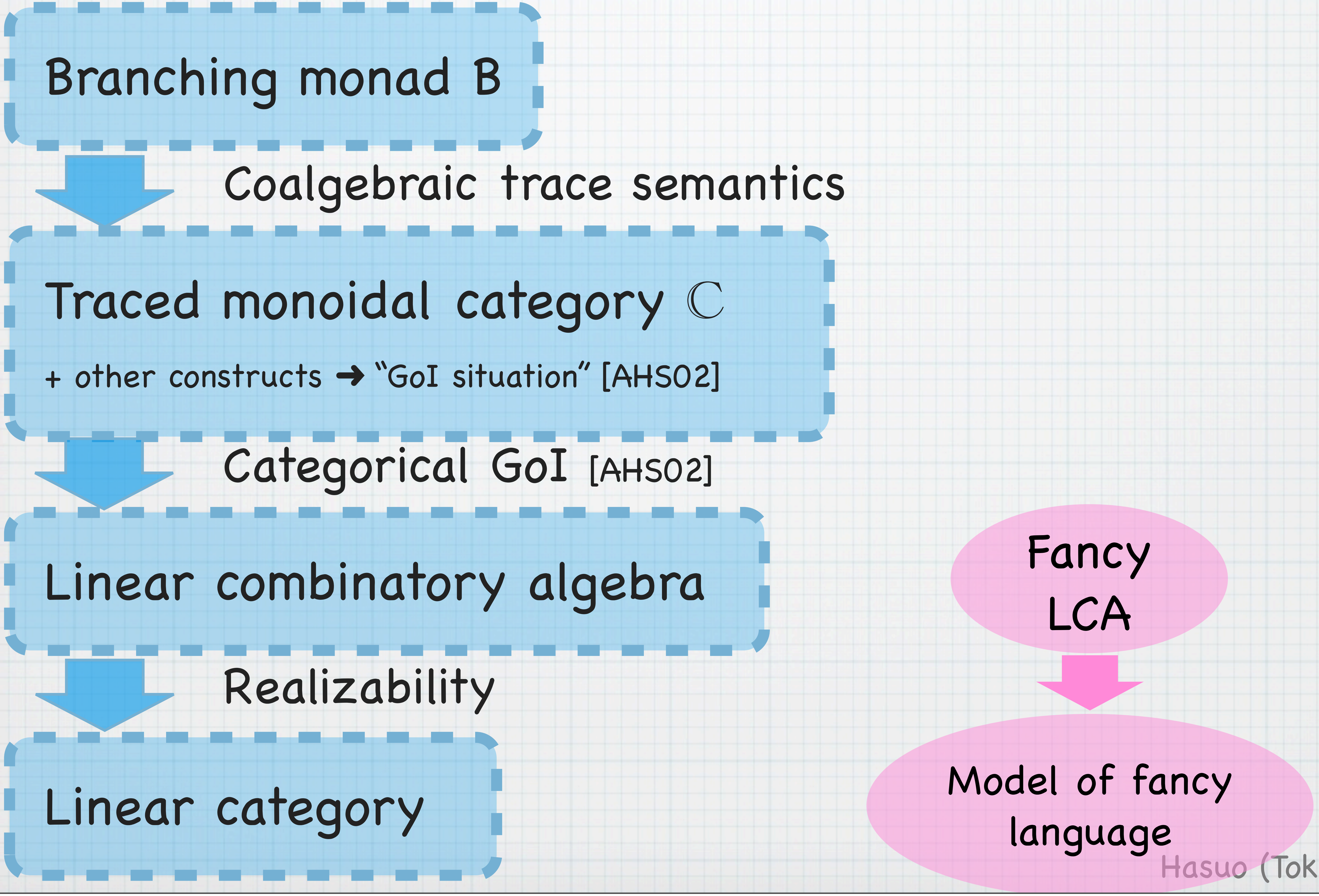

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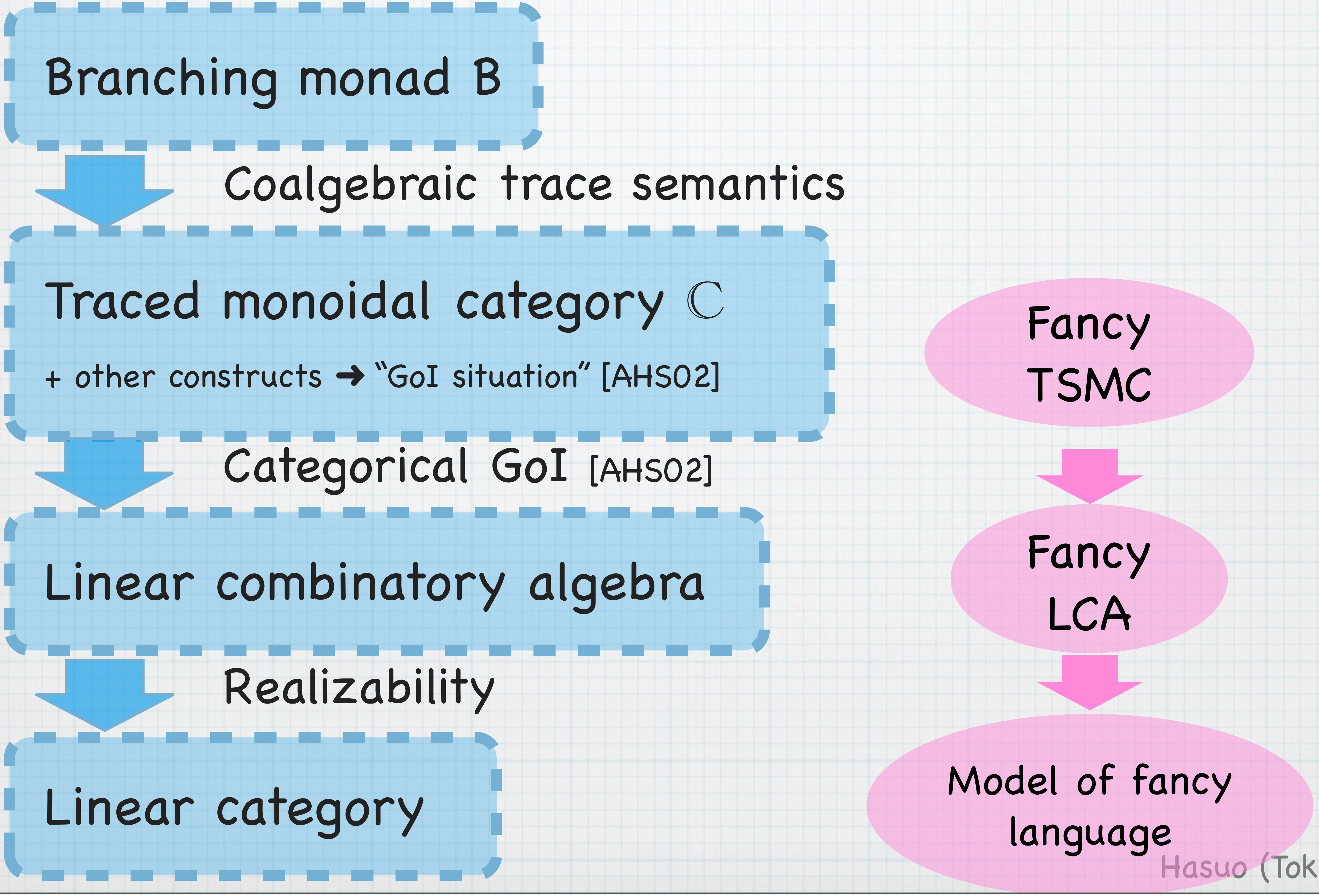

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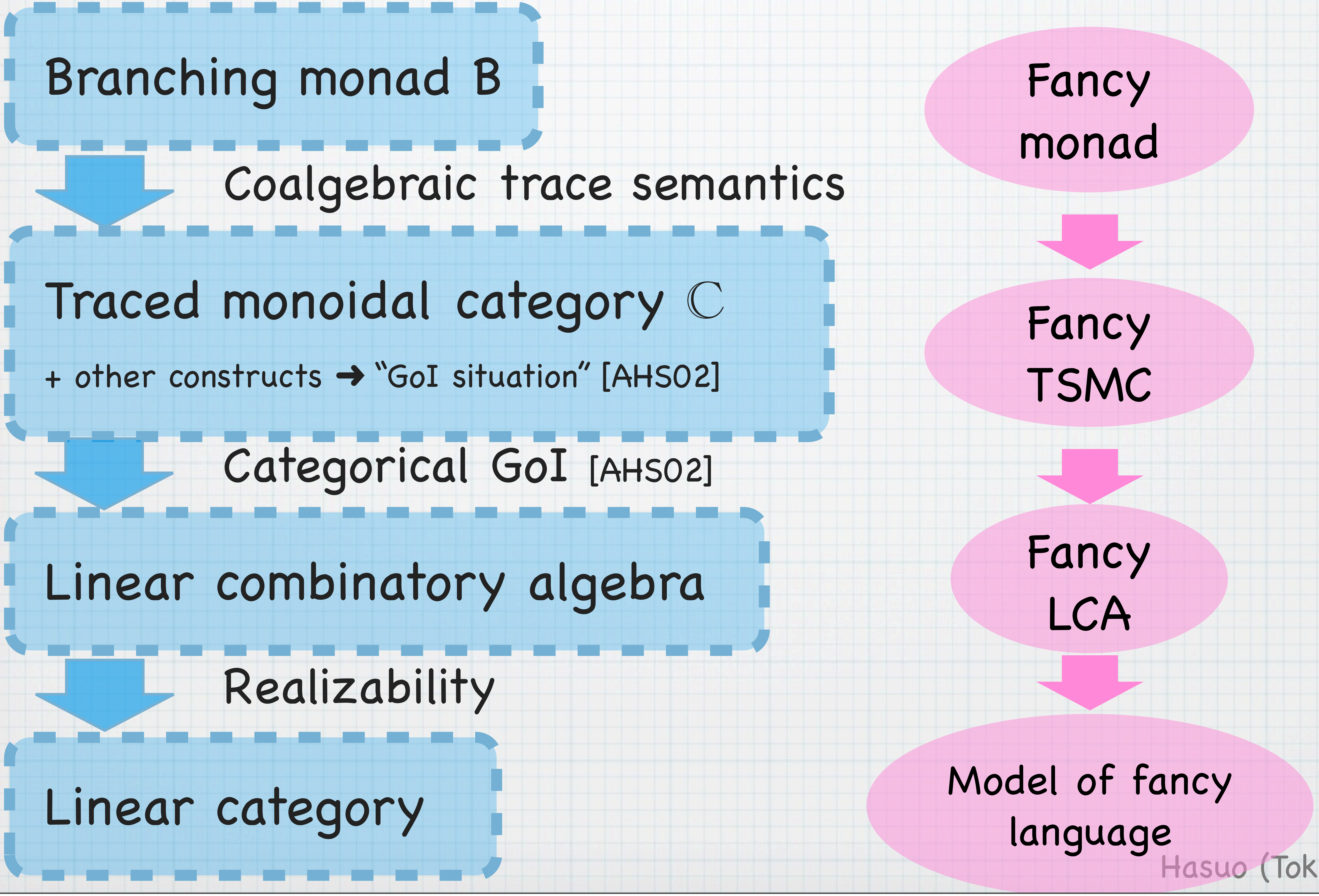

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Model of fancy language

What is Fancy, Nowadays?

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- Biology?

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- Biology?
- Hybrid systems?
  - Both discrete and continuous data, typically in **cyber-physical systems** (CPS)
  - → Our approach via **non-standard analysis**
    [Suenaga, IH, ICALP'11] [IH, Suenaga, CAV'12]
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What is Fancy, Nowadays?

- Biology?
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    [Suenaga, IH, ICALP'11] [IH, Suenaga, CAV'12]
    [Suenaga, Sekine, IH, POPL'13]
- Quantum?
  - Yes this worked!

Part 3

Phil Scott.
Tutorial on Geometry of
Interaction, FMCS 2004.
Page 47/47

Future Directions

- GoI 2 : Non-converging algms (untyped λ-calc / PCF)
  - uses more topological info on operatn algs
- GoI 3: uses additives & additive proof nets —
- GoI 4 (last month): von Neumann algebras: $Ex(f, \tau)$ for f arb (not necessarily coming from proof)
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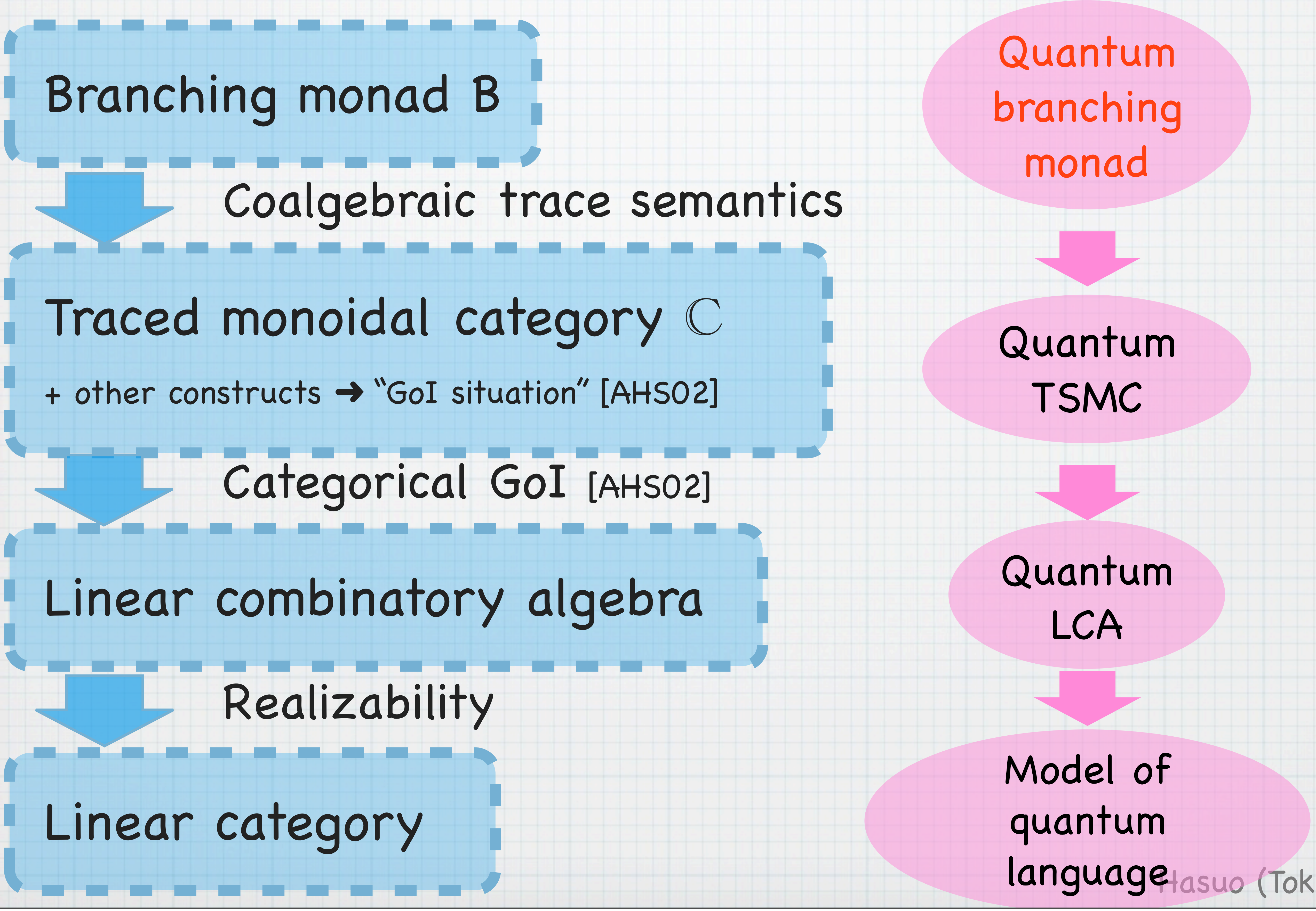

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Quantum branching monad

Quantum TSMC

Quantum LCA

Model of quantum language

The Quantum Branching Monad

$$\boldsymbol{\mathcal{Q}Y} = \left\{ \boldsymbol{c : Y \to \prod_{m,n\in\mathbb{N}} \mathbf{QO}_{m,n}} \;\middle|\; \text{the trace condition} \right\}$$

The Quantum Branching M

$$\mathbf{QO}_{m,n} := \left\{ \begin{array}{l} \text{quantum operations,} \\ \text{from dim. } m \text{ to dim. } n \end{array} \right\}$$

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$$\sum_{y \in Y} \sum_{n \in \mathbb{N}} \mathbf{tr}\big[(c(y))_{m,n}(\rho) \big] \le 1 \;,$$

$$\forall m \in \mathbb{N},\ \forall \rho \in D_m .$$

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$$\forall m \in \mathbb{N},\ \forall \rho \in D_m.$$

- Compare with

$$\mathcal{P}Y = \left\{ c : Y \to 2 \right\}$$

$$\mathcal{D}Y = \left\{ c : Y \to [0,1] \;\middle|\; \sum_{y\in Y} c(y) \leq 1 \right\}$$

The Quantum Branching M

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The Quantum Branching Monad

$$\mathcal{Q}Y = \left\{ c : Y \to \prod_{m,n\in\mathbb{N}} \mathbf{QO}_{m,n} \;\middle|\; \underline{\text{the trace condition}} \right\}$$

$$\sum_{y\in Y}\sum_{n\in\mathbb{N}} \mathbf{tr}\left[\left(c(y)\right)_{m,n}(\rho)\right] \le 1 \; , \qquad \forall m \in \mathbb{N},\ \forall \rho \in D_m .$$

$$\begin{array}{c} X \xrightarrow{f} Y \ \text{ in } \mathcal{K}\ell(\mathcal{Q}) \\ \hline\hline X \xrightarrow{f} \mathcal{Q}Y \ \text{ in } \mathbf{Sets} \end{array}$$

- Given $x \in X$, $y \in Y$, $m \in \mathbb{N}$, $n \in \mathbb{N}$ determines a quantum operation

$$\Big(f(x)(y) \Big)_{m,n} \; : \; D_m \to D_n$$

- Subject to the **trace condition**

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Any opr. on quantum data:

combination of

- preparation
- unitary transf.
- measurement

The Quantum Branching Monad

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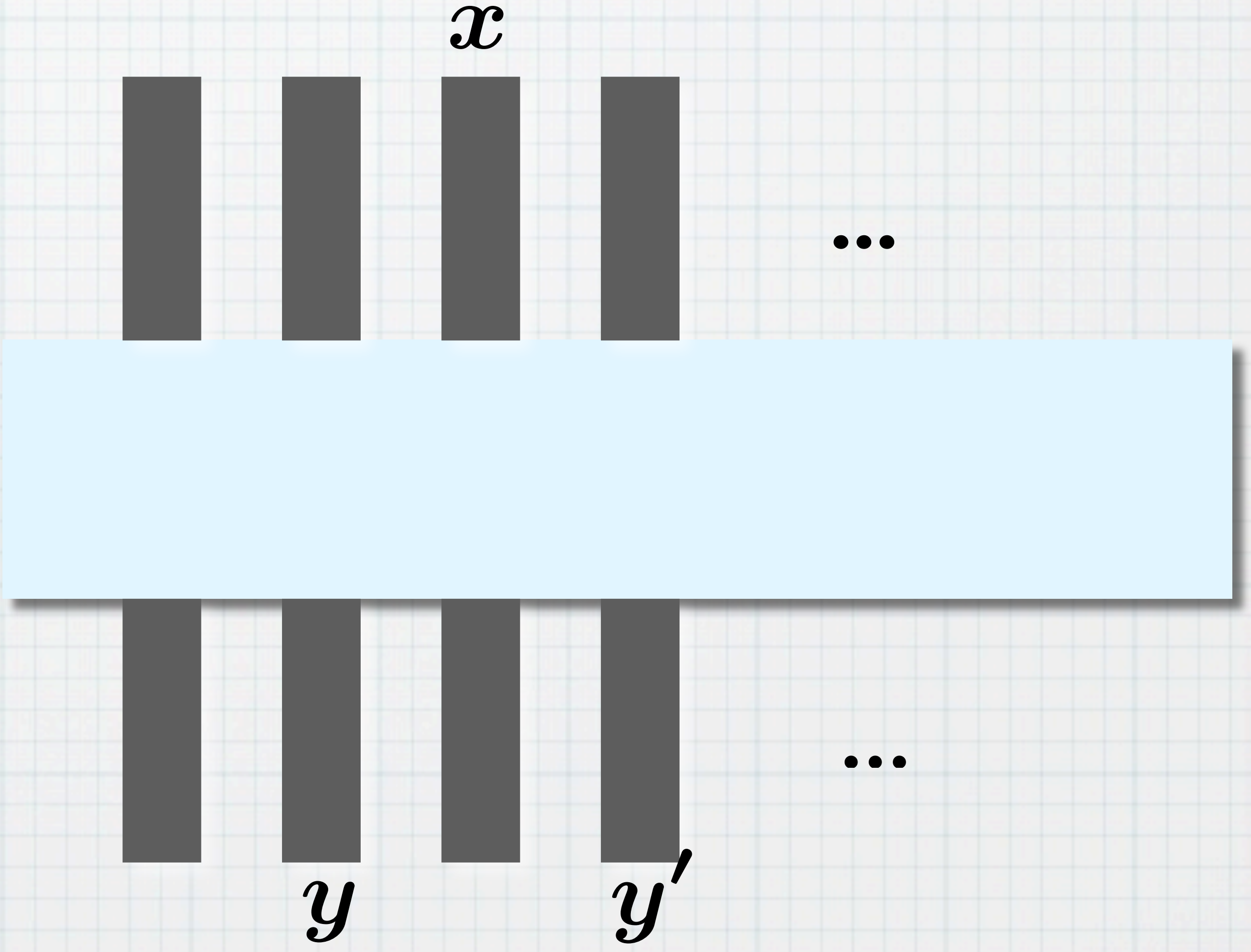

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entrance

exit

in dim.

out dim.

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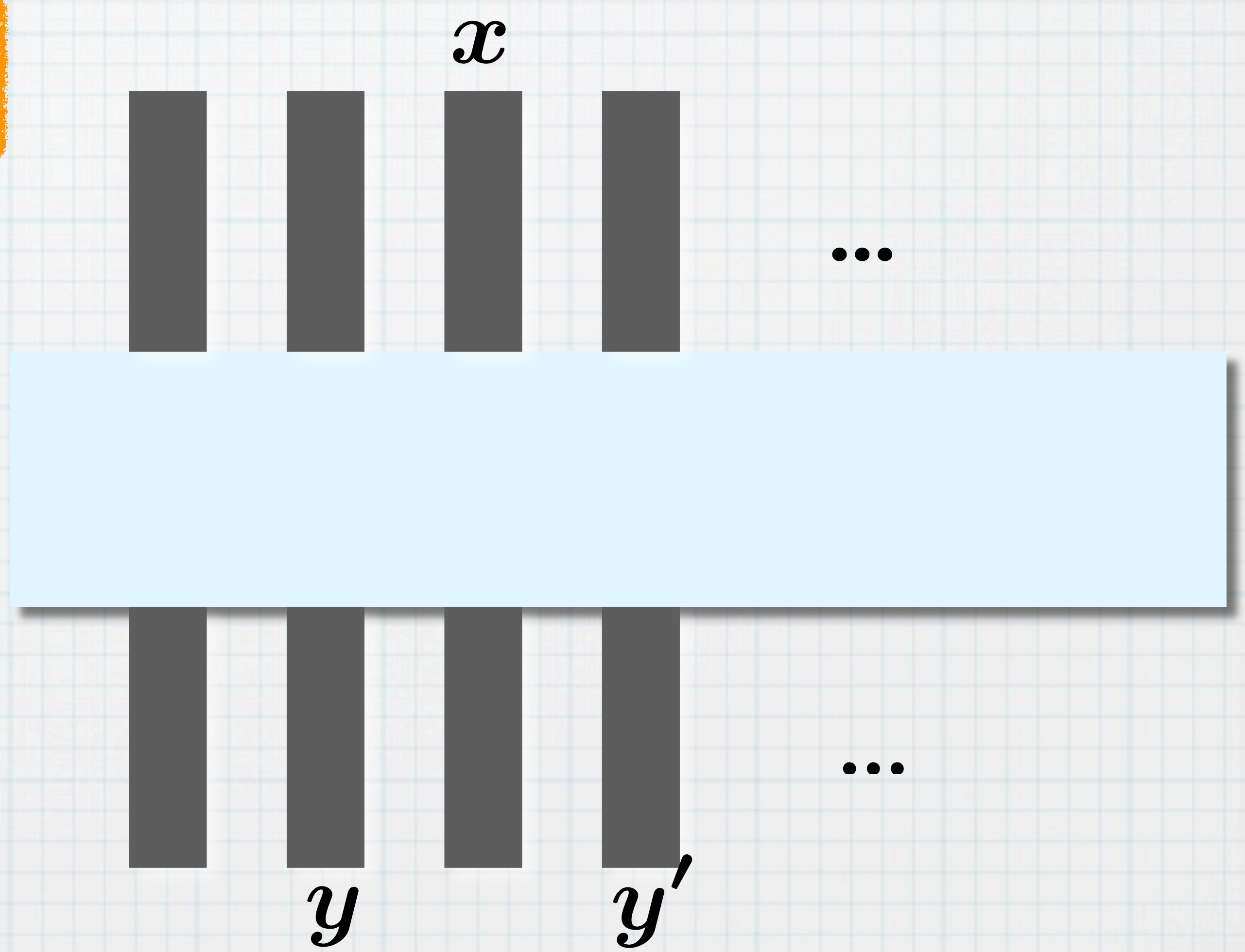

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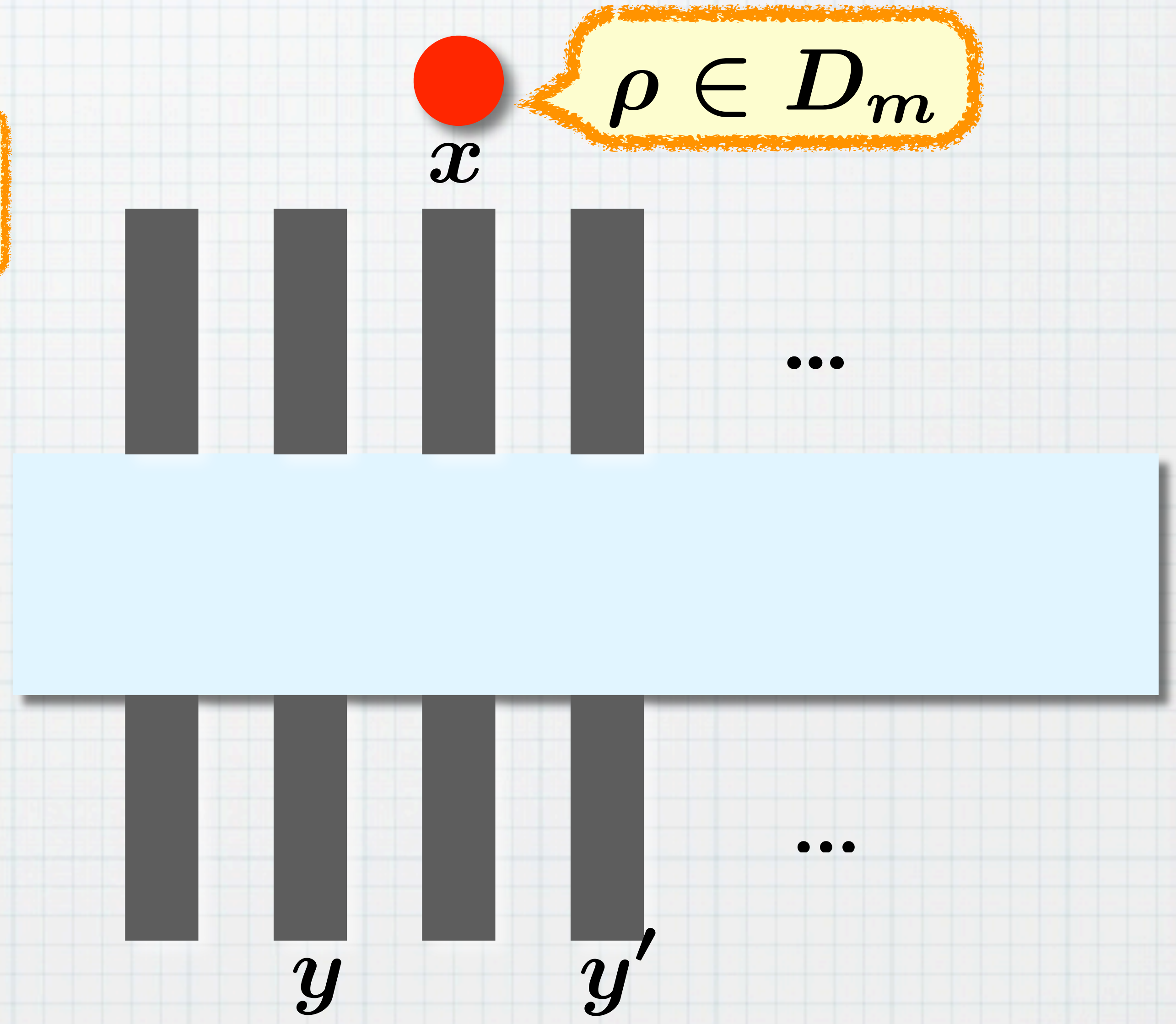

$\rho \in D_m$

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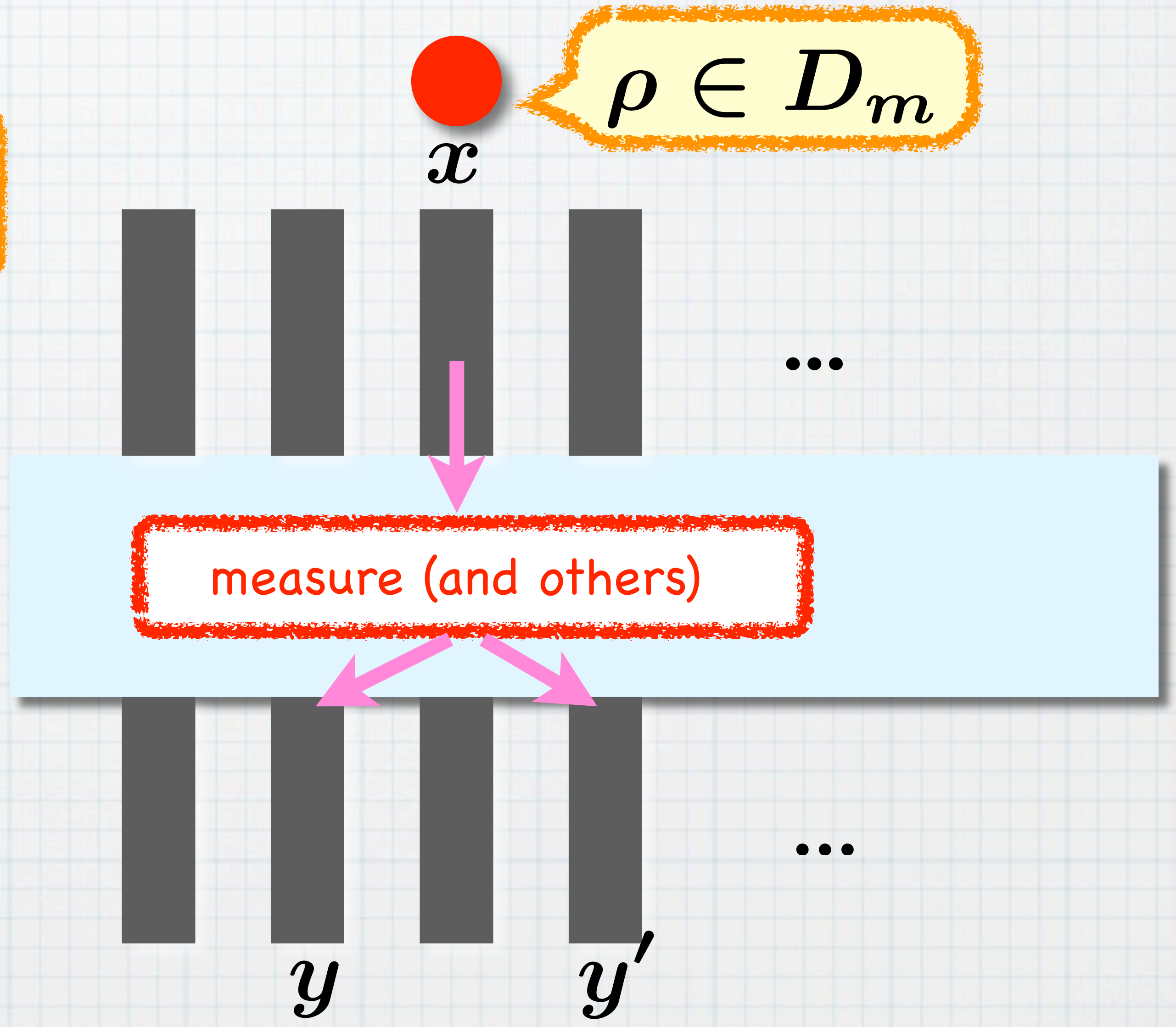

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entrance exit in dim. out dim.

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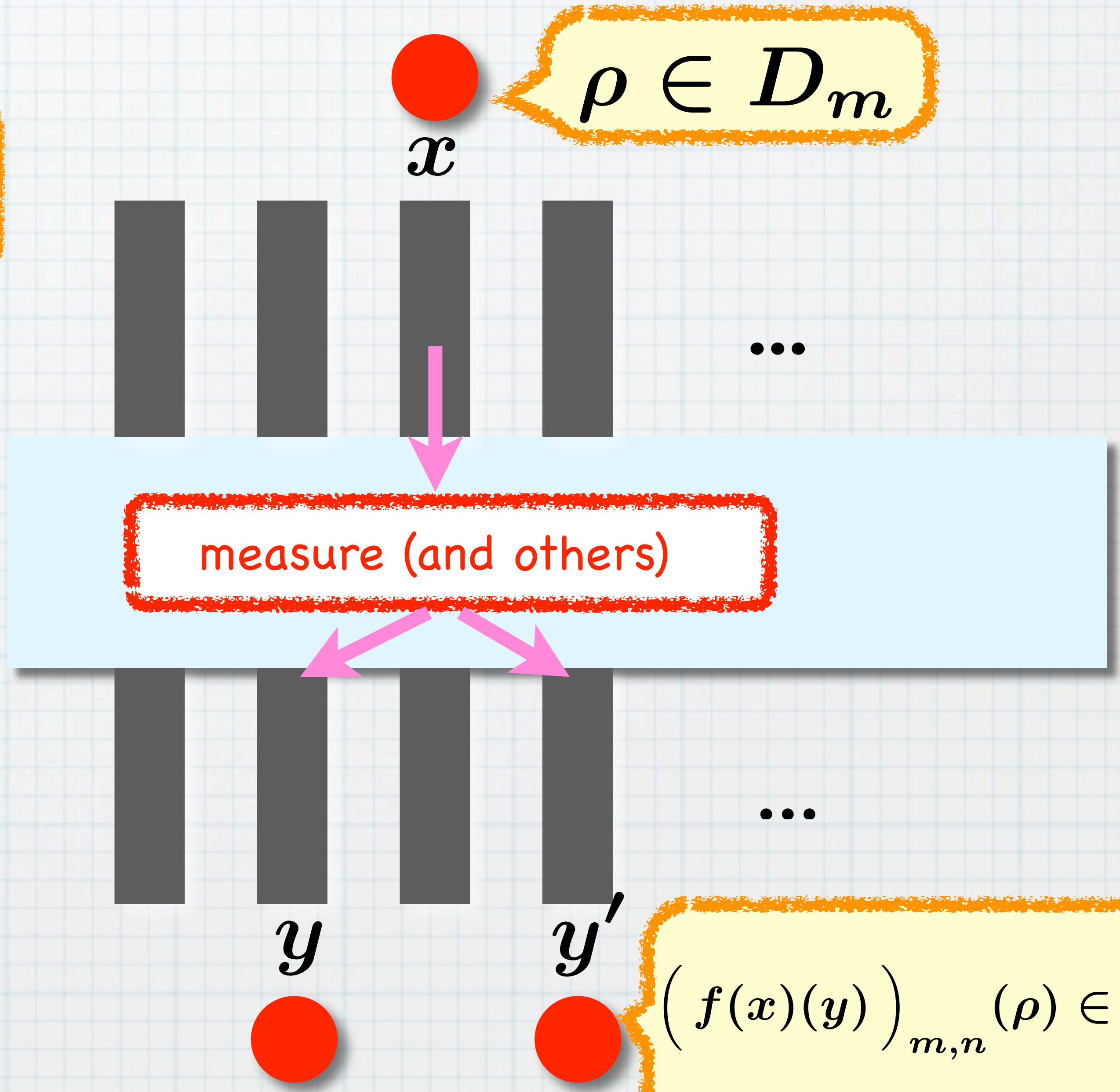

$\rho \in D_m$

$\big(f(x)(y)\big)_{m,n}(\rho) \in D_n$

for each n

The Quantum Branching Monad

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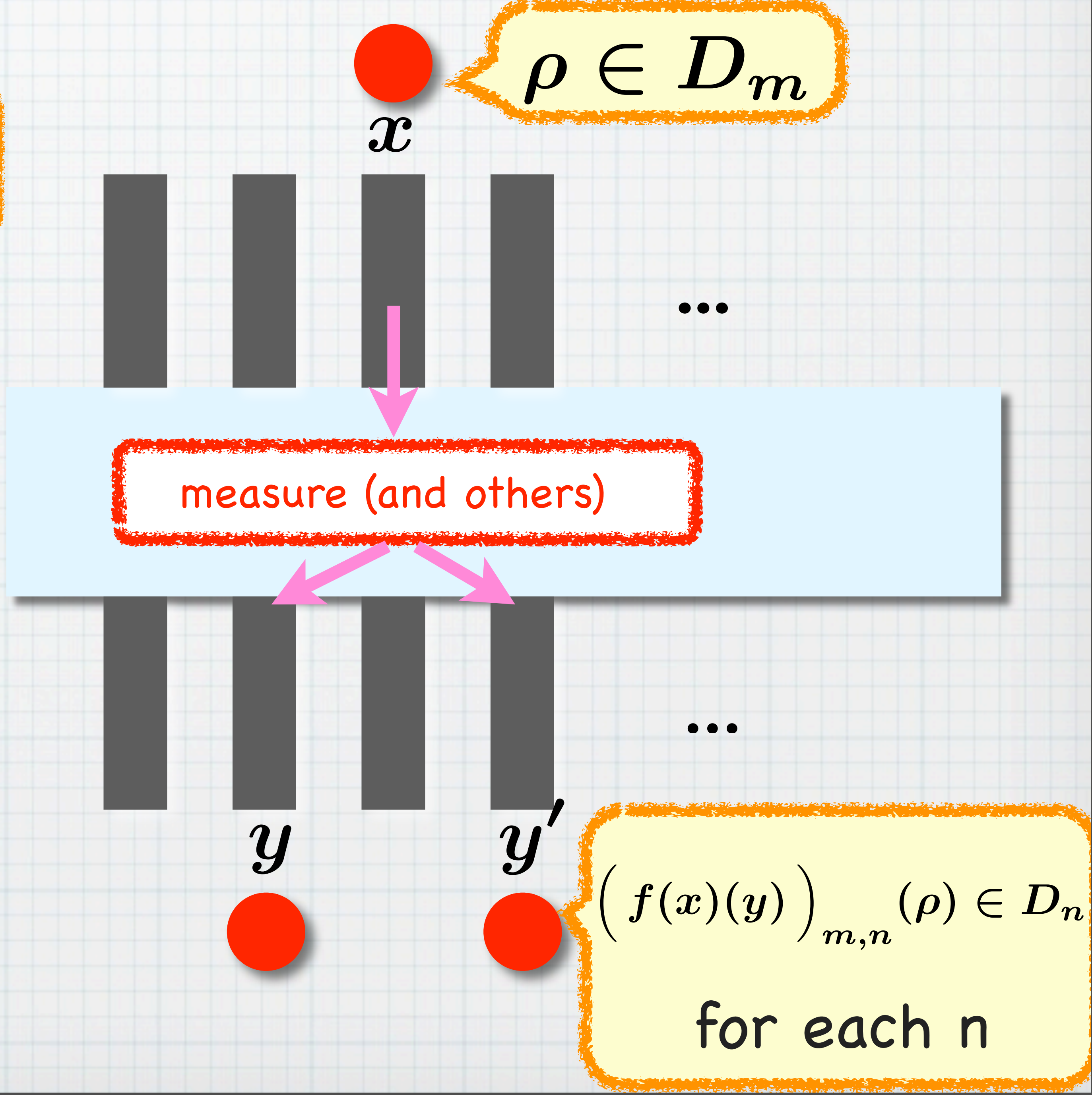
- Given $\overset{\text{entrance}}{x \in X}$, $\overset{\text{exit}}{y \in Y}$, $\overset{\text{in dim.}}{m \in \mathbb{N}}$, $\overset{\text{out dim.}}{n \in \mathbb{N}}$ determines a quantum operation $\big(f(x)(y)\big)_{m,n}$
- **trace cond.:**

$$\sum_{y,n} \Pr\left(\begin{array}{c}\text{Token led}\\ \text{to } y\\ \text{with dim. } n\end{array}\right) \le 1$$

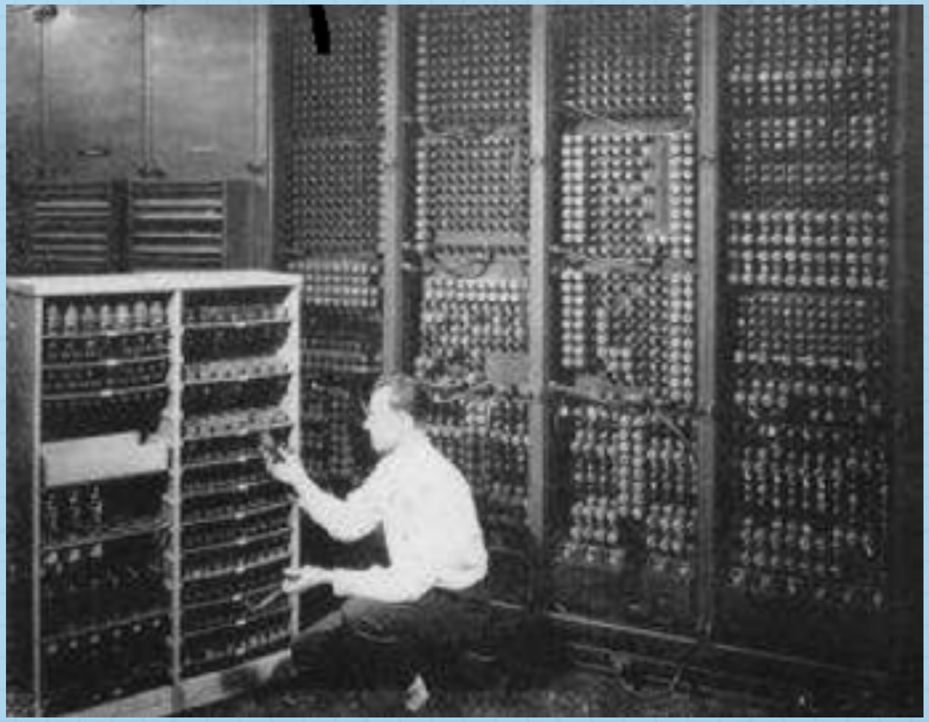


$\big(f(x)(y)\big)_{m,n}(\rho) \in D_n$

for each n

“Quantum Data, Classical Control”

Illustration by N. Hoshino



“Quantum Data, Classical Control”

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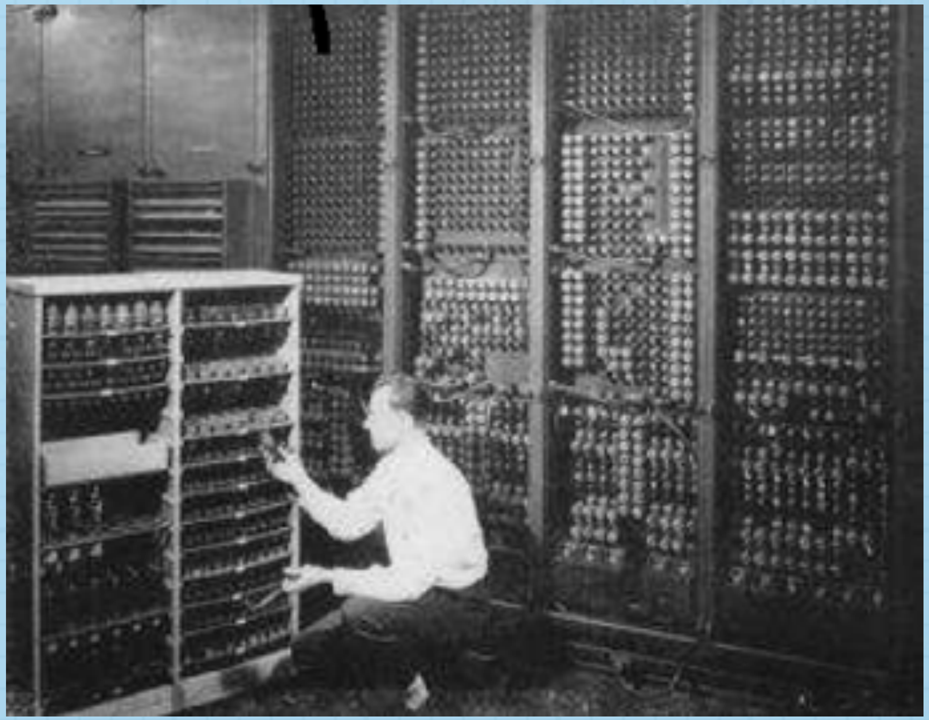
Quantum data

$$\frac{1}{\sqrt{2}}$$

Classical control



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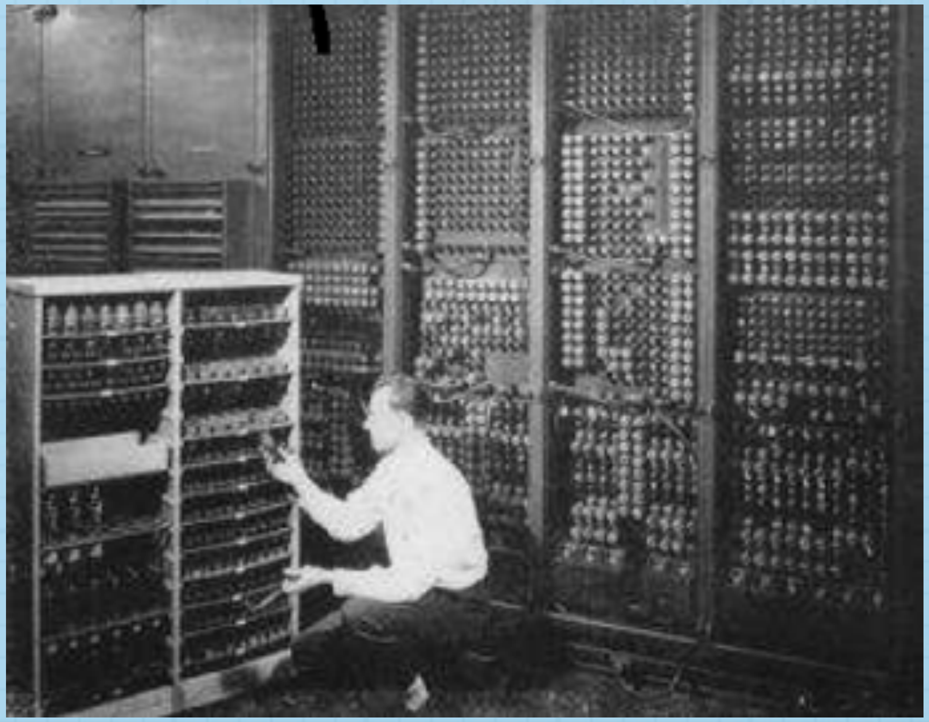
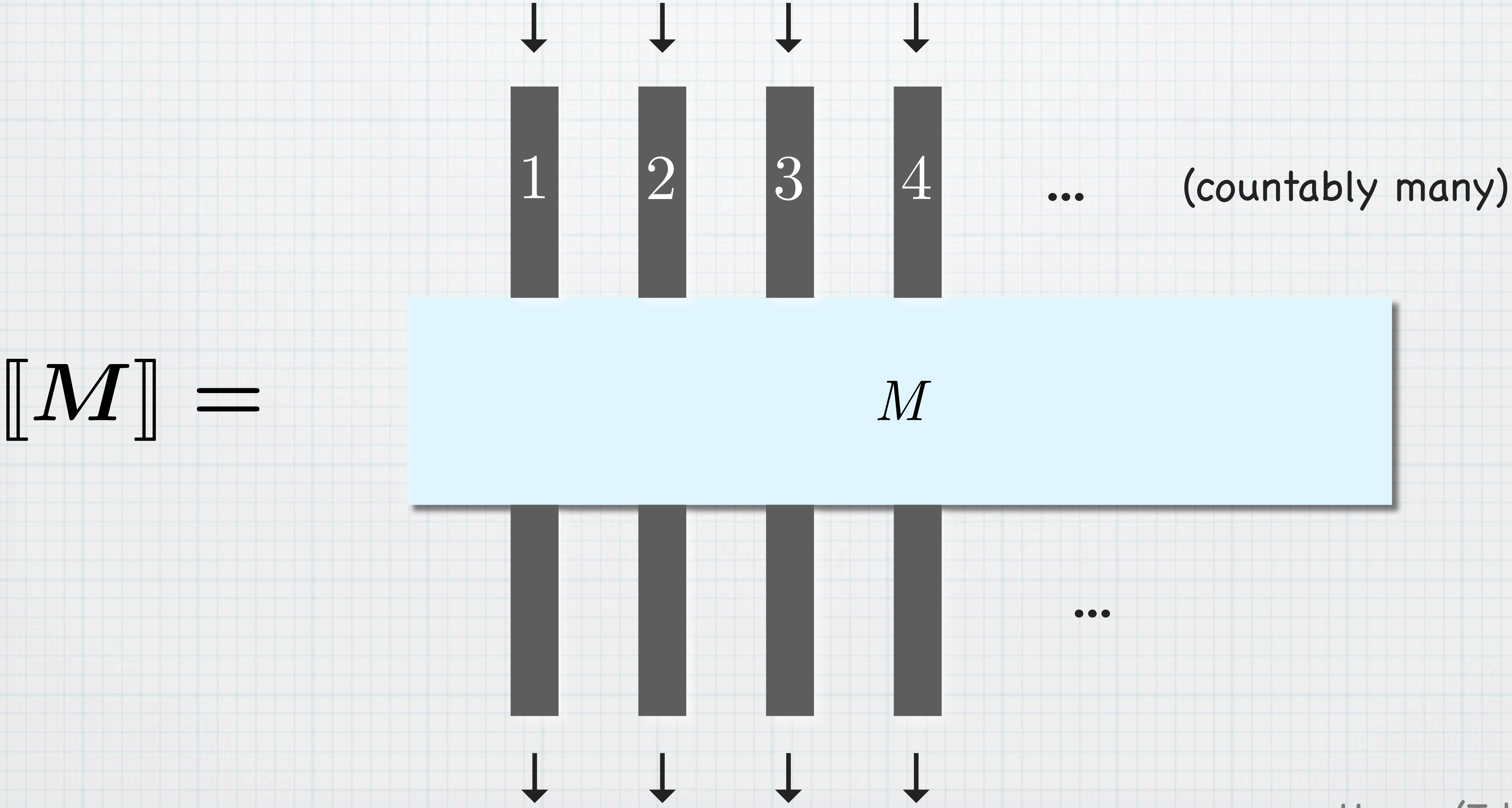


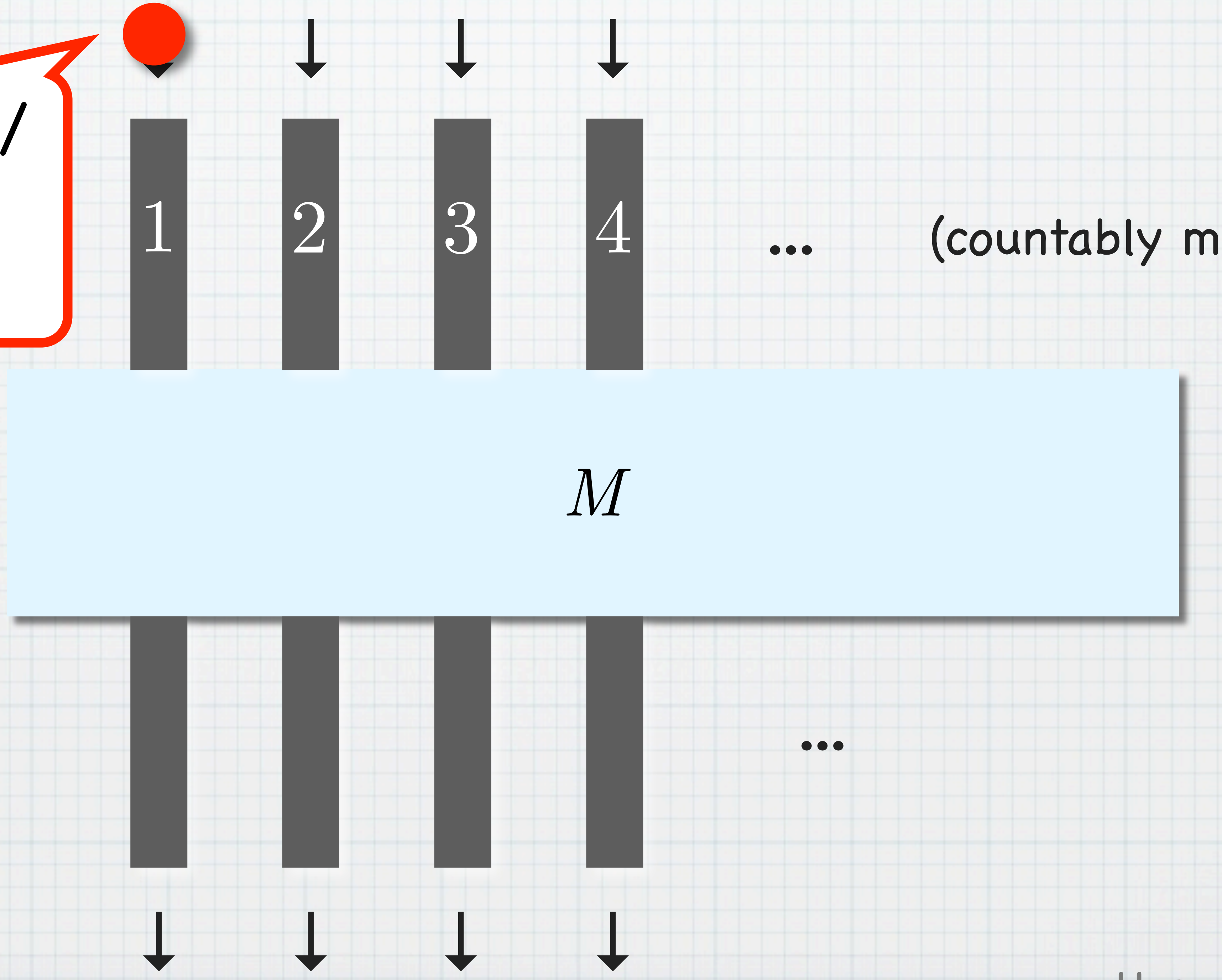
Quantum

Geometry of Interaction



Quantum
Geometry of Interaction

Not just a token/ particle, but quantum state!

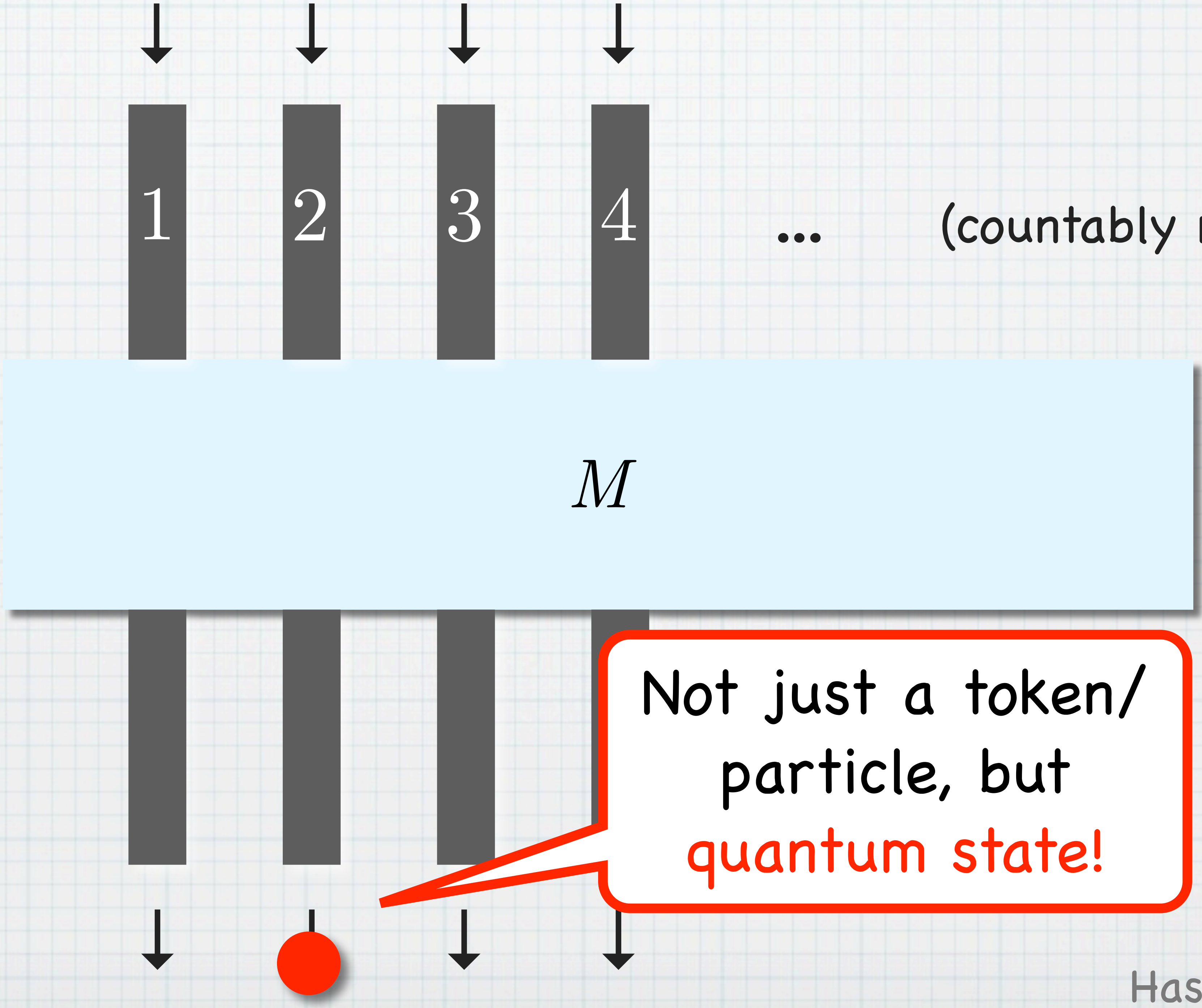


(countably many)

$$[\![M]\!] =$$

Quantum
Geometry of Interaction

$[\![M]\!] =$



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Hasuo (Tokyo)

Quantum
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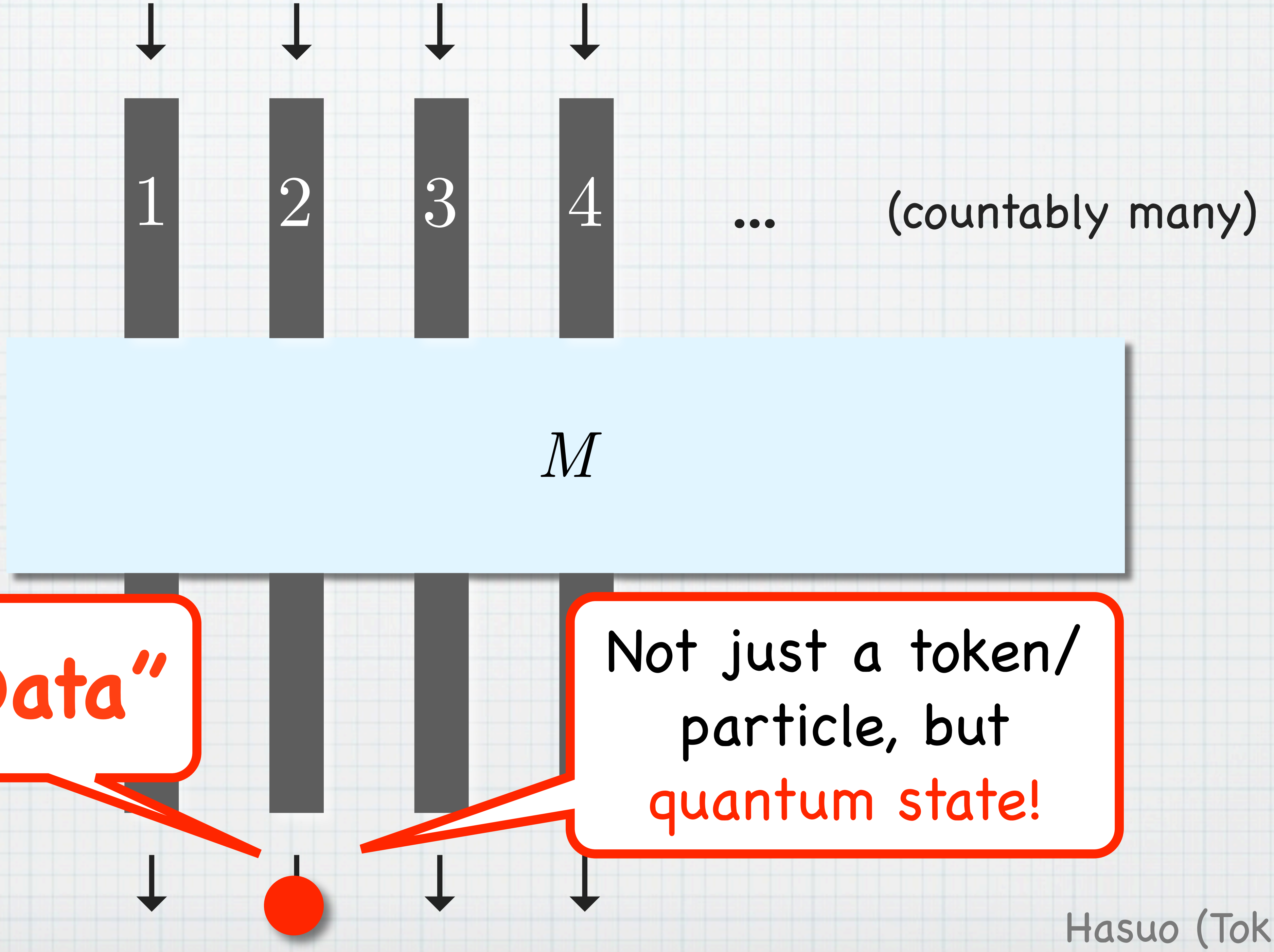

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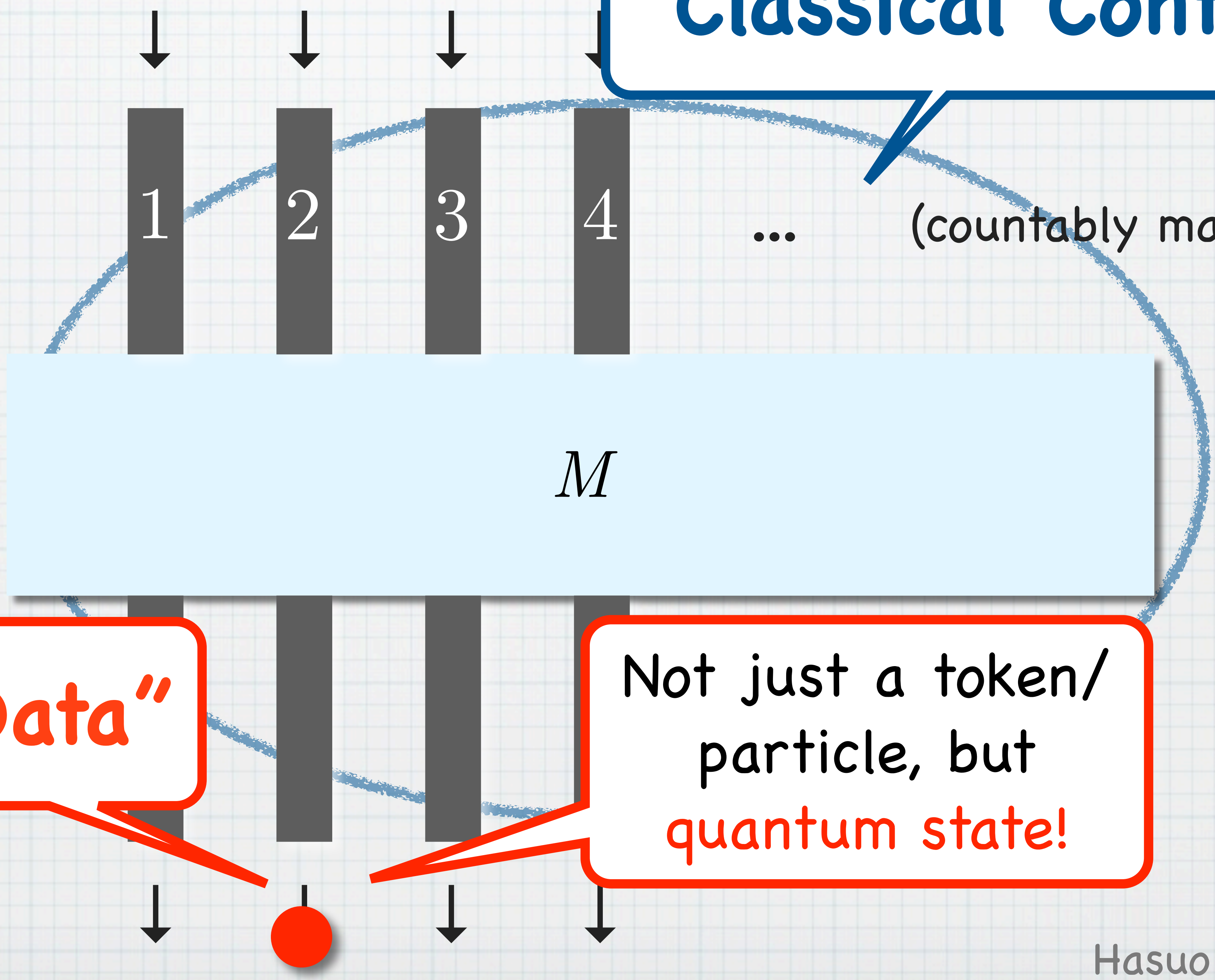


"Classical Control"

(countably many)

"Quantum Data"

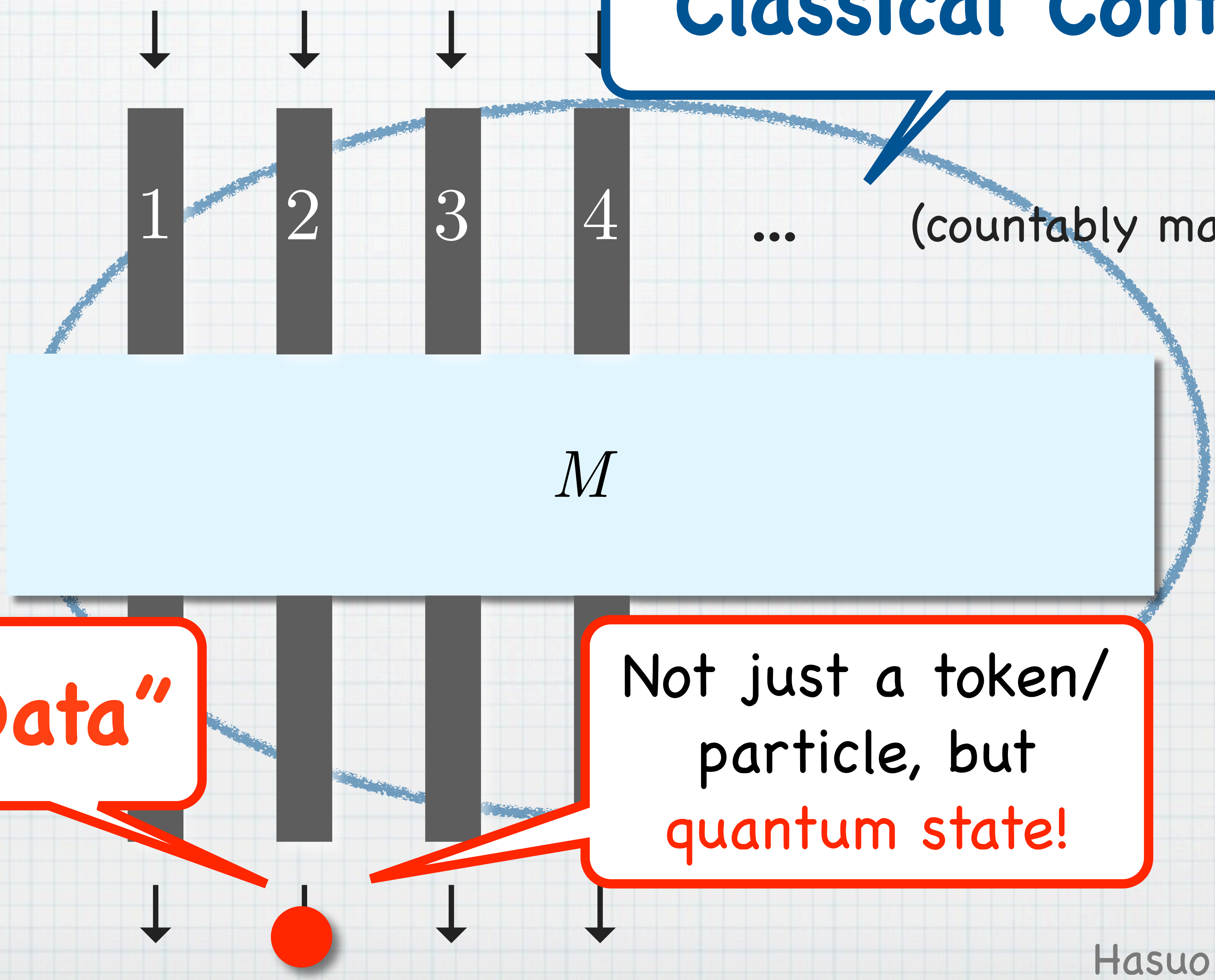
Not just a token/ particle, but quantum state!

Qua
Geometry o

- “in which pipe”
- (measurement ➜ case-distinction) leads a token to different pipes

“Classical Control”

$$[\![M]\!] =$$



(countably many)

“Quantum Data”

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End of the Story?

- No! All the technicalities are yet to come:
  - CPS-style interpretation (for partial measurement)
  - Result type: a final coalgebra in $\mathbf{PER}_Q$
  - **Admissible PERs** for recursion
  - …
- On the next occasion :-)

Results

- The monad Q qualifies as a "branching monad"
- The quantum GoI workflow leads to a linear category $\mathbf{PER}_Q$
- From which we construct an adequate denotational model for a quantum λ-calculus (a variant of Selinger & Valiron's)

Three "Traces"

$$\begin{tikzcd} \boldsymbol{FX} \arrow[r, dashed, "\boldsymbol{F}\mathbf{beh}(\boldsymbol{c})"] & \boldsymbol{FZ} \\ \boldsymbol{X} \arrow[u, "\boldsymbol{c}"] \arrow[r, dashed, "\mathbf{beh}(\boldsymbol{c})"'] & \boldsymbol{Y} \arrow[u, "\text{final}"'] \end{tikzcd}$$

Coinduction in $\boldsymbol{Kl}(\mathbf{B})$

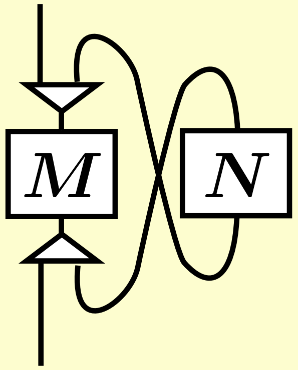
Coalgebraic **Trace** Semantics

appl.

Traced monoidal category

Categorical GoI

[Abramsky, Haghverdi, Scott]



Quantum λ-calculus

Measurements by **tracing out** matrices

Conclusions & Future Work

- Coalgebraic technologies in **interaction-based denotational semantics**
  - GoI, games (AJM/HO), token machines, ...
- Dynamic/operational stuff: not only in concurrency theory!
- Simplifying our model; lang. w/ "quantum store"
  - Ongoing w/ N. Hoshino, T. Roussel, C. Faggian

Thank you for your attention!

Ichiro Hasuo (Dept. CS, U Tokyo)

http://www-mmm.is.s.u-tokyo.ac.jp/~ichiro/

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