Categorical Geometry of Interaction and Application to Higher-Order Quantum Computation

Based on: IH & N. Hoshino, **Semantics of Higher-Order Quantum Computation via Geometry of Interaction,** Proc. LICS 2011. (Extended ver. coming soon)

Ichiro Hasuo
University of Tokyo (JP)

Naohiko Hoshino
RIMS, Kyoto University (JP)

Supported by Aihara Innovative Mathematical Modelling Project, FIRST Program, JSPS/CSTP

東京大学
THE UNIVERSITY OF TOKYO

京都大学
KYOTO UNIVERSITY

GaLoP (Queen Mary) 2013/7/19

Highlights

- **Categorical GoI** [Abramsky, Haghverdi, Scott]
  - Categorical axiomatization of "when we can run a **GoI business**"
  - **not** like "category of games"

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  - **not** like "category of games"
- Combined with **coalgebras** [Rutten, Jacobs, ...]
  - Nice operational flavor!
- Application: **quantum λ-calculus** [Selinger, Valiron, van Tonder, ...]
  - "The categorical GoI workflow"

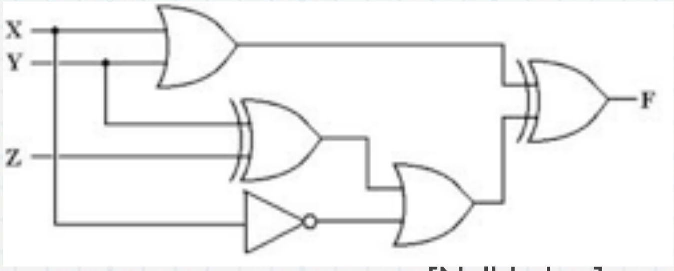
Part 1

Functional QPL: Some Contexts

Quantum Programming Language

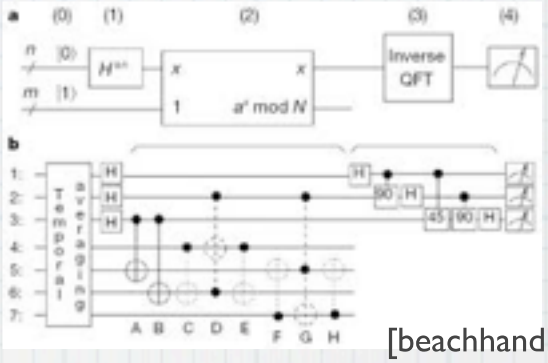
Classical	Quantum
(Boolean) circuit	Quantum circuit

(Boolean) circuit



[Null-Lobur]

Quantum circuit



[beachhandball.es]

Programming language

```
int i,j;
int factorial(int k)
{
    j=1;
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        j=j*i;
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}
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Quantum Programming Language

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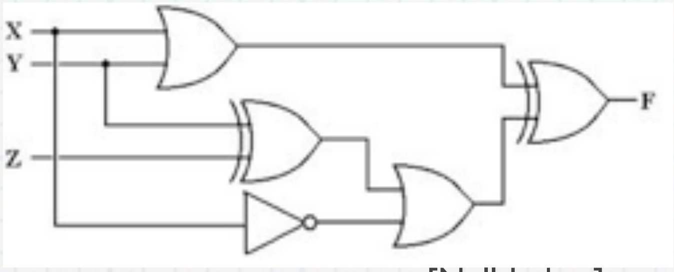
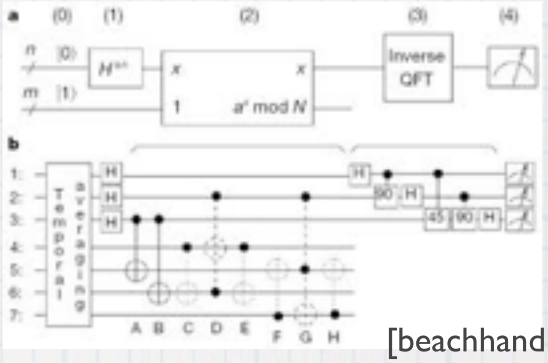
(Boolean) circuit	 [Null-Lobur]	Quantum circuit	 [beachhandball.es]

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$$\begin{aligned}\mathbf{telep} = \quad & \mathit{let}\ \langle x, y\rangle = \mathbf{EPR} * \mathit{in} \\ & \quad \mathit{let}\ f = \mathbf{BellMeasure}\ x\ \mathit{in} \\ & \quad \mathit{let}\ g = \mathbf{U}\ y \\ & \quad \mathit{in}\ \langle f, g\rangle.\end{aligned}$$

[Selinger-Valiron]

Quantum Programming Language

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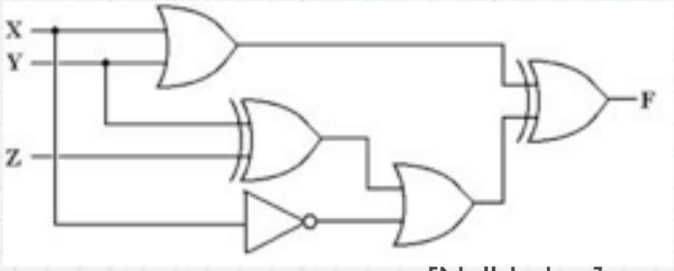

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Quantum

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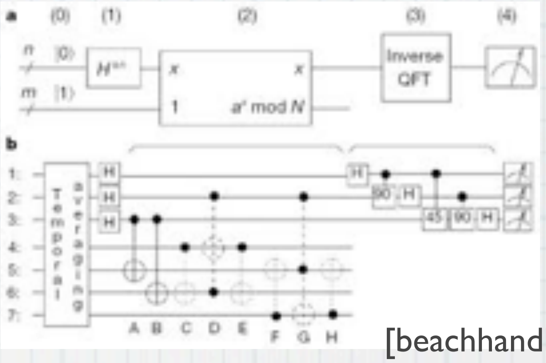

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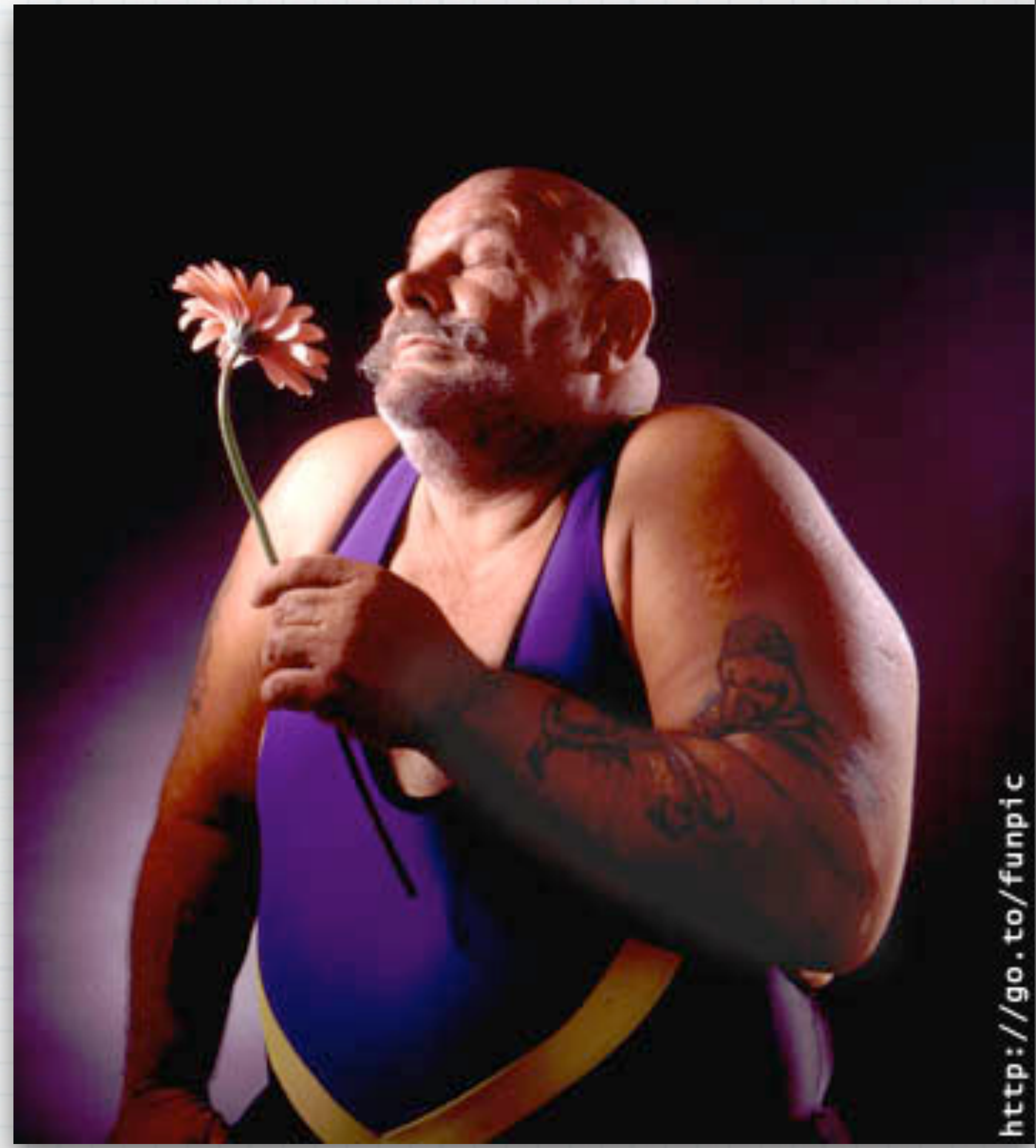
[Selinger-Valiron]

- For discovery of **algorithms**
- For **reasoning, verification**

Functional Quantum Programming Language

Functional Quantum Programming Language

- A **real man**'s programming style



http://go.to/funpic

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Functional Quantum Programming Language

- A **real man's** programming style
- Heavily used in the **financial sector**
- ...

ICFP'11 Sponsers (Tokyo, Sep 2011)

Functional Quantum Programming Language

- A **real man's** programming style
- Heavily used in the **financial sector**
- ...
- **Mathematically nice and clean**
  - Aids semantical study
  - **Transfer** from classical to quantum

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Functional QPL: Syntax

- **Linear λ-calculus**

  + quantum primitives [van Tonder, Selinger, Valiron, ...]

- Linearity for **no-cloning**
  - "Input can be used only once"
  - Not allowed/typable:
  - Duplicable (classical) data: by the **!-modality**

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Friday, July 19, 13

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"arbitrary many copies"

Functional QPL: Semantics

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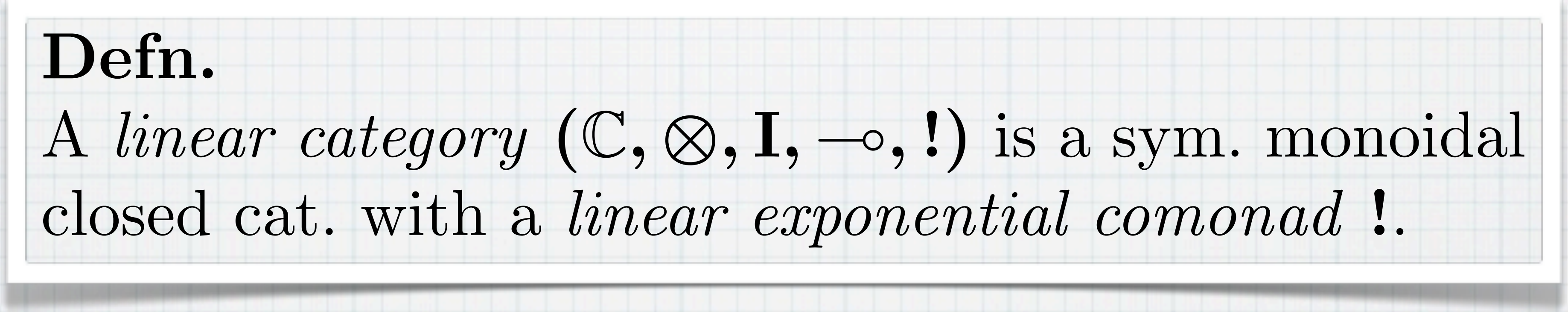
- Denotational semantics

  - **Linear category**: [Benton & Wadler, Bierman]
    (axioms for) a categorical model of linear λ-calculus



Defn.
A *linear category* $(\mathbb{C}, \otimes, \mathbf{I}, \multimap, \mathbf{!})$ is a sym. monoidal closed cat. with a *linear exponential comonad* $\mathbf{!}$.

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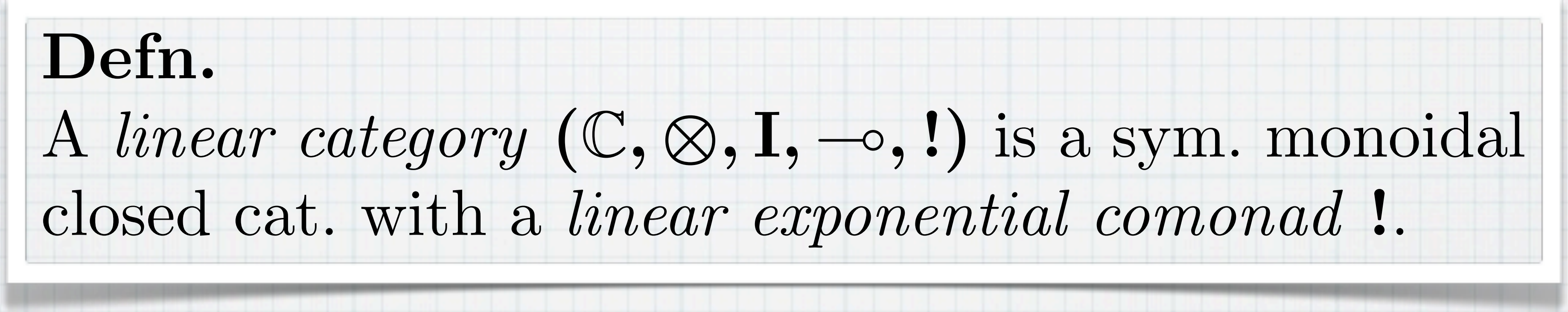

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  - For functional QPL? Is **Hilb** (or alike) a linear cat.?

Functional QPL: Semantics

- **Hilb** (or alike) is **not** a linear category

- **Challenge**: coexistence of **quantum** and **classical** data
- Only partial results
  - [Selinger & Valiron, '08]:
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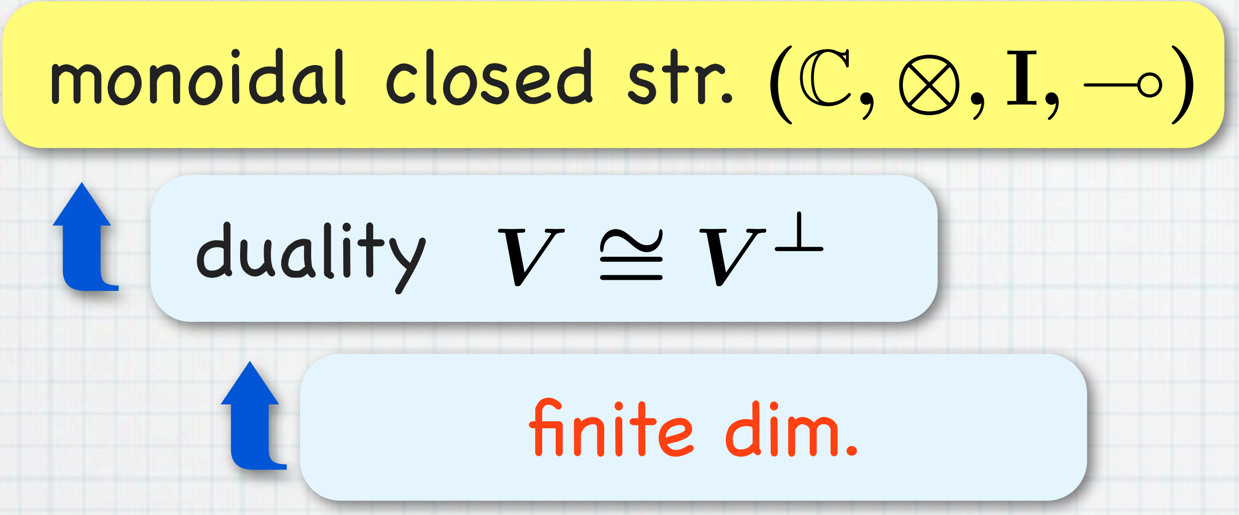

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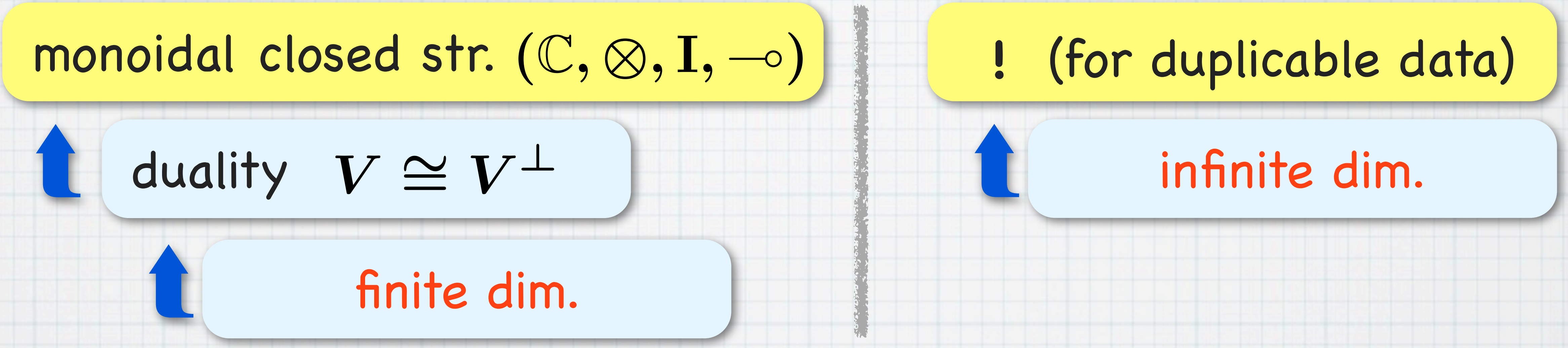

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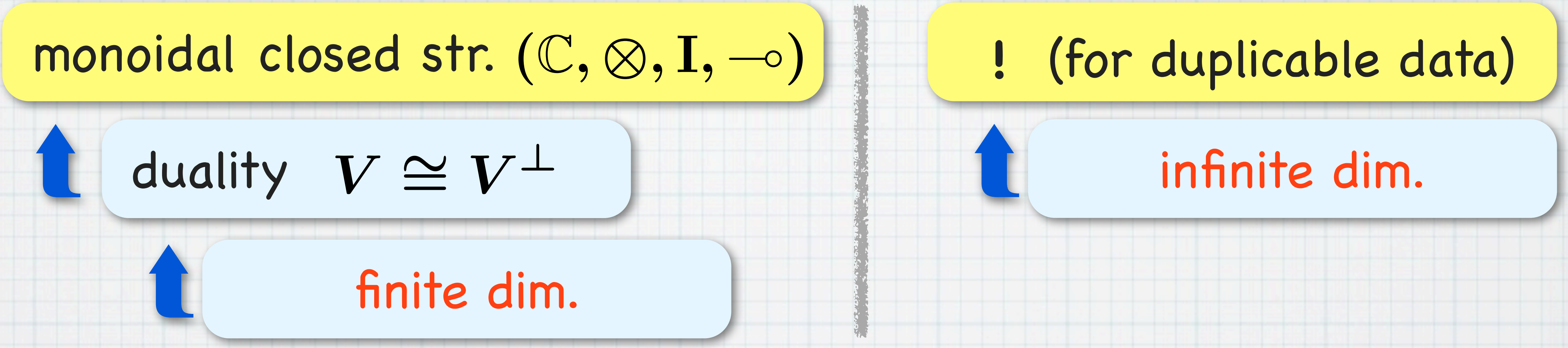

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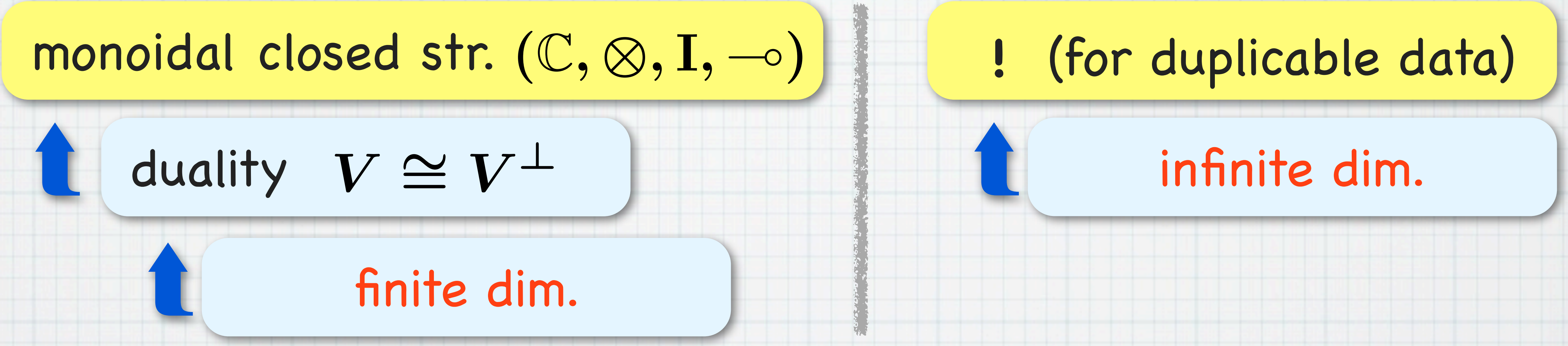

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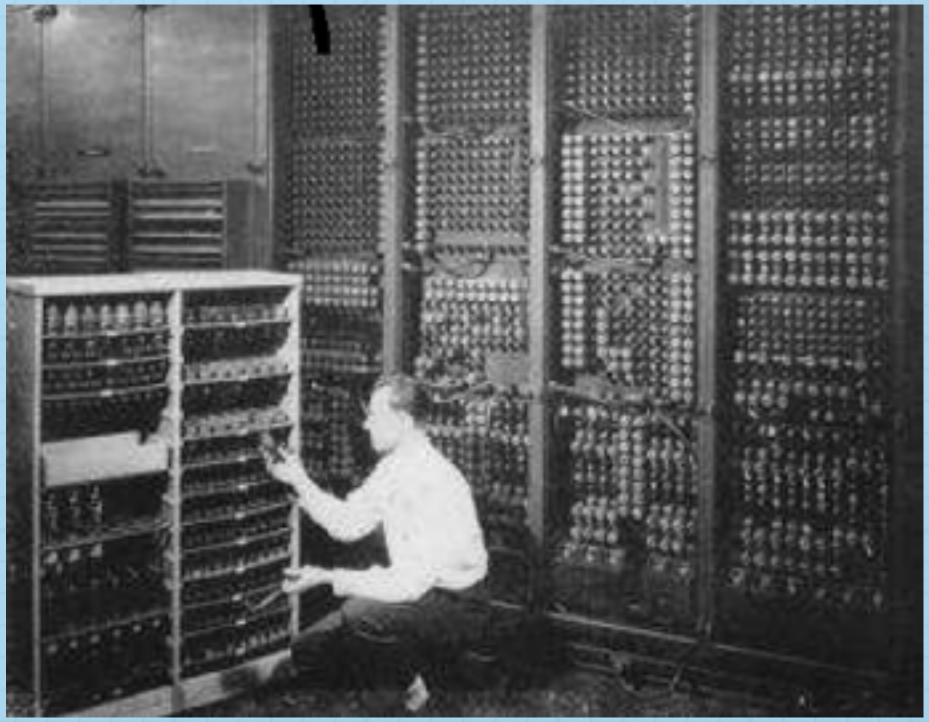
“Quantum Data, Classical Control”

Illustration by N. Hoshino

Quantum data

Classical control



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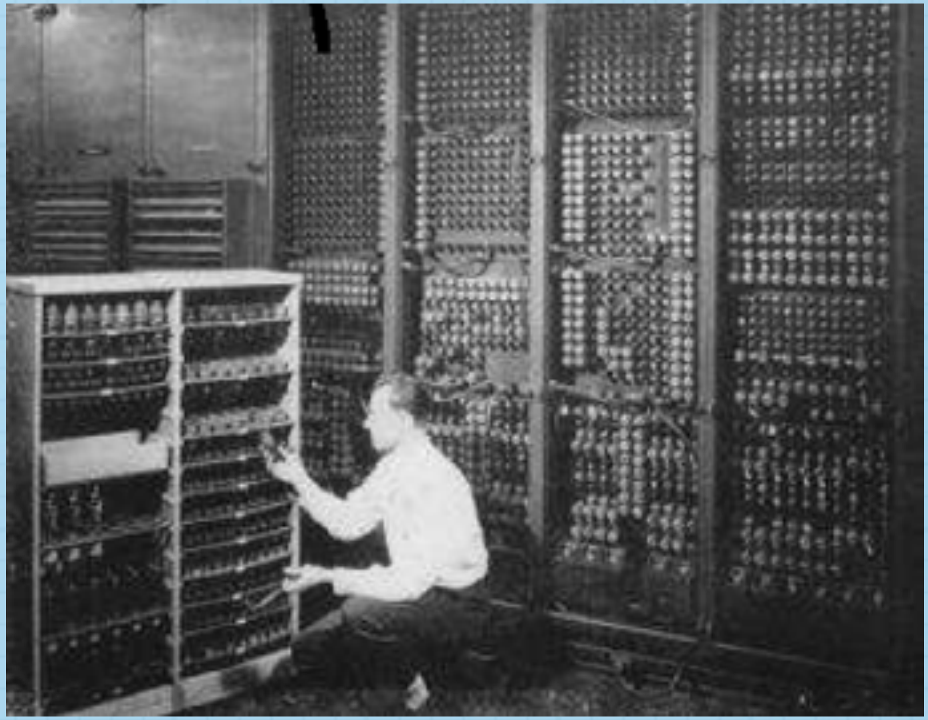

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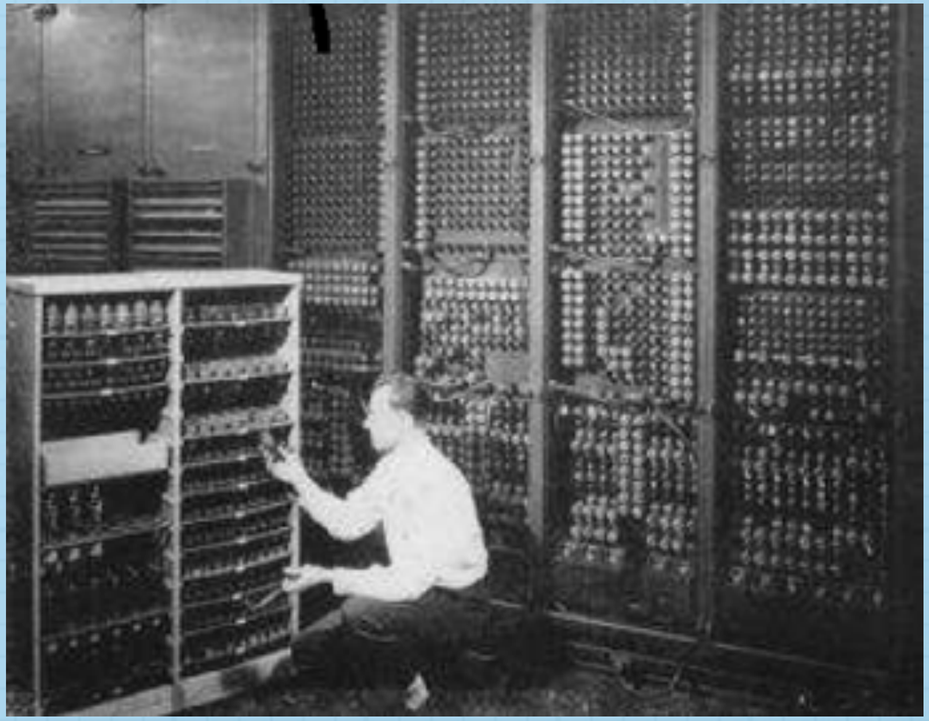

What We Do

- **GoI (Geometry of Interaction)** [Girard '89]

  An "implementation" of **classical control**

$$\mathbf{tr}(f) =$$

$$f_{XY} \sqcup \left(\coprod_{n\in\mathbb{N}} f_{ZY} \circ (f_{ZZ})^n \circ f_{XZ} \right)$$

What We Do

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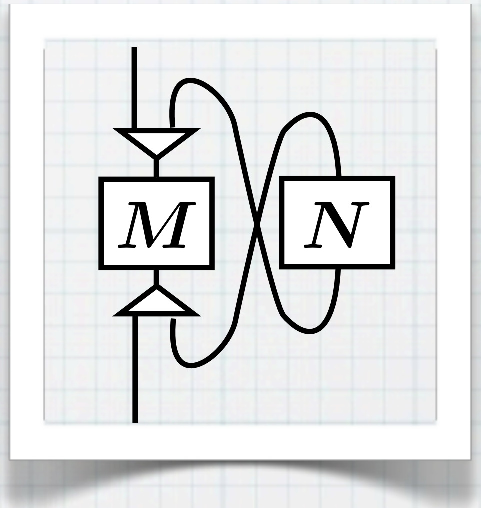

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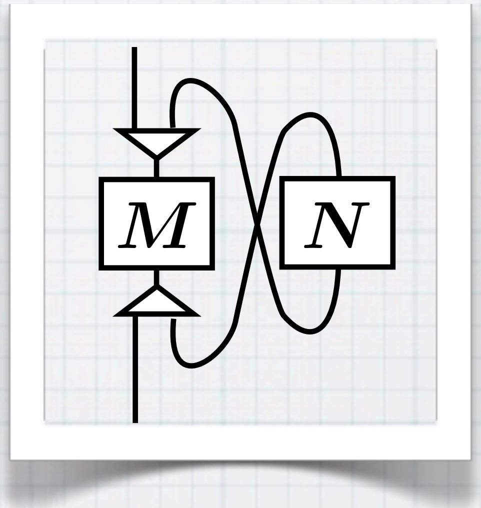
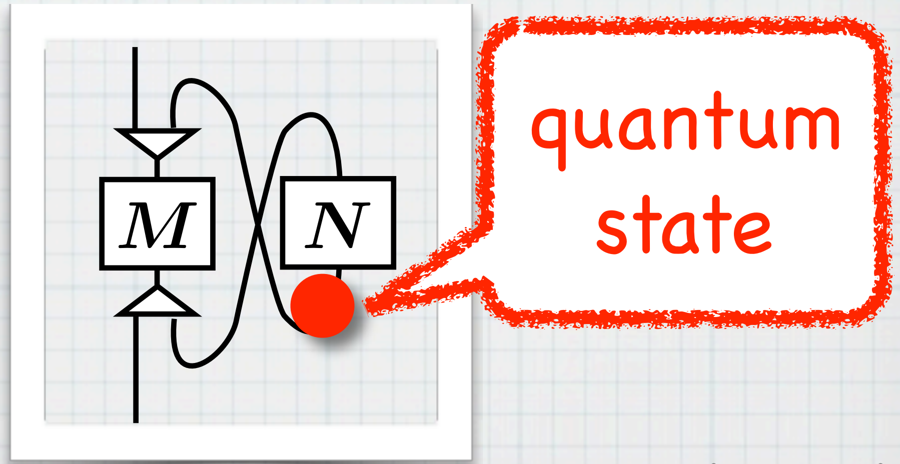


- We add a **quantum layer** to GoI
  - ➜ "Quantum data, classical control"
  - Used: theory of coalgebra

    [Hasuo, Jacobs, Sokolova '07] [Jacobs '10]



Part 2

The Categorical GoI Workflow

GoI:
Geometry of Interaction

- J.-Y. Girard, at Logic Colloquium '88

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Geometry of Interaction

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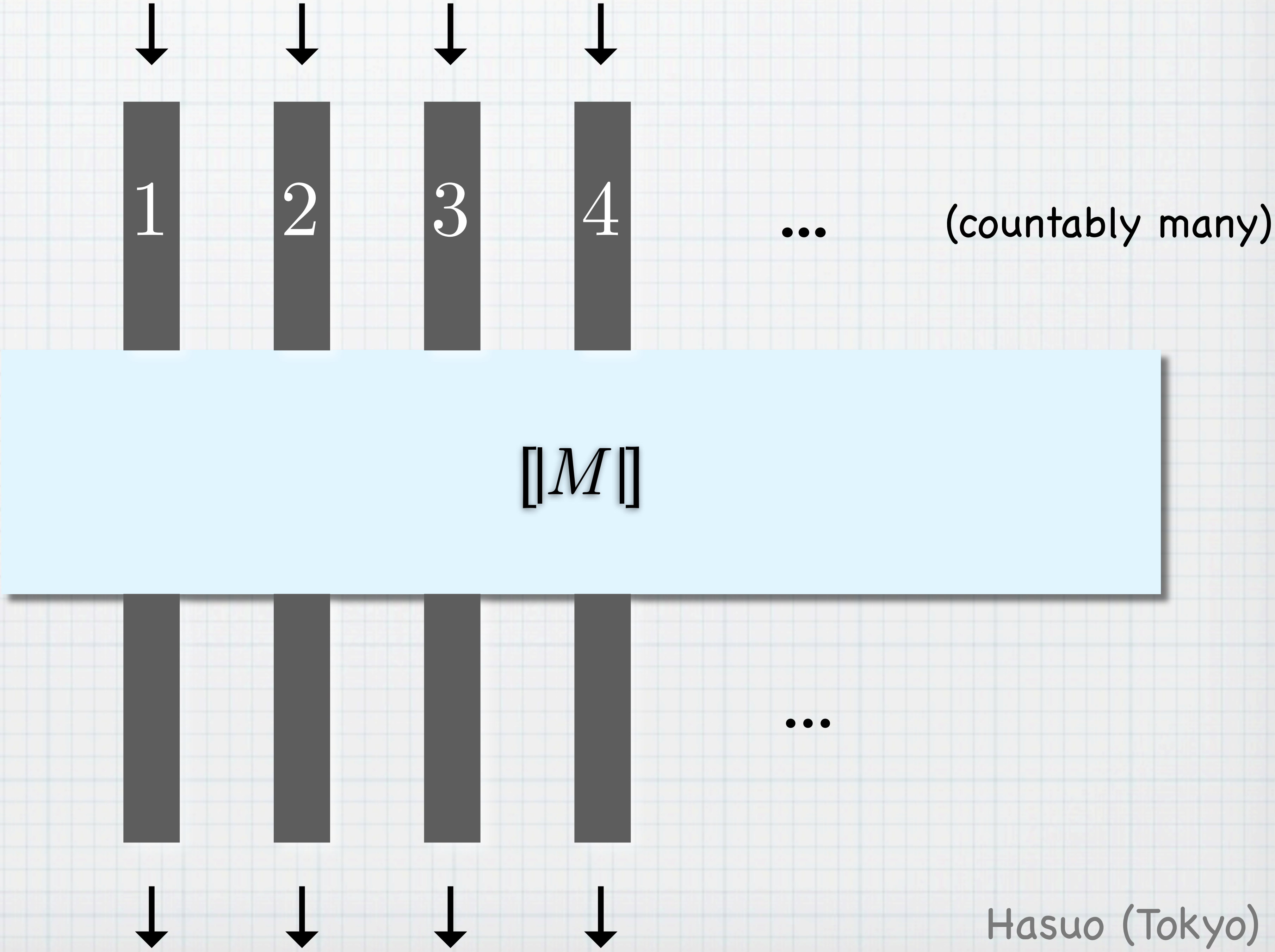
- Provides denotational semantics $[\![M]\!]$ for linear λ-term M
- In this talk:
  - Its categorical formulation [Abramsky, Haghverdi, Scott '02]
  - "The GoI Animation"

The GoI Animation

$$[\![M]\!] \;=\; (\,\mathbb{N} \rightharpoonup \mathbb{N},\;\text{a partial function}\,)$$

$$=\;\text{“piping”}$$



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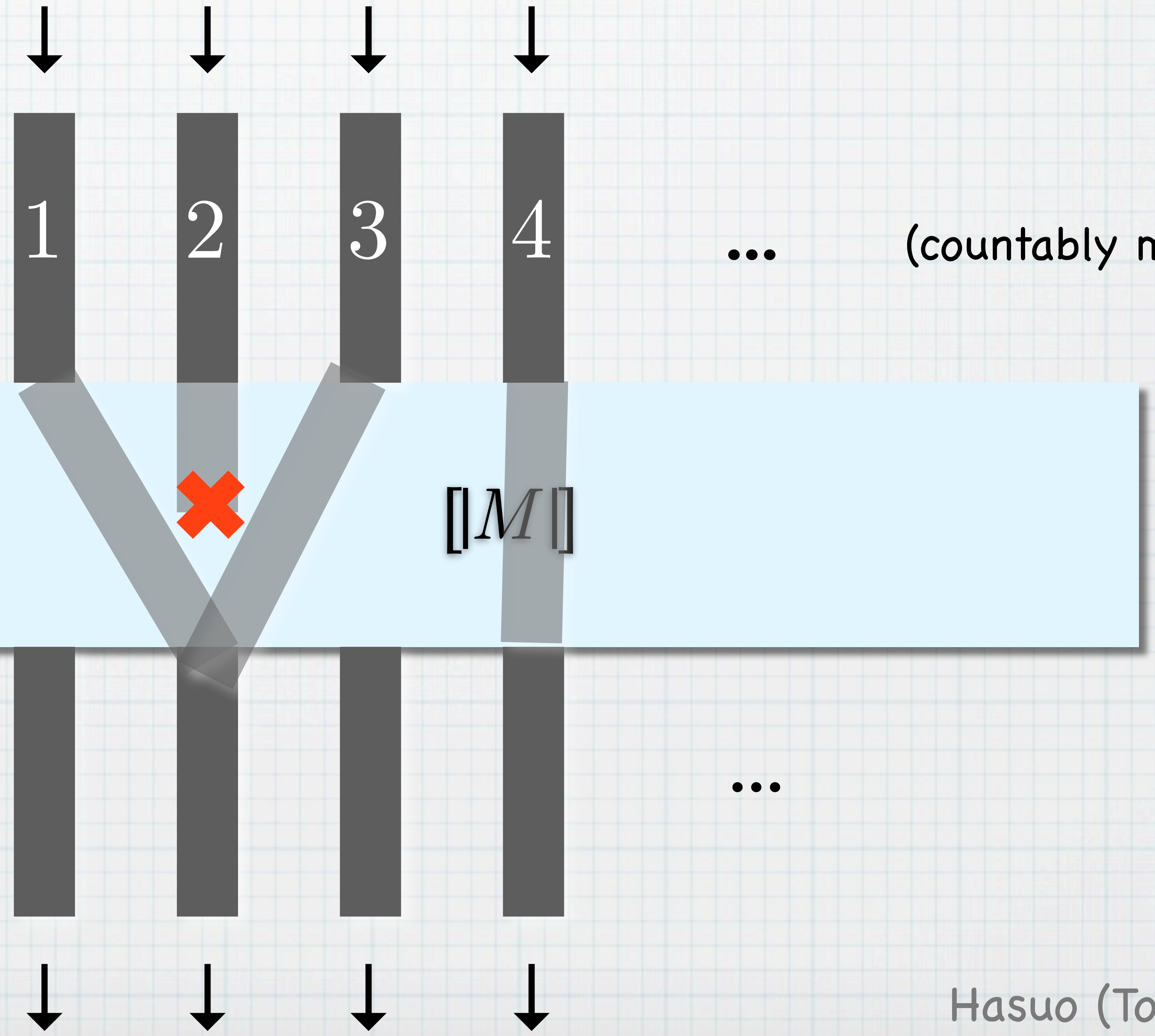

(countably many)

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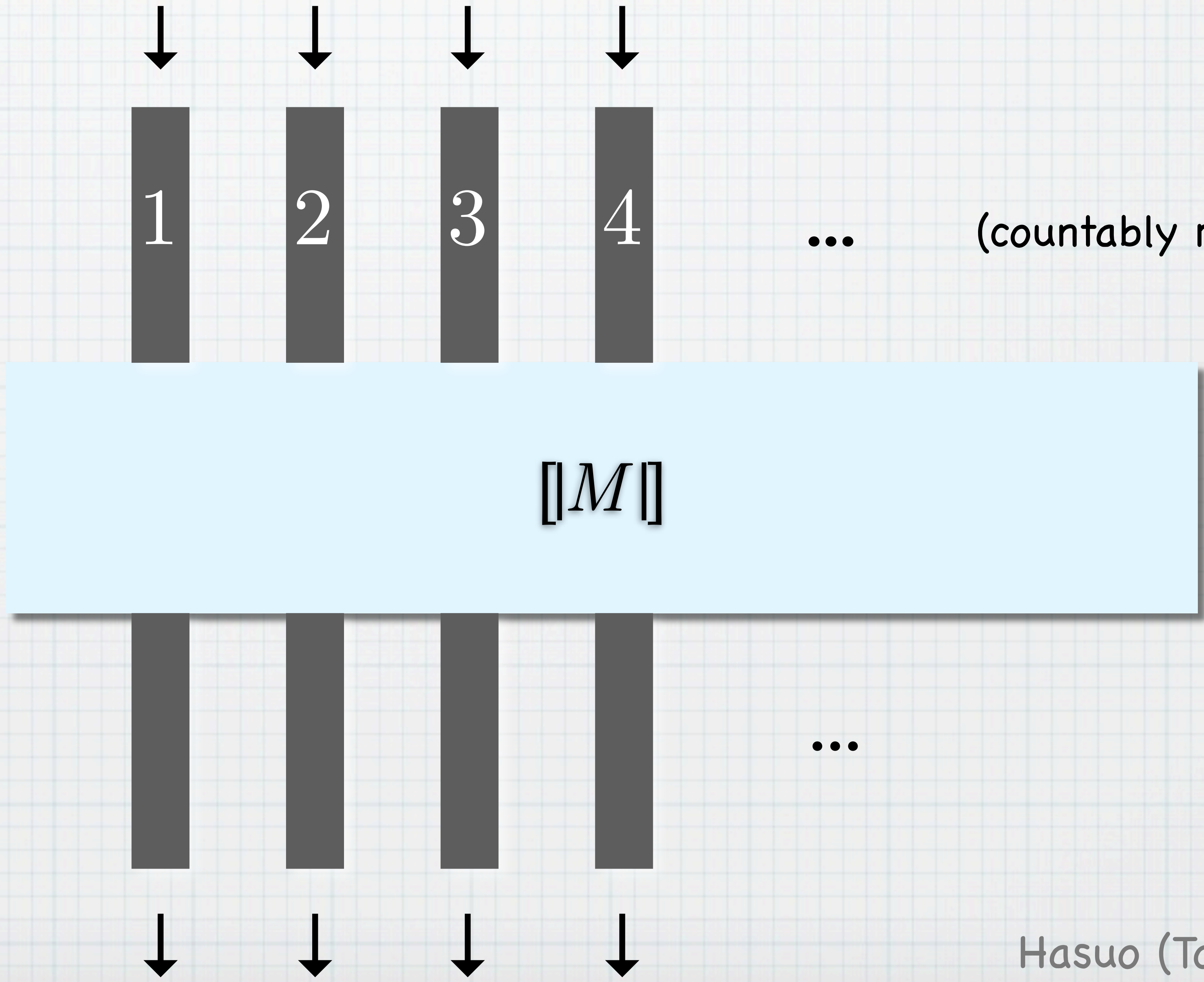

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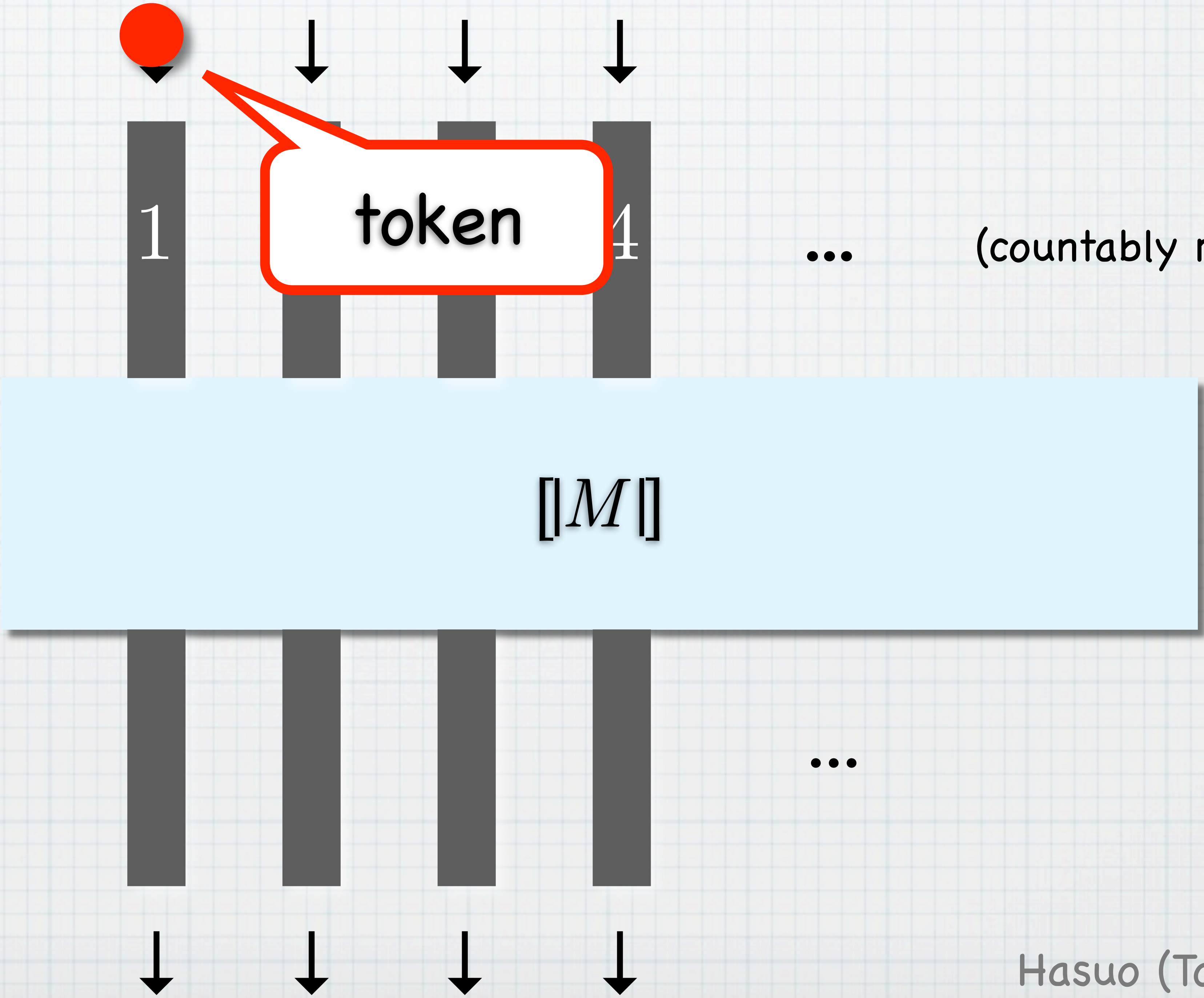

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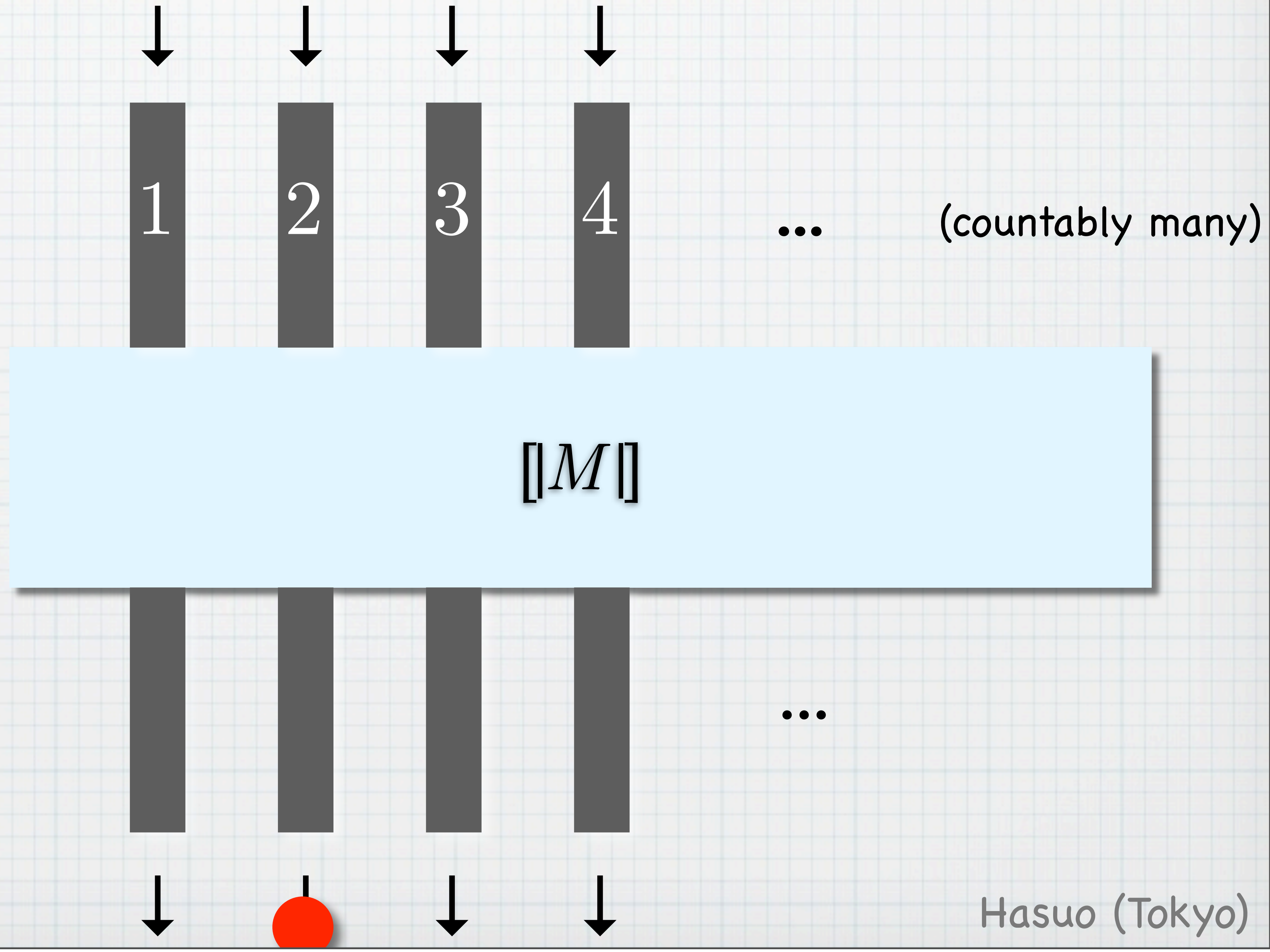

1

2

3

4

...

(countably many)

$[\![M]\!]$

...

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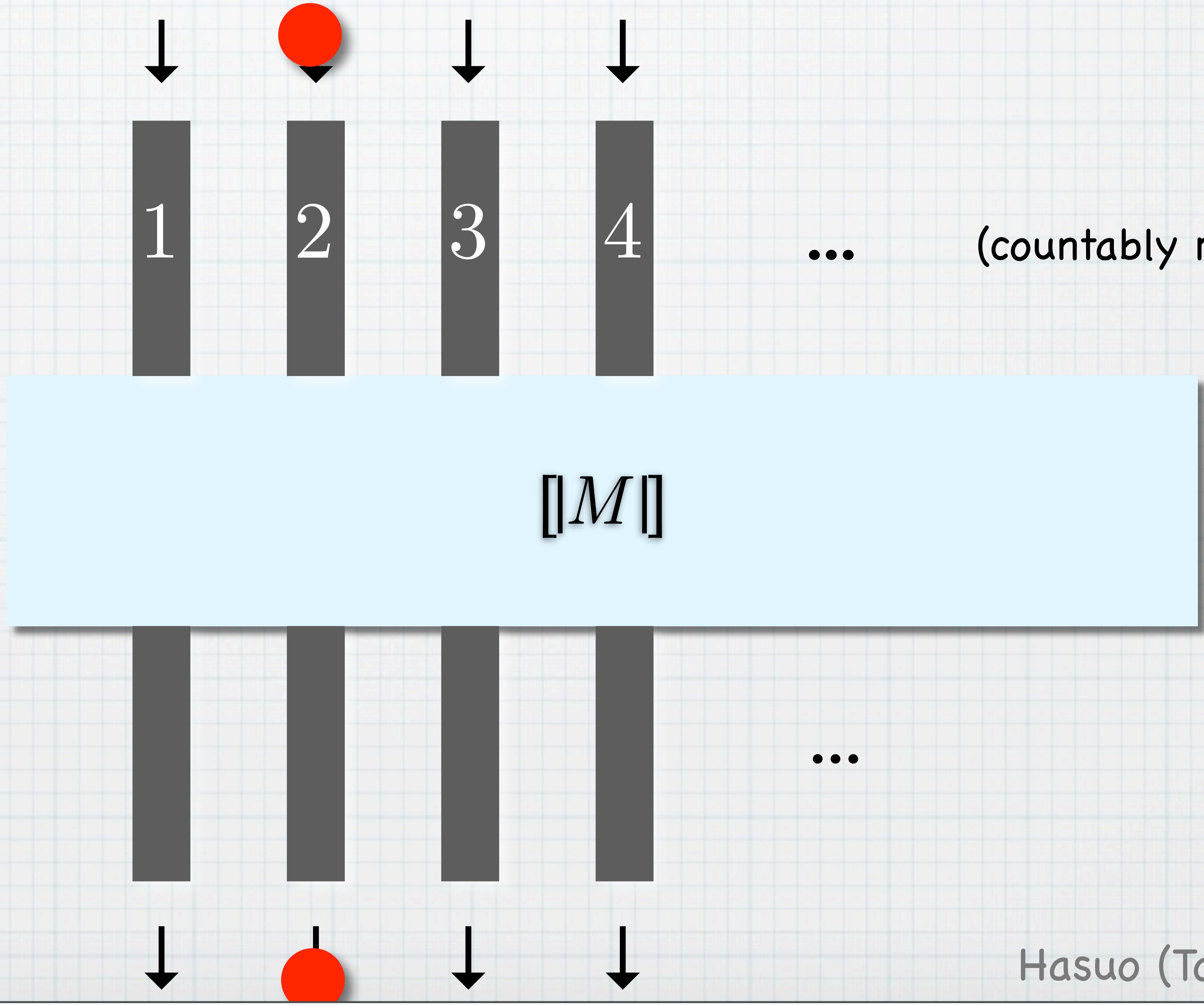

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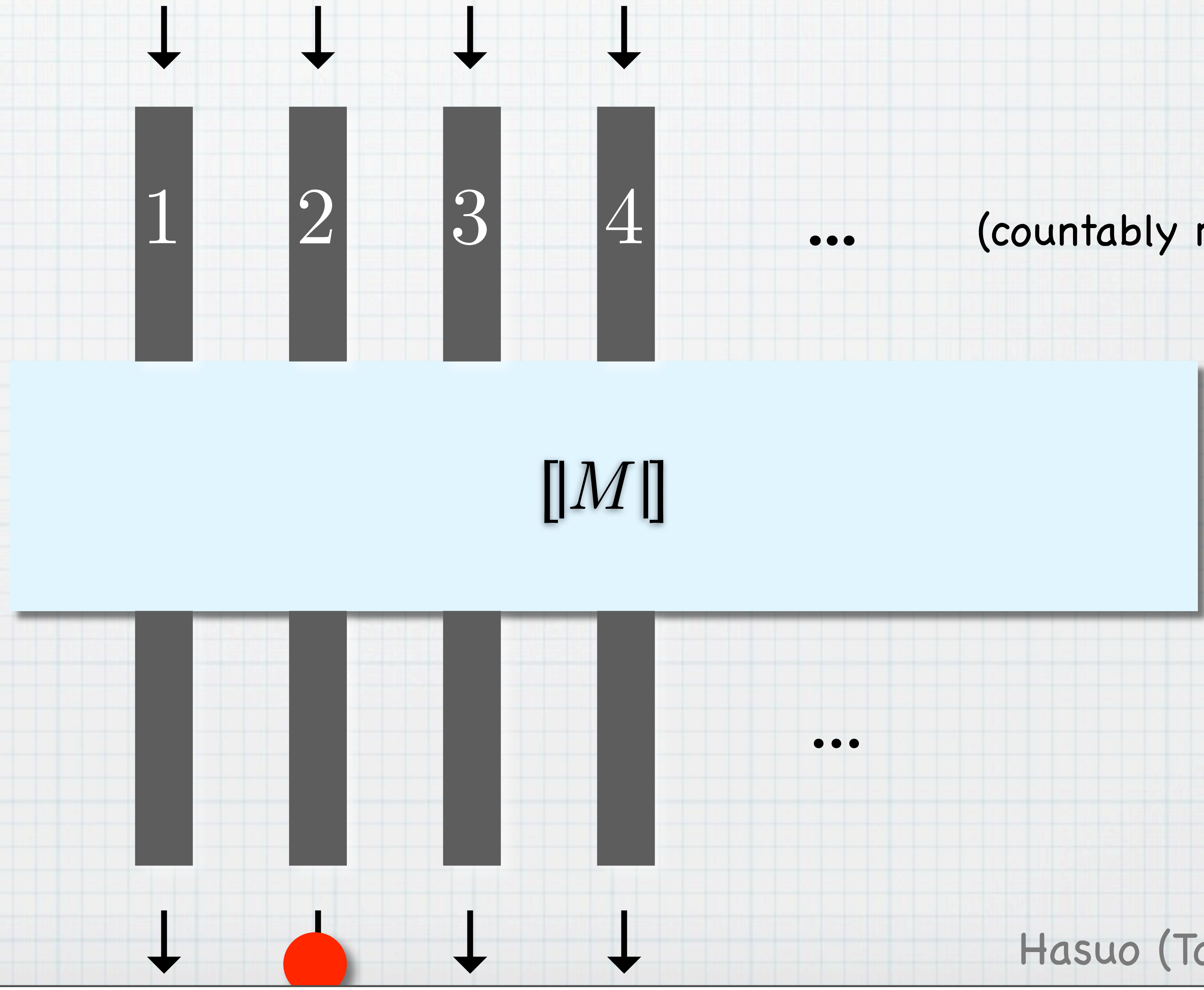

(countably many)

The GoI Animation

- Function application $[\![MN]\!]$
  - by “parallel composition + hiding”

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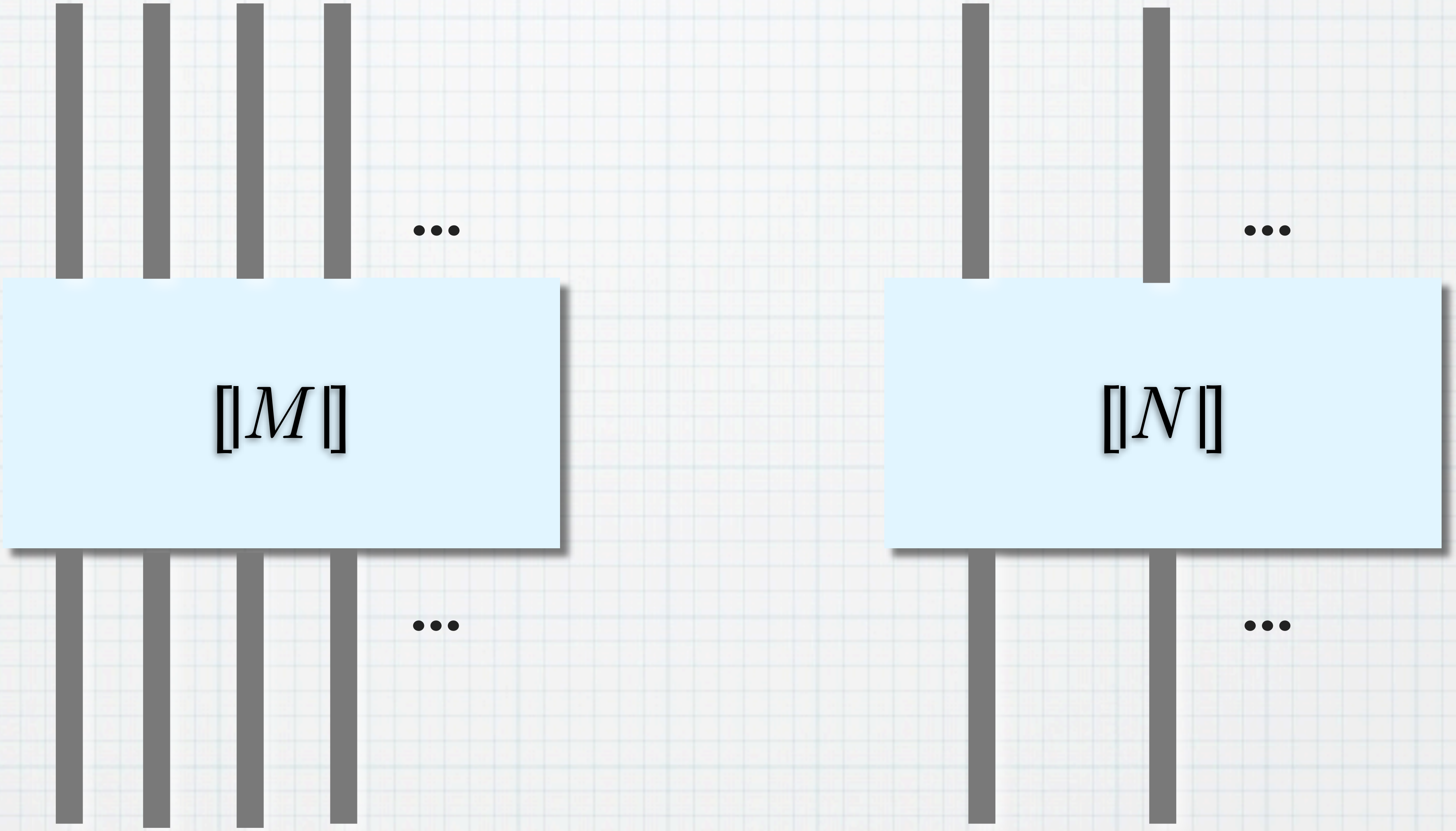
$$[\![\boldsymbol{M}\boldsymbol{N}]\!] =$$



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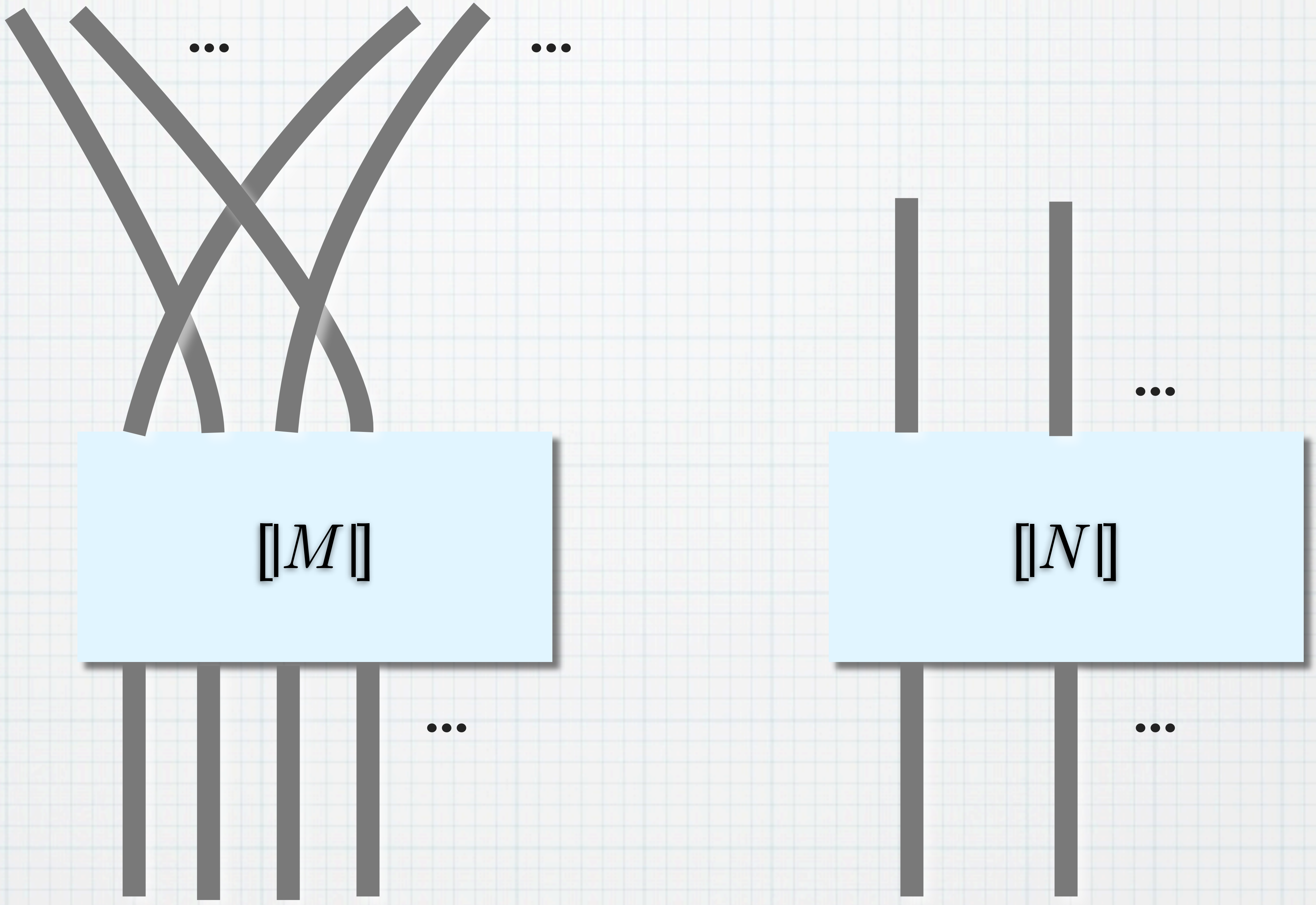

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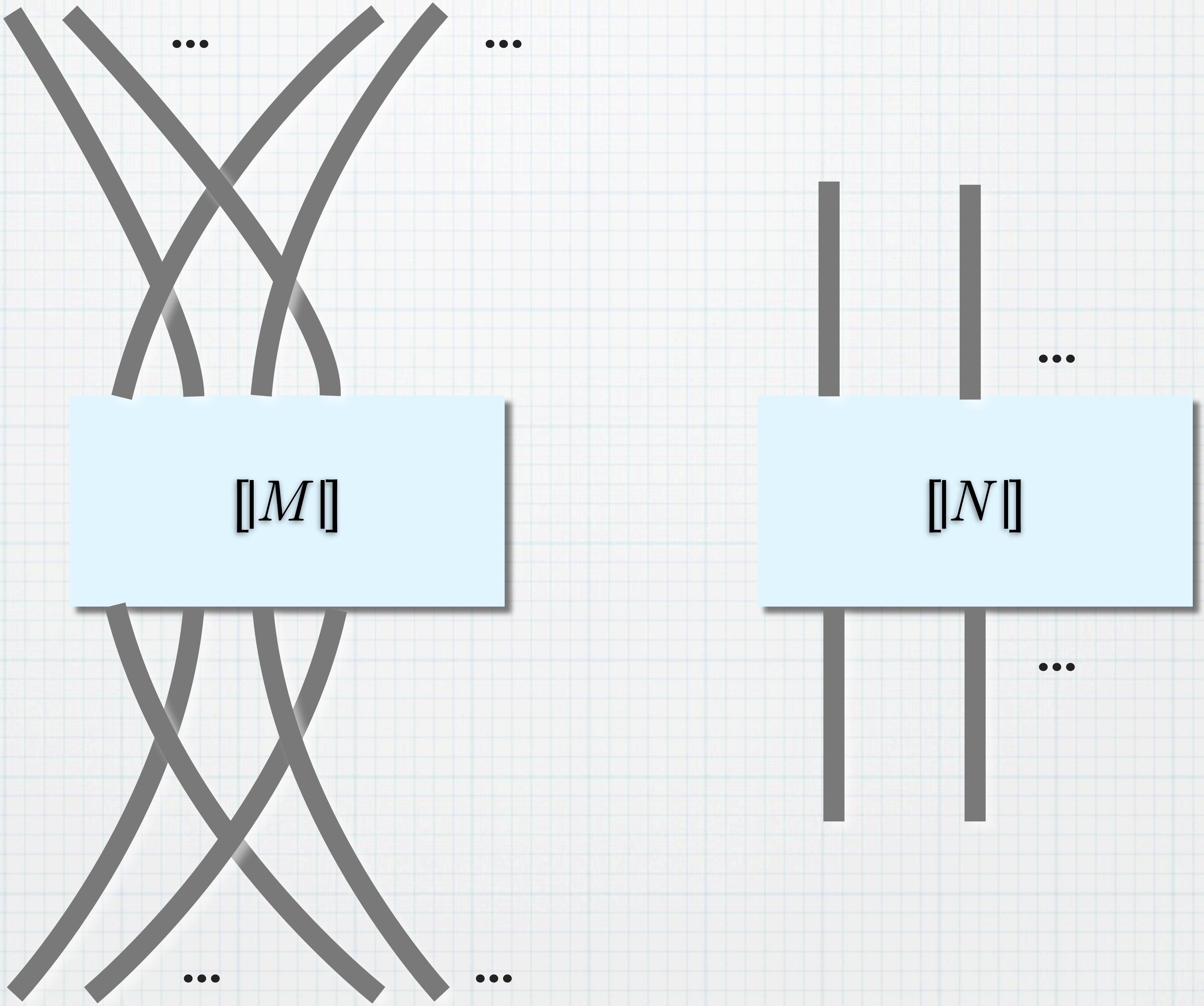

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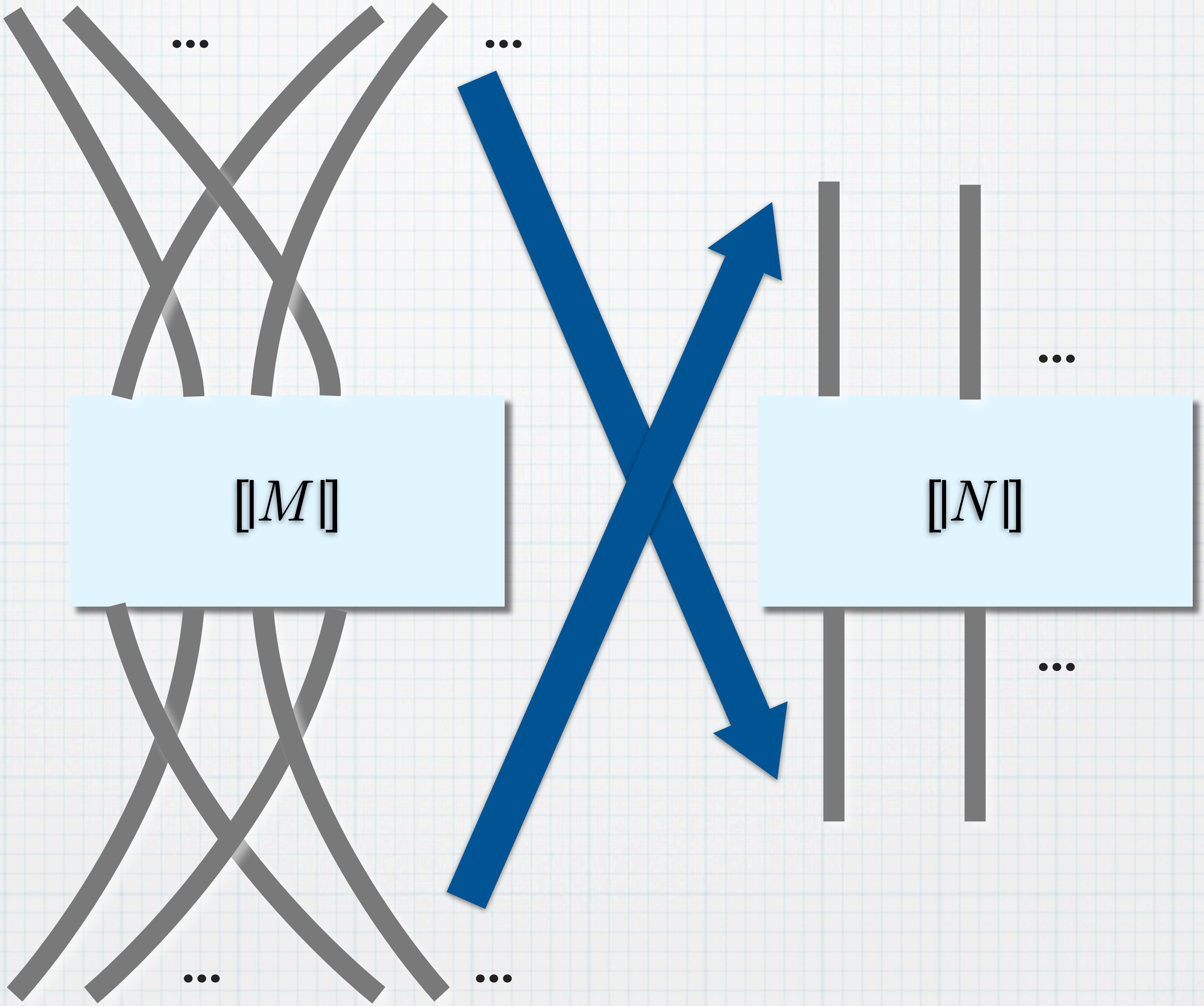

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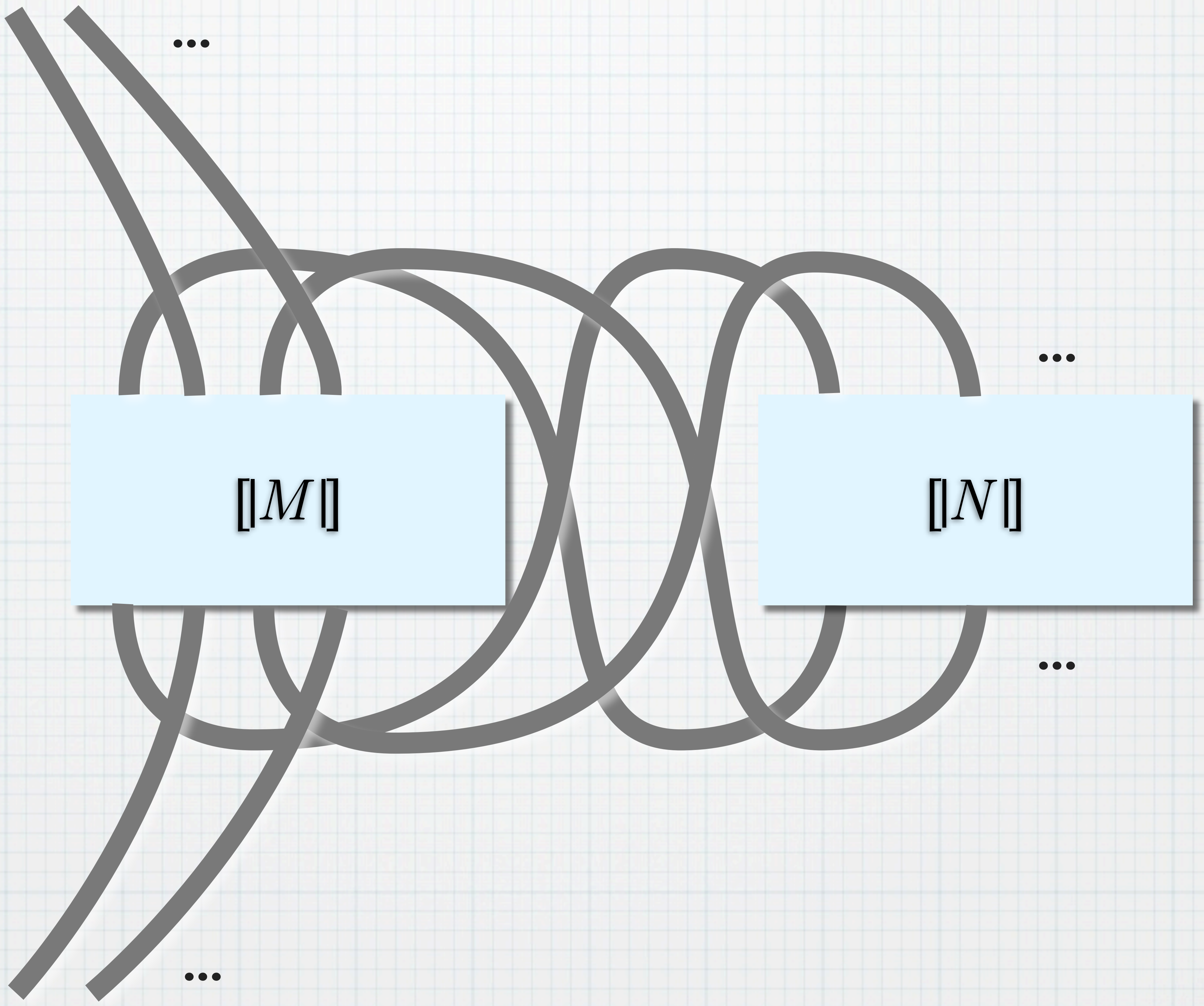

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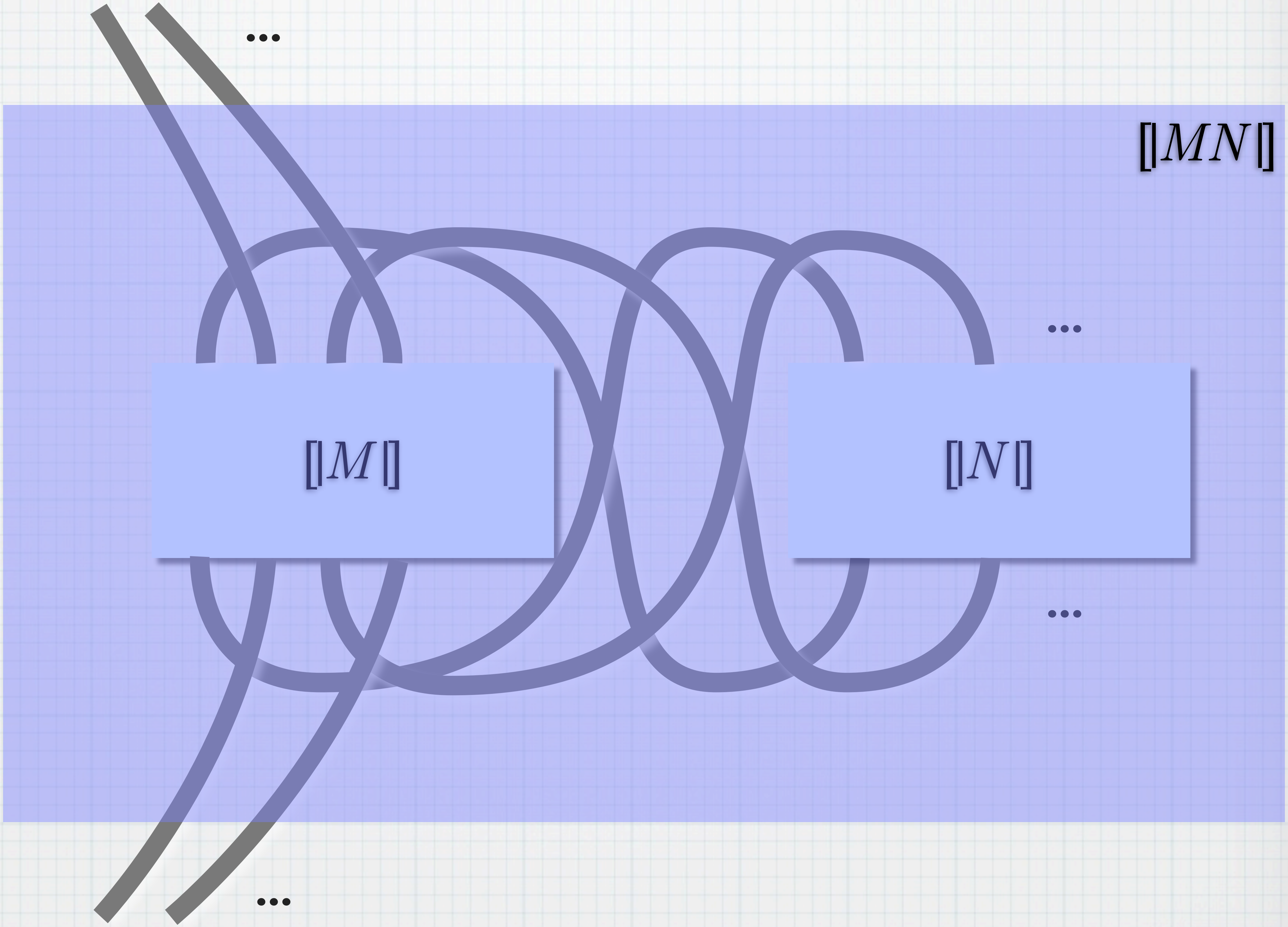

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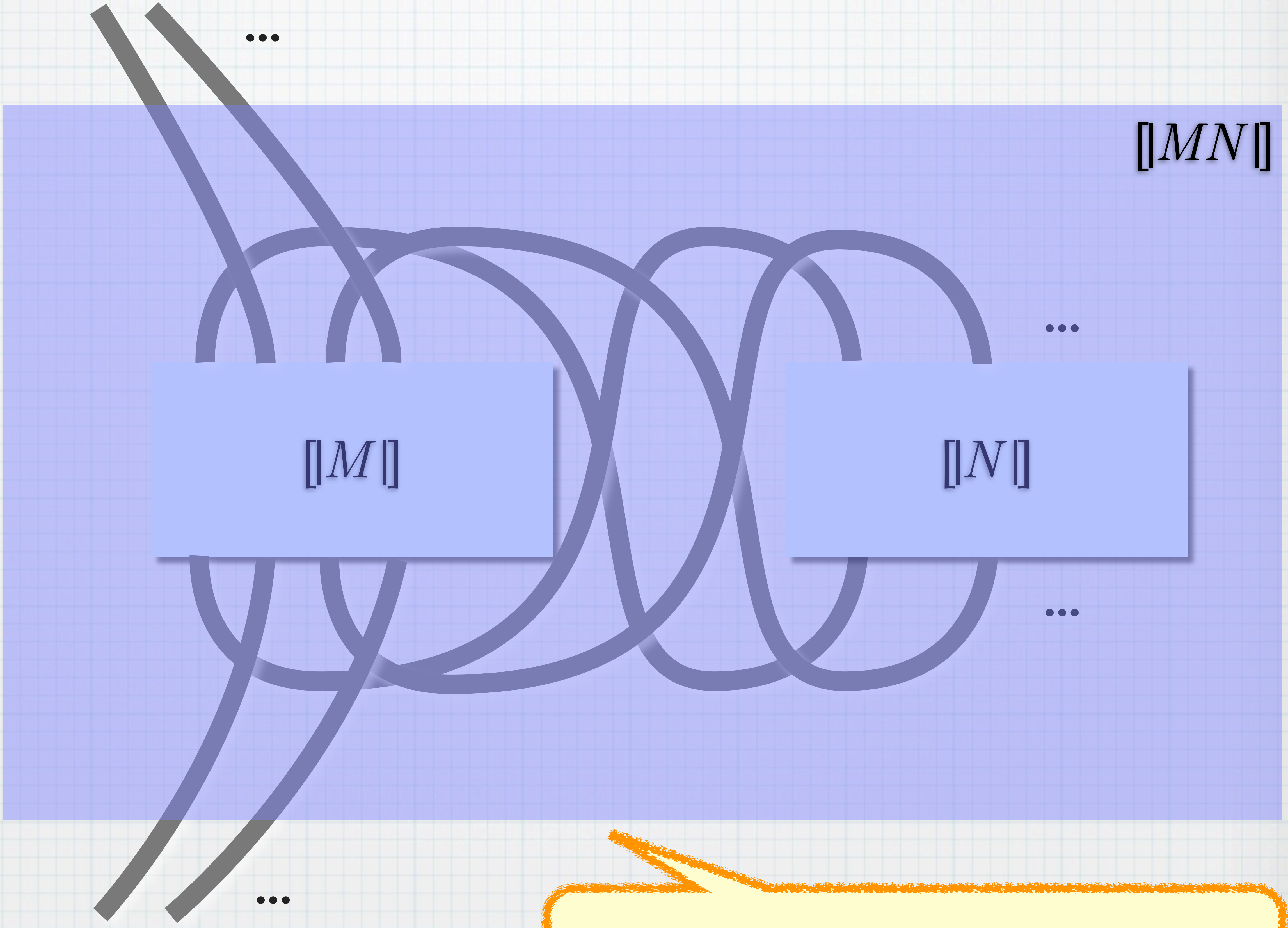

"parallel composition + hiding"
(cf. AJM games)

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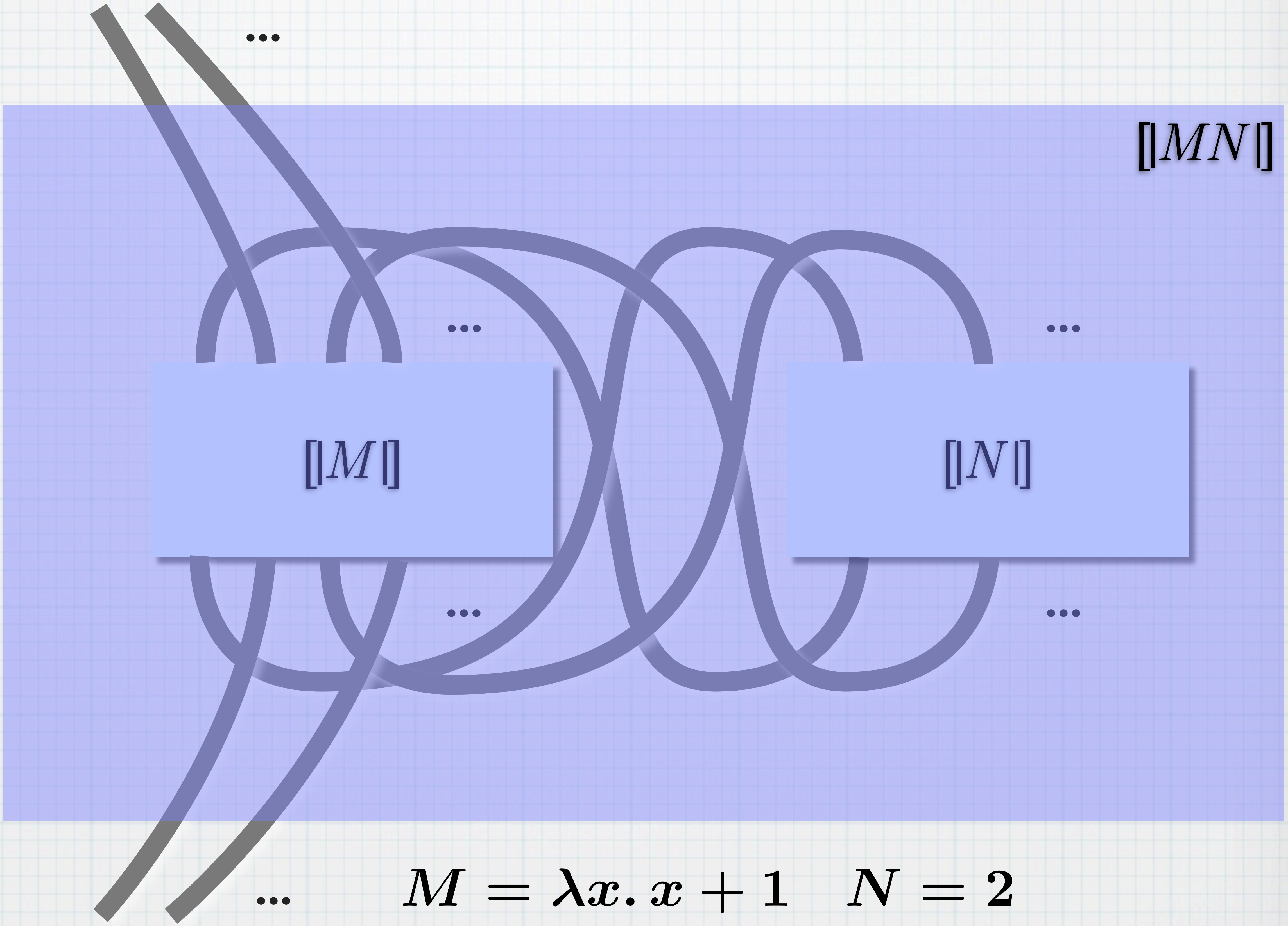
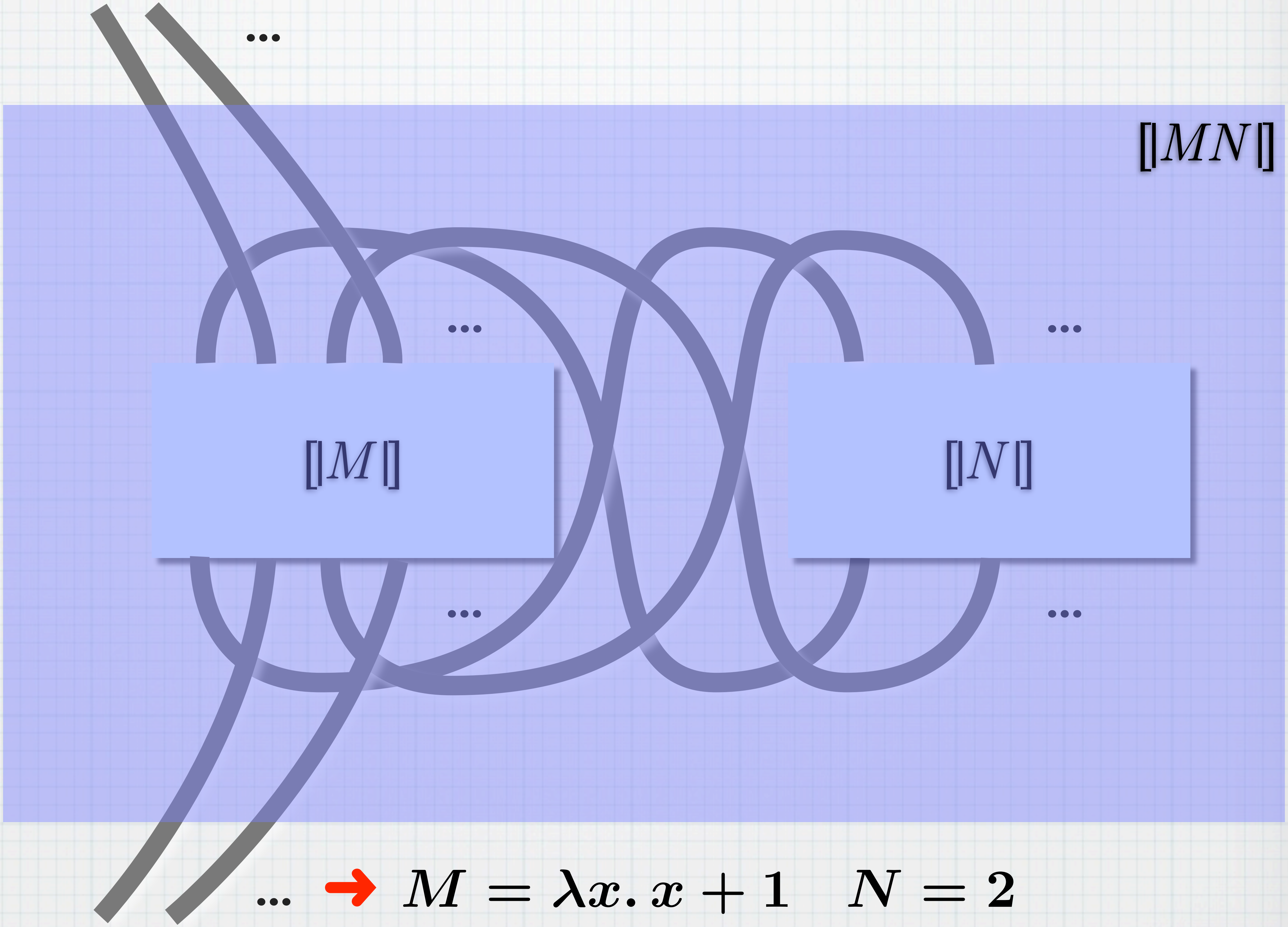


$$\boldsymbol{M = \lambda x.\, x + 1} \qquad \boldsymbol{N = 2}$$
$$\boldsymbol{M = \lambda x.\, 1} \qquad \boldsymbol{N = 2}$$
$$\boldsymbol{M = \lambda f.\, f 1} \qquad \boldsymbol{N = \lambda x.\, (x + 1)}$$

$[\![MN]\!]$

$=$



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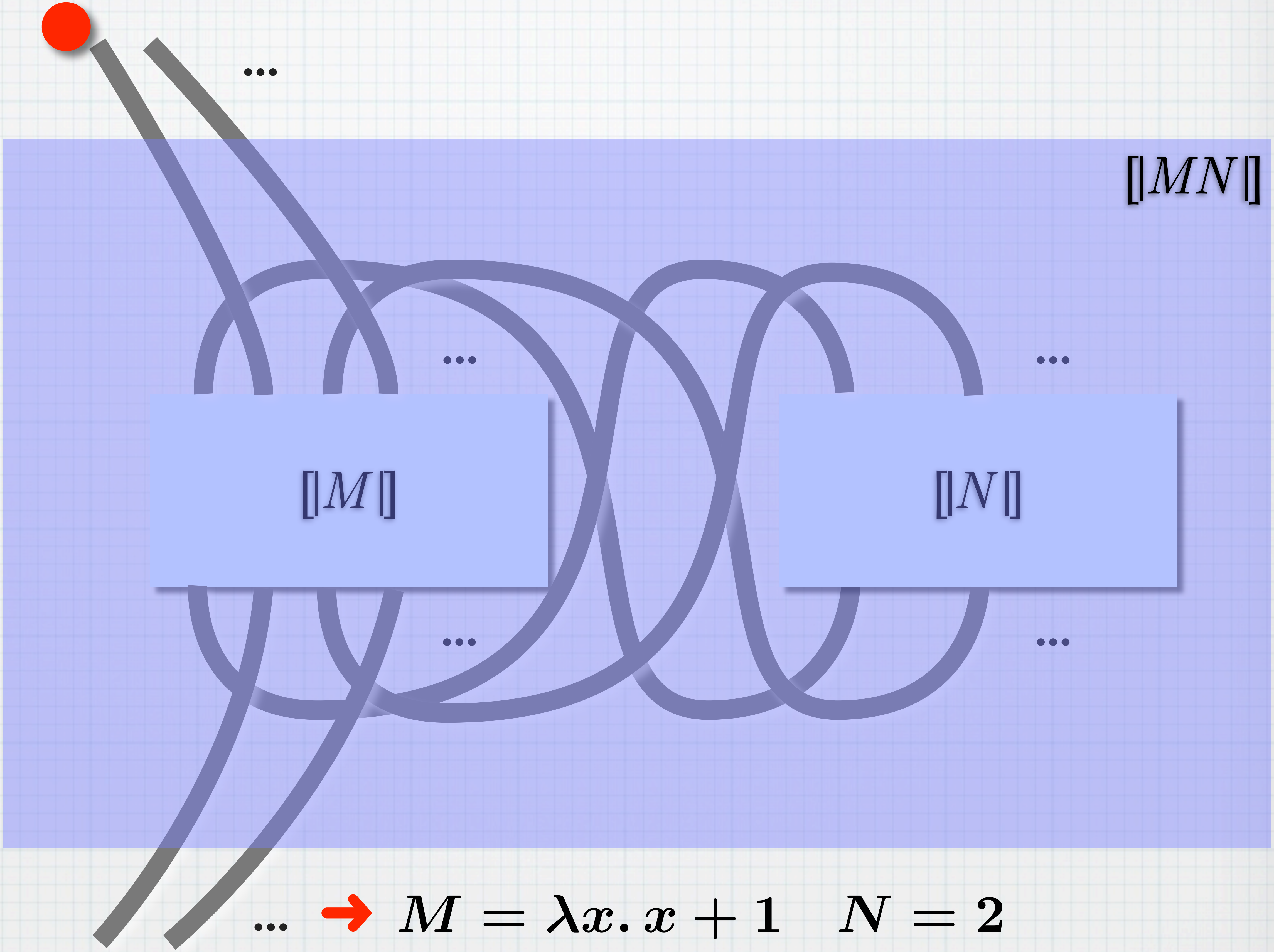

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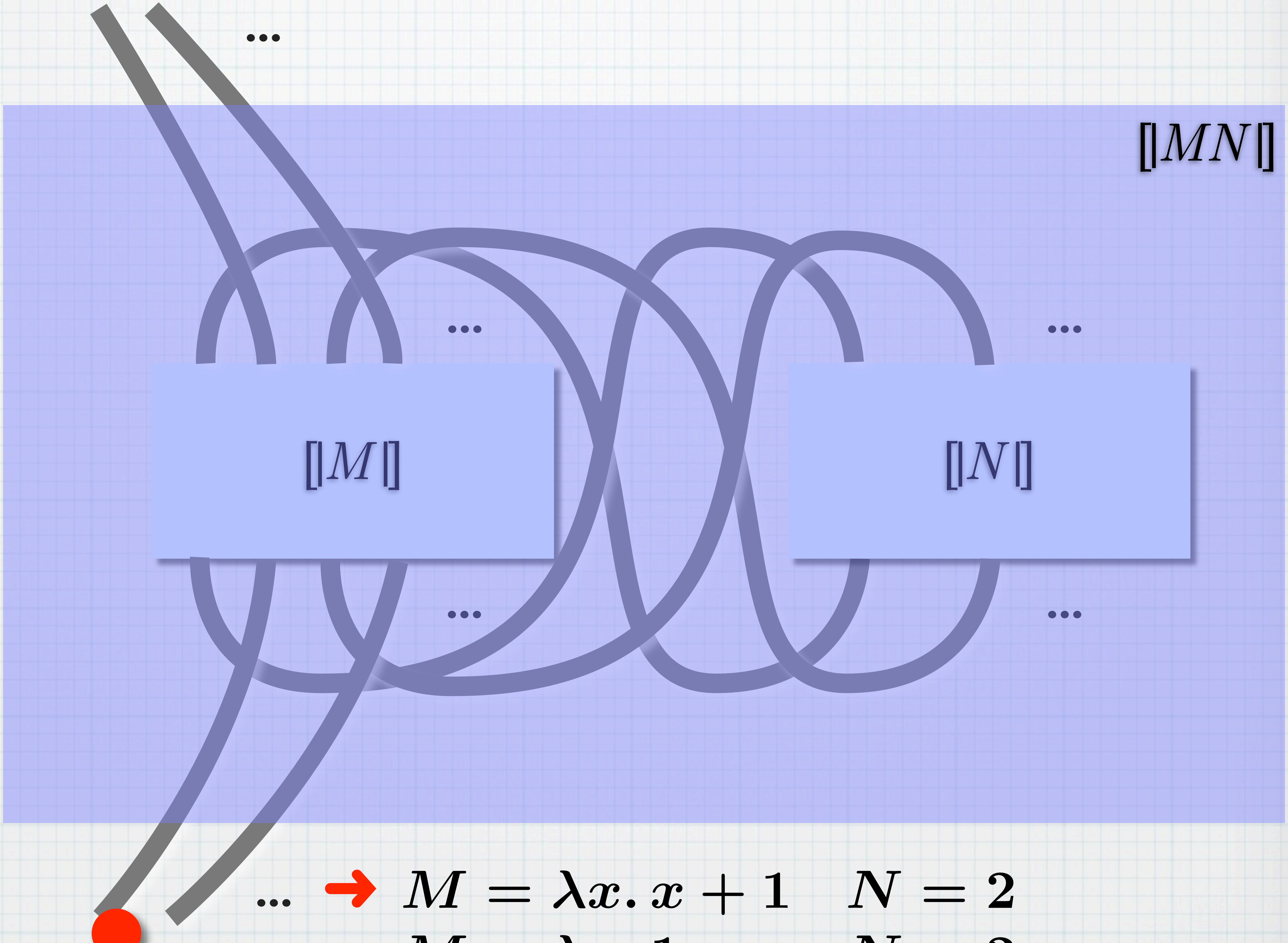

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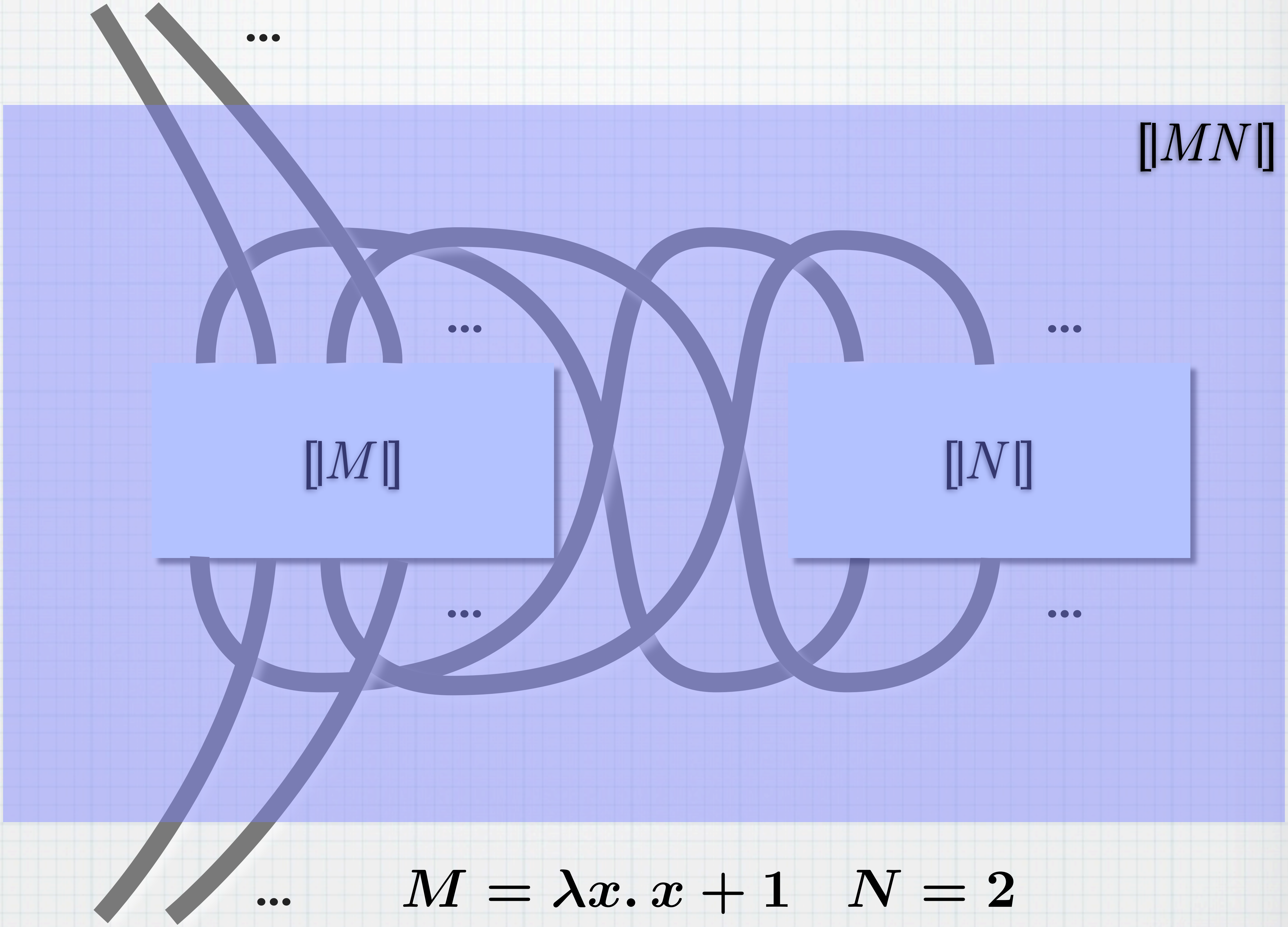

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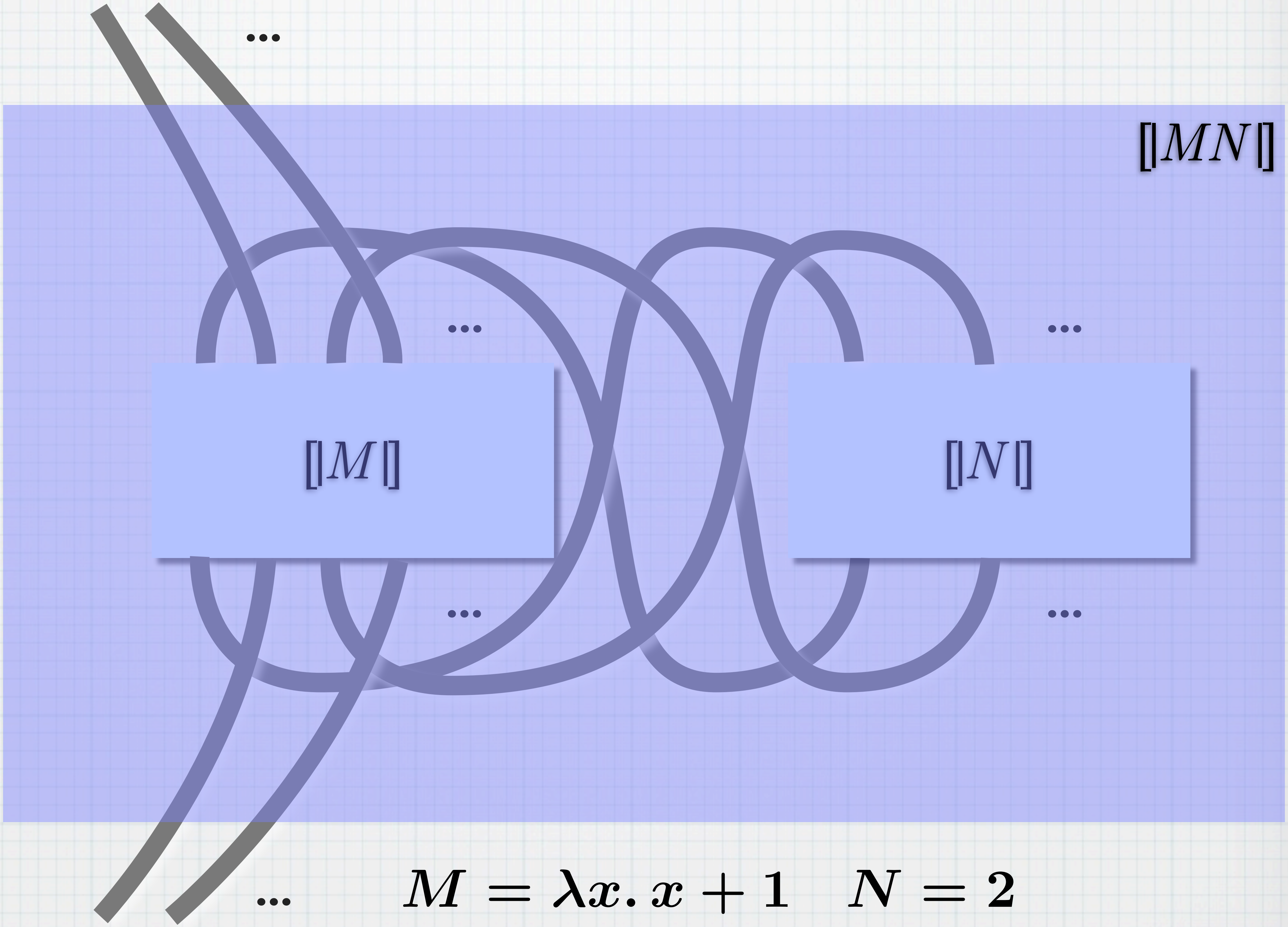

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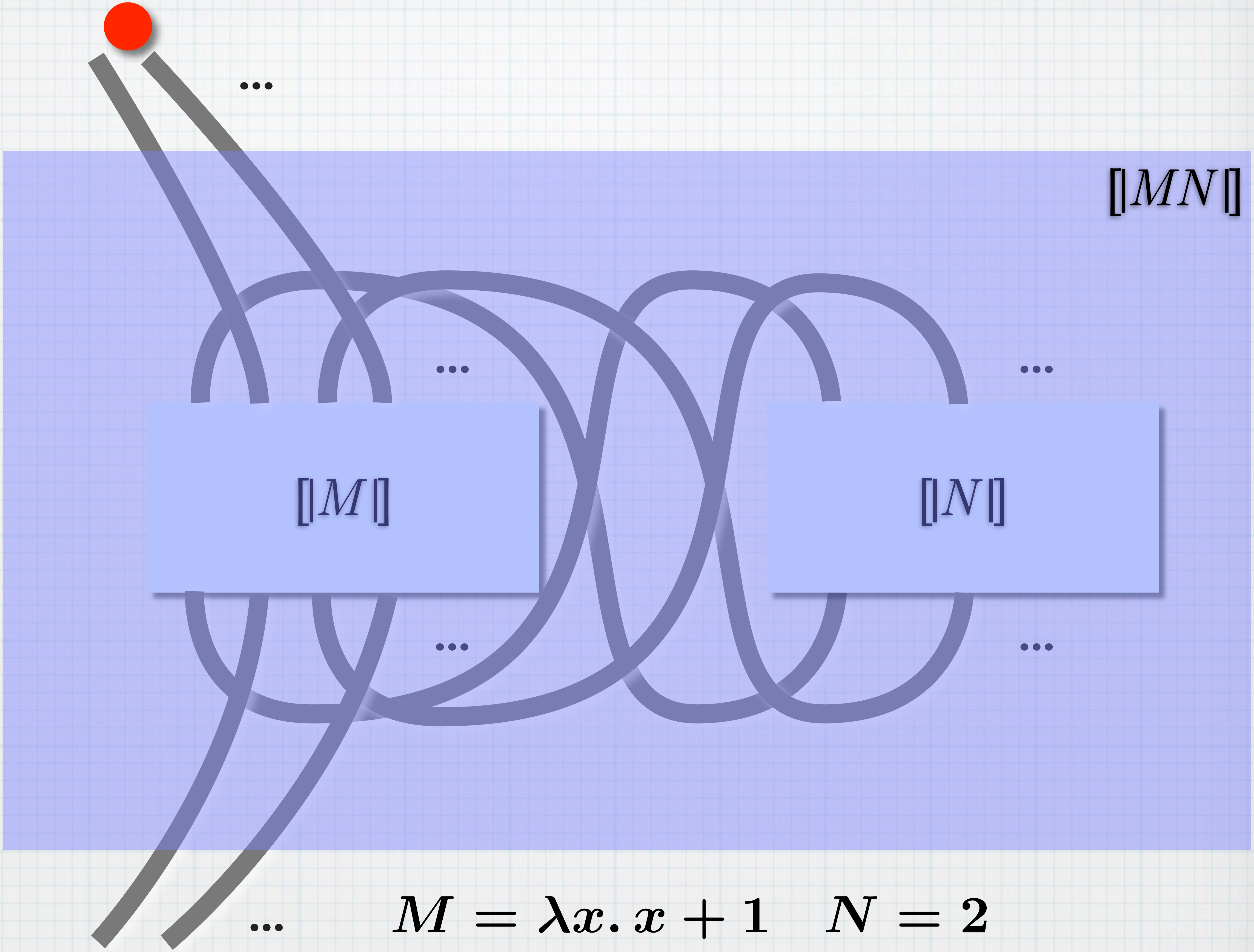

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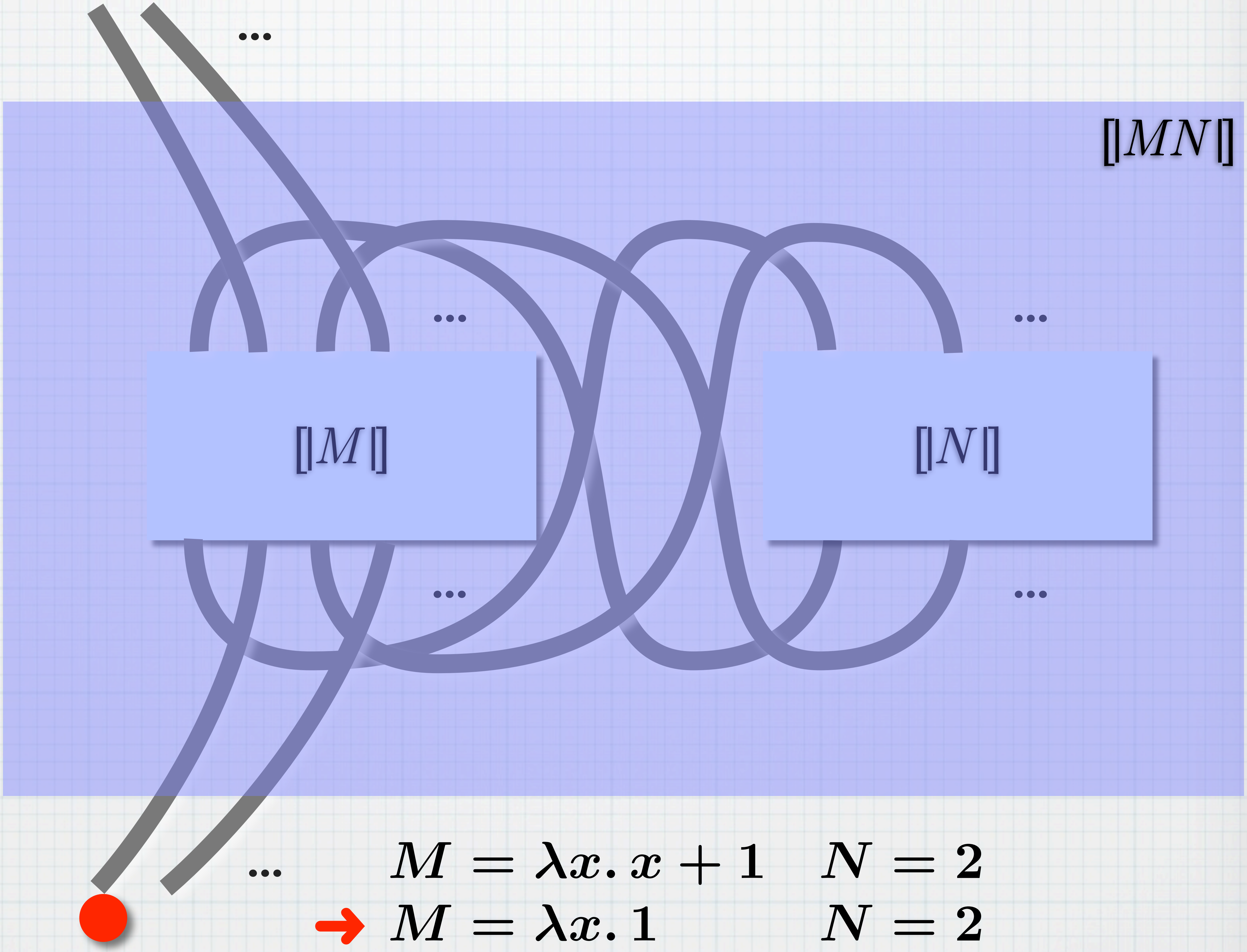

$$\begin{aligned} M &= \lambda x.\, x+1 & N &= 2\\ \rightarrow M &= \lambda x.\,1 & N &= 2\\ M &= \lambda f.\, f1 & N &= \lambda x.\,(x+1)\end{aligned}$$

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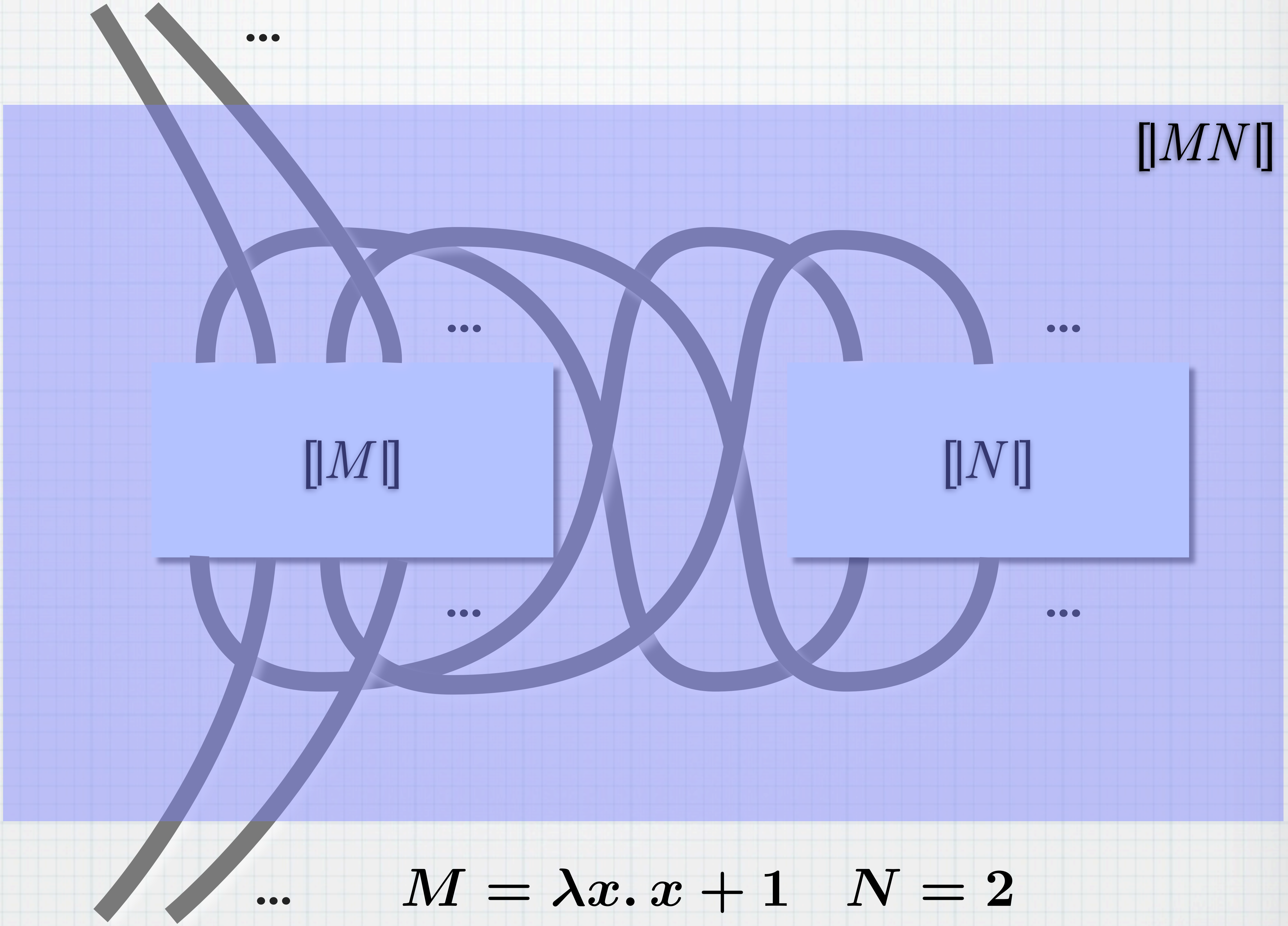

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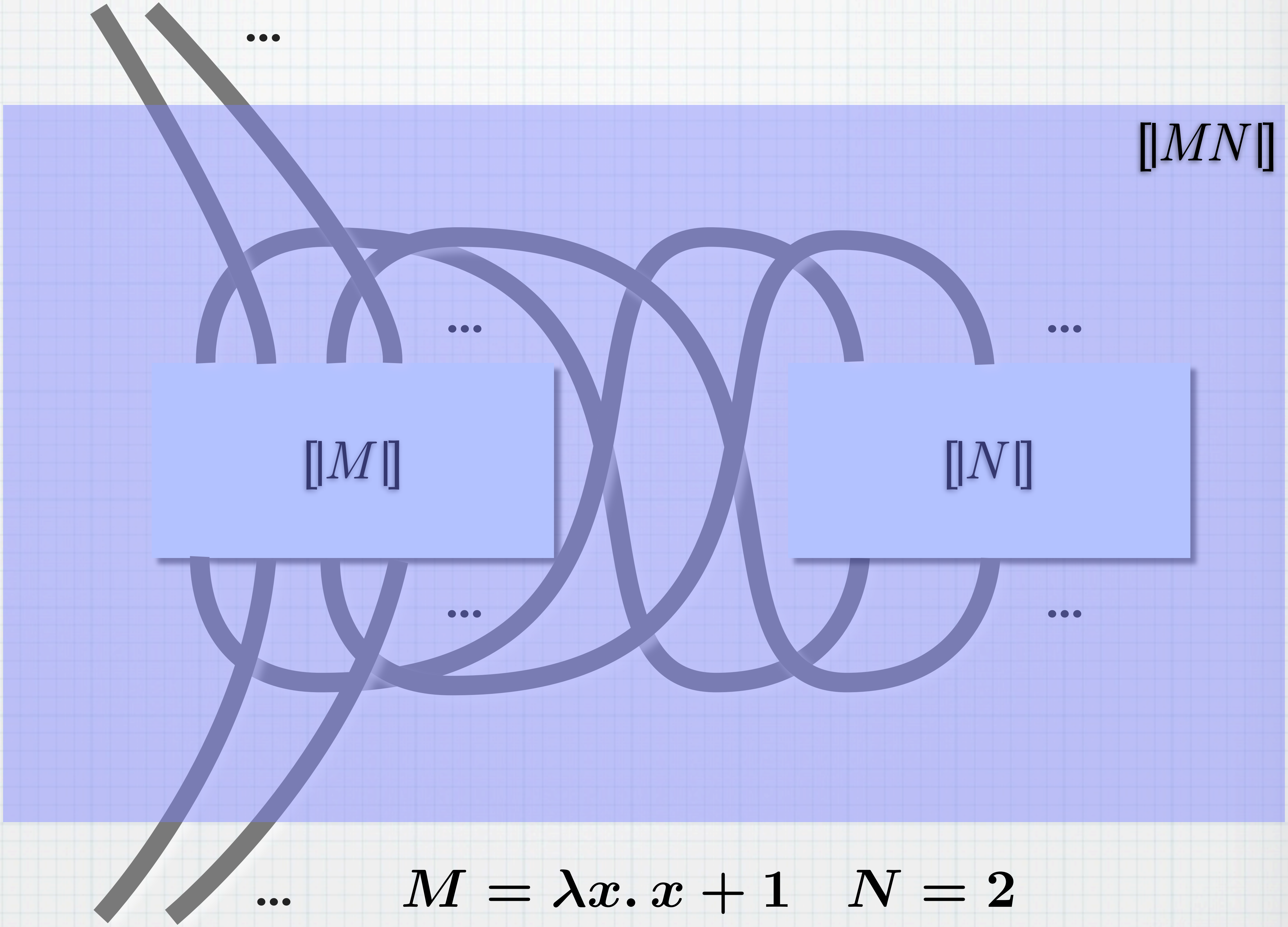

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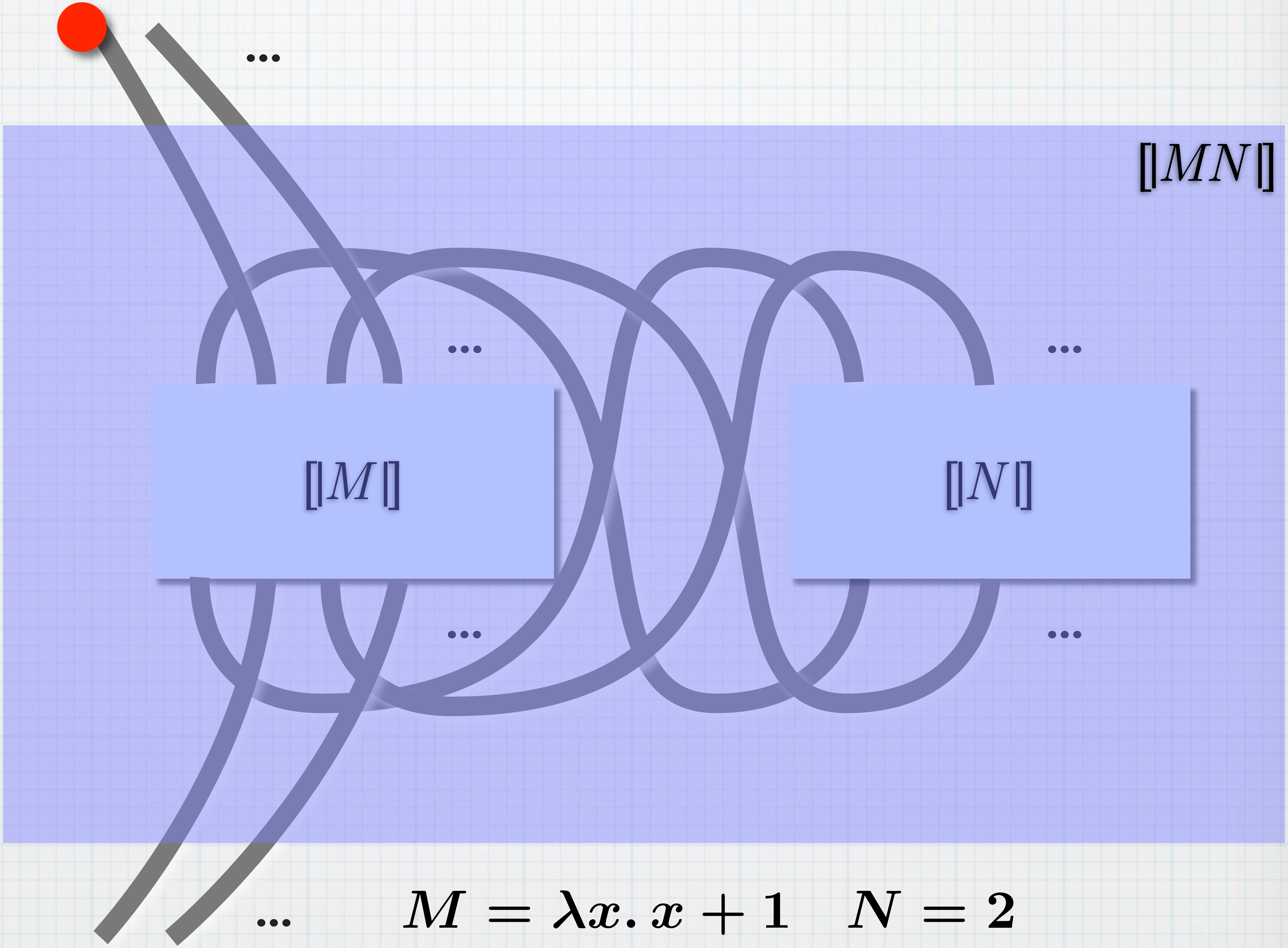

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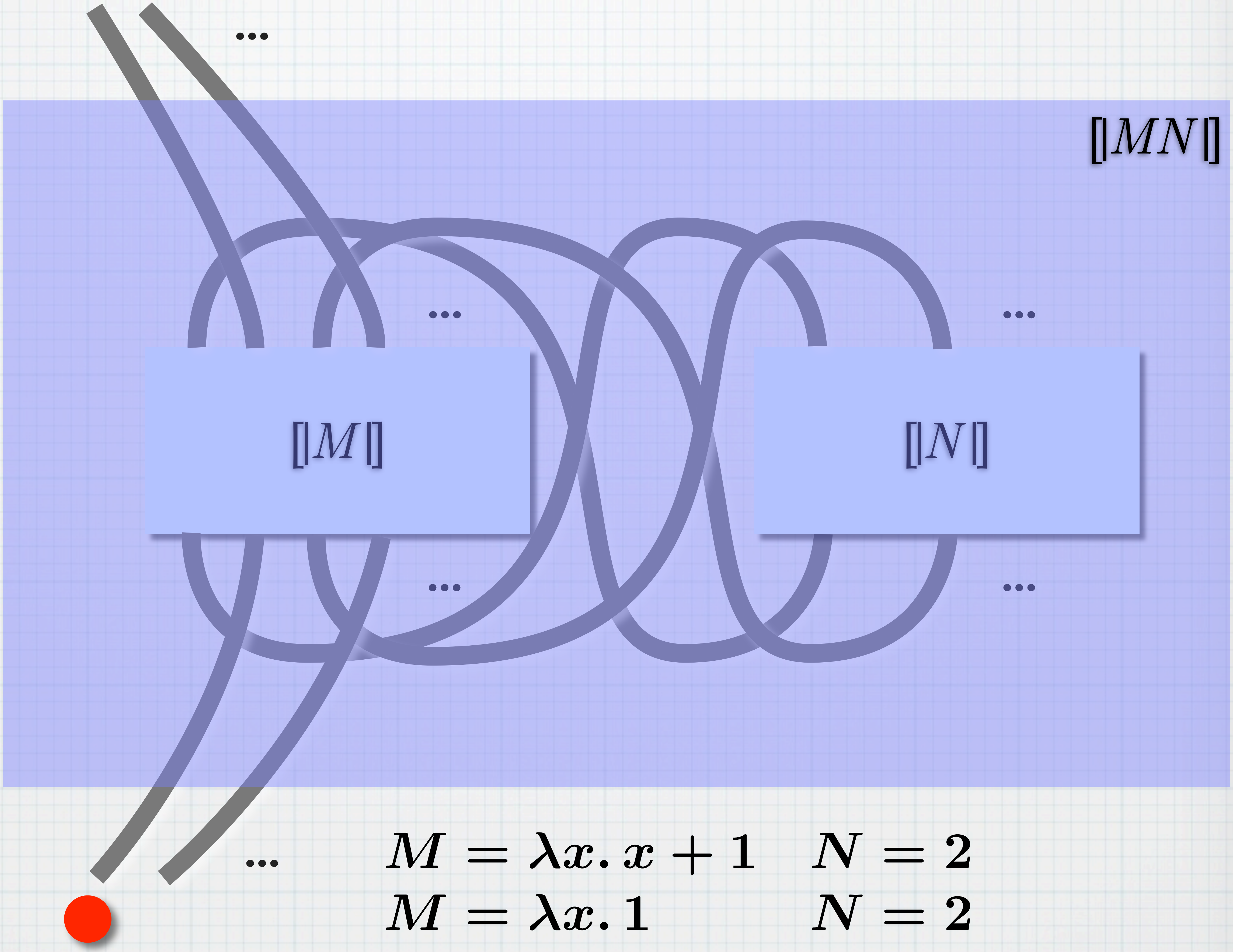

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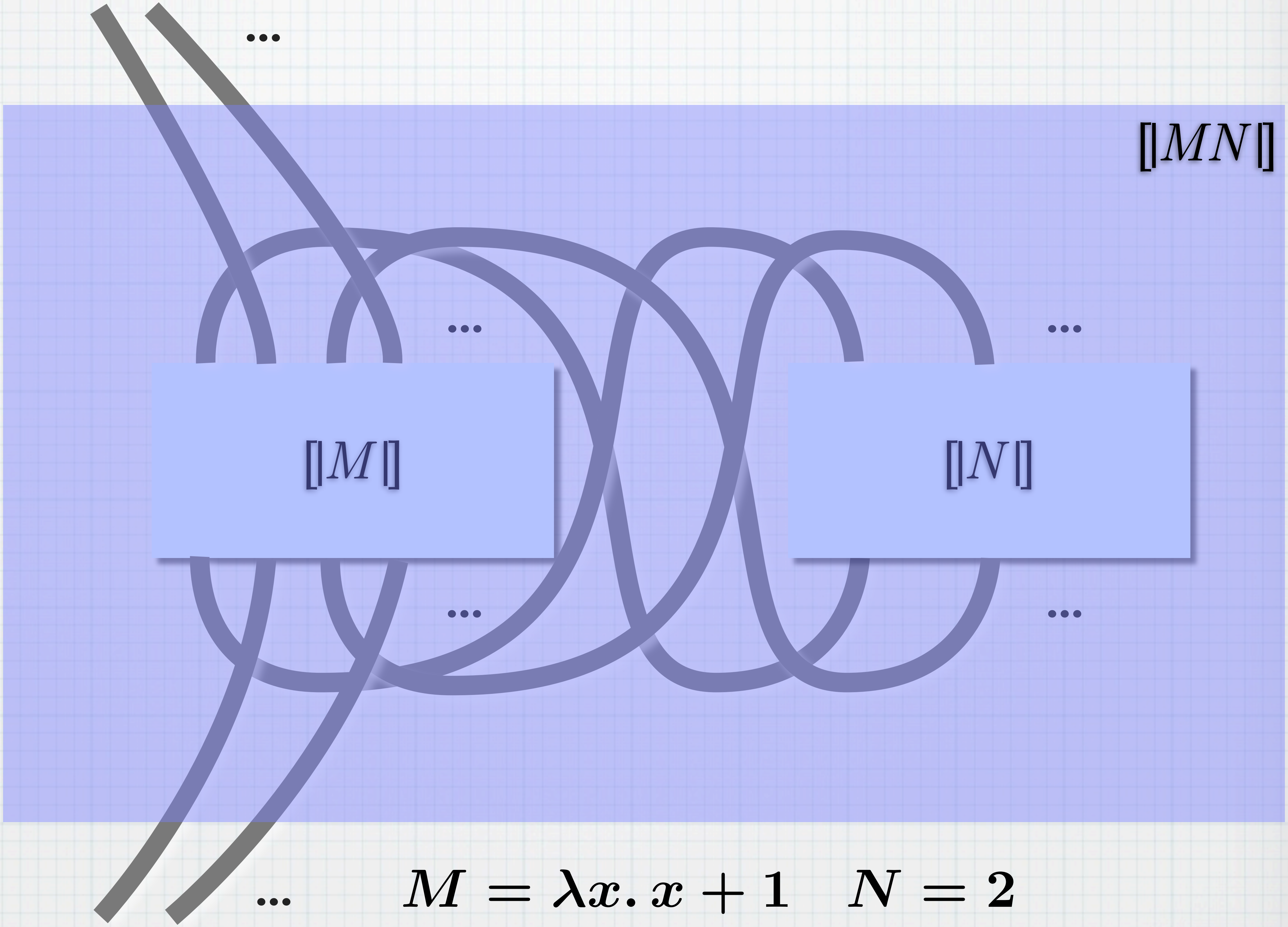

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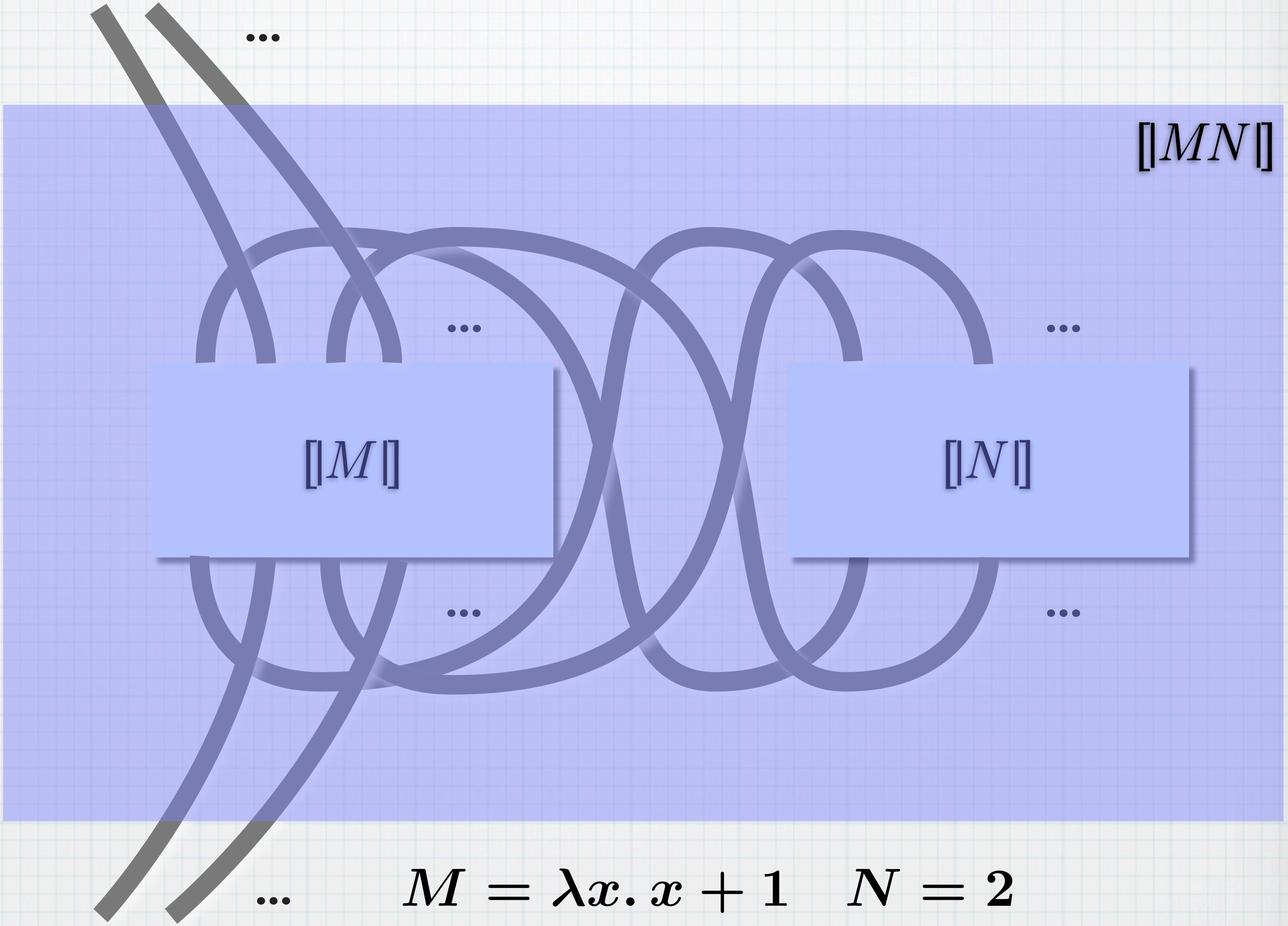

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$A \to B$ as $!A \multimap B$

Hasuo (Tokyo)

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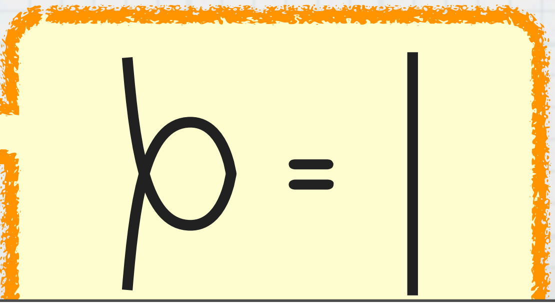

(Tokyo)

GoI:
Geometry of

- **C*-algebra** presentation
- **Token machine** presentation

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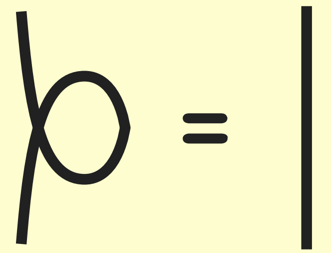


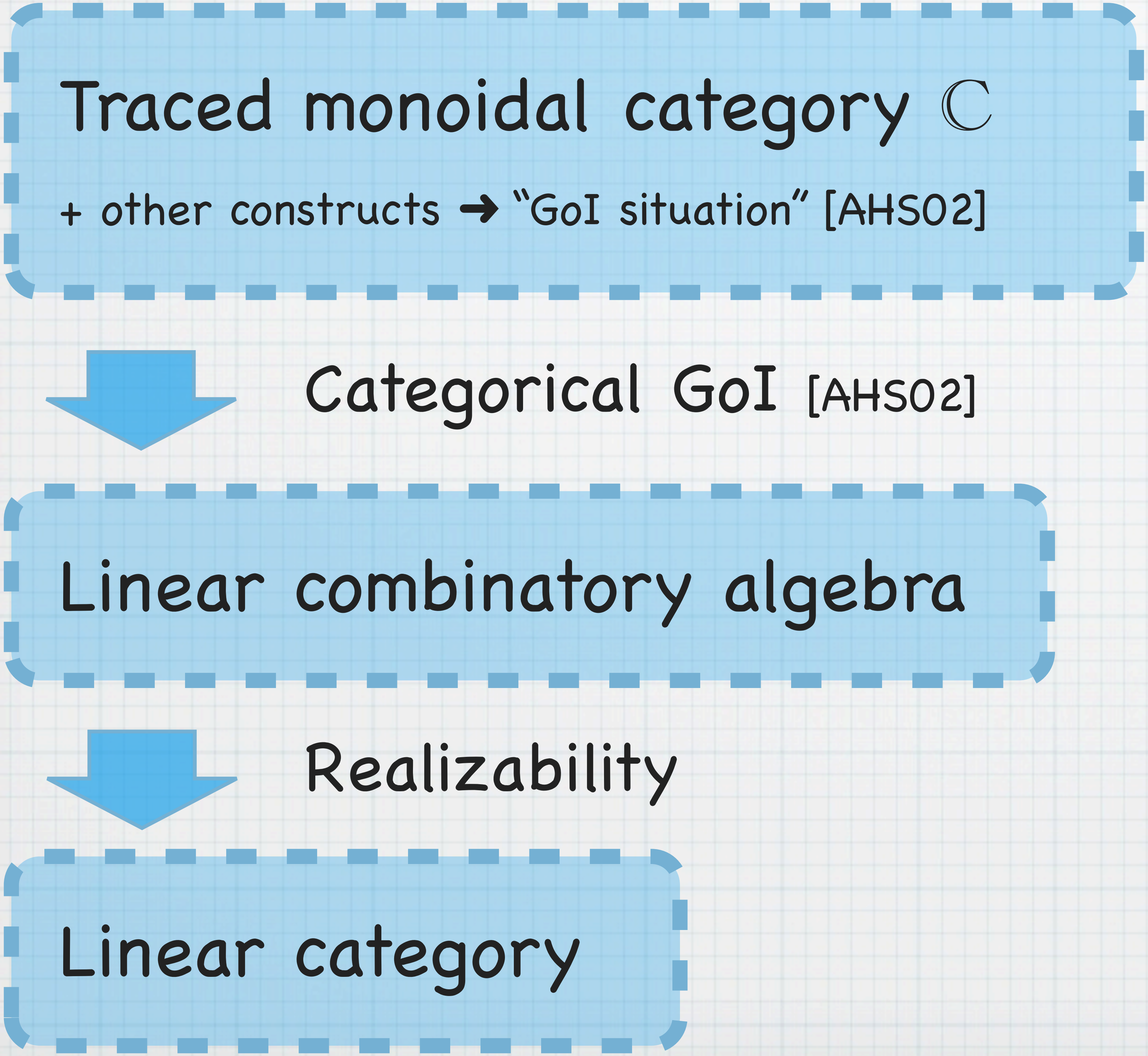
(Tokyo)

Categorical GoI

- Axiomatics of GoI in the categorical language

- Our main reference:
  - [AHS02] S. Abramsky, E. Haghverdi, and P. Scott, "Geometry of interaction and linear combinatory algebras," MSCS 2002
  - Especially its technical report version (Oxford CL), since it's a bit more detailed

The Categorical GoI Workflow



Traced monoidal category $\mathbb{C}$

+ other constructs ➜ "GoI situation" [AHS02]

Categorical GoI [AHS02]

Linear combinatory algebra

Realizability

Linear category

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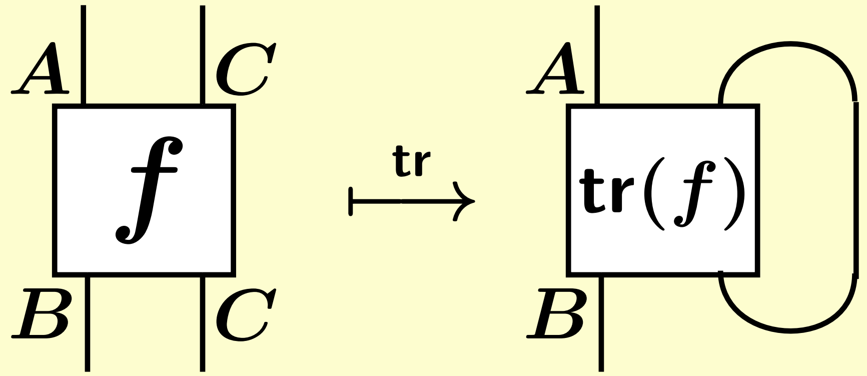

⬇ Categorical GoI [AHS02]

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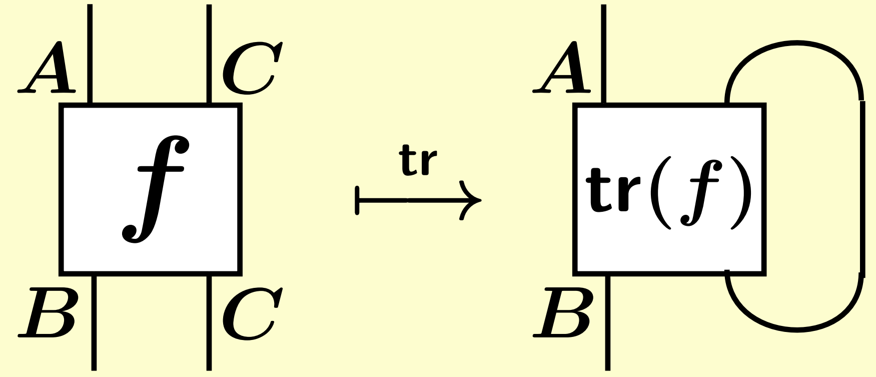

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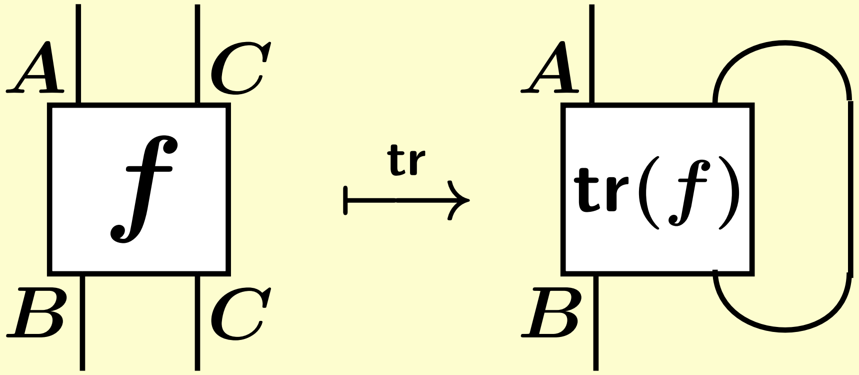

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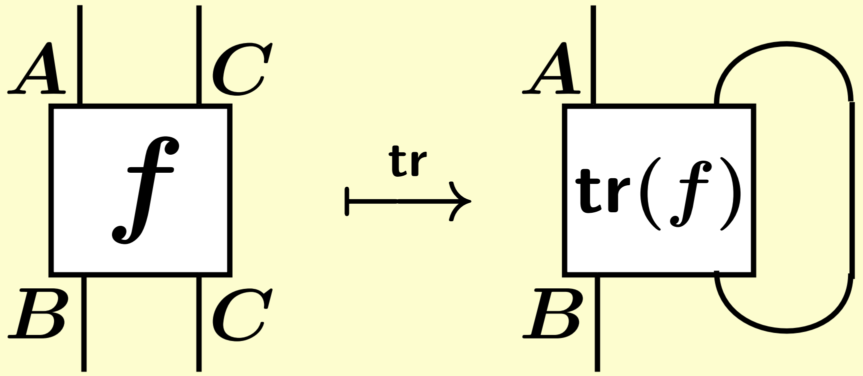

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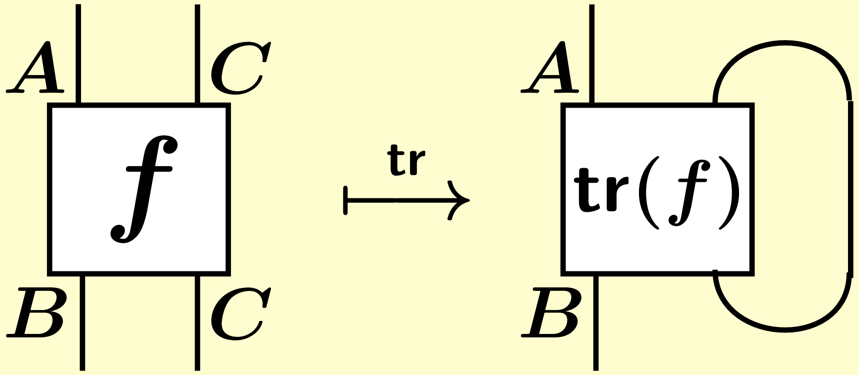

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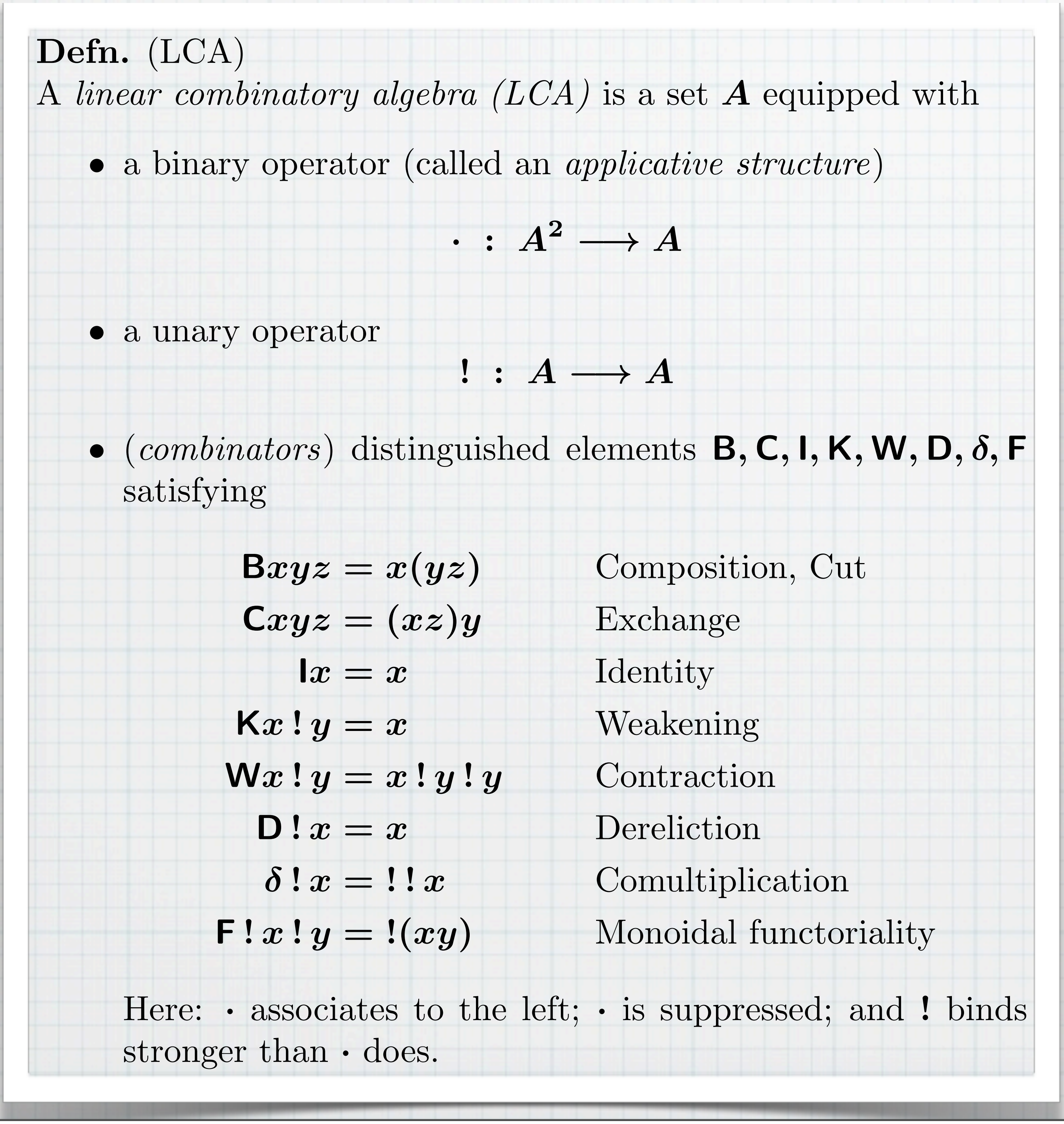
Linear Combinatory Algebra (LCA)



Defn. (LCA)
A *linear combinatory algebra (LCA)* is a set $\boldsymbol{A}$ equipped with

- a binary operator (called an *applicative structure*)

$$\cdot \;:\; \boldsymbol{A}^2 \longrightarrow \boldsymbol{A}$$

- a unary operator

$$! \;:\; \boldsymbol{A} \longrightarrow \boldsymbol{A}$$

- (*combinators*) distinguished elements $\mathsf{B}, \mathsf{C}, \mathsf{I}, \mathsf{K}, \mathsf{W}, \mathsf{D}, \delta, \mathsf{F}$ satisfying

$\mathsf{B}xyz = x(yz)$	Composition, Cut
$\mathsf{C}xyz = (xz)y$	Exchange
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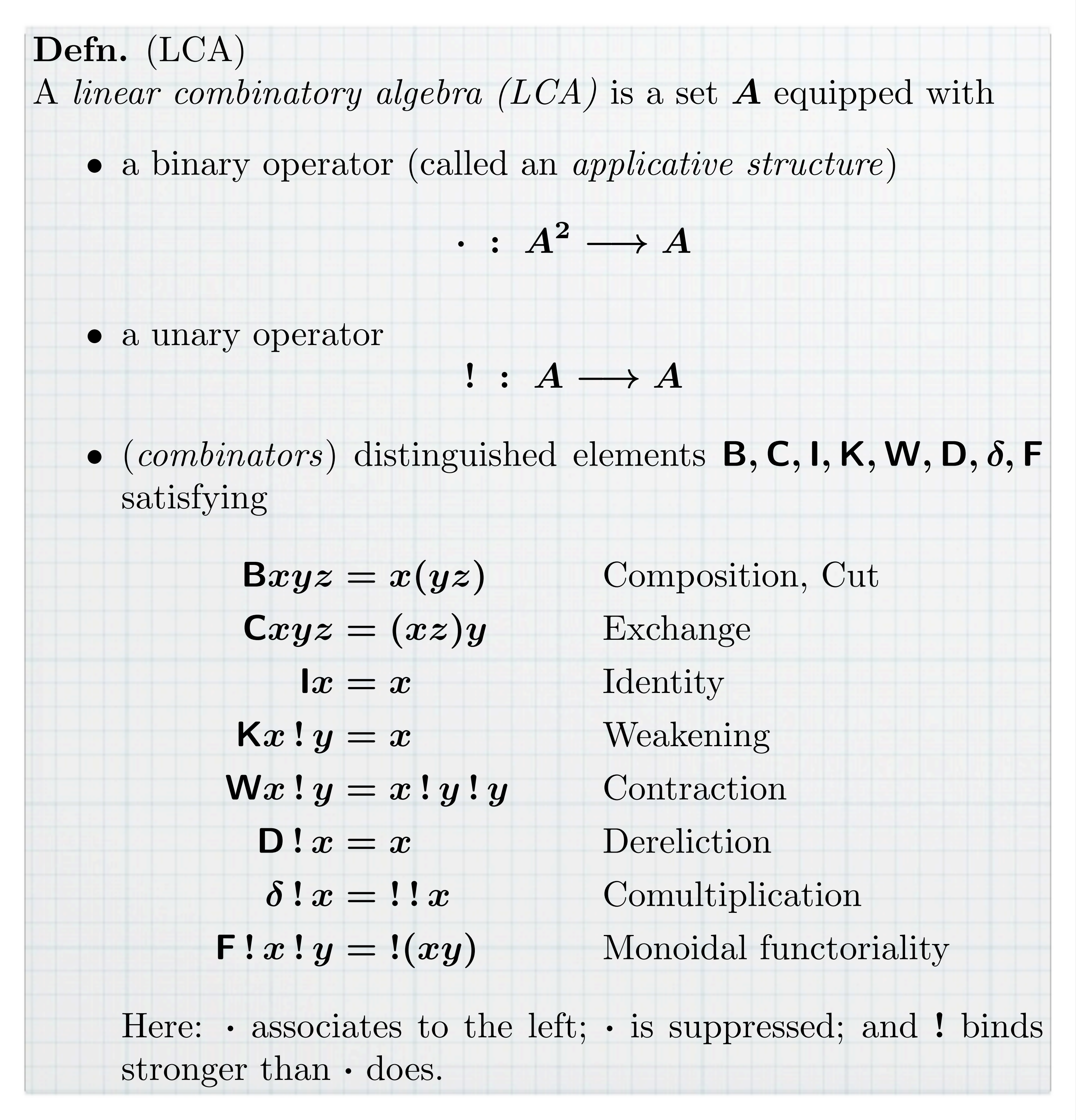
Linear Combinatory Algebra (LCA)

What
we want (outcome)



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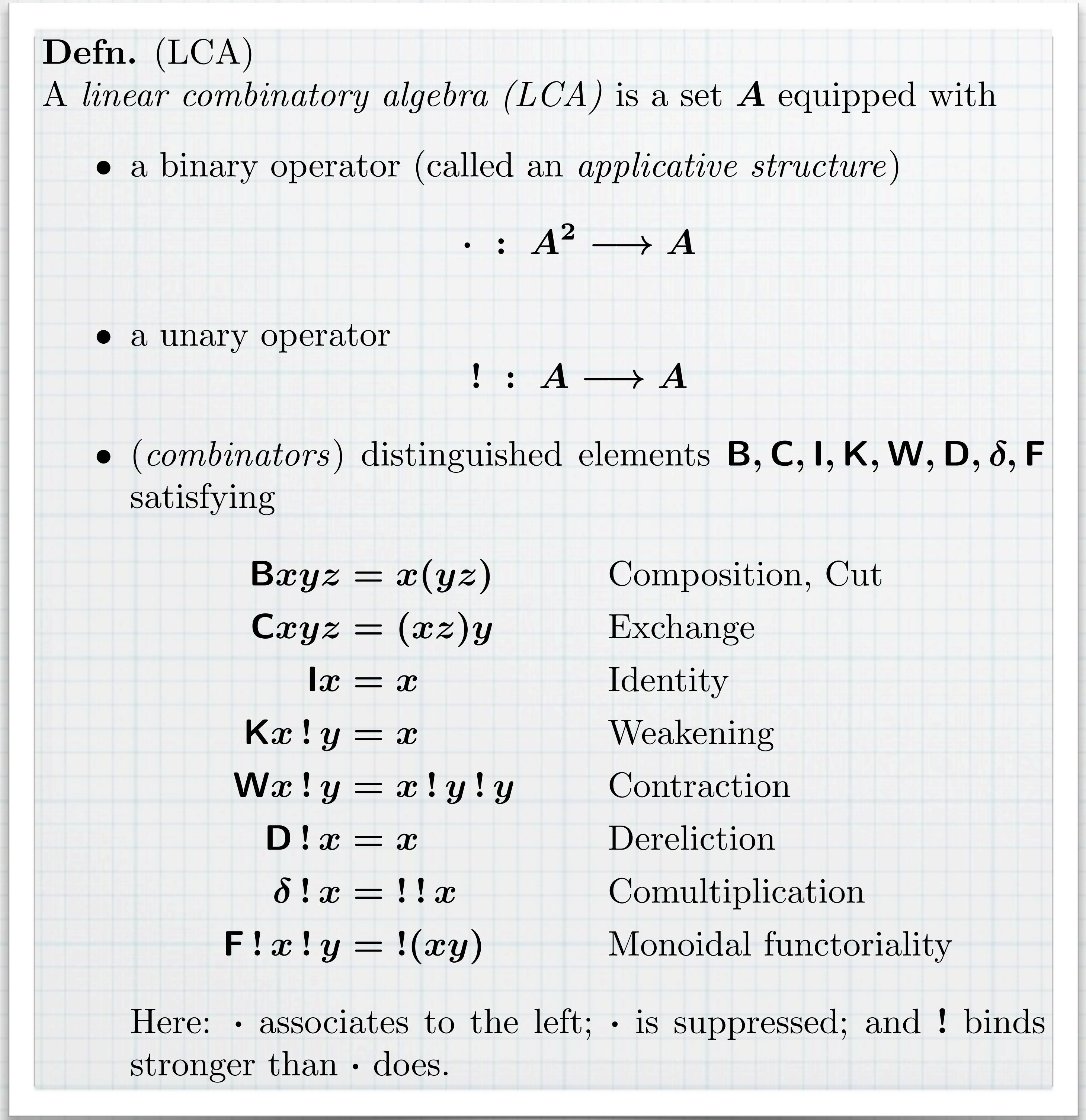

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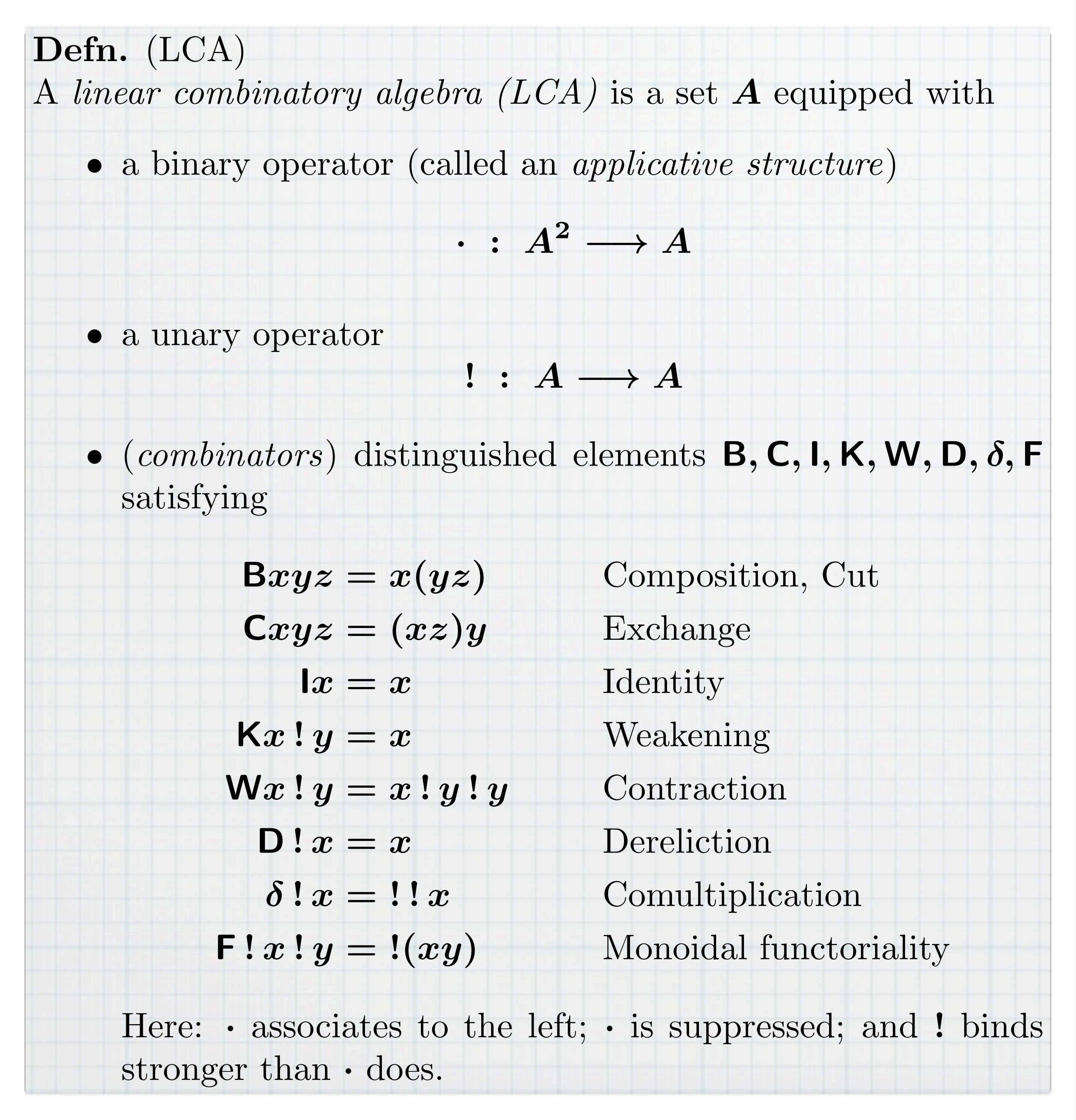

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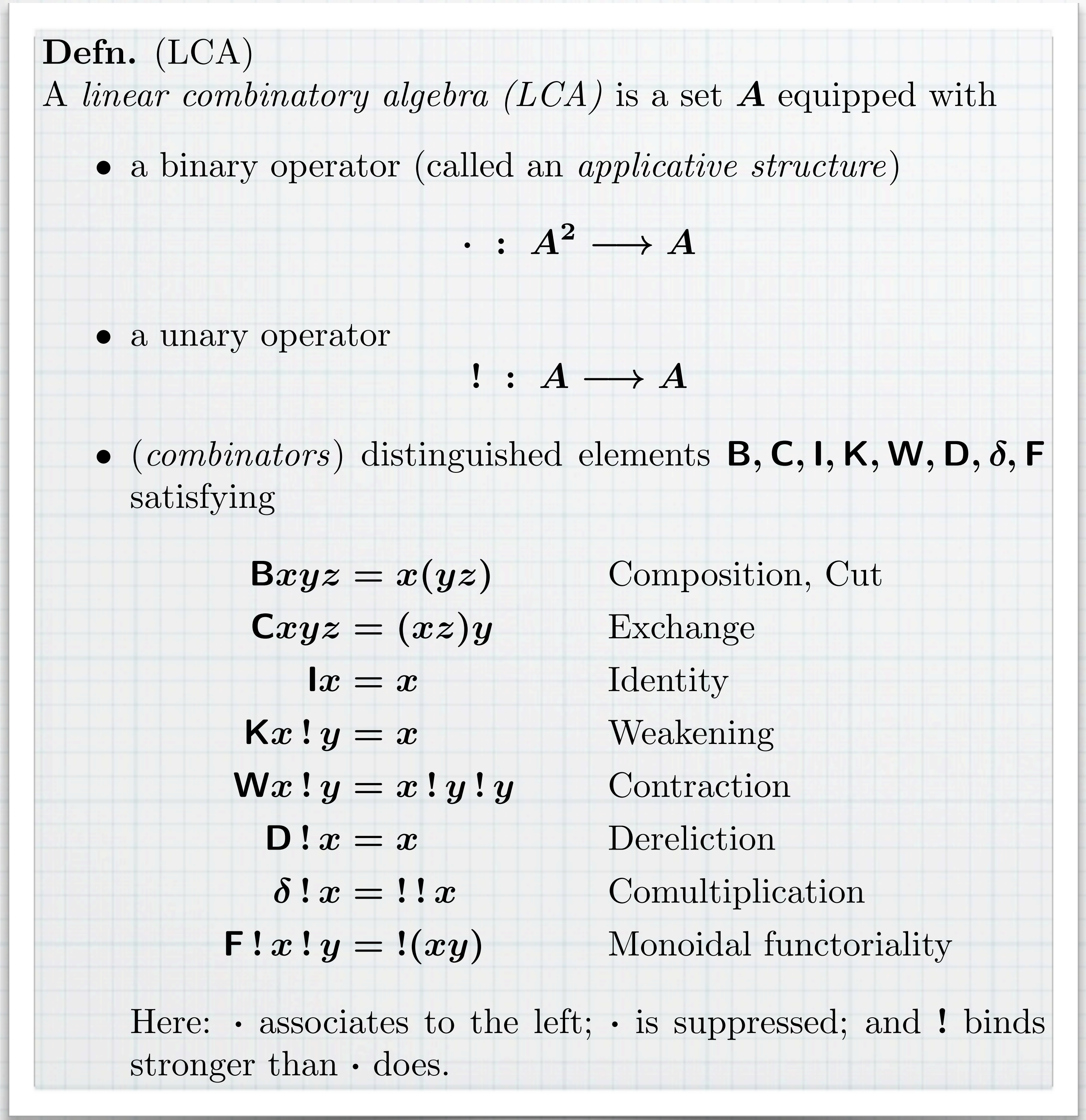

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- Model of untyped linear λ
- a ∈ A ≈ closed linear λ-term
- No **S** or **K** (linear!)
- Combinatory completeness: e.g.

$$\lambda xyz.\, zxy$$

designates an elem. of A

What we use (ingredient)

GoI situation

Defn. (GoI situation [AHS02])
A *GoI situation* is a triple $(\mathbb{C}, \boldsymbol{F}, \boldsymbol{U})$ where

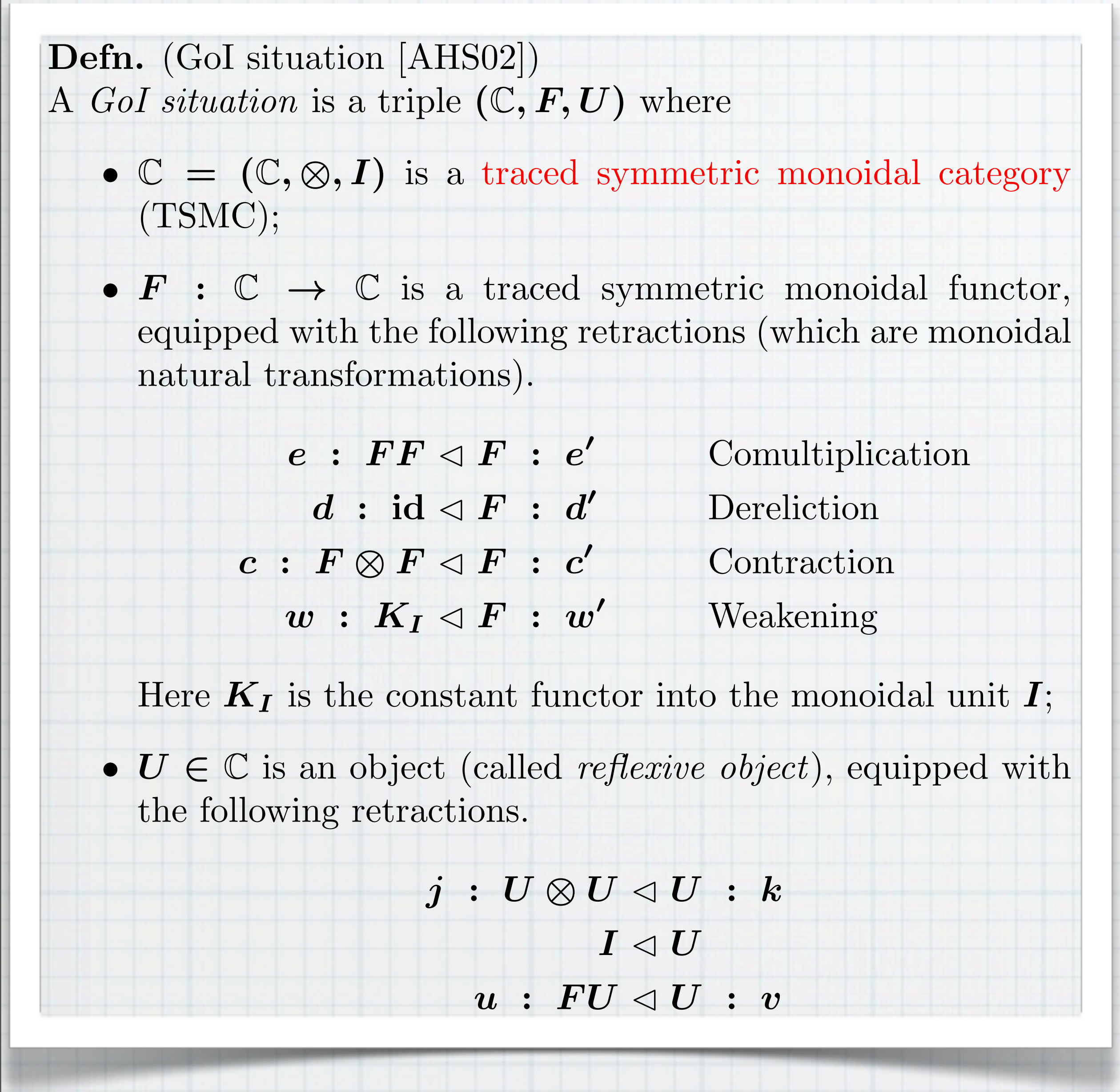
- $\mathbb{C} = (\mathbb{C}, \otimes, \boldsymbol{I})$ is a traced symmetric monoidal category (TSMC);
- $\boldsymbol{F} : \mathbb{C} \to \mathbb{C}$ is a traced symmetric monoidal functor, equipped with the following retractions (which are monoidal natural transformations).

  $$\begin{array}{rcrclcl} \boldsymbol{e} & : & \boldsymbol{FF} & \lhd & \boldsymbol{F} & : & \boldsymbol{e'} & \quad \text{Comultiplication} \\ \boldsymbol{d} & : & \textbf{id} & \lhd & \boldsymbol{F} & : & \boldsymbol{d'} & \quad \text{Dereliction} \\ \boldsymbol{c} & : & \boldsymbol{F} \otimes \boldsymbol{F} & \lhd & \boldsymbol{F} & : & \boldsymbol{c'} & \quad \text{Contraction} \\ \boldsymbol{w} & : & \boldsymbol{K_I} & \lhd & \boldsymbol{F} & : & \boldsymbol{w'} & \quad \text{Weakening} \end{array}$$

  Here $\boldsymbol{K_I}$ is the constant functor into the monoidal unit $\boldsymbol{I}$;
- $\boldsymbol{U} \in \mathbb{C}$ is an object (called *reflexive object*), equipped with the following retractions.

  $$\begin{array}{rcrclcl} \boldsymbol{j} & : & \boldsymbol{U} \otimes \boldsymbol{U} & \lhd & \boldsymbol{U} & : & \boldsymbol{k} \\ & & \boldsymbol{I} & \lhd & \boldsymbol{U} & & \\ \boldsymbol{u} & : & \boldsymbol{FU} & \lhd & \boldsymbol{U} & : & \boldsymbol{v} \end{array}$$

GoI situation



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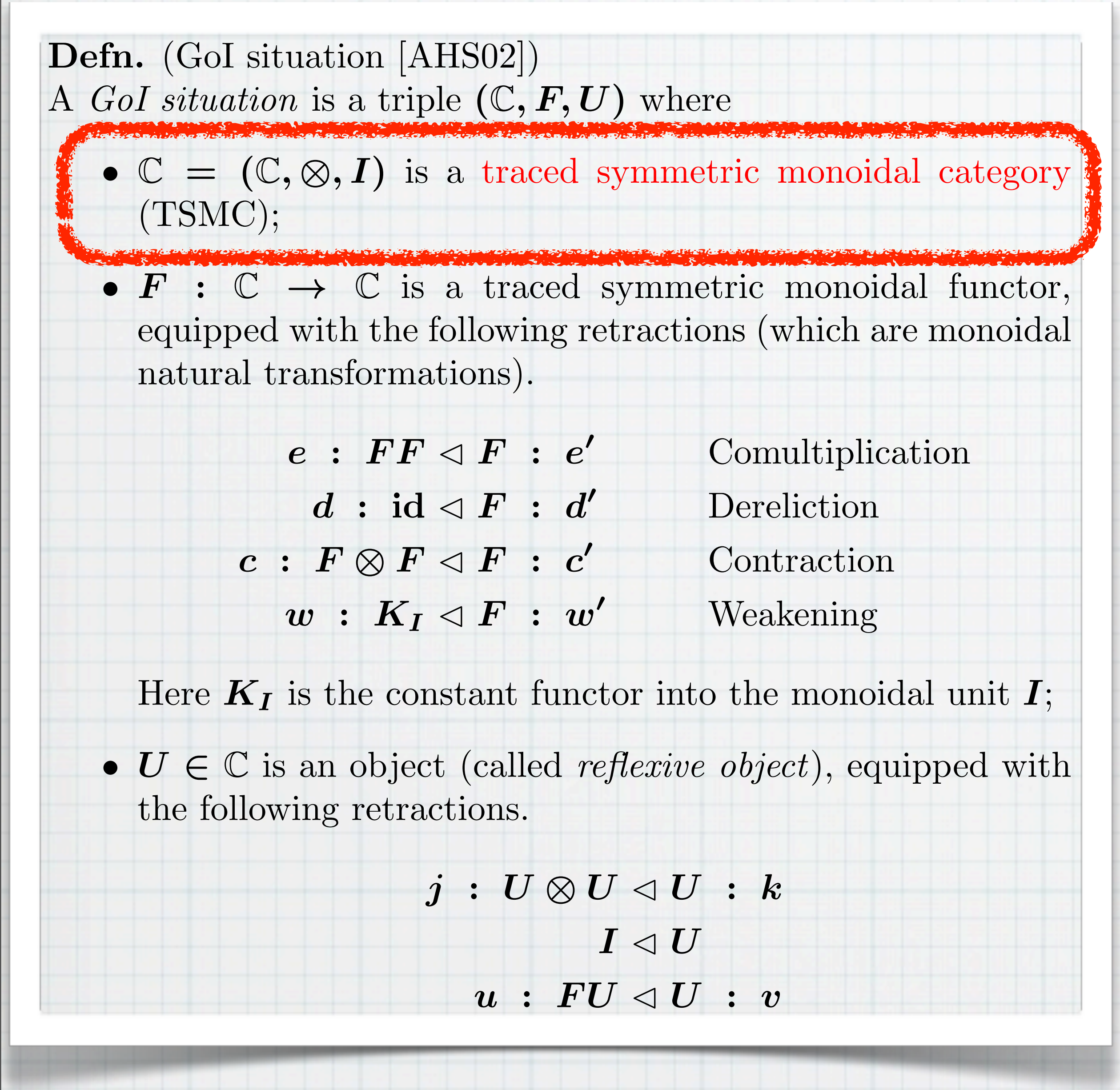
  \end{array}$$

- Monoidal category $(\mathbb{C}, \otimes, \boldsymbol{I})$
  - String diagrams

GoI situation



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- **Monoidal category** $(\mathbb{C}, \otimes, I)$
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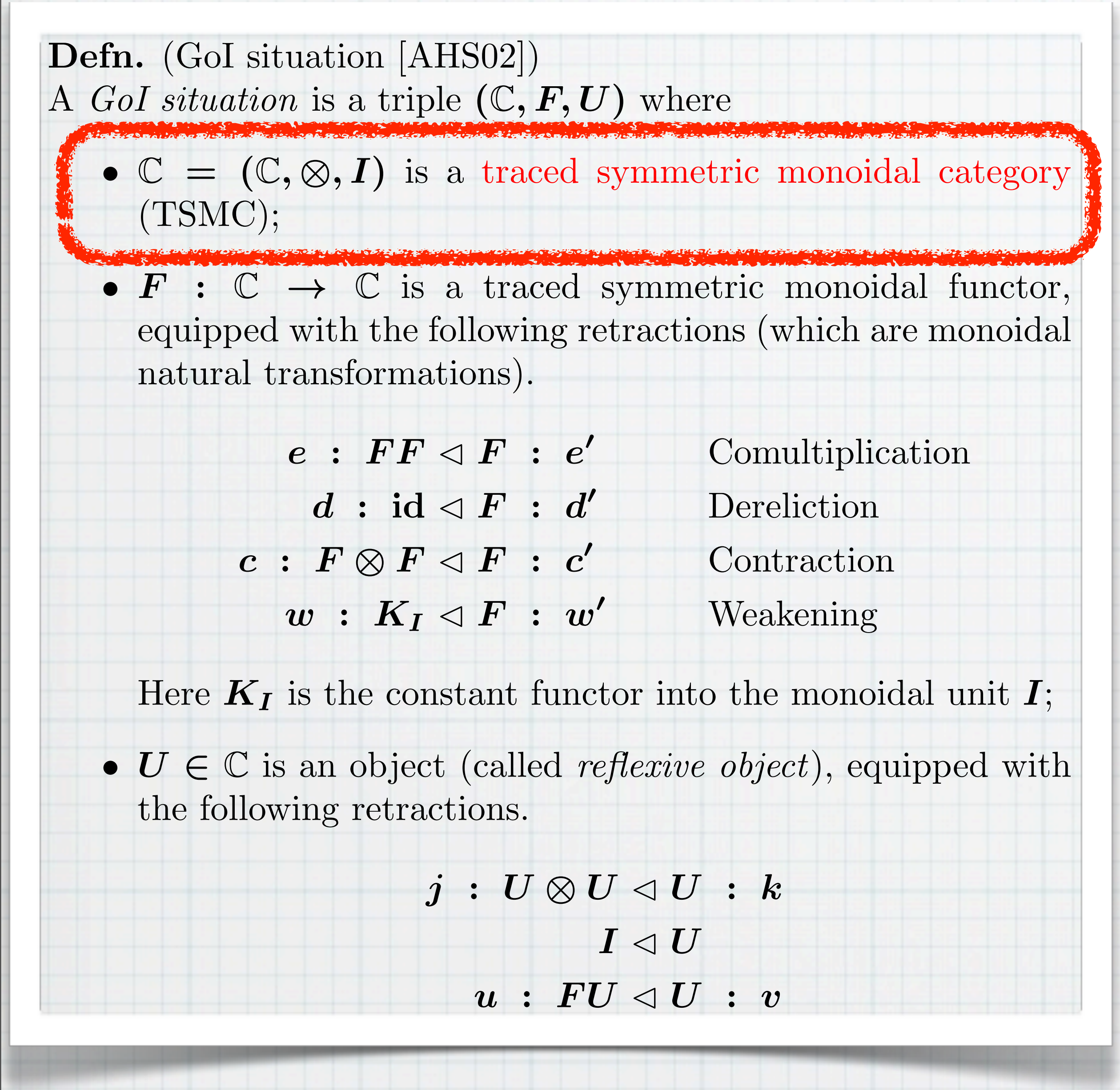
GoI situation



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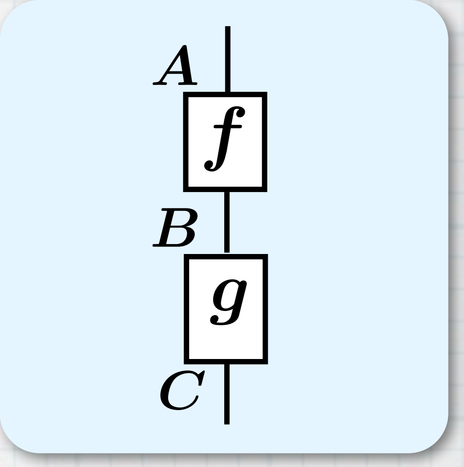
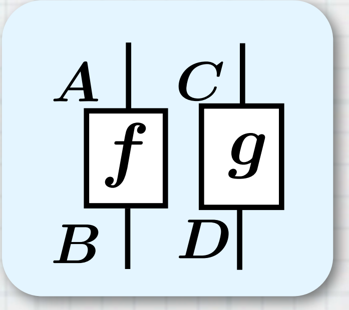
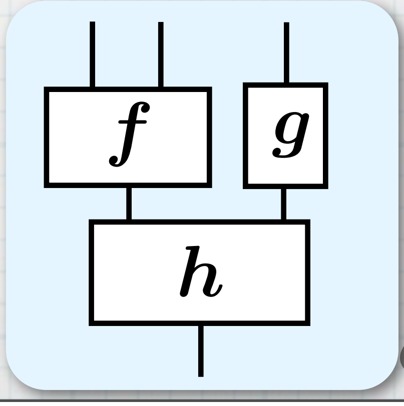
$$\frac{A \xrightarrow{f} B \qquad B \xrightarrow{g} C}{A \xrightarrow{g \circ f} C}$$



$$\frac{A \xrightarrow{f} B \qquad C \xrightarrow{g} D}{A \otimes C \xrightarrow{f \otimes g} B \otimes D}$$



$$h \circ (f \otimes g)$$



o)

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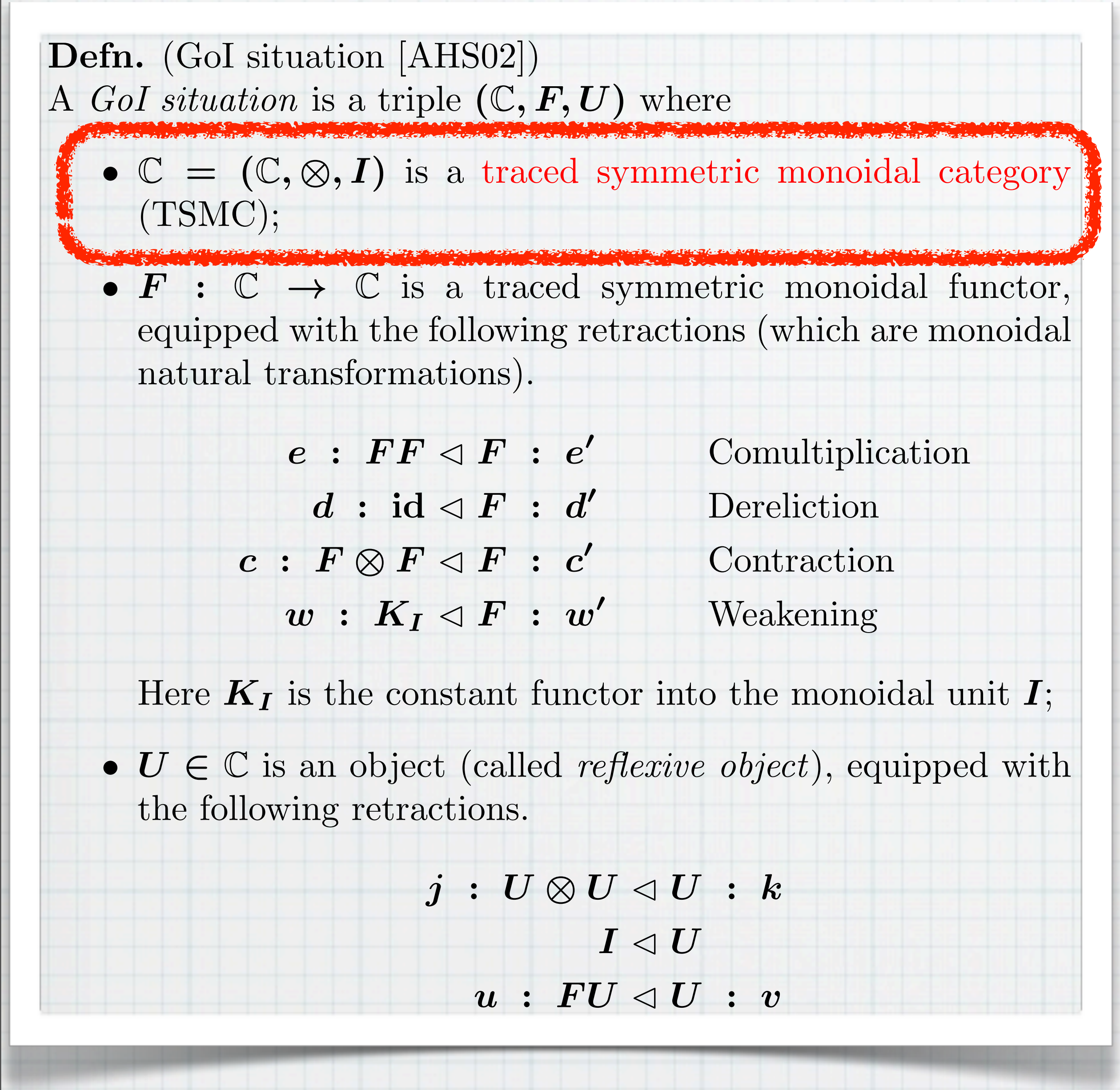

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- **Traced** monoidal category
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$$\frac{A \otimes C \xrightarrow{f} B \otimes C}{A \xrightarrow{\mathrm{tr}(f)} B}$$

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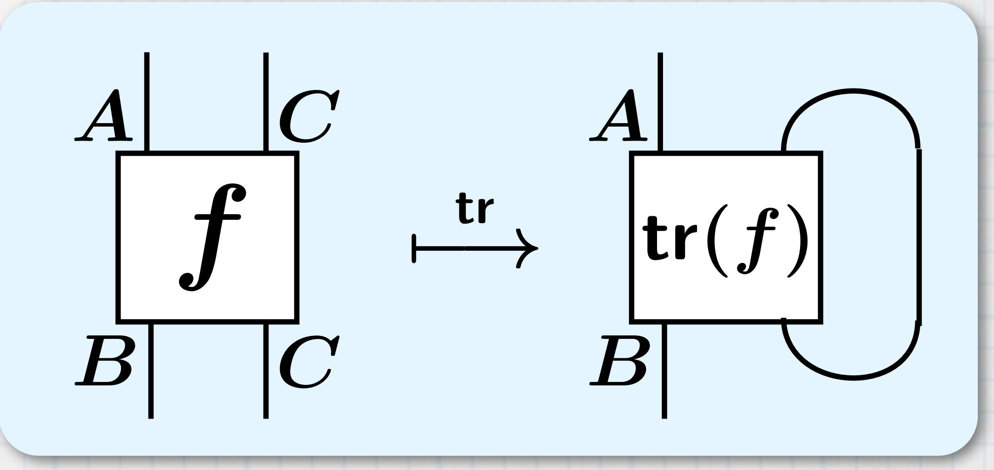
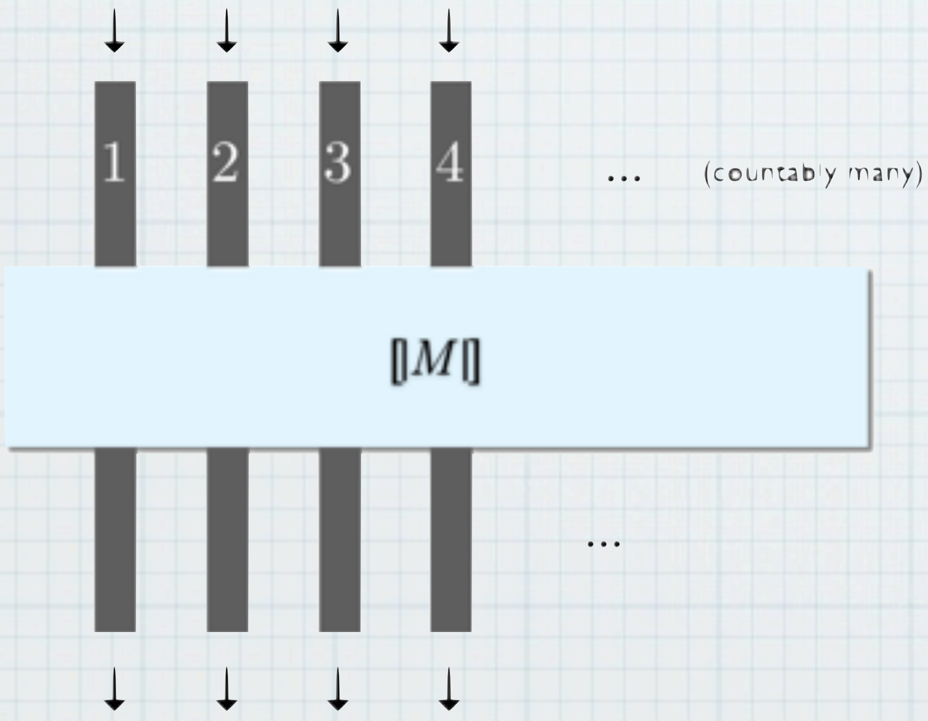
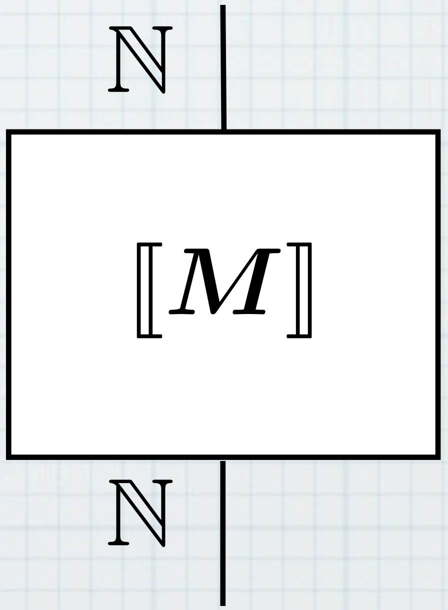


String Diagram vs. "Pipe Diagram"

- I use two ways of depicting partial functions $\mathbb{N} \rightharpoonup \mathbb{N}$



Pipe diagram



String diagram

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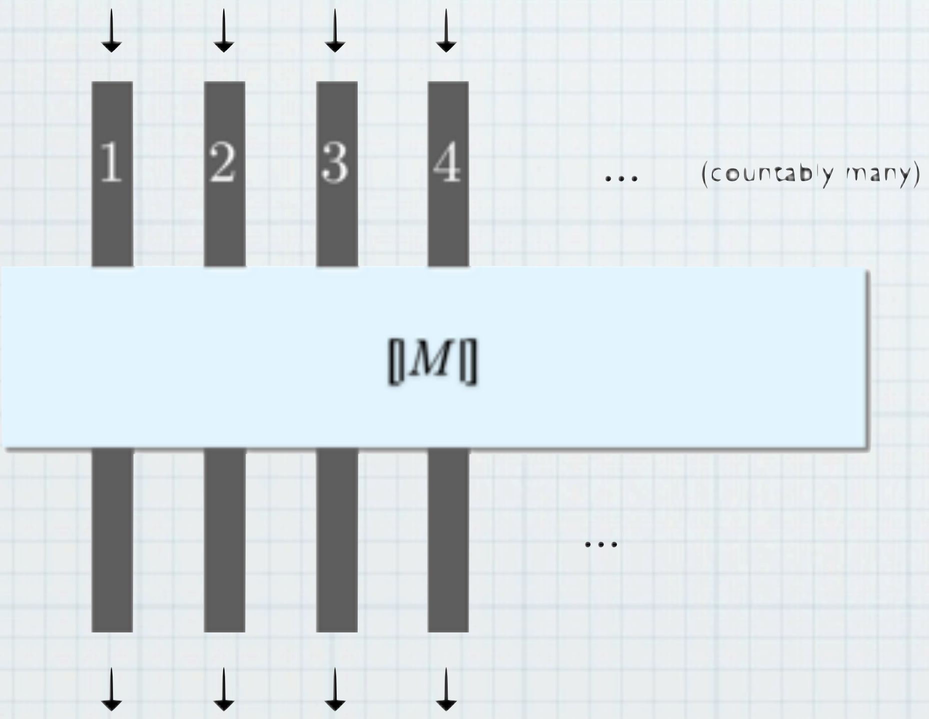
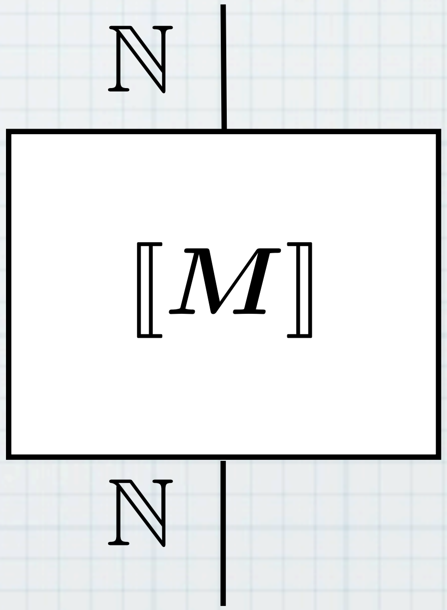
In the monoidal category $(\mathbf{Pfn}, +, 0)$



Pipe diagram



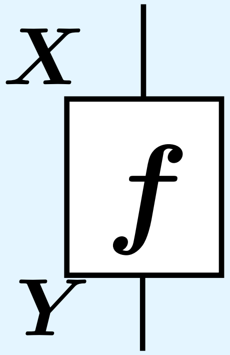
String diagram

Traced Sym. Monoidal Category

$(\mathbf{Pfn}, +, 0)$

- Category **Pfn** of partial functions
  - **Obj.** A set X
  - **Arr.** A partial function

$$\begin{array}{c} X \to Y \ \text{ in } \mathbf{Pfn} \\ \hline\hline X \rightharpoonup Y, \text{ partial function} \end{array}$$



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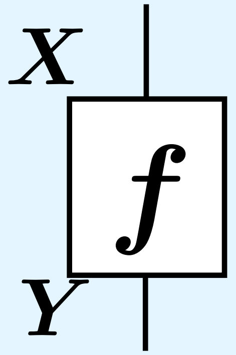
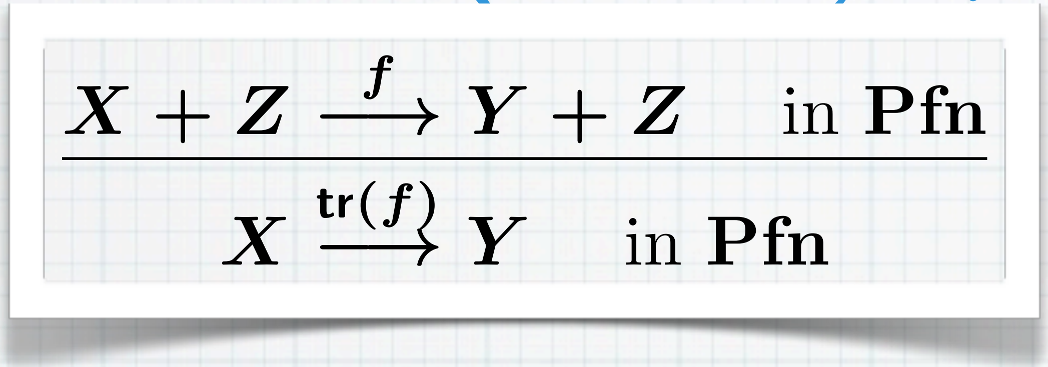


- is traced symmetric monoidal

Traced Sym. Monoidal Category

$$(\mathbf{Pfn}, +, 0)$$



$$\frac{X + Z \xrightarrow{f} Y + Z \qquad \text{in } \mathbf{Pfn}}{X \xrightarrow{\mathsf{tr}(f)} Y \qquad \text{in } \mathbf{Pfn}}$$

How?

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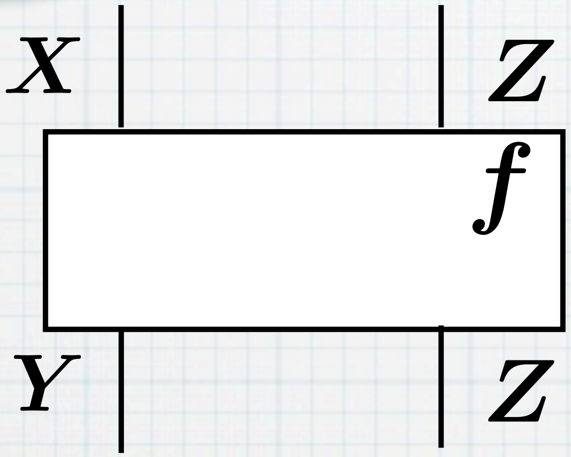
- 

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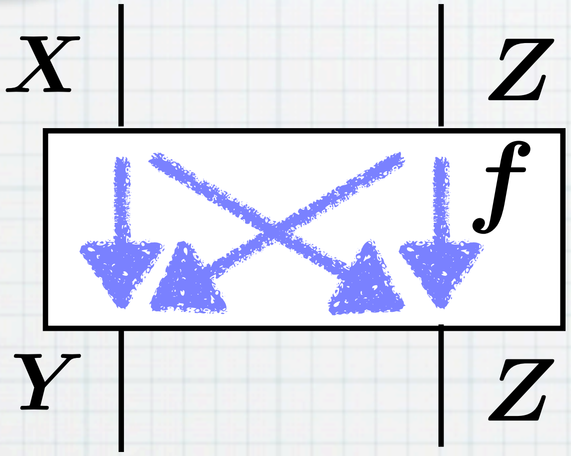
- 

Hasuo (Tokyo)

Traced Sym. Monoidal Category

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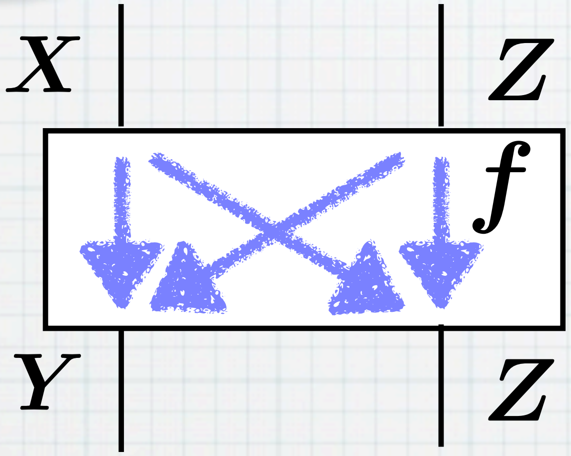
- 

$$f_{XY}(x) := \begin{cases} f(x) & \text{if } f(x) \in Y \\ \bot & \text{o.w.} \end{cases}$$

Similar for f_{XZ}, f_{ZY}, f_{ZZ}

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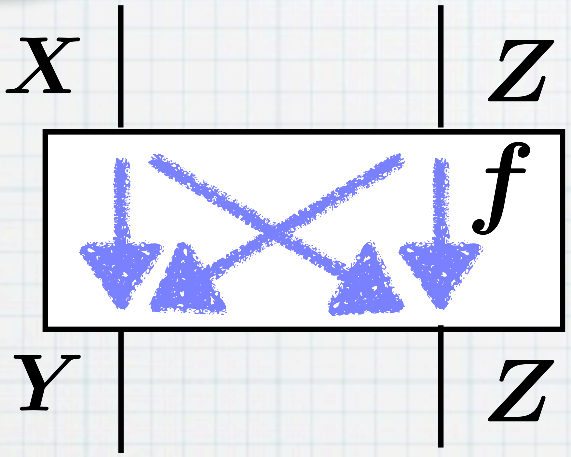
- 

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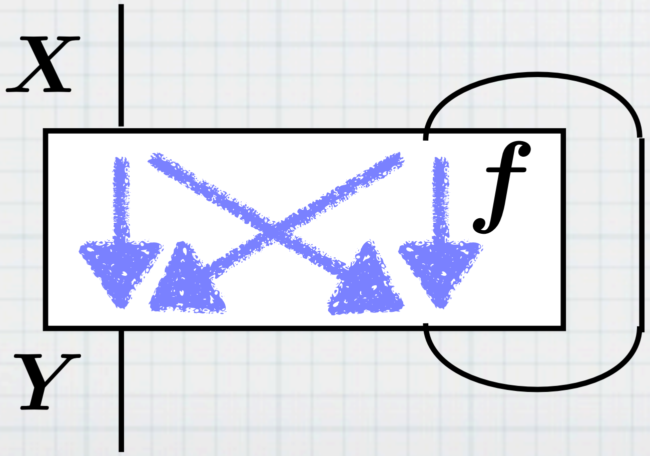

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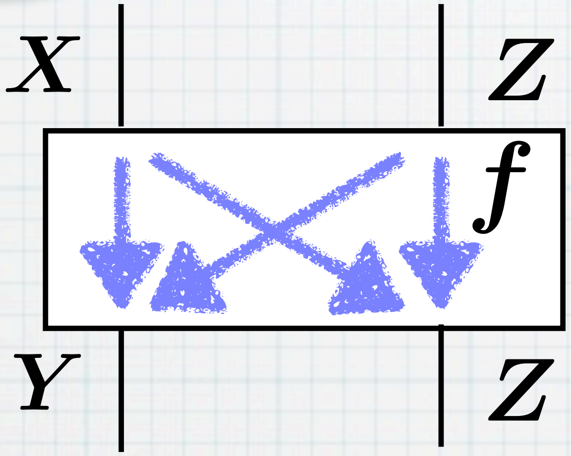
- 

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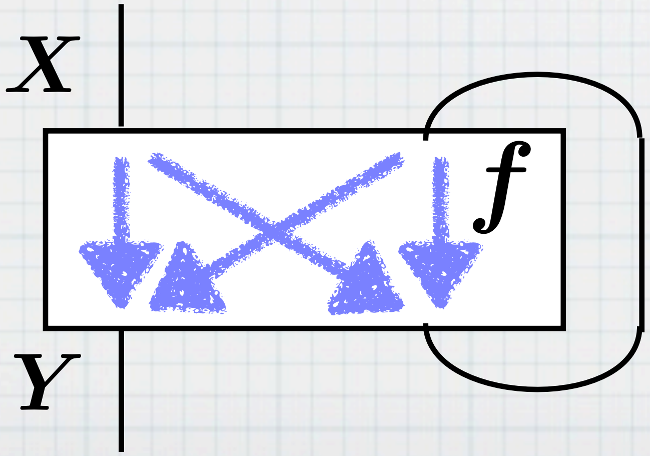

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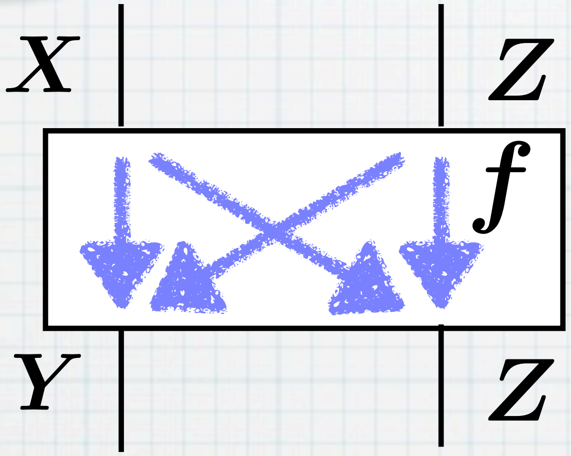
- 

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- **Execution formula** (Girard)
- Partiality is essential (infinite loop)

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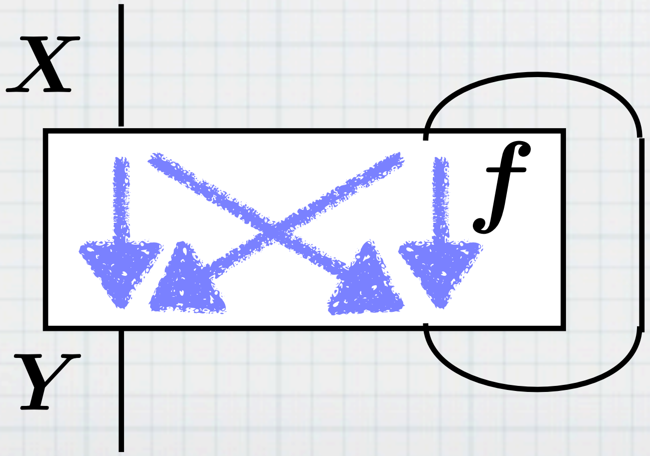

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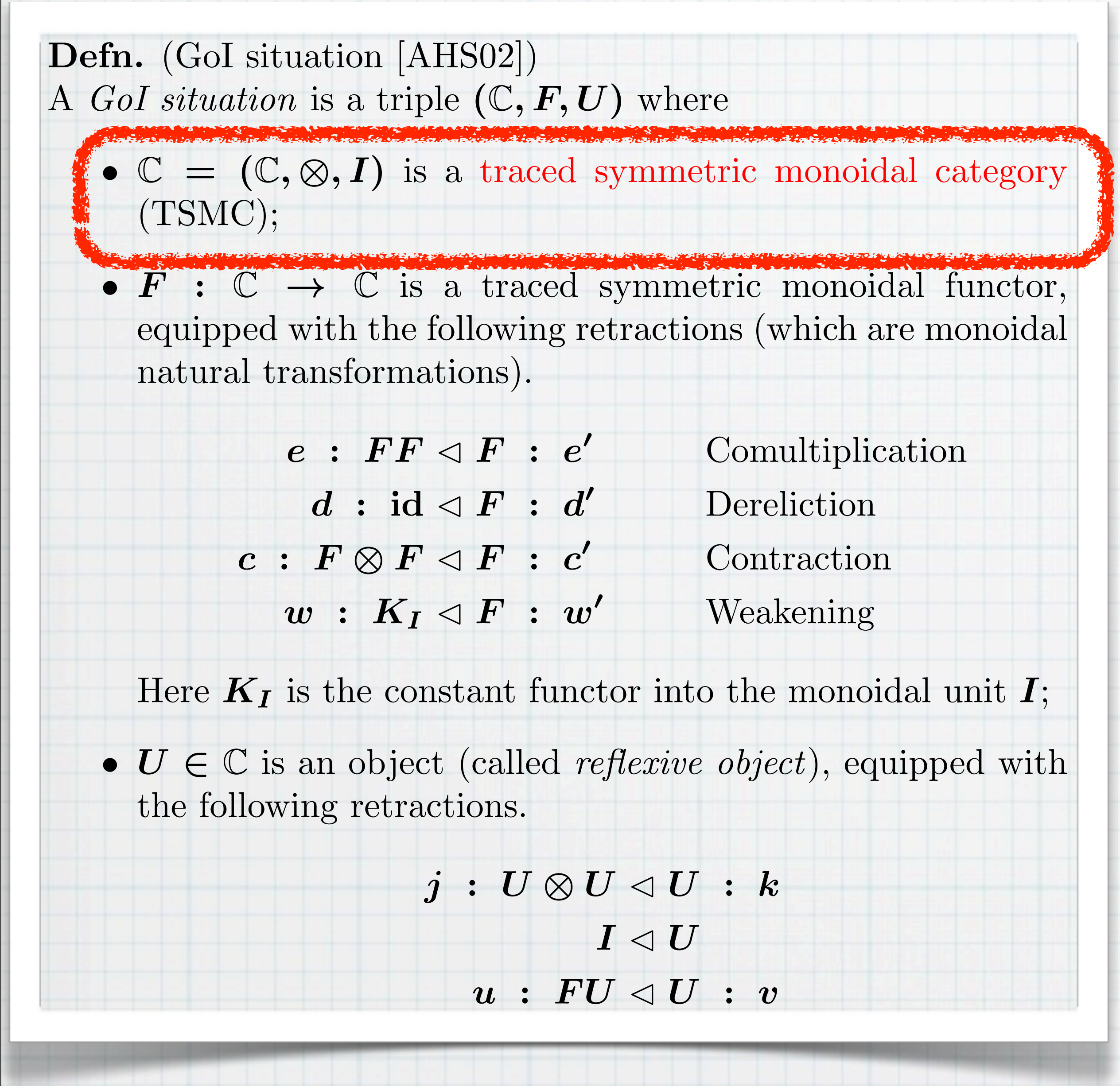
GoI situation



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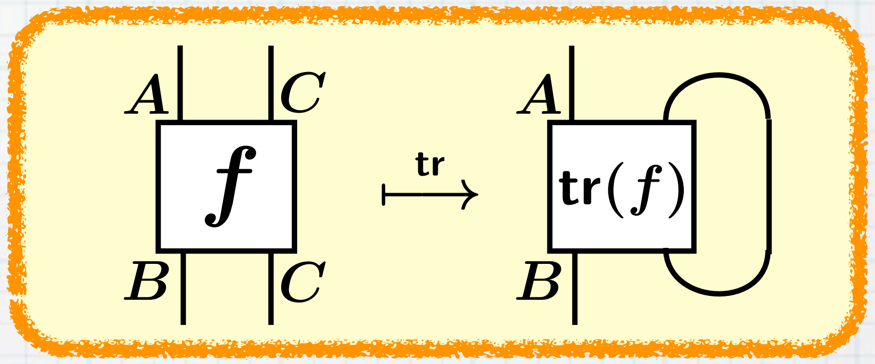
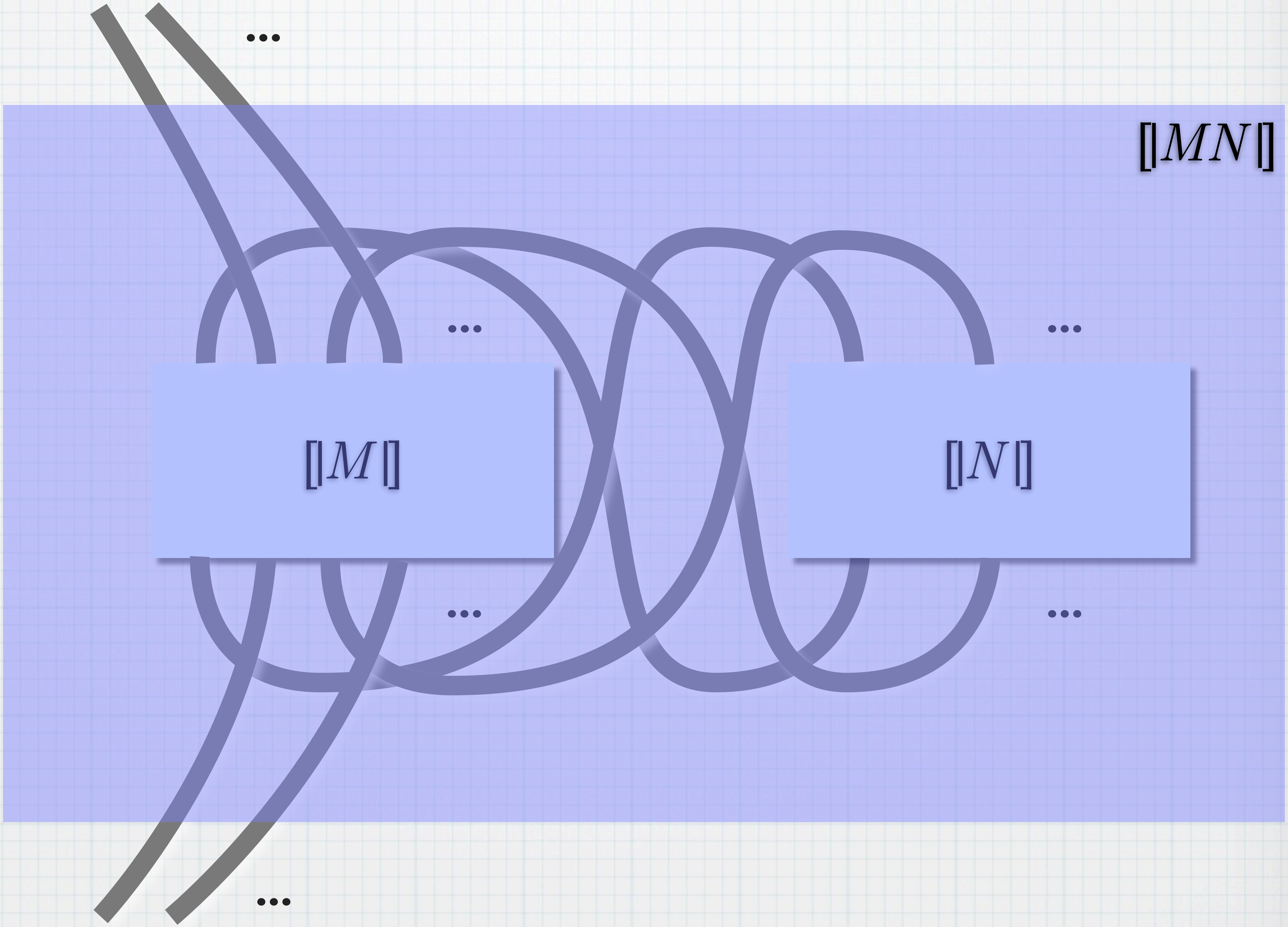


  - Why for GoI?

$[\![\boldsymbol{M}\boldsymbol{N}]\!] =$



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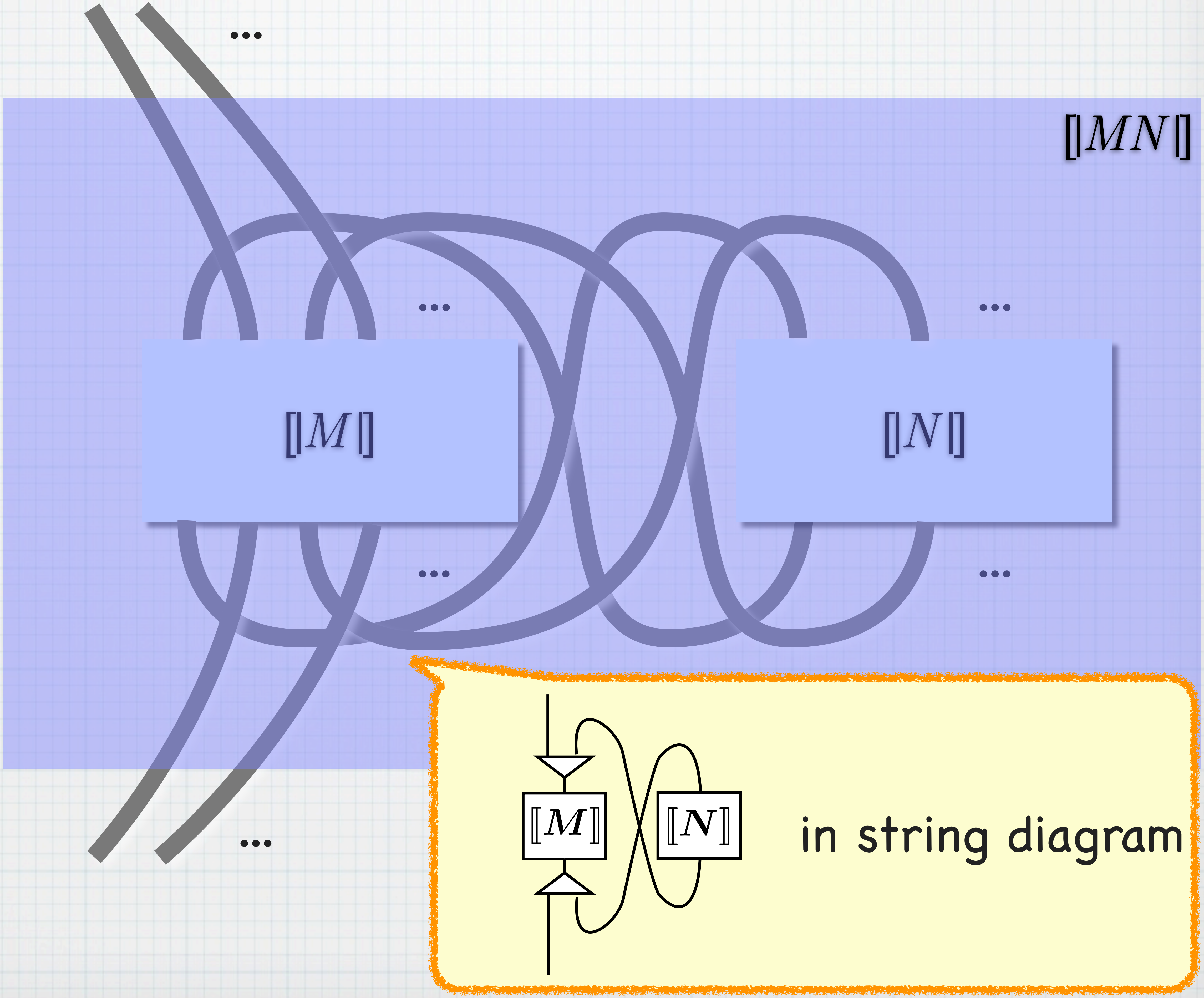
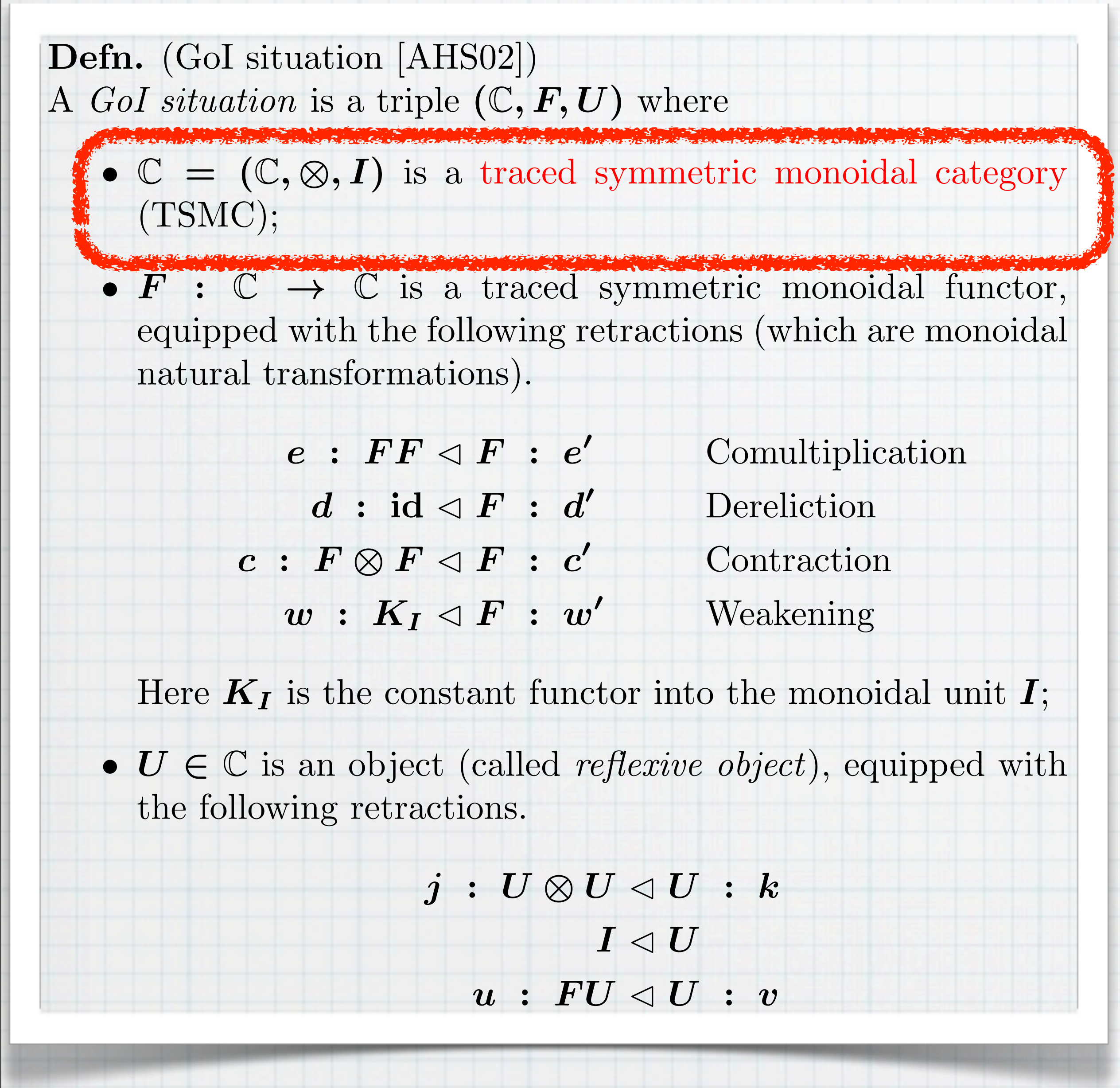


in string diagram

GoI situation



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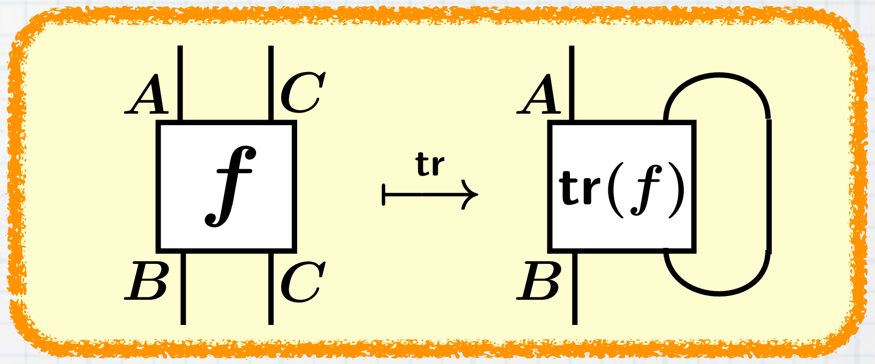

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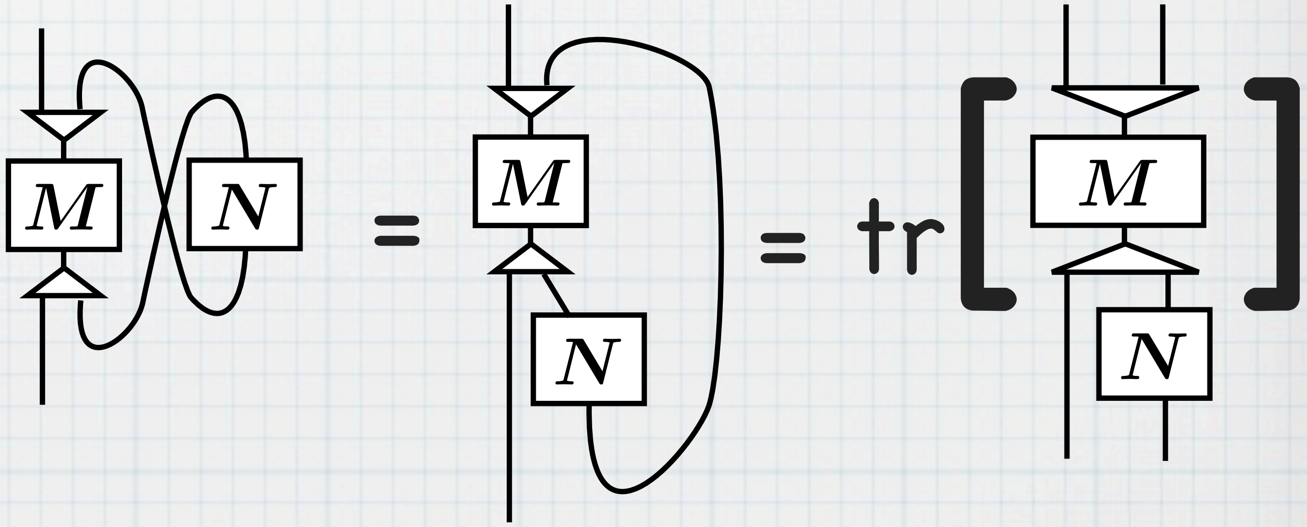

  - Leading example: **Pfn**

GoI situation

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Defn. (Retraction)
A *retraction* from X to Y,

$$f : X \lhd Y : g \ ,$$

is a pair of arrows

$$\begin{tikzcd} X \arrow[loop left, "\mathrm{id}"] \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

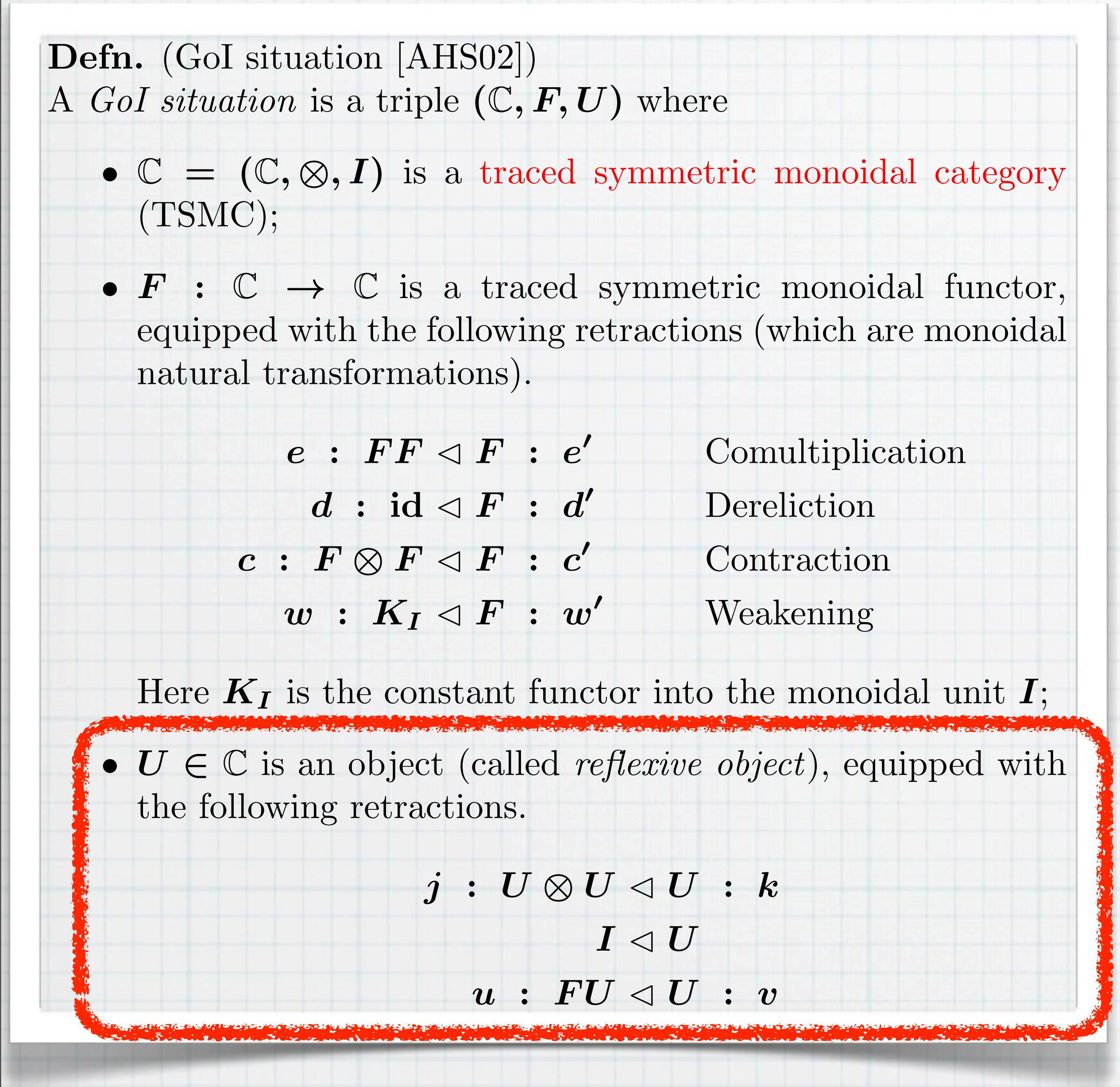
"embedding"

"projection"

such that $g \circ f = \mathrm{id}_X$.

- Functor F
  - For obtaining $! : A \to A$

GoI situation



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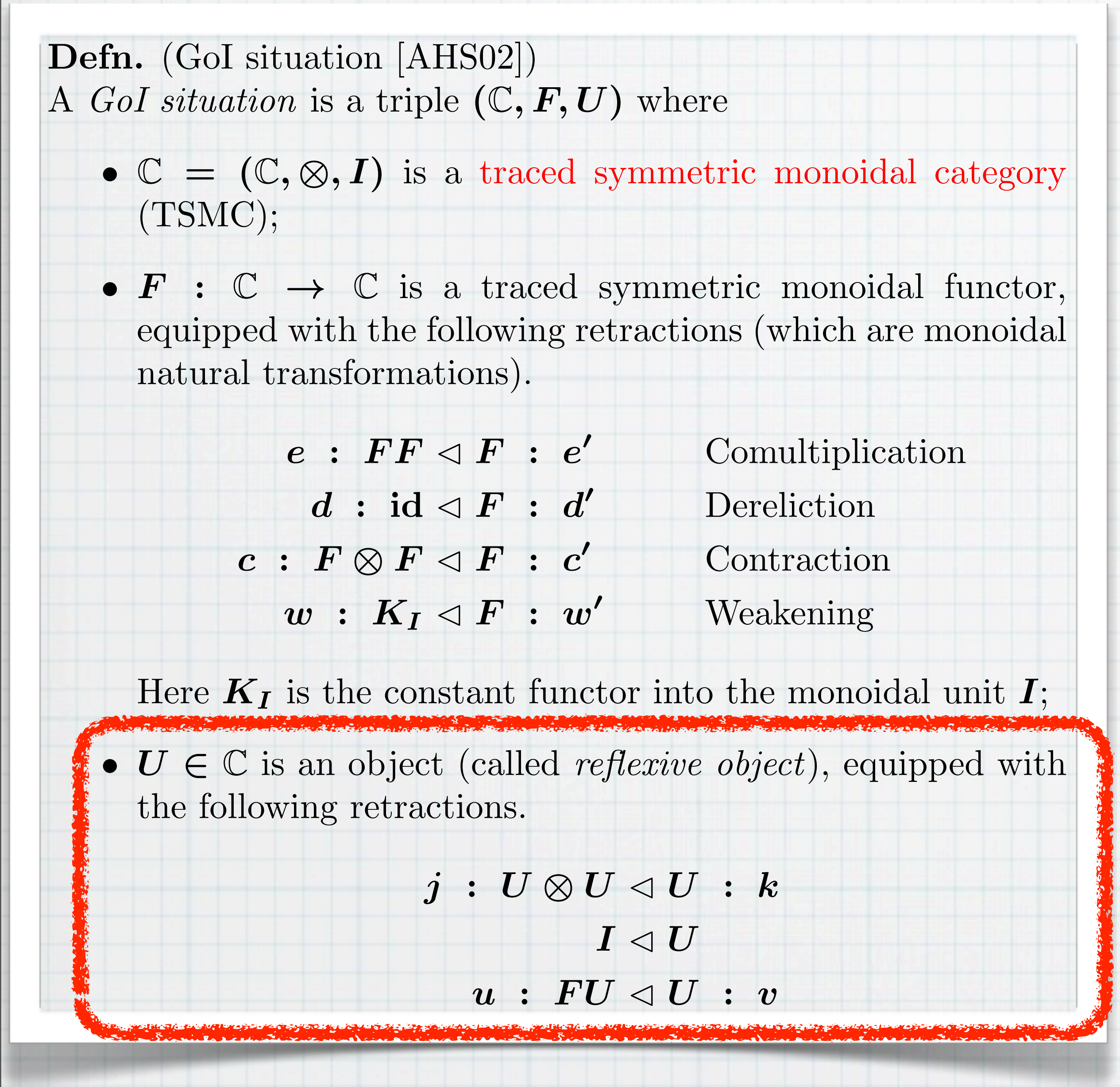
- The reflexive object U
- Retr. $$\begin{tikzcd} \boldsymbol{U} \otimes \boldsymbol{U} \arrow[r, bend left, "\boldsymbol{j}"] & \boldsymbol{U} \arrow[l, bend left, "\boldsymbol{k}"] \end{tikzcd}$$

GoI situation



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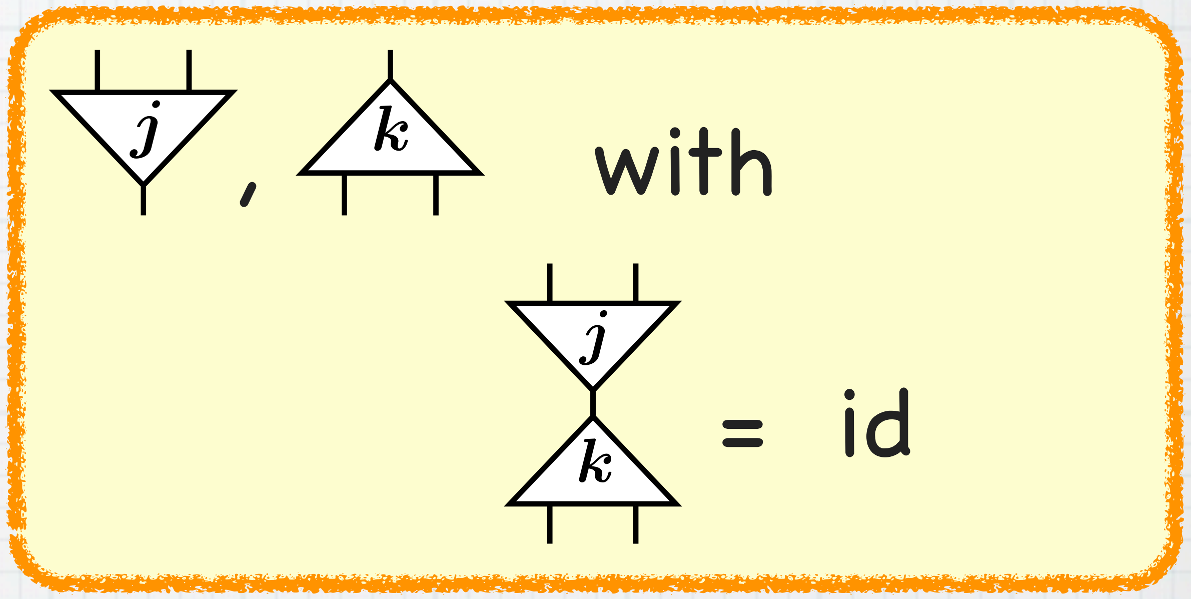
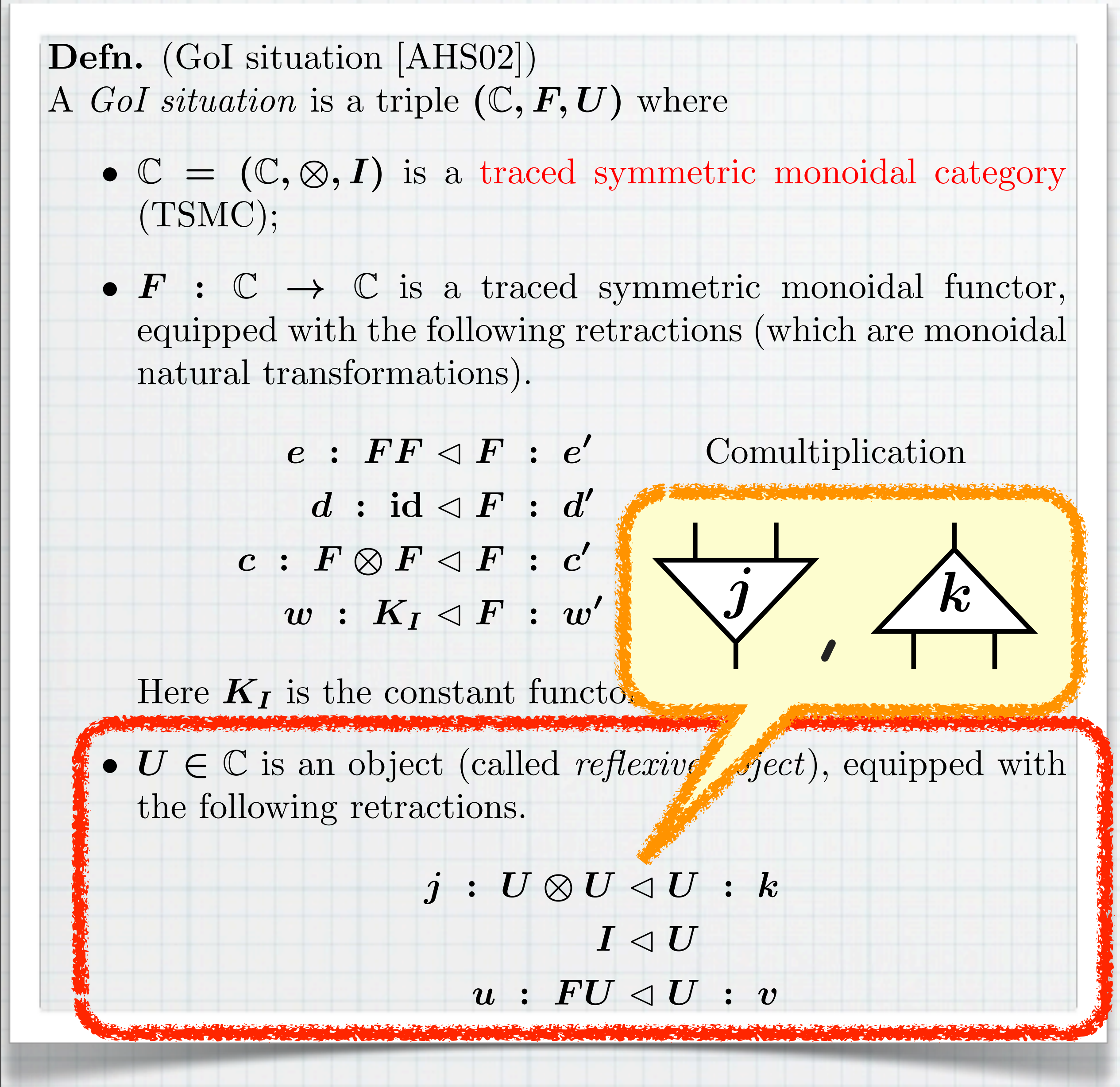
- Retr. $$\begin{tikzcd} U \otimes U \arrow[r, bend left, "j"] & U \arrow[l, bend left, "k"] \end{tikzcd}$$



, with

= id

GoI situation



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$$\begin{aligned} \boldsymbol{e} &: \boldsymbol{FF} \lhd \boldsymbol{F} : \boldsymbol{e}' \qquad \text{Comultiplication} \\ \boldsymbol{d} &: \mathbf{id} \lhd \boldsymbol{F} : \boldsymbol{d}' \\ \boldsymbol{c} &: \boldsymbol{F} \otimes \boldsymbol{F} \lhd \boldsymbol{F} : \boldsymbol{c}' \\ \boldsymbol{w} &: \boldsymbol{K_I} \lhd \boldsymbol{F} : \boldsymbol{w}' \end{aligned}$$

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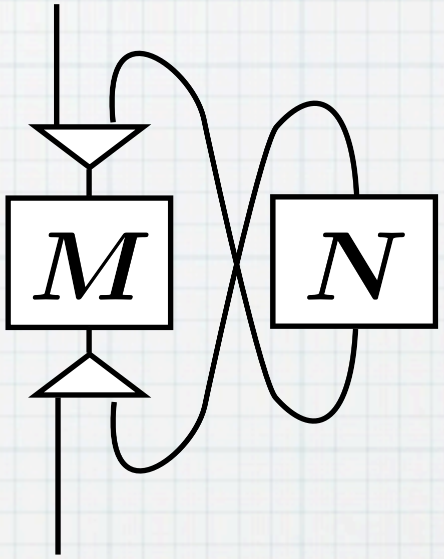
- $\boldsymbol{U} \in \mathbb{C}$ is an object (called *reflexive object*), equipped with the following retractions.

$$\begin{aligned} \boldsymbol{j} : \boldsymbol{U} \otimes \boldsymbol{U} &\lhd \boldsymbol{U} : \boldsymbol{k} \\ \boldsymbol{I} &\lhd \boldsymbol{U} \\ \boldsymbol{u} : \boldsymbol{FU} &\lhd \boldsymbol{U} : \boldsymbol{v} \end{aligned}$$

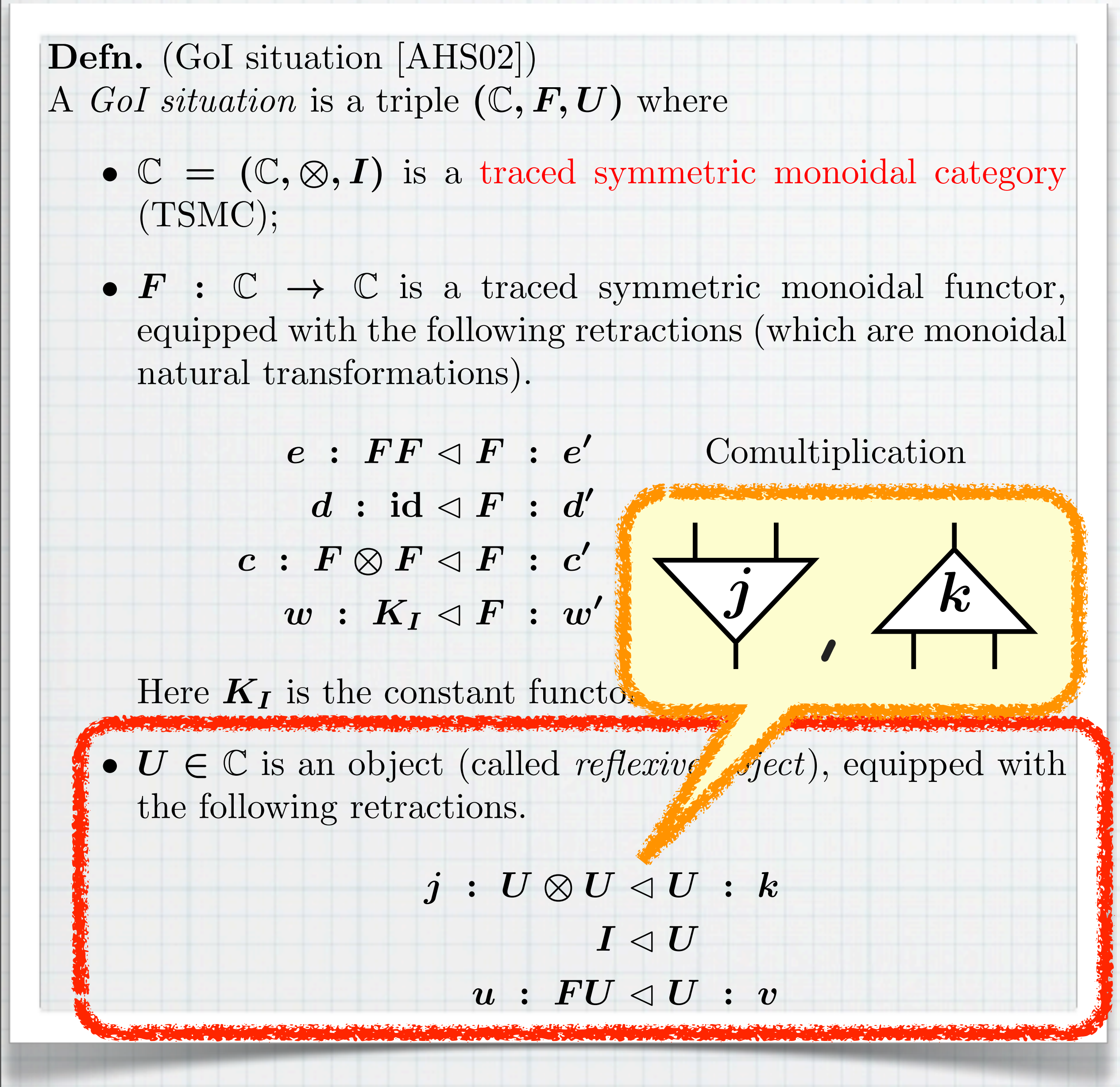
- The reflexive object U

- Why for GoI?



- Example in **Pfn**:

GoI situation



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$$\begin{aligned} e &: FF \lhd F : e' && \text{Comultiplication} \\ d &: \mathrm{id} \lhd F : d' \\ c &: F \otimes F \lhd F : c' \\ w &: K_I \lhd F : w' \end{aligned}$$

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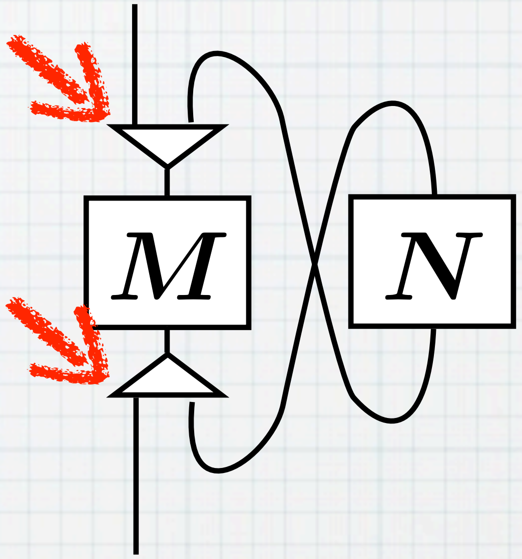
- The reflexive object U
- Why for GoI?



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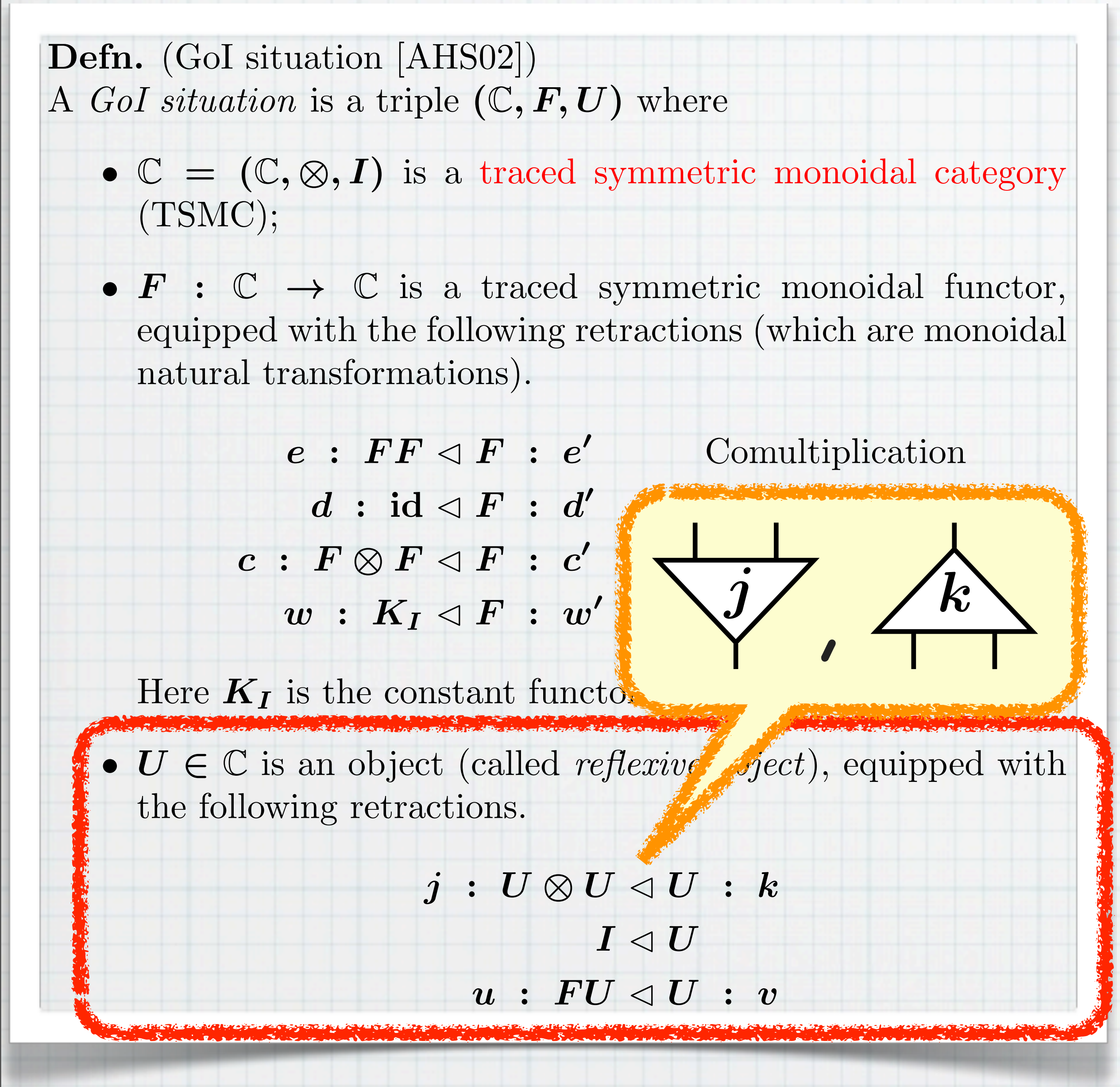

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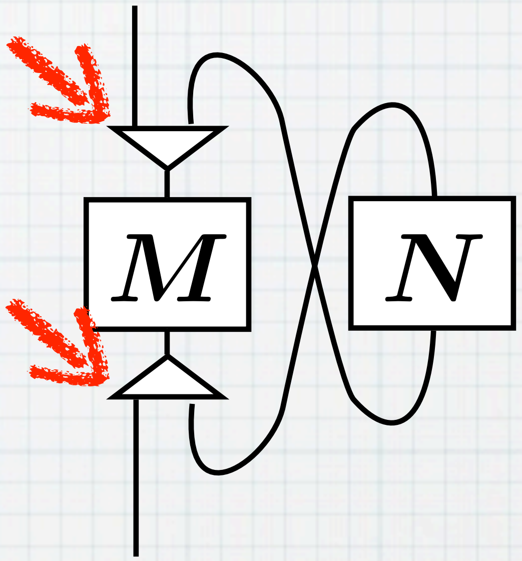
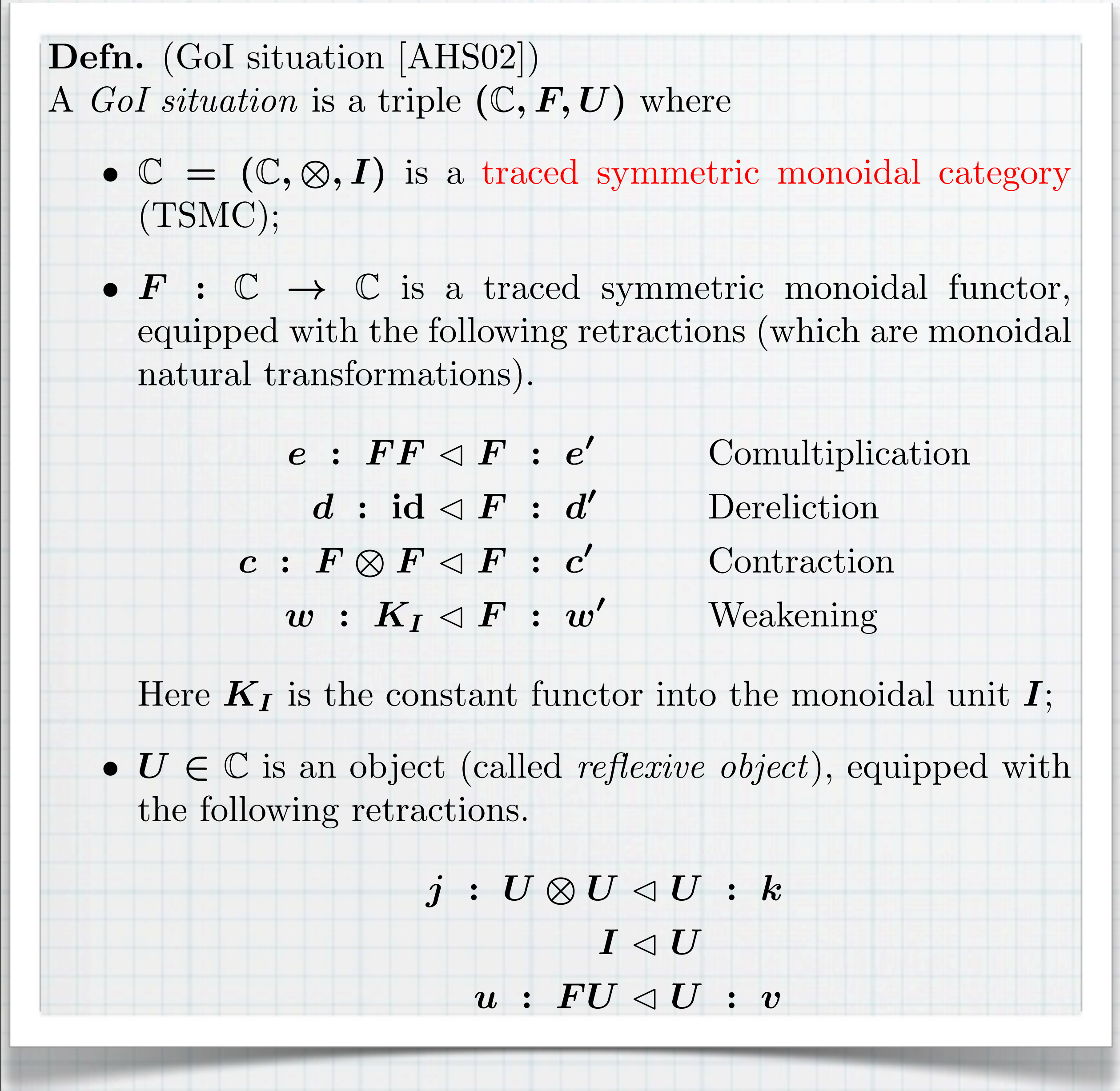


  - Example in **Pfn**:

$$\mathbb{N} \in \mathbf{Pfn}, \quad \text{with}$$
$$\mathbb{N} + \mathbb{N} \cong \mathbb{N},$$
$$\mathbb{N} \cdot \mathbb{N} \cong \mathbb{N}$$

GoI Situation: Summary



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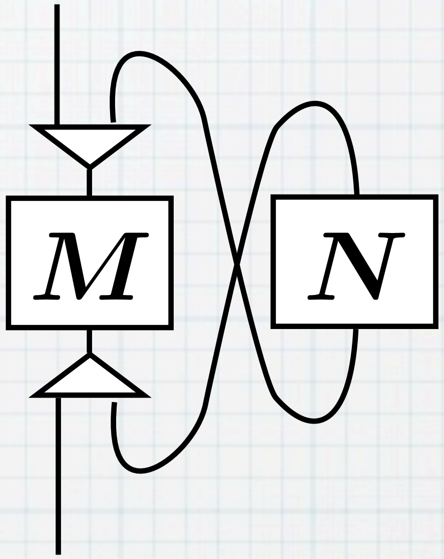
- Categorical axiomatics of the "GoI animation"



- Example:

$$(\mathbf{Pfn}, \mathbb{N} \cdot \_\,, \mathbb{N})$$

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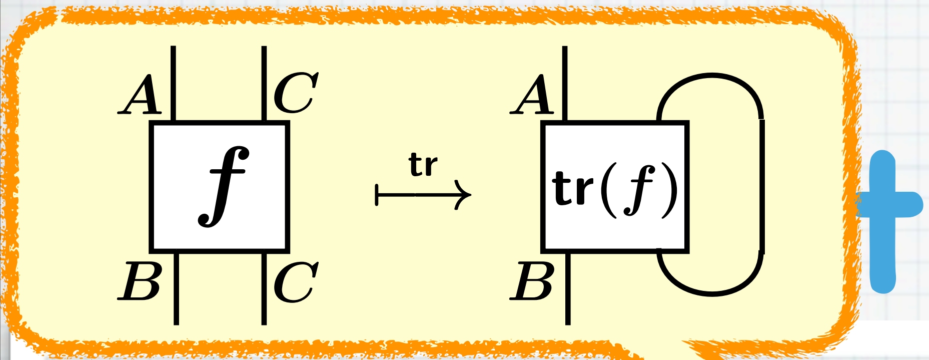

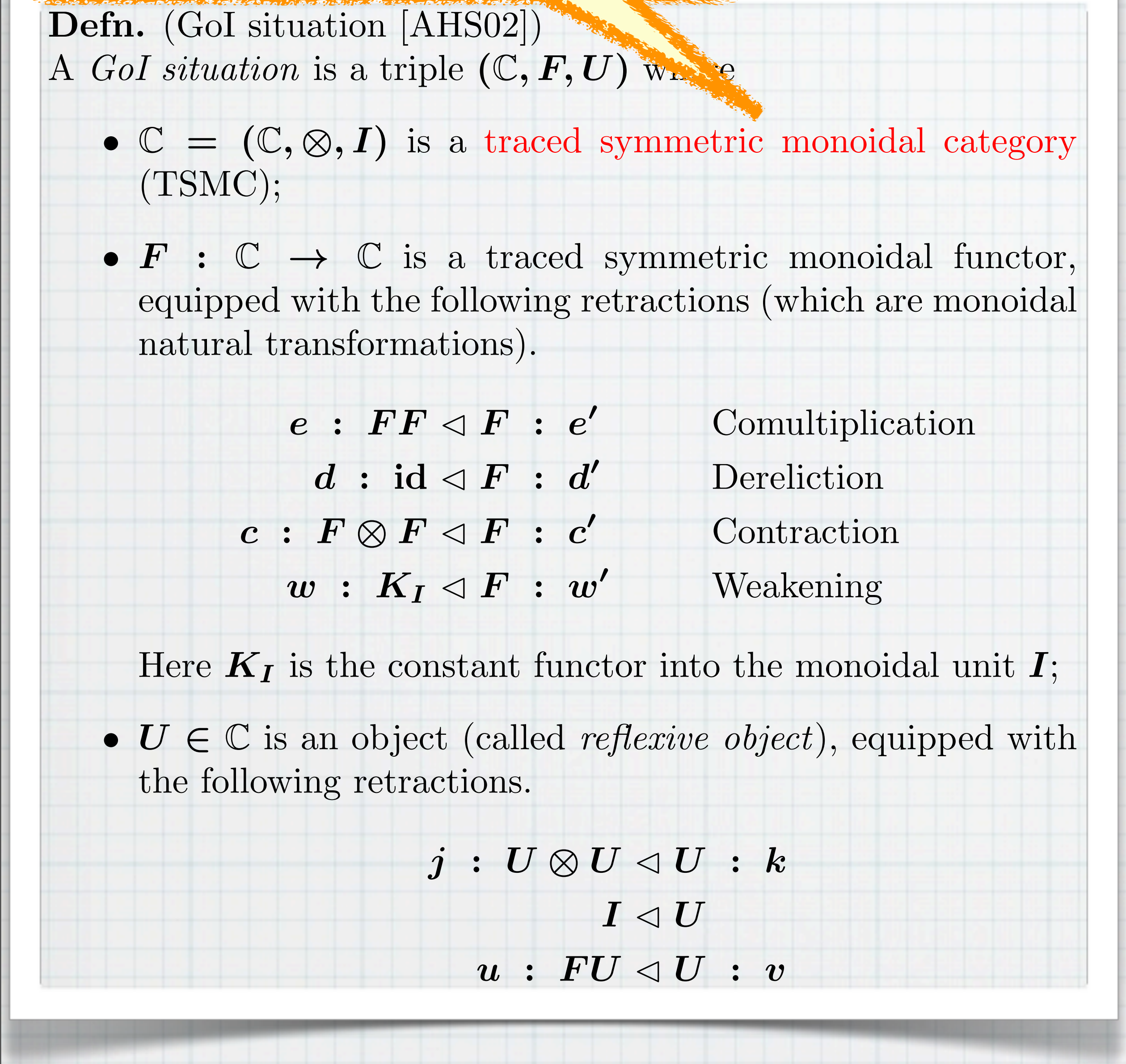

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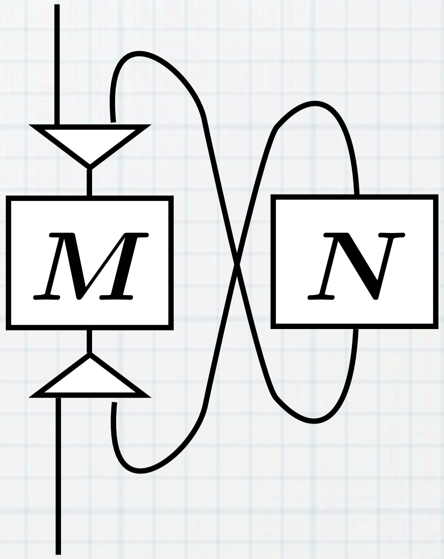

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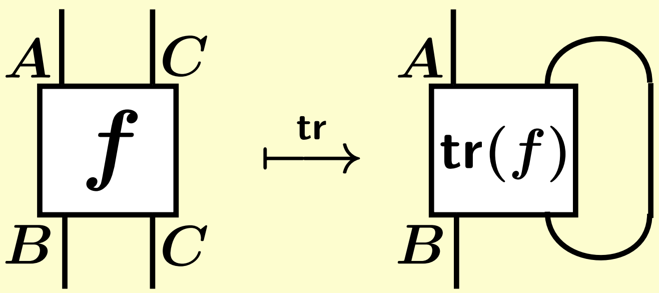

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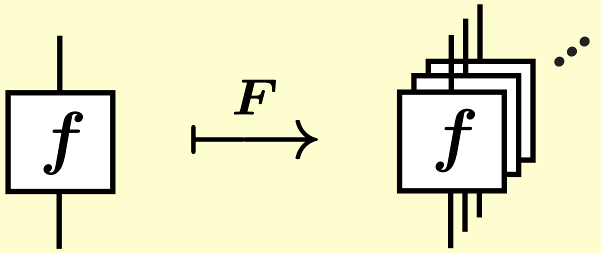
For ! , via



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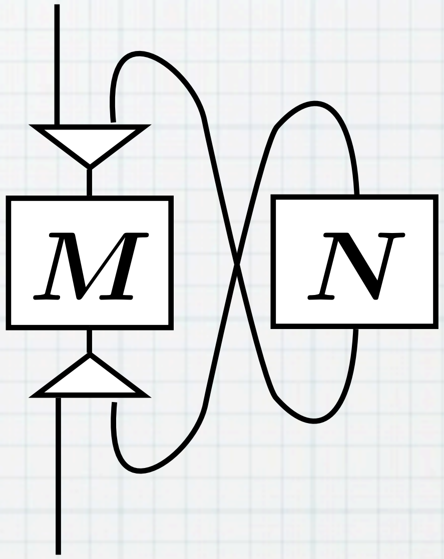

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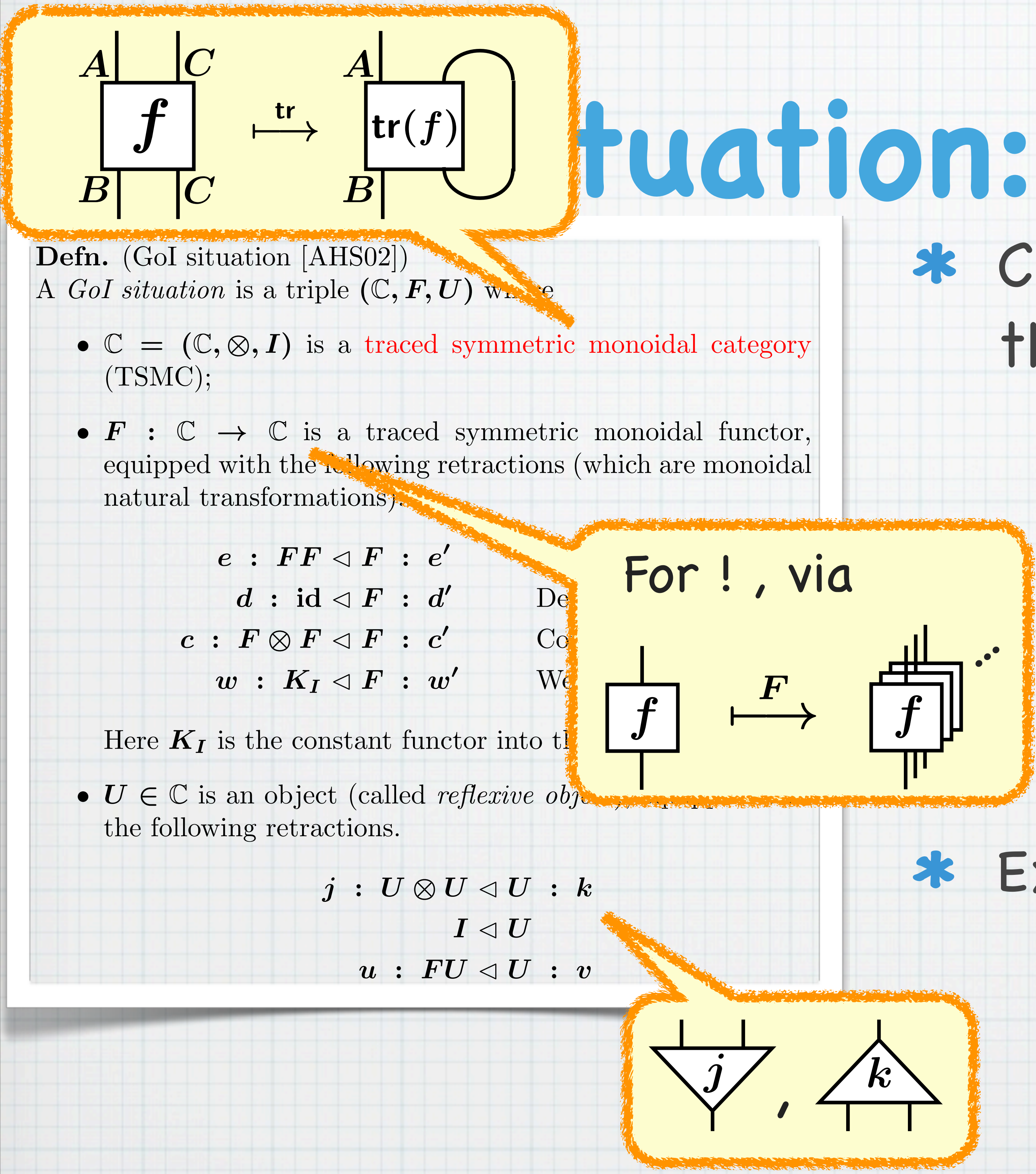

A
C
f
B
C
tr
A
tr(f)
B

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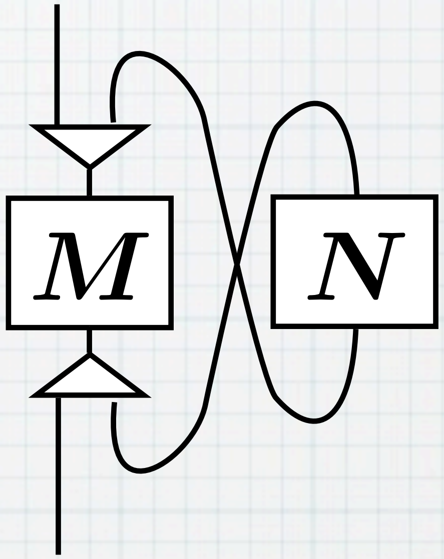

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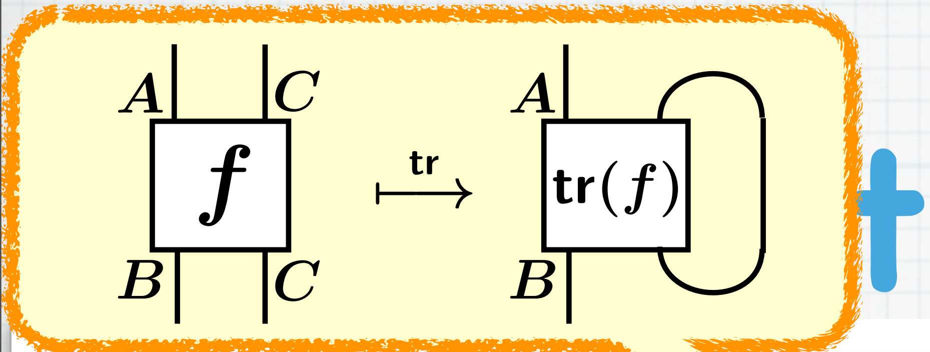

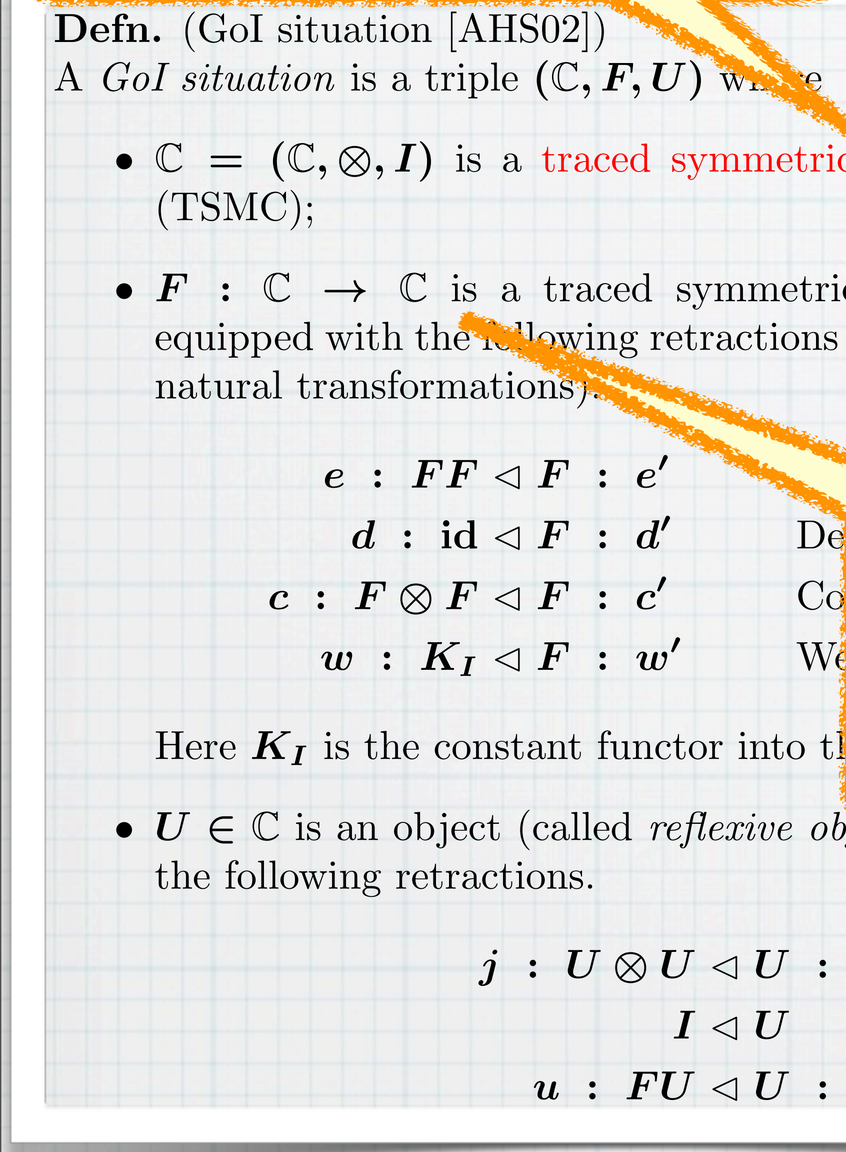

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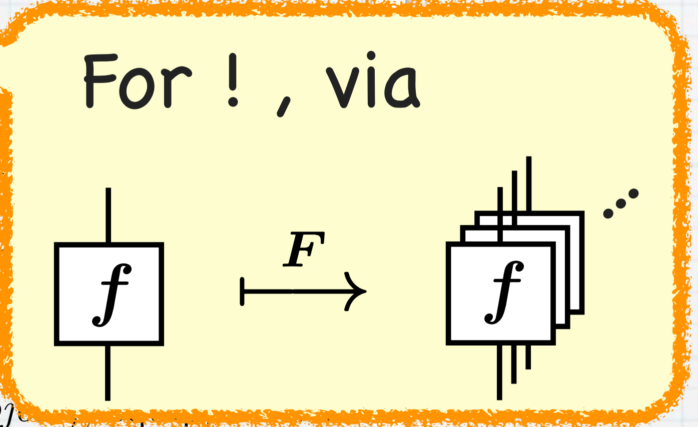

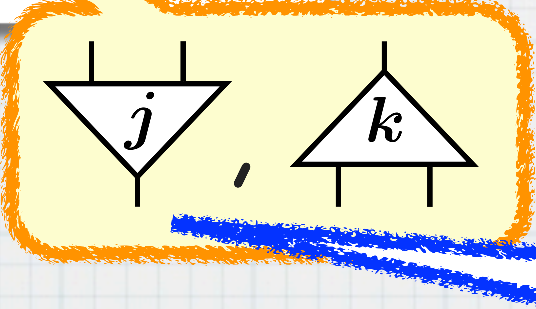

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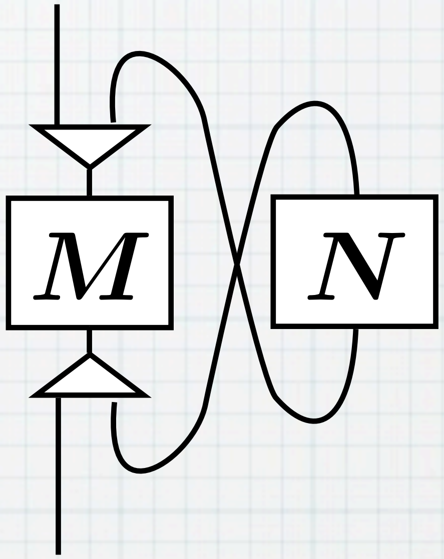

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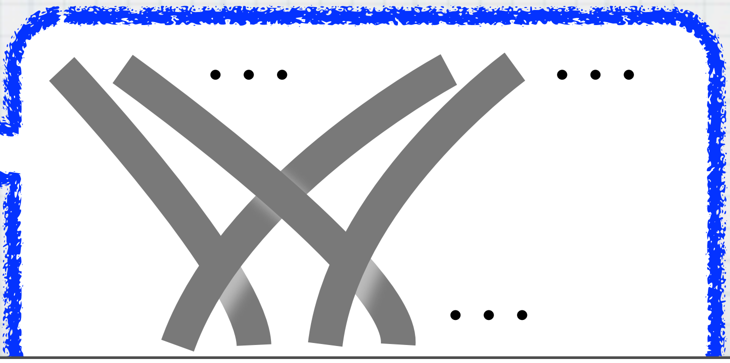

Categorical GoI: Constr. of an LCA

Thm. ([AHS02])
Given a GoI situation $(\mathbb{C}, \boldsymbol{F}, \boldsymbol{U})$, the homset

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carries a canonical LCA structure.

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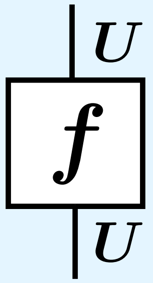

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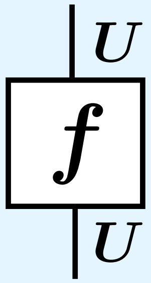
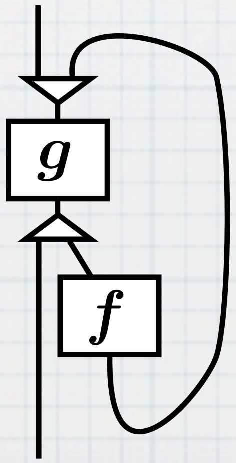


$\in \mathbb{C}(\boldsymbol{U}, \boldsymbol{U})$

- $\boldsymbol{g} \cdot \boldsymbol{f}$

$$:= \mathbf{tr}\Big((\boldsymbol{U} \otimes \boldsymbol{f}) \circ \boldsymbol{k} \circ \boldsymbol{g} \circ \boldsymbol{j} \Big)$$

=



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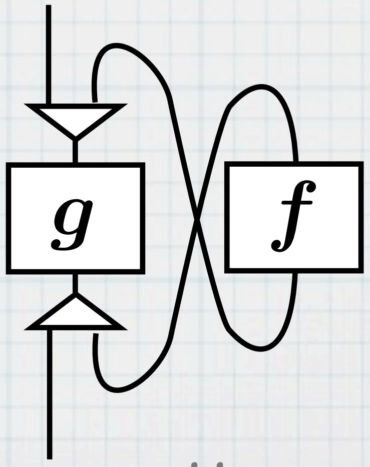

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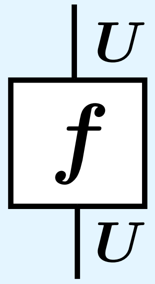
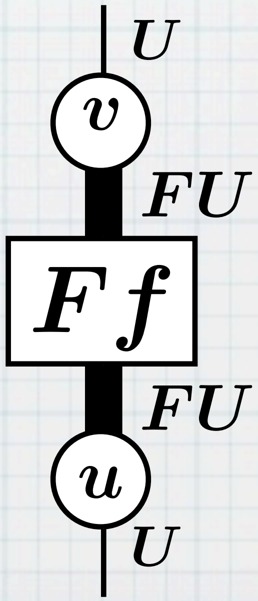


$\in \mathbb{C}(\boldsymbol{U}, \boldsymbol{U})$

- $!\boldsymbol{f} \;:=\; \boldsymbol{u} \circ \boldsymbol{F}\boldsymbol{f} \circ \boldsymbol{v}$

=



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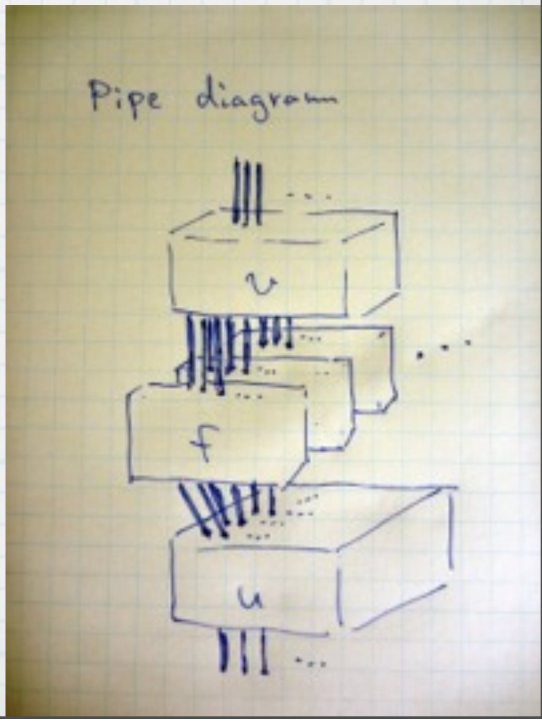

Categorical GoI: Constr. of an LCA

- Combinator $Bxyz = x(yz)$

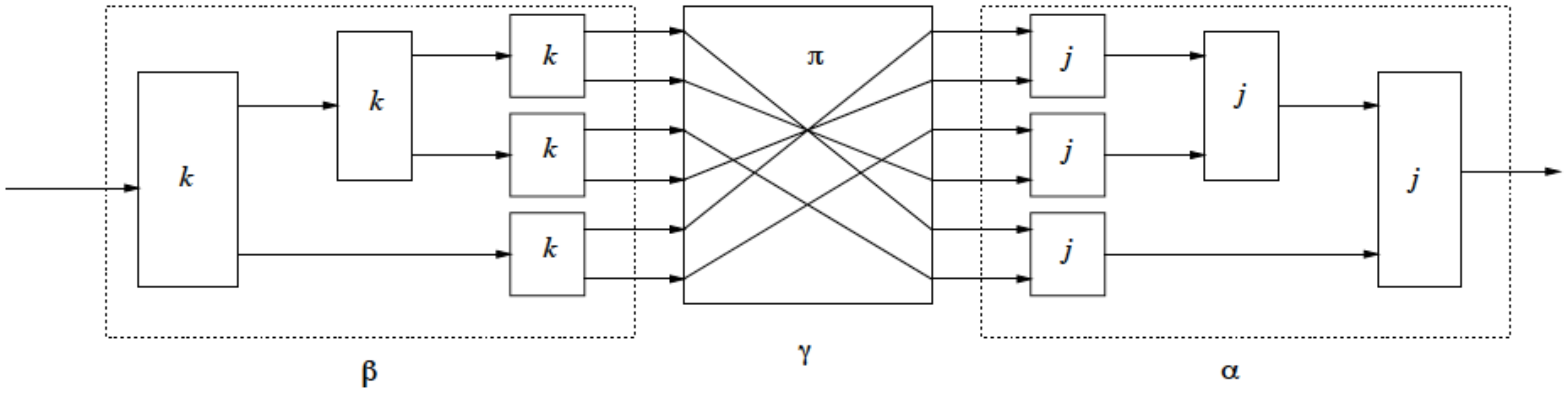

Figure 7: Composition Combinator B

from [AHS02]

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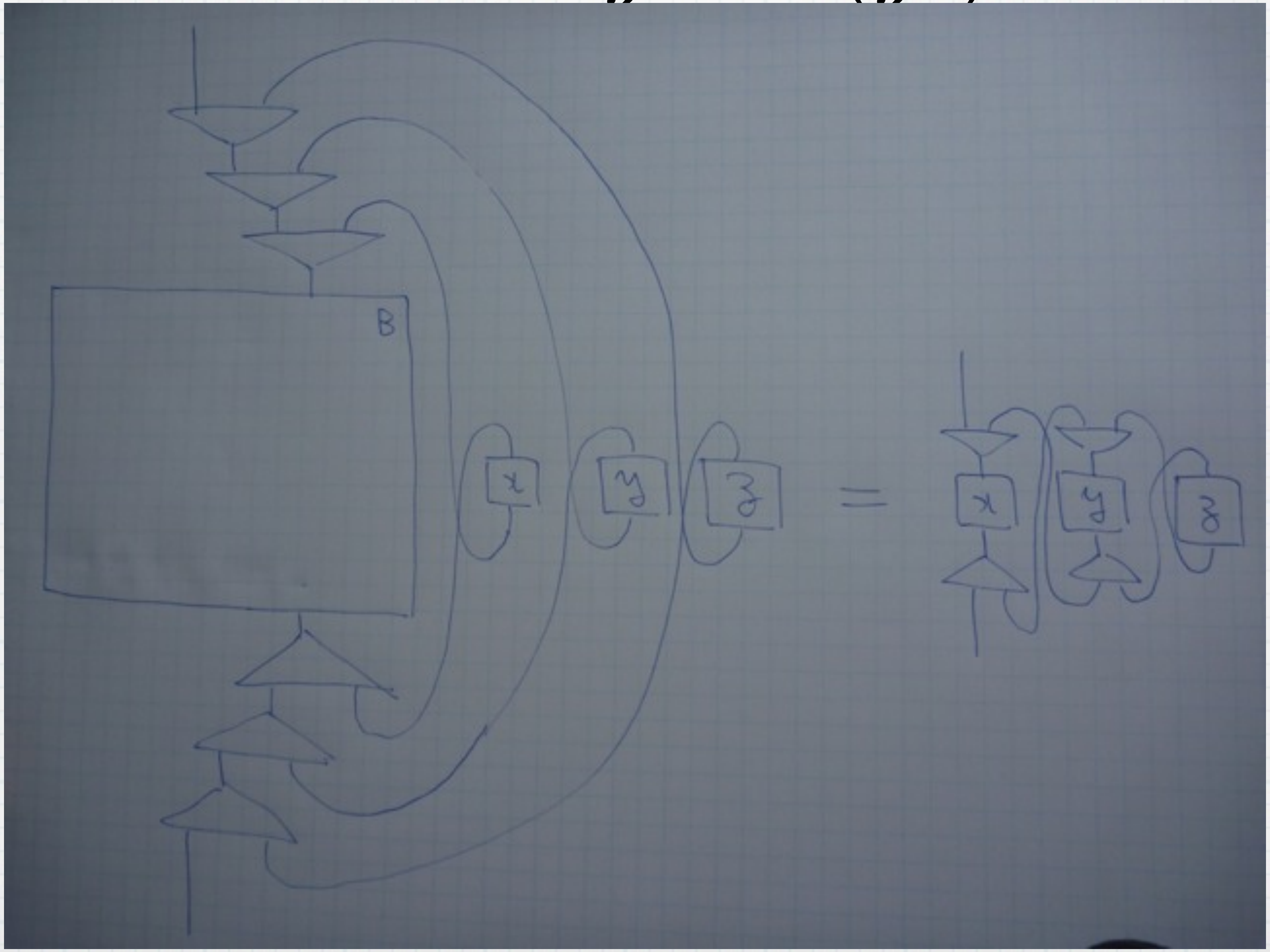
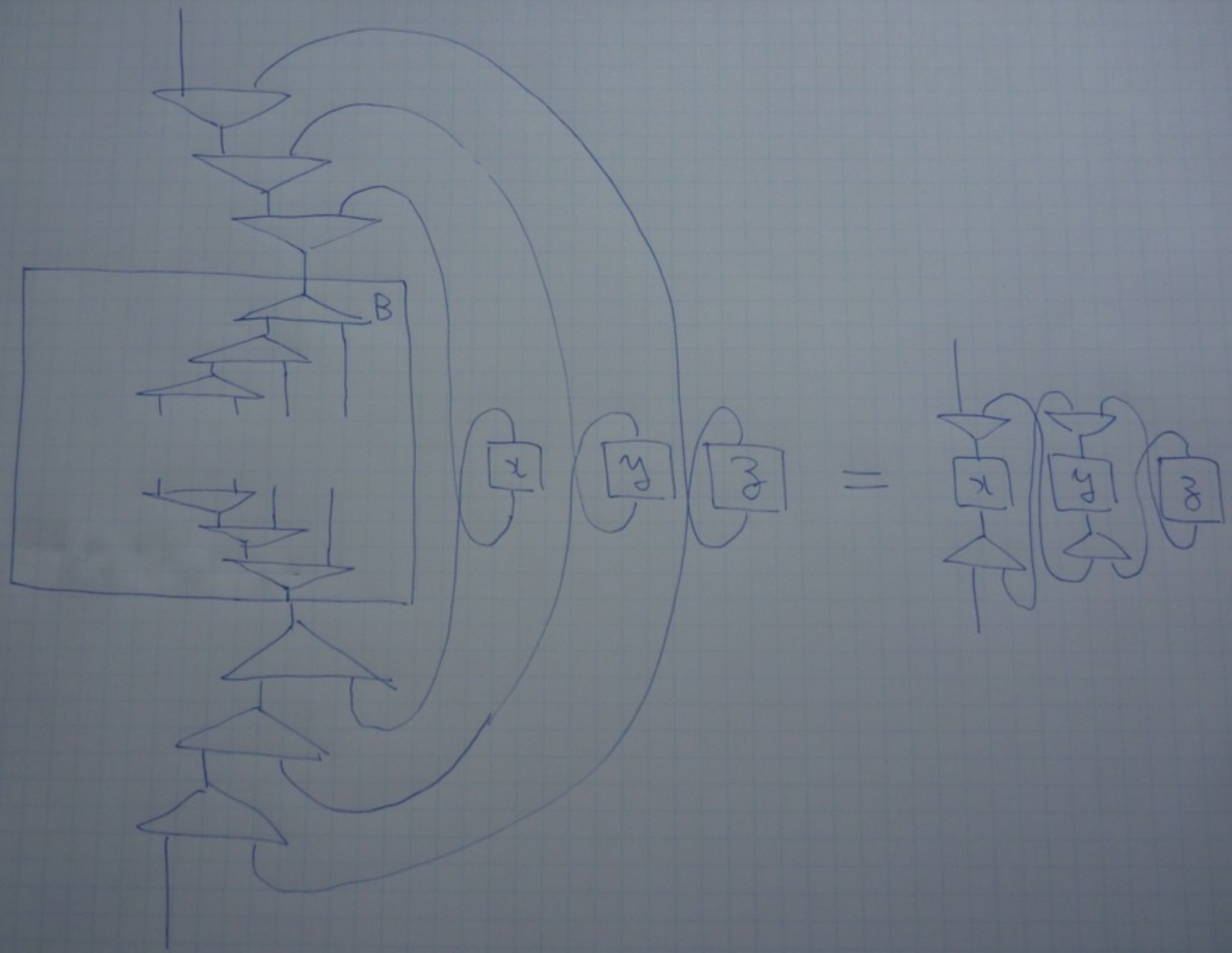
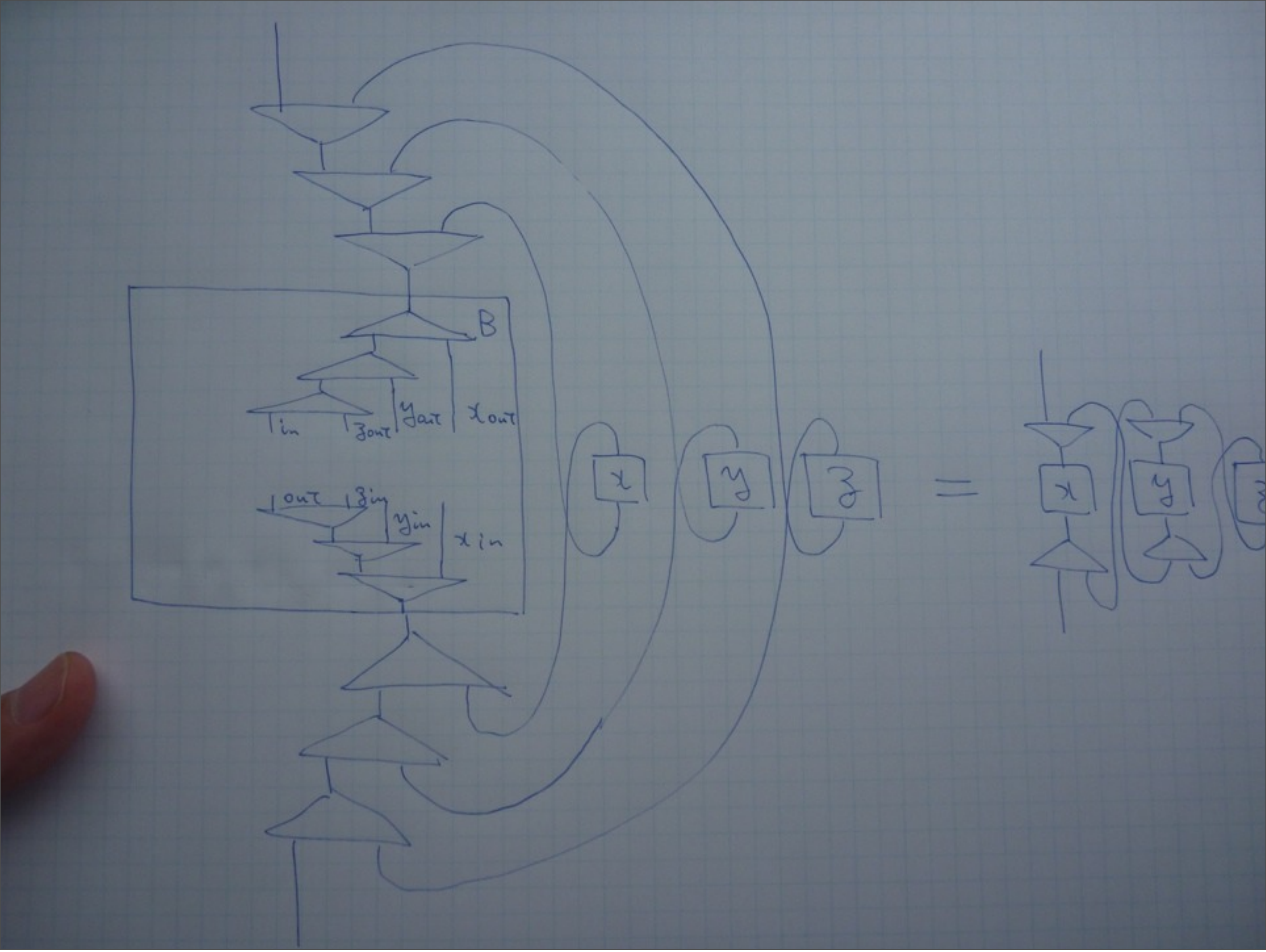
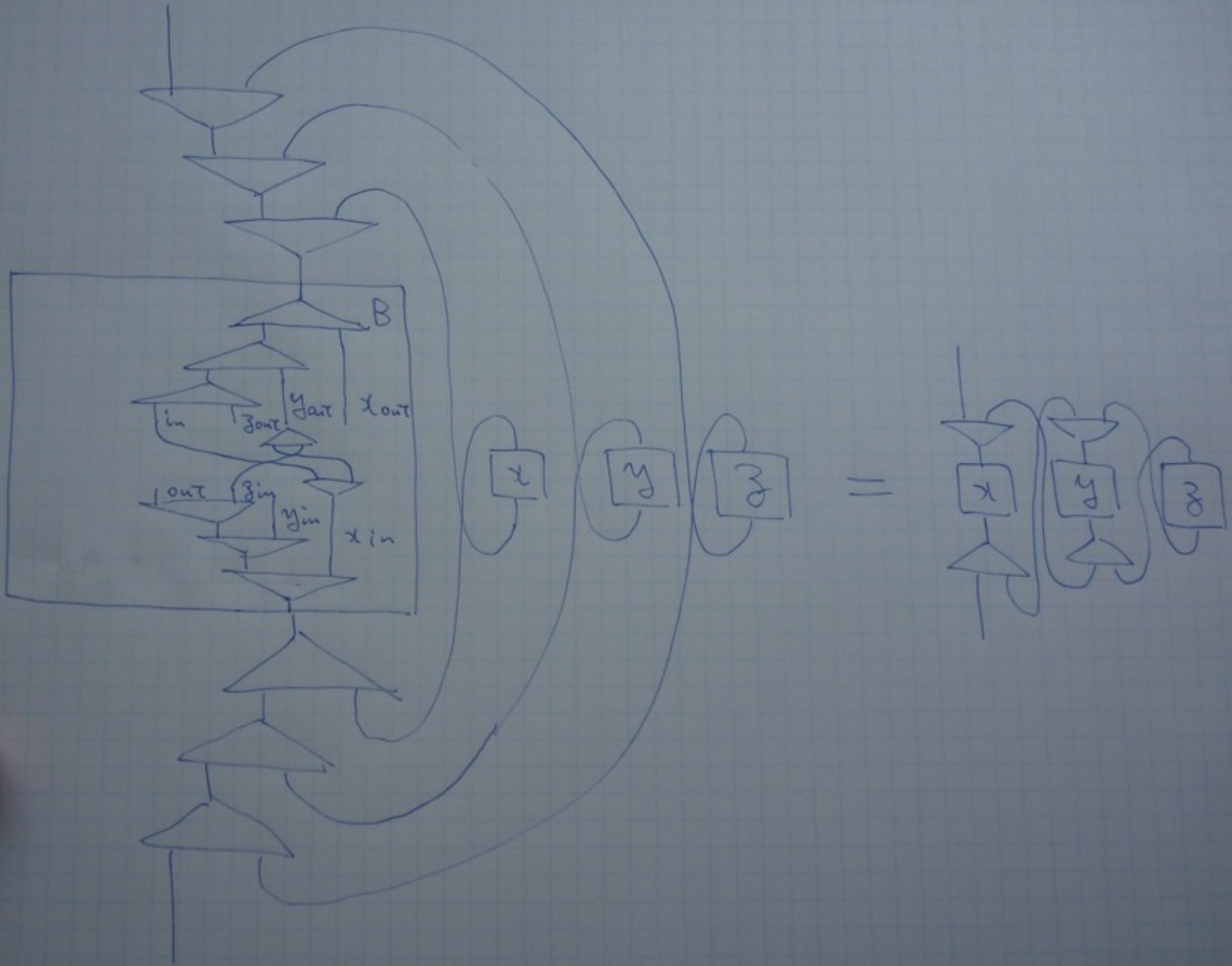
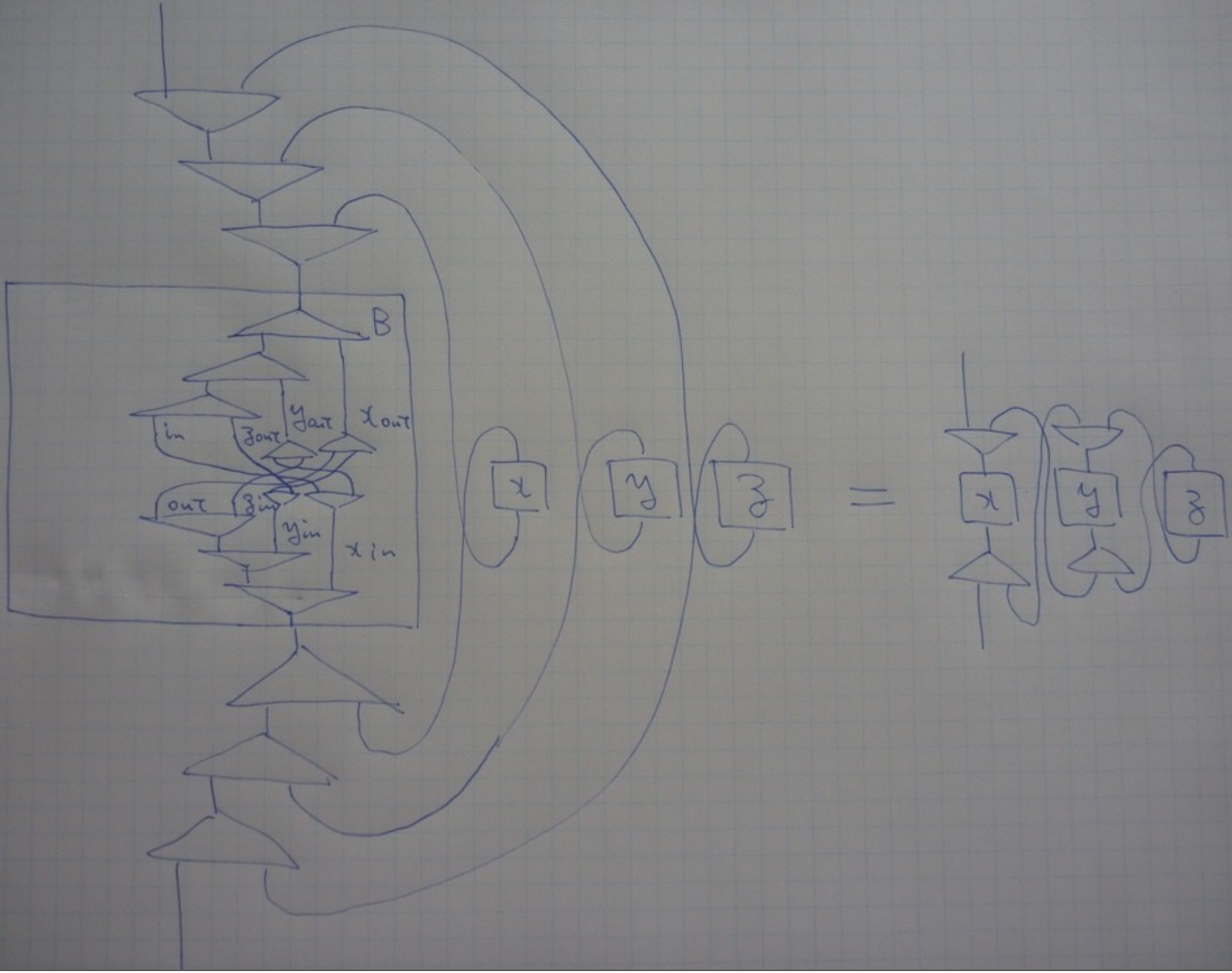



B
x
y
z
=
x
y
z


B
in
zout
yout
xout
out
zin
yin
xin
x
y
z
=
x
y
z


B
in
zout
yout
xout
out
zin
yin
xin
x
y
z
=
x
y
z


B
in
z out
y out
x out
out
z in
y in
x in
x
y
z
=
x
y
z

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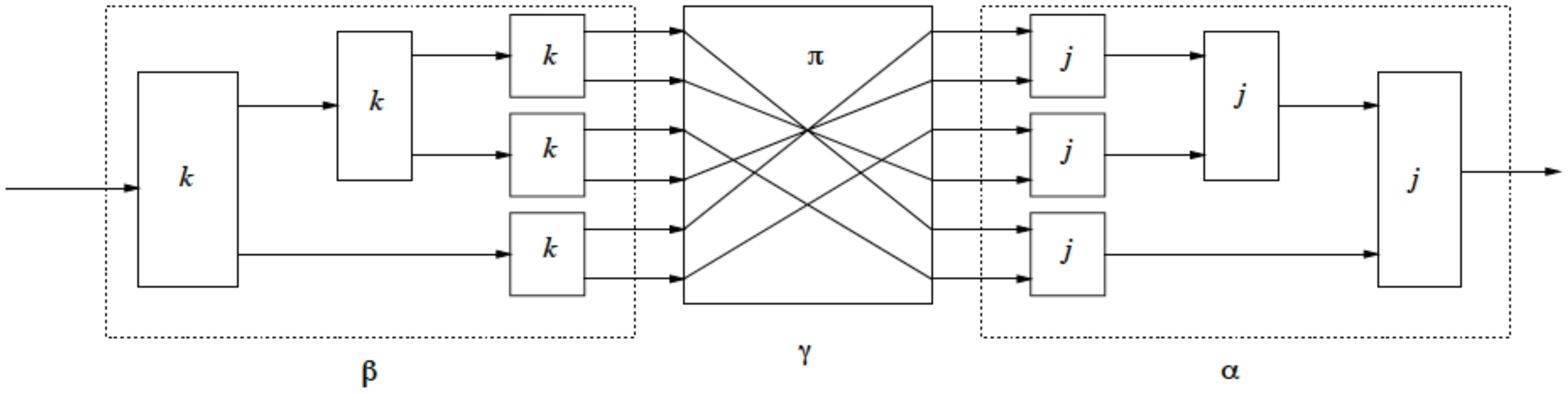

Figure 7: Composition Combinator B

from [AHS02]

Categorical GoI: Constr. of an LCA

- Combinator $Bxyz = x(yz)$

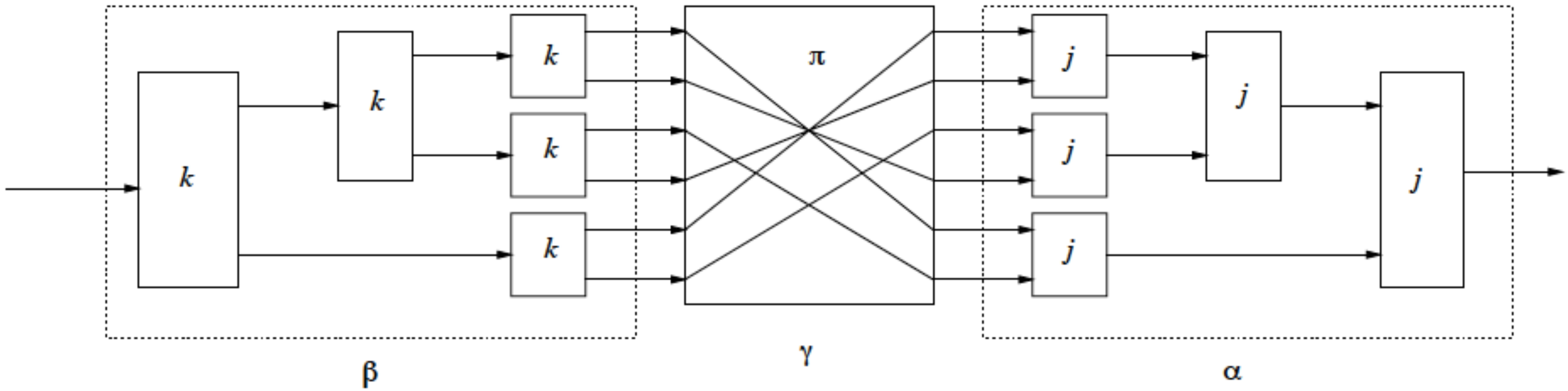

Figure 7: Composition Combinator B

from [AHS02]

Nice dynamic interpretation of (linear) computation!!

Summary: Categorical GoI

Defn. (GoI situation [AHS02])
A *GoI situation* is a triple $(\mathbb{C}, \boldsymbol{F}, \boldsymbol{U})$ where

- $\mathbb{C} = (\mathbb{C}, \otimes, \boldsymbol{I})$ is a traced symmetric monoidal category (TSMC);
- $\boldsymbol{F} : \mathbb{C} \to \mathbb{C}$ is a traced symmetric monoidal functor, equipped with the following retractions (which are monoidal natural transformations).

$$\begin{array}{rcll} \boldsymbol{e} : \boldsymbol{FF} \lhd \boldsymbol{F} : \boldsymbol{e'} & & & \text{Comultiplication} \\ \boldsymbol{d} : \mathbf{id} \lhd \boldsymbol{F} : \boldsymbol{d'} & & & \text{Dereliction} \\ \boldsymbol{c} : \boldsymbol{F} \otimes \boldsymbol{F} \lhd \boldsymbol{F} : \boldsymbol{c'} & & & \text{Contraction} \\ \boldsymbol{w} : \boldsymbol{K_I} \lhd \boldsymbol{F} : \boldsymbol{w'} & & & \text{Weakening} \end{array}$$

  Here $\boldsymbol{K_I}$ is the constant functor into the monoidal unit $\boldsymbol{I}$;
- $\boldsymbol{U} \in \mathbb{C}$ is an object (called *reflexive object*), equipped with the following retractions.

$$\begin{array}{c} \boldsymbol{j} : \boldsymbol{U} \otimes \boldsymbol{U} \lhd \boldsymbol{U} : \boldsymbol{k} \\ \boldsymbol{I} \lhd \boldsymbol{U} \\ \boldsymbol{u} : \boldsymbol{FU} \lhd \boldsymbol{U} : \boldsymbol{v} \end{array}$$

Thm. ([AHS02])
Given a GoI situation $(\mathbb{C}, \boldsymbol{F}, \boldsymbol{U})$, the homset

$$\mathbb{C}(\boldsymbol{U}, \boldsymbol{U})$$

carries a canonical LCA structure.

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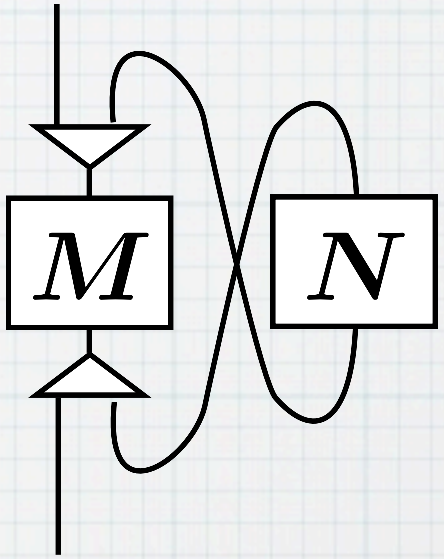
Why Categorical Generalization?: Examples Other Than Pfn [AHS02]

- Strategy: find a TSMC!



- "Wave-style" examples
  - ⊗ is Cartesian product(-like)
    - in which case,

      trace ≈ fixed point operator [Hasegawa/Hyland]
  - An example: $\big(\,(\omega\text{-}\mathbf{Cpo}, \times, 1),\ (\_)^{\mathbb{N}},\ A^{\mathbb{N}}\,\big)$
  - (... less of a dynamic flavor)

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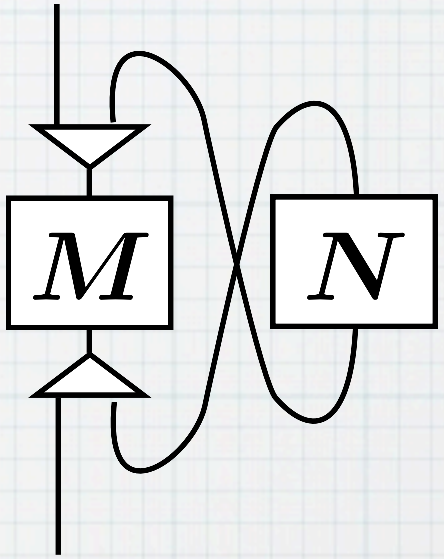
Why Categorical Generalization?: Examples Other Than $\mathbf{Pfn}$ [AHS02]

- "Particle-style" examples
  - Obj. X∈C is set-like; ⊗ is coproduct-like
    - The GoI animation is valid
  - Examples:
    - Partial functions $\big(\,(\mathbf{Pfn},+,\mathbf{0}),\ \mathbb{N}\cdot\_\,,\ \mathbb{N}\,\big)$
    - Binary relations $\big(\,(\mathbf{Rel},+,\mathbf{0}),\ \mathbb{N}\cdot\_\,,\ \mathbb{N}\,\big)$
    - "Discrete stochastic relations" $\big(\,(\mathbf{DSRel},+,\mathbf{0}),\ \mathbb{N}\cdot\_\,,\ \mathbb{N}\,\big)$



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$$\begin{array}{c} X \to Y \ \text{ in } \mathbf{Pfn} \\ \hline\hline X \rightharpoonup Y, \text{ partial function} \\ \hline\hline X \to \mathcal{L}Y \ \text{ in } \mathbf{Sets} \end{array} \quad \text{where } \mathcal{L}Y = \{\bot\} + Y$$

- **Rel** (relations)

$$\begin{array}{c} X \to Y \ \text{ in } \mathbf{Rel} \\ \hline\hline R \subseteq X \times Y, \text{ relation} \\ \hline\hline X \to \mathcal{P}Y \ \text{ in } \mathbf{Sets} \end{array} \quad \text{where } \mathcal{P} \text{ is the powerset monad}$$

- **DSRel**

$$\begin{array}{c} X \to Y \ \text{ in } \mathbf{DSRel} \\ \hline\hline X \to \mathcal{D}Y \ \text{ in } \mathbf{Sets} \end{array}$$

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Why Catego

Examples Other Than Pfn [AHS02]

Categories of sets and
(functions with different branching/partiality)

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Why Catego
Examples Other Than **Fun** [AHS02]

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(Potential) non-termination

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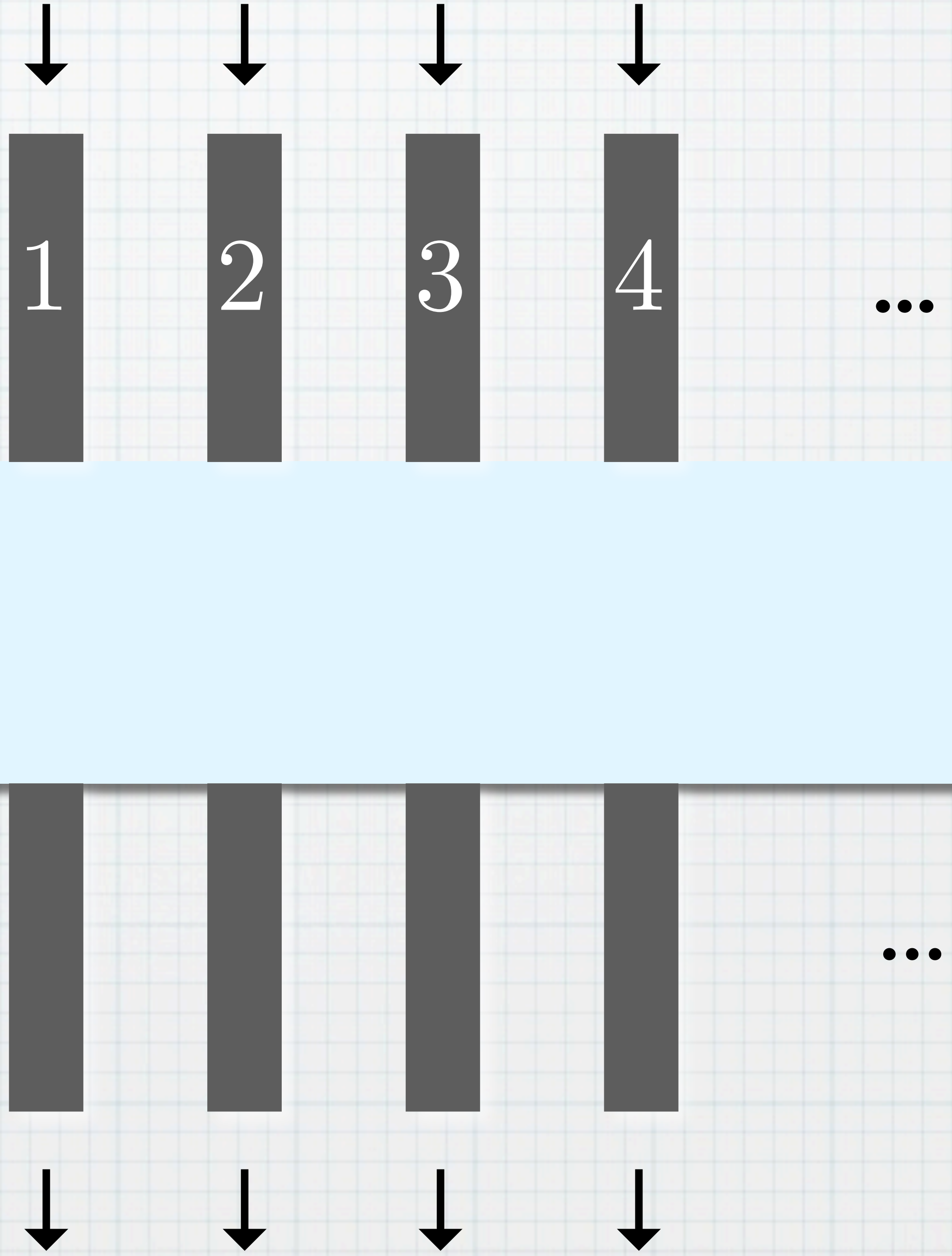
Probabilistic branching

Different Branching in The GoI Animation

- **Pfn** (partial functions)
  - Pipes can be stuck
- **Rel** (relations)
  - Pipes can branch
- **DSRel**
  - Pipes can branch probabilistically



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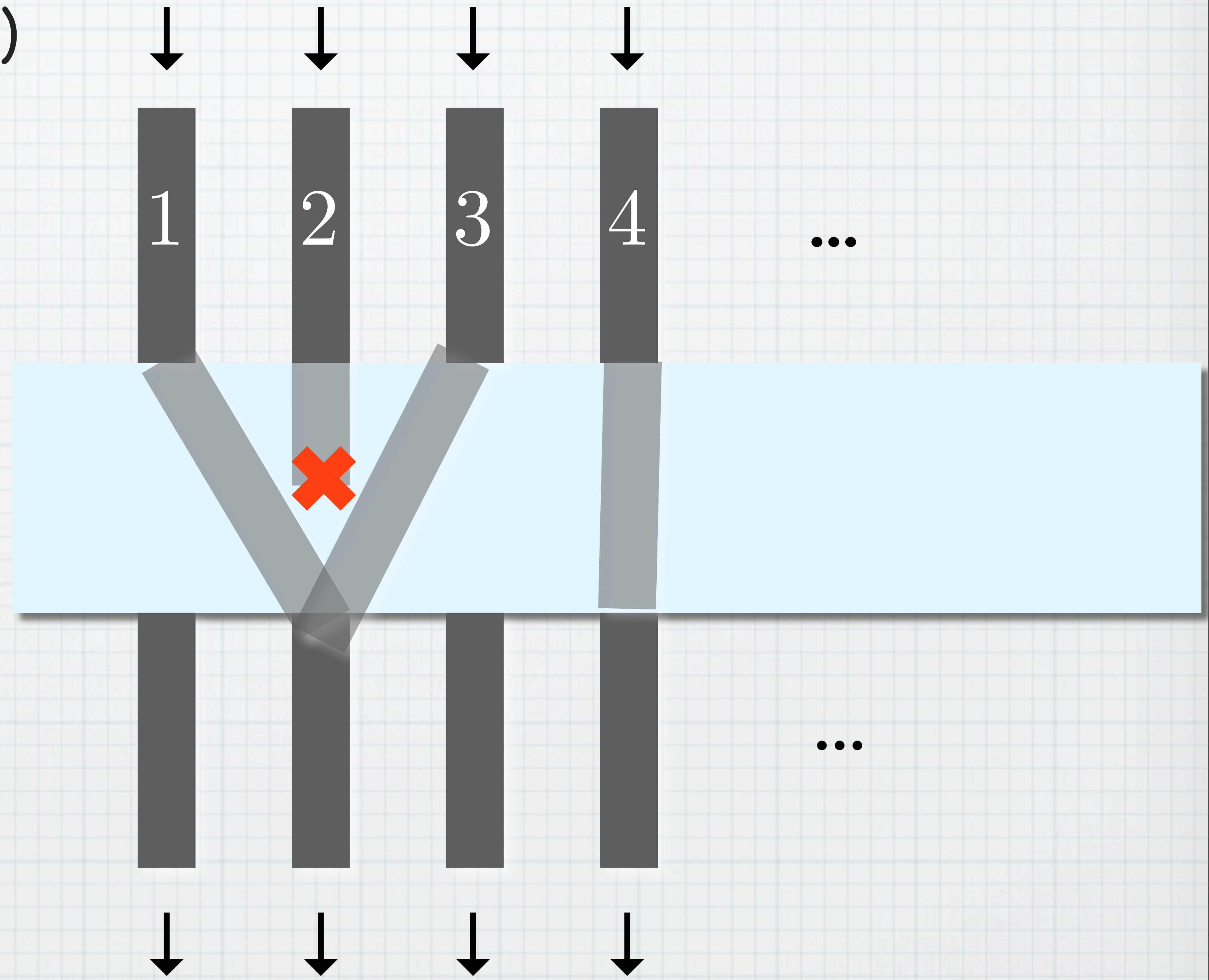

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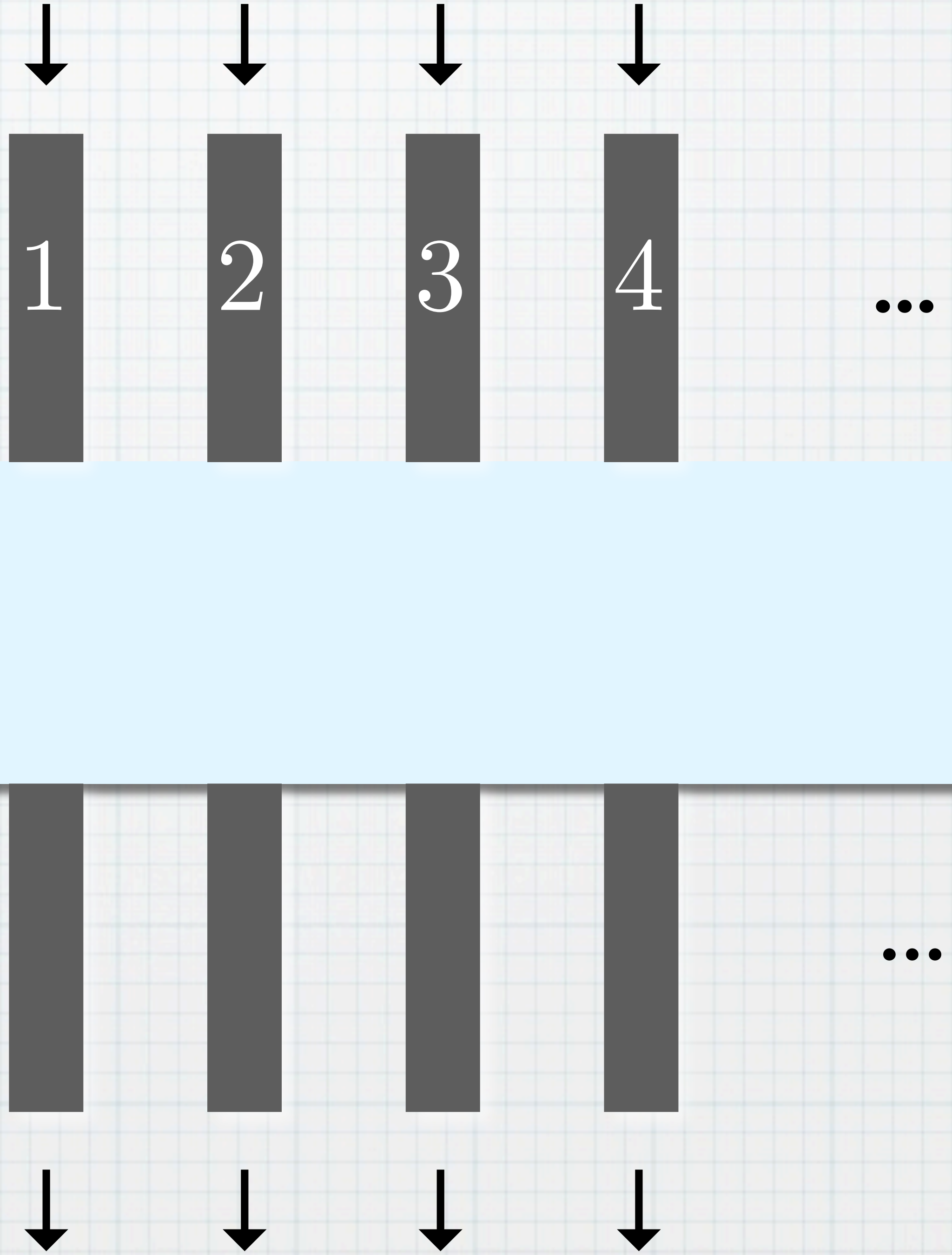

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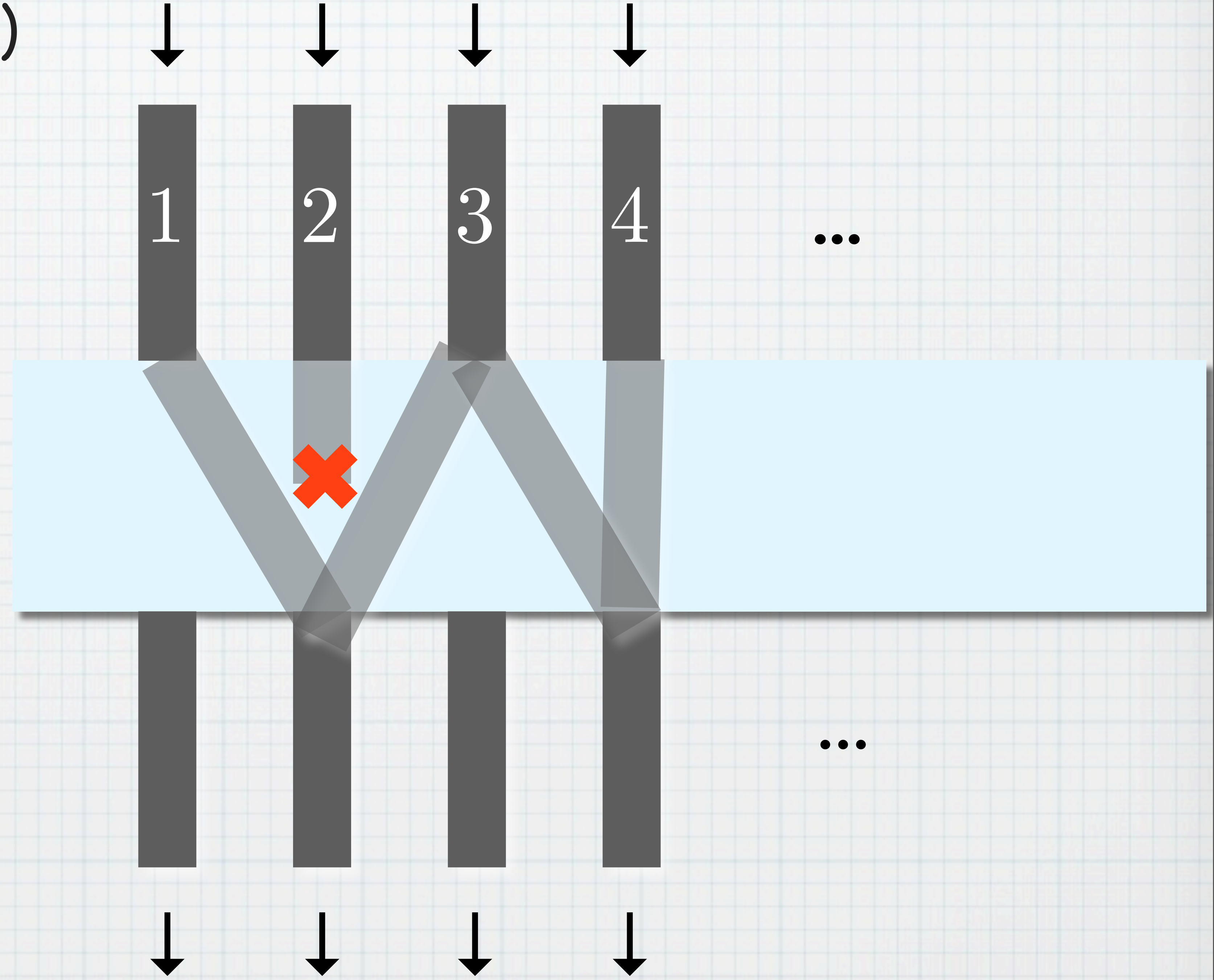

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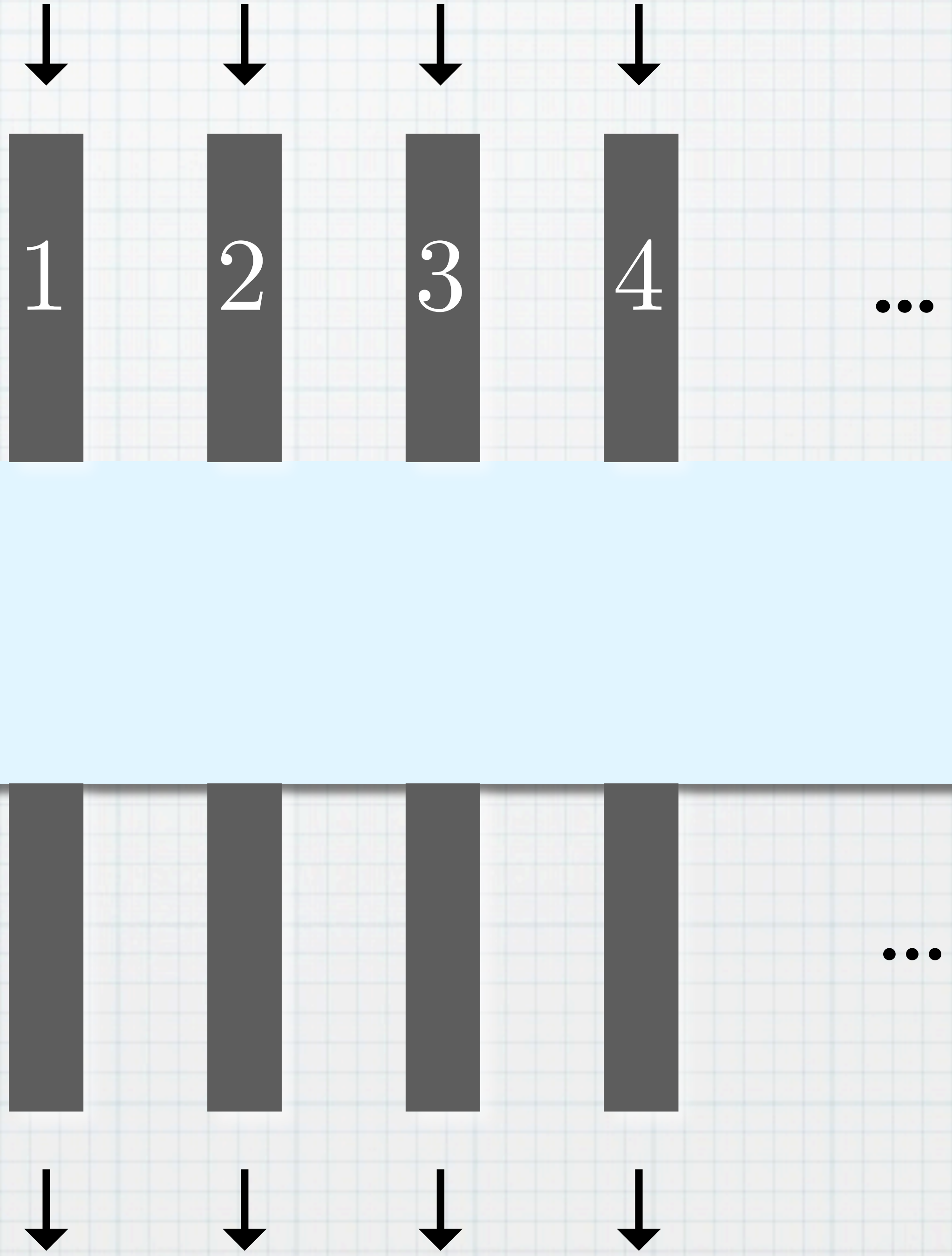

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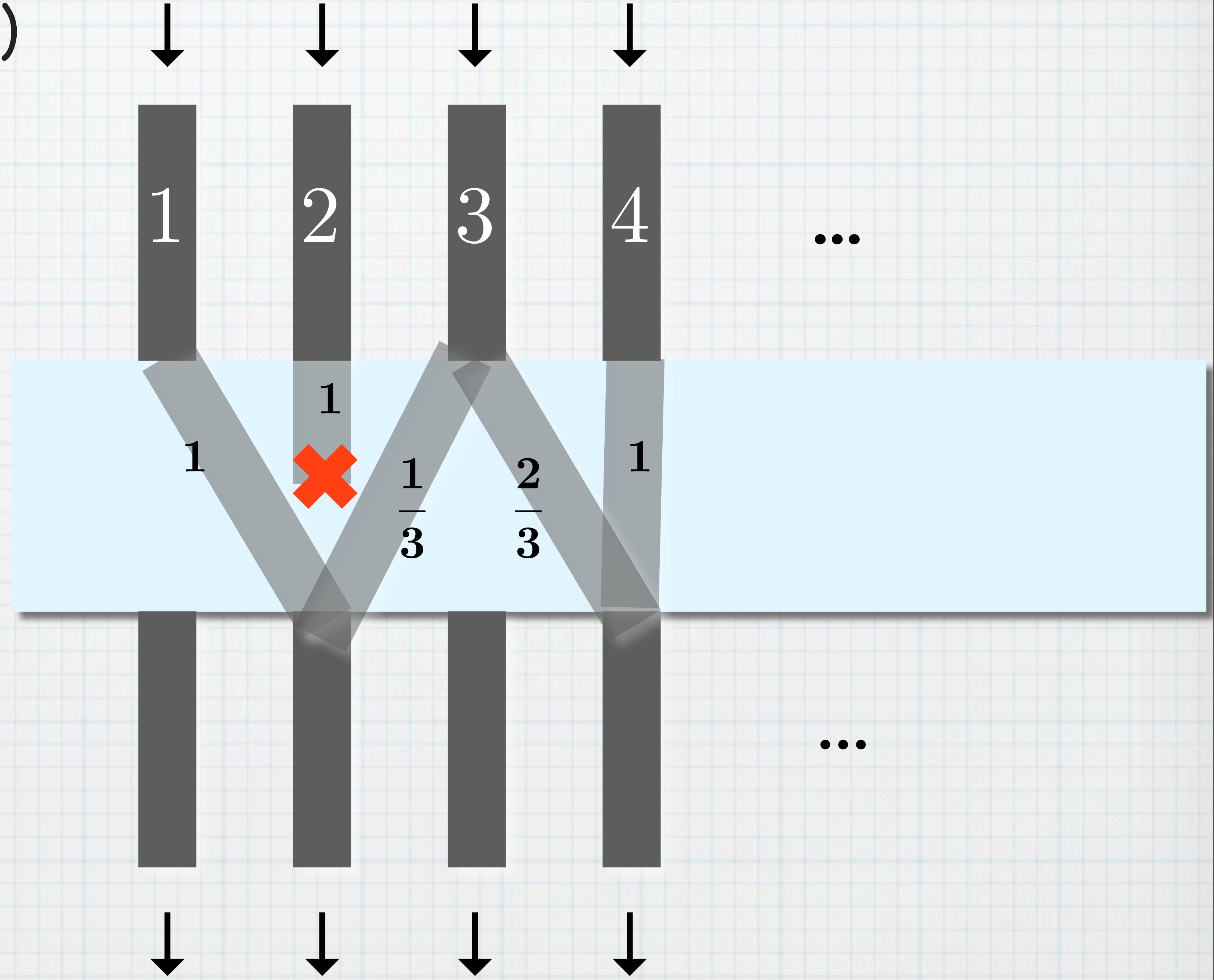

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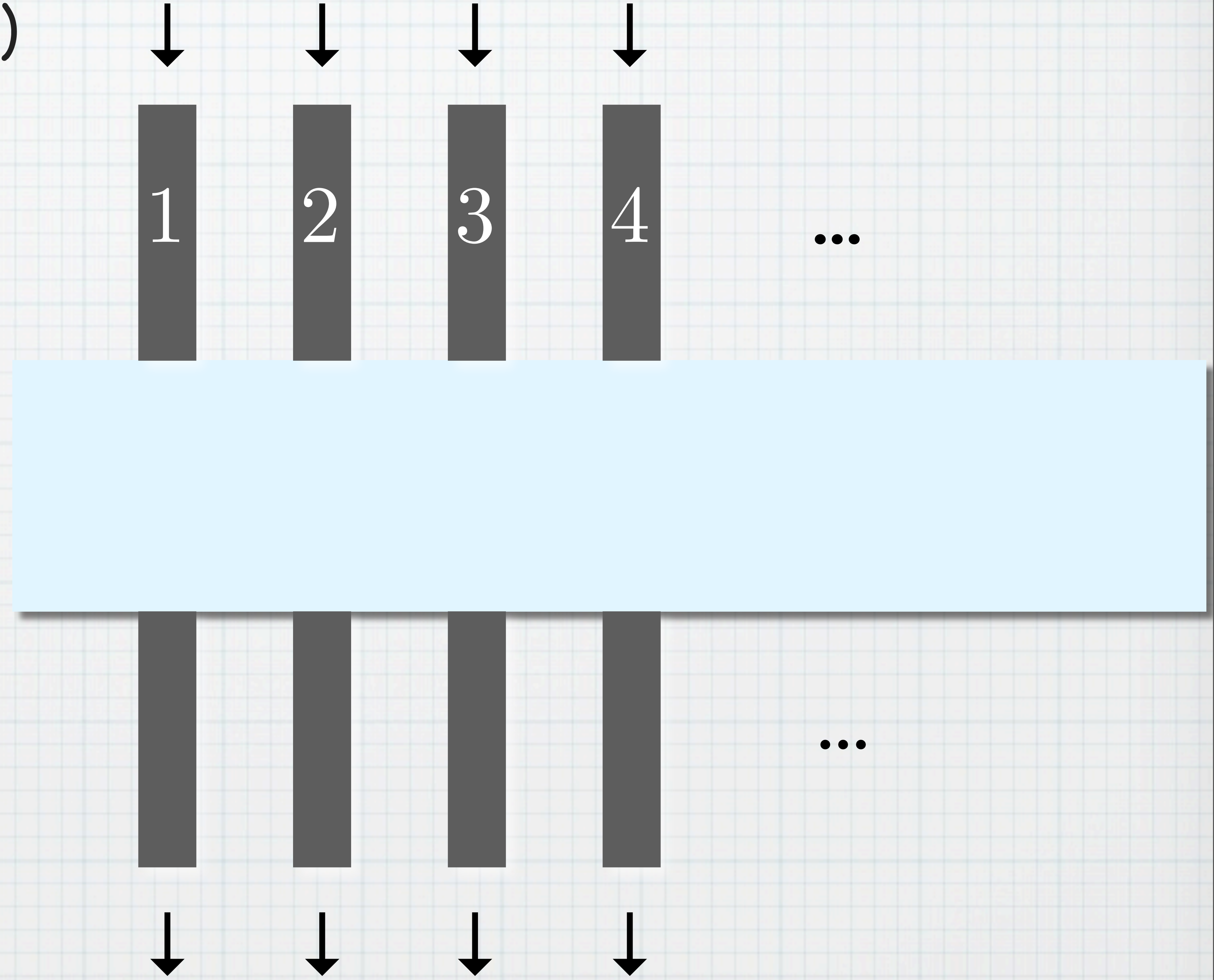

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Why Catego
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Probabilistic branching

Why Catego
Examples

Kl(*B*) for different branching monads *B*

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Probabilistic branching

The Coauthor

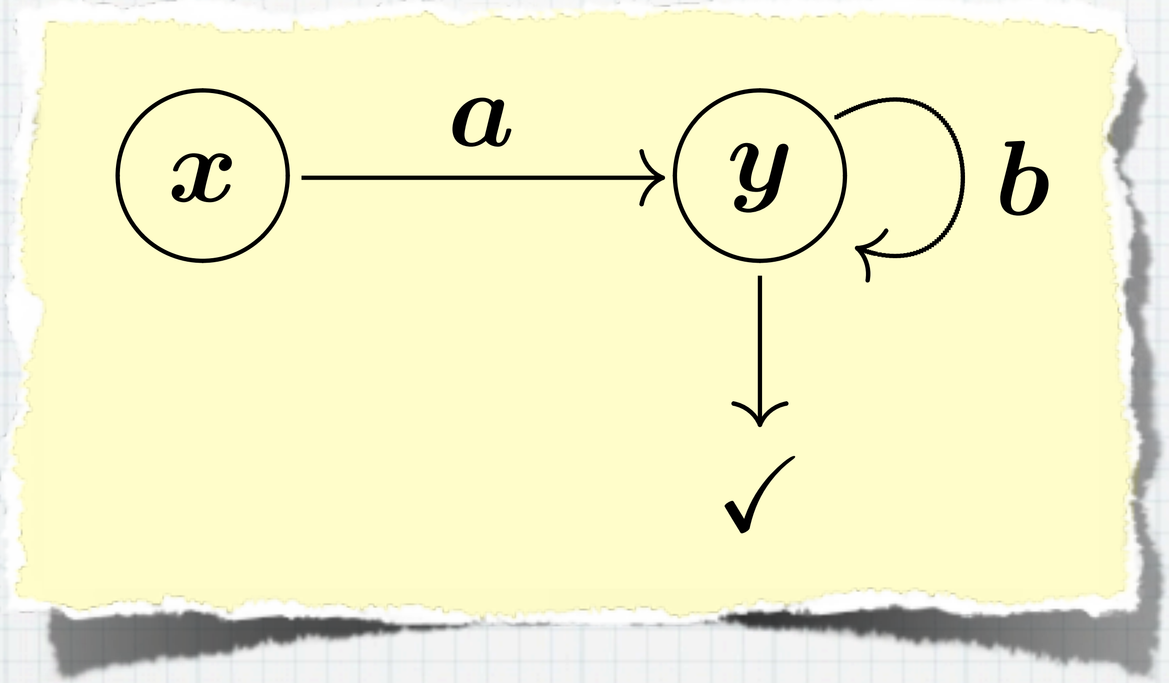
- Naohiko Hoshino
  - DSc (Kyoto, 2011)
    - Supervisor: Masahito "Hassei" Hasegawa
  - Currently at RIMS, Kyoto U.
    - http://www.kurims.kyoto-u.ac.jp/~naophiko/

Intermission

(If time allows)

Coalgebraic Trace Semantics

Trace Semantics of Systems



$$\mathbf{tr}(x) = \{a, ab, abb, \dots\} = ab^*$$

- **Non-deterministic** branching: sign. functor is $\mathcal{P}(1 + \Sigma \times \_)$

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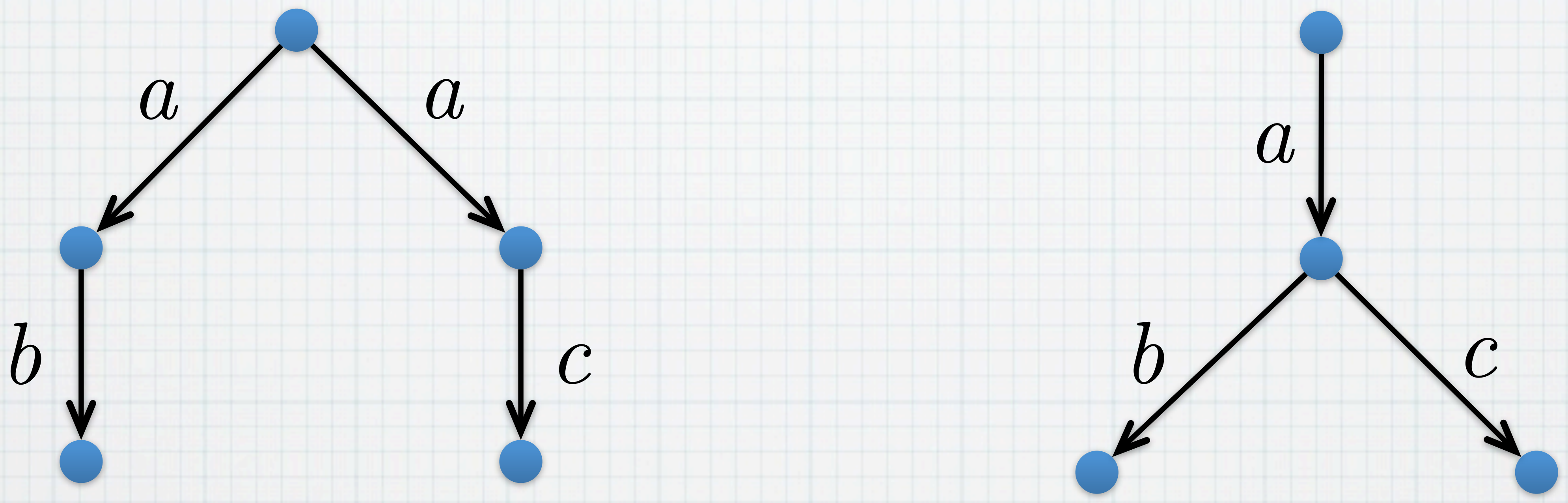
Bisimilarity vs. Trace Sem.



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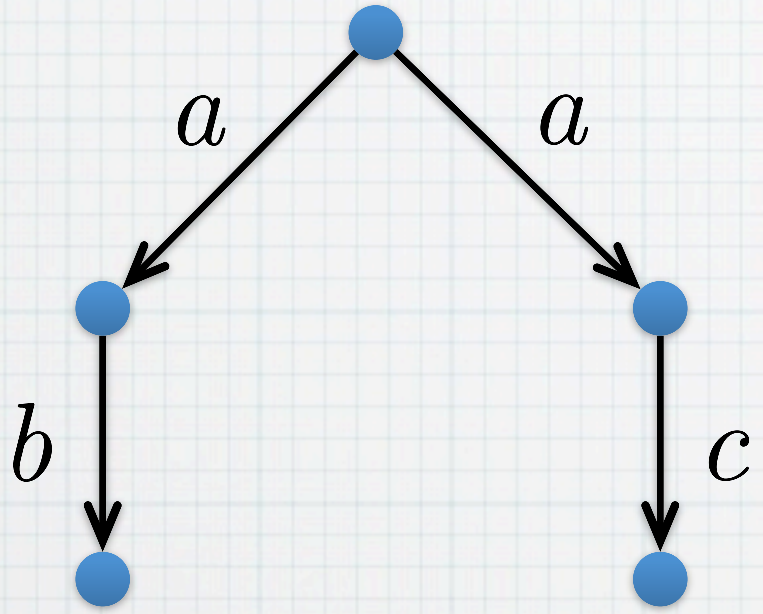

$\neq$

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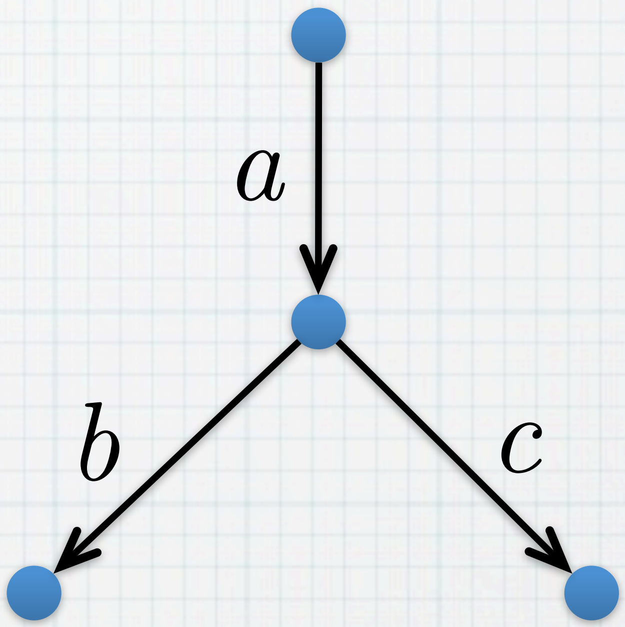

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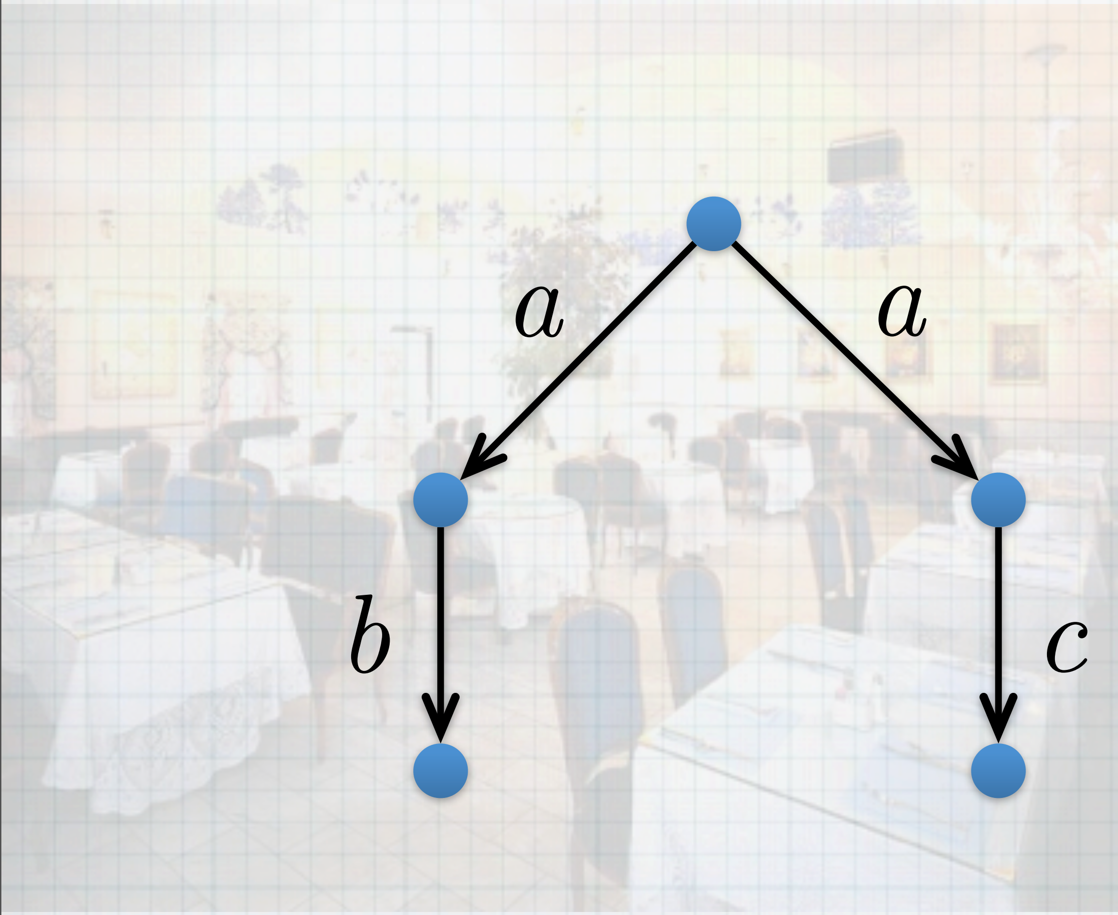

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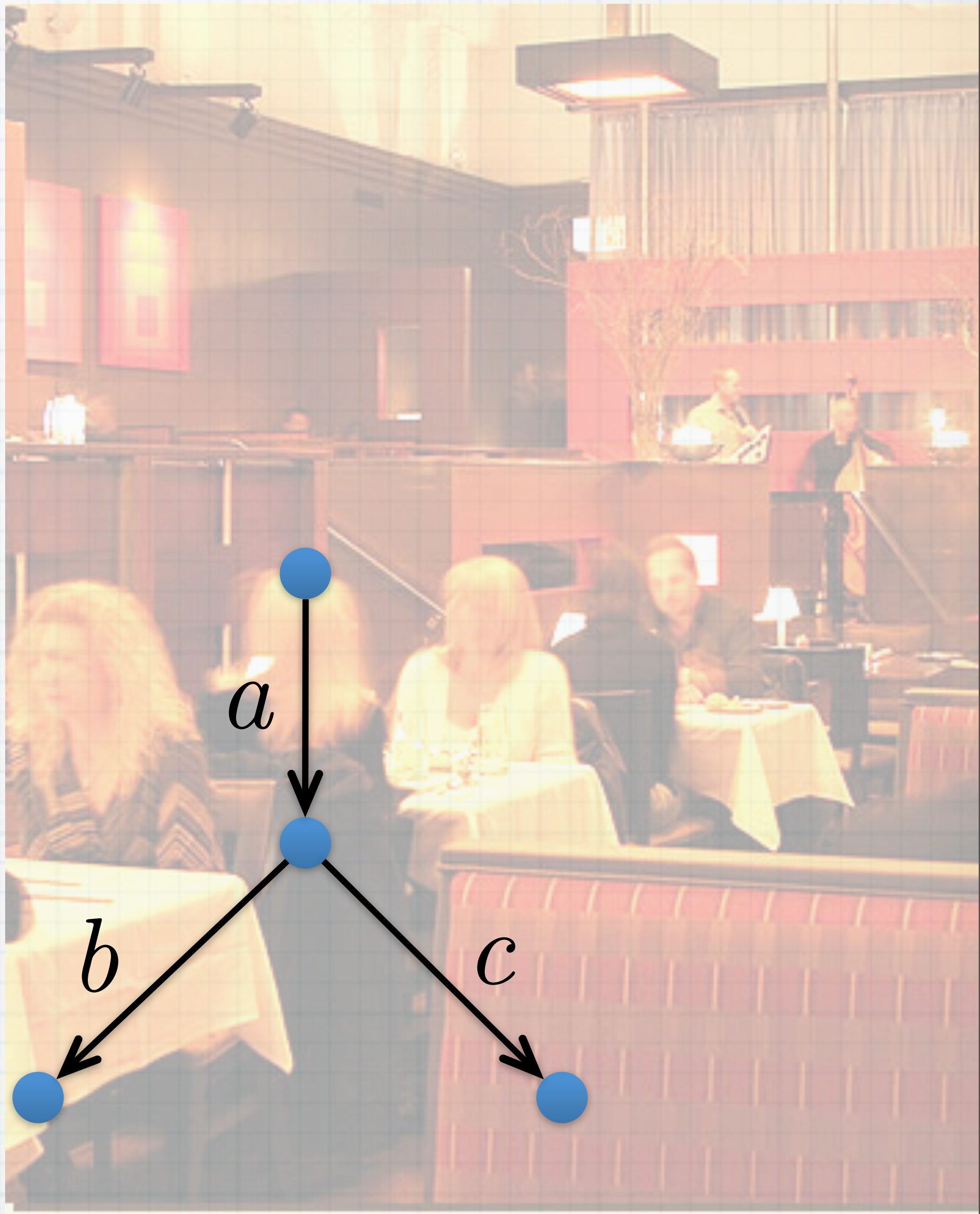

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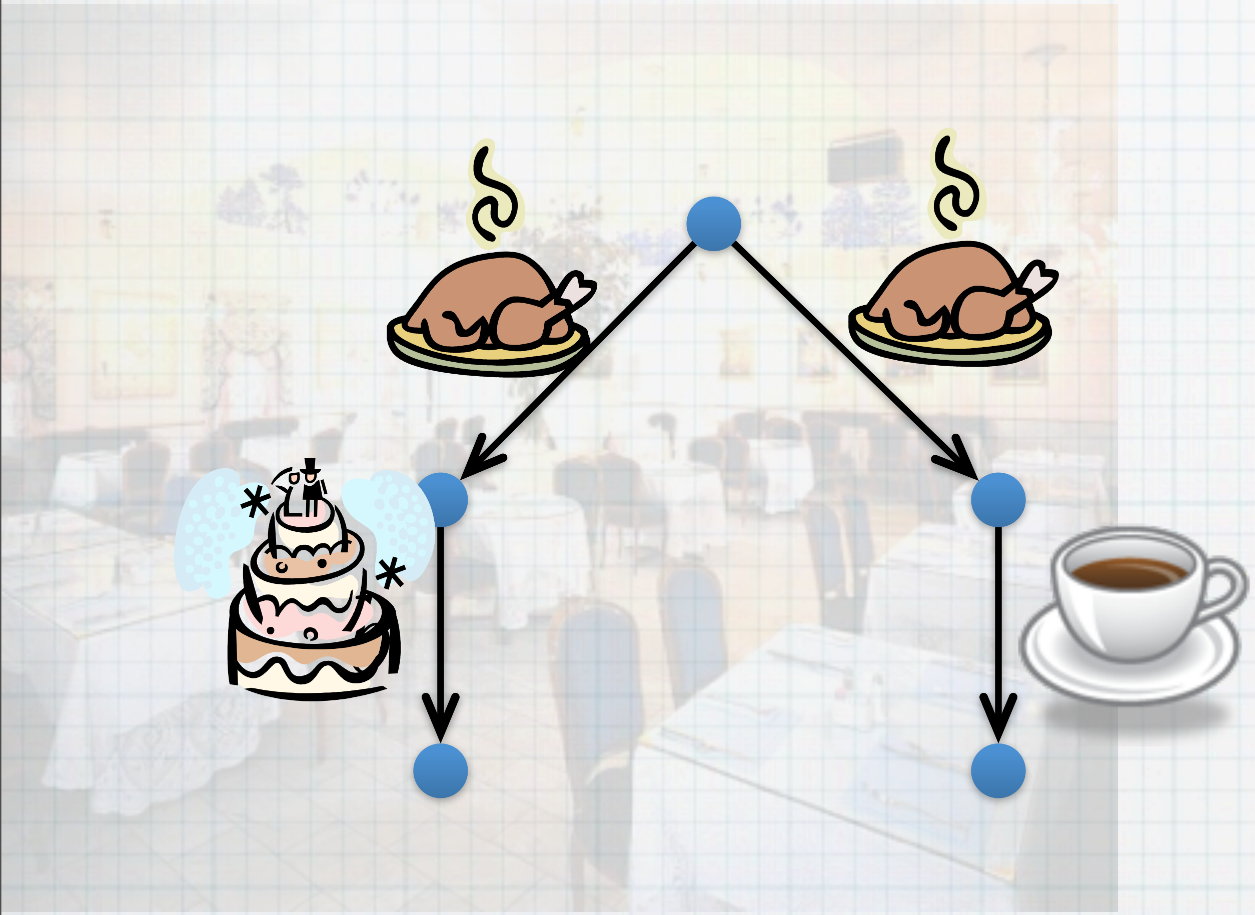

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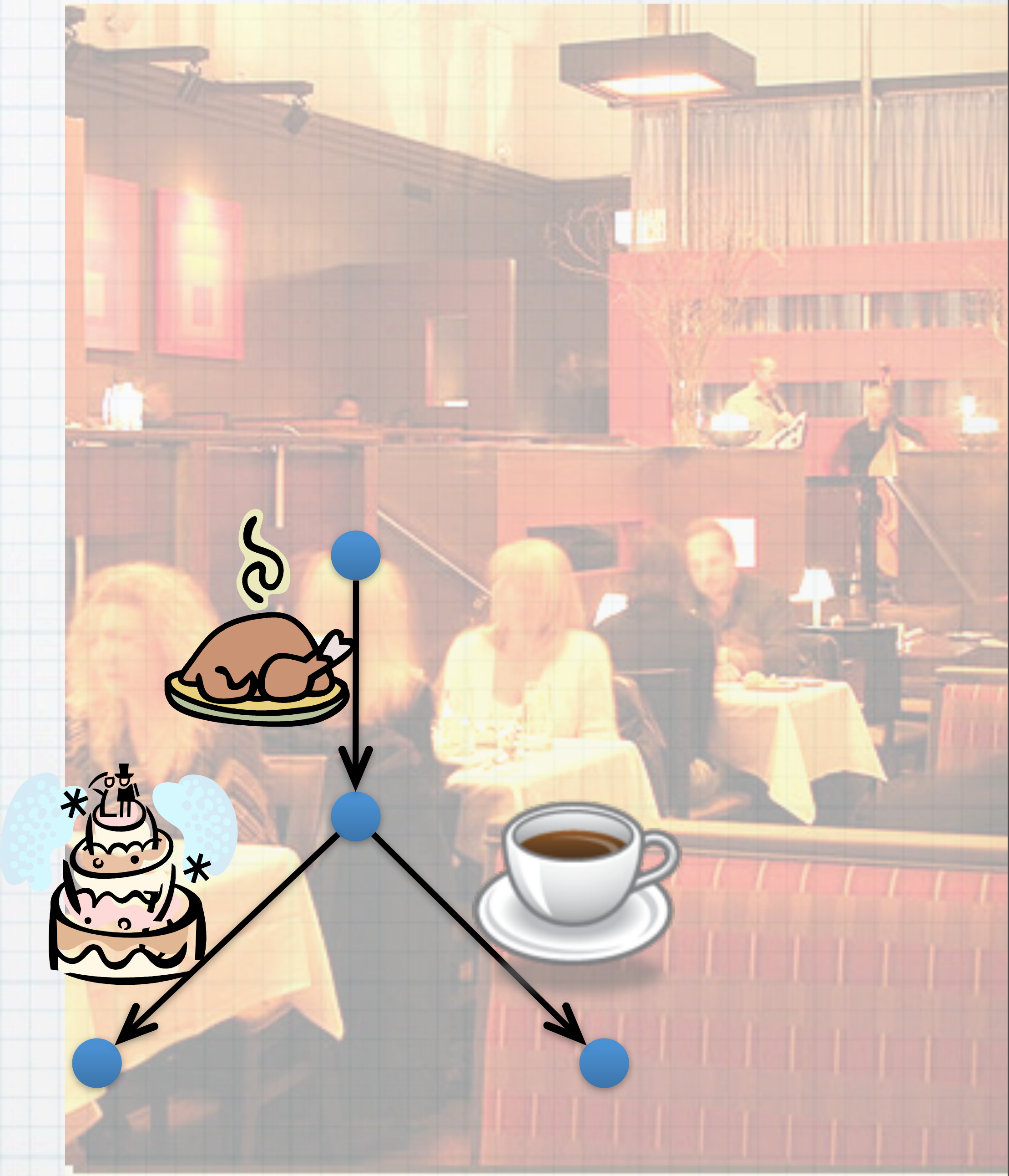


Bisimilarity

Branching structure matters.

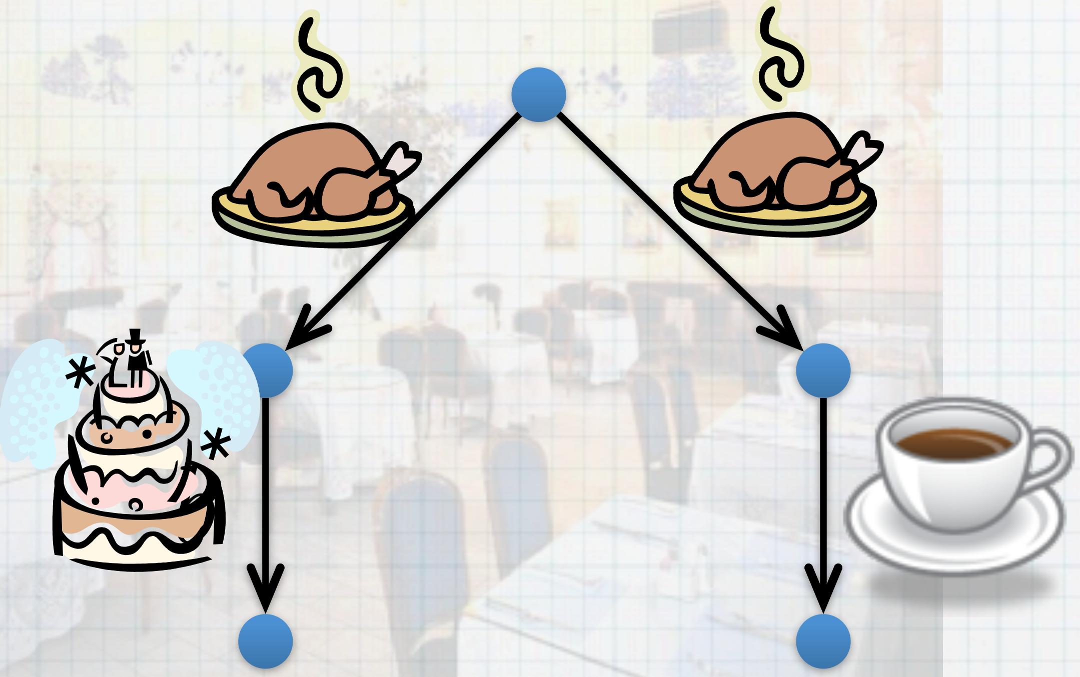
Can I choose later?

Trace semantics

Branching structure does not matter.

Anyway we'll get the same sets of food.

Bisimilarity vs. Trace Sem.



$\neq$

$=$

Also by final coalgebra?

$$\begin{tikzcd} FX \arrow[r, dashed, "F\mathbf{beh}(c)"] & FZ \\ X \arrow[u, "c"] \arrow[r, dashed, "\mathbf{beh}(c)"'] & Y \arrow[u, "\text{final}"'] \end{tikzcd}$$

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Coinduction in a Kleisli Category

[IH, Jacobs, Sokolova, '07]

$$\begin{array}{c} X \longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\;\; Y \quad \text{in } \mathcal{K}\ell(B) \\ \hline\hline X \longrightarrow BY \quad \text{in } \mathbf{Sets} \end{array}$$

Thm. Let F be an endofunctor, and B be a monad, both on **Sets**. Assume:

1. We have a distributive law $\lambda : FB \Rightarrow BF$.
2. The functor F preserves ω-colimits, yielding an initial algebra $\begin{array}{c} FA \\ \cong\downarrow\alpha \\ A \end{array}$.
3. The Kleisli category $\mathcal{K}\ell(B)$ is $\mathbf{Cpo}_\perp$-enriched and composition in $\mathcal{K}\ell(B)$ is left-strict.

Then:

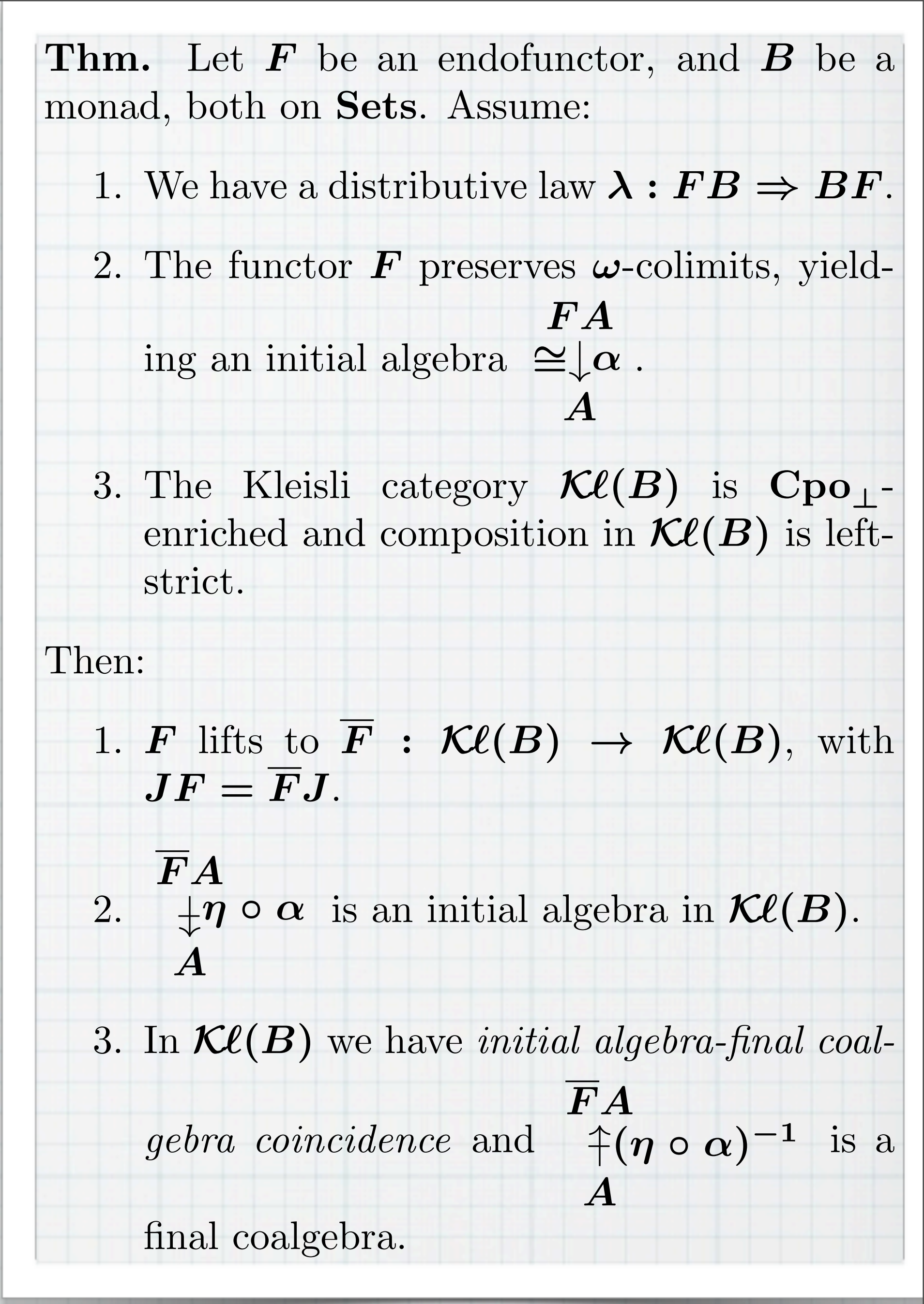
1. F lifts to $\overline{F} : \mathcal{K}\ell(B) \to \mathcal{K}\ell(B)$, with $JF = \overline{F}J$.
2. $\begin{array}{c} \overline{F}A \\ \downarrow\!\!\!\!+\,\eta\circ\alpha \\ A \end{array}$ is an initial algebra in $\mathcal{K}\ell(B)$.
3. In $\mathcal{K}\ell(B)$ we have *initial algebra-final coalgebra coincidence* and $\begin{array}{c} \overline{F}A \\ \uparrow\!\!\!\!+\,(\eta\circ\alpha)^{-1} \\ A \end{array}$ is a final coalgebra.

Coinduction in a Kleisli Category

[IH, Jacobs, Sokolova, '07]

$$\frac{X \relbar\joinrel\mapstochar\joinrel\rightarrow Y \quad \text{in } \mathcal{K}\ell(B)}{X \longrightarrow BY \quad \text{in } \mathbf{Sets}}$$

- Initial algebra lifts from Sets to $Kl(B)$
  - diagram chasing [Johnstone]



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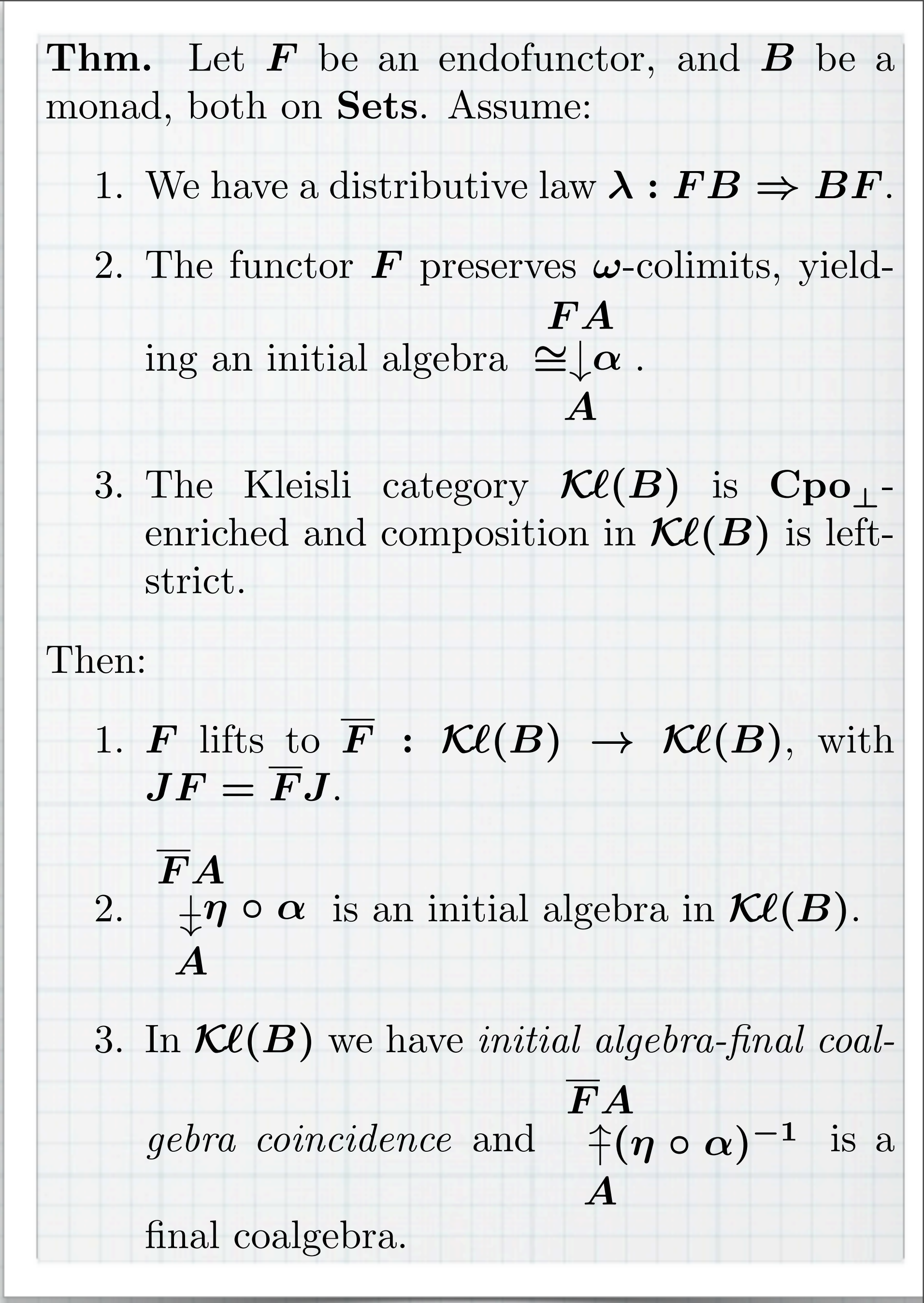

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- Initial algebra lifts from **Sets** to $Kl(B)$
  - diagram chasing [Johnstone]
- In $Kl(B)$ we have IA-FC coincidence
  - typical of "domain-theoretic" categories
  - "Algebraically compact" [Freyd]

Coinduction in a Kleisli Category

- E.g. $B = \mathcal{P},\ F = 1 + \Sigma \times (\_\,)$

$$\begin{tikzcd}
1 + \Sigma \times X \arrow[r, dashed, "\shortmid" marking, "1 + \Sigma \times \mathbf{tr}(c)"] & 1 + \Sigma \times \Sigma^* & \text{in } \mathcal{K}\ell(\mathcal{P}) \\
X \arrow[u, "\shortmid" marking, "c"] \arrow[r, dashed, "\shortmid" marking, "\mathbf{tr}(c)"'] & \Sigma^* \arrow[u, "\shortmid" marking, "\text{final}"'] &
\end{tikzcd}$$

- Separation between B and F

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\end{tikzcd}
\quad \text{in } \mathcal{K}\ell(\mathcal{P})$$

$$\begin{tikzcd}
\mathcal{P}(1 + \Sigma \times X) \\
X \arrow[u, "c"]
\end{tikzcd}
\quad \text{in } \mathbf{Sets}$$

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in $\mathcal{K}\ell(\mathcal{P})$

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in **Sets**

induced by

$$\begin{tikzcd}
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in **Sets**

$$\begin{tikzcd}
\mathcal{P}(1 + \Sigma \times X) \\
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\end{tikzcd}$$

in **Sets**

$$X \xrightarrow{\mathbf{tr}(c)} \mathcal{P}(\Sigma^*)$$

- Separation between B and F

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in $\mathcal{K}\ell(\mathcal{P})$

$$\begin{tikzcd}
\mathcal{P}(1 + \Sigma \times X) \\
X \arrow[u, "c"]
\end{tikzcd}$$

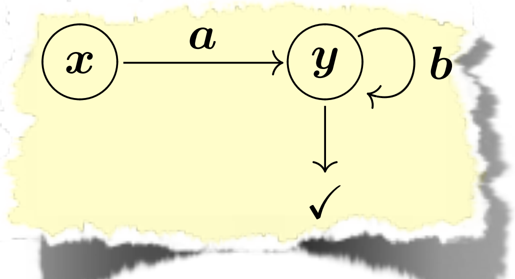
in **Sets**

induced by

$$\begin{tikzcd}
1 + \Sigma \times \Sigma^* \arrow[d, "\text{initial}"] \\
\Sigma^*
\end{tikzcd}$$

in **Sets**

$$X \xrightarrow{\mathbf{tr}(c)} \mathcal{P}(\Sigma^*)$$



$\mathbf{tr}(x) = \{a, ab, abb, \dots\} = ab^*$

- Separation between B and F

Coinduction in a Kleisli Category

- E.g. $B = \mathcal{P},\ F = 1 + \Sigma \times (\_)$

$$\begin{tikzcd}
1 + \Sigma \times X \arrow[r, dashed, "1 + \Sigma \times \mathbf{tr}(c)"] & 1 + \Sigma \times \Sigma^* \\
X \arrow[u, "c"] \arrow[r, dashed, "\mathbf{tr}(c)"'] & \Sigma^* \arrow[u, "\text{final}"']
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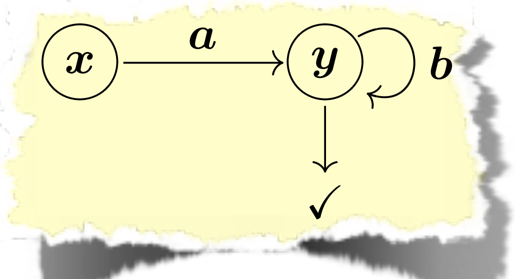

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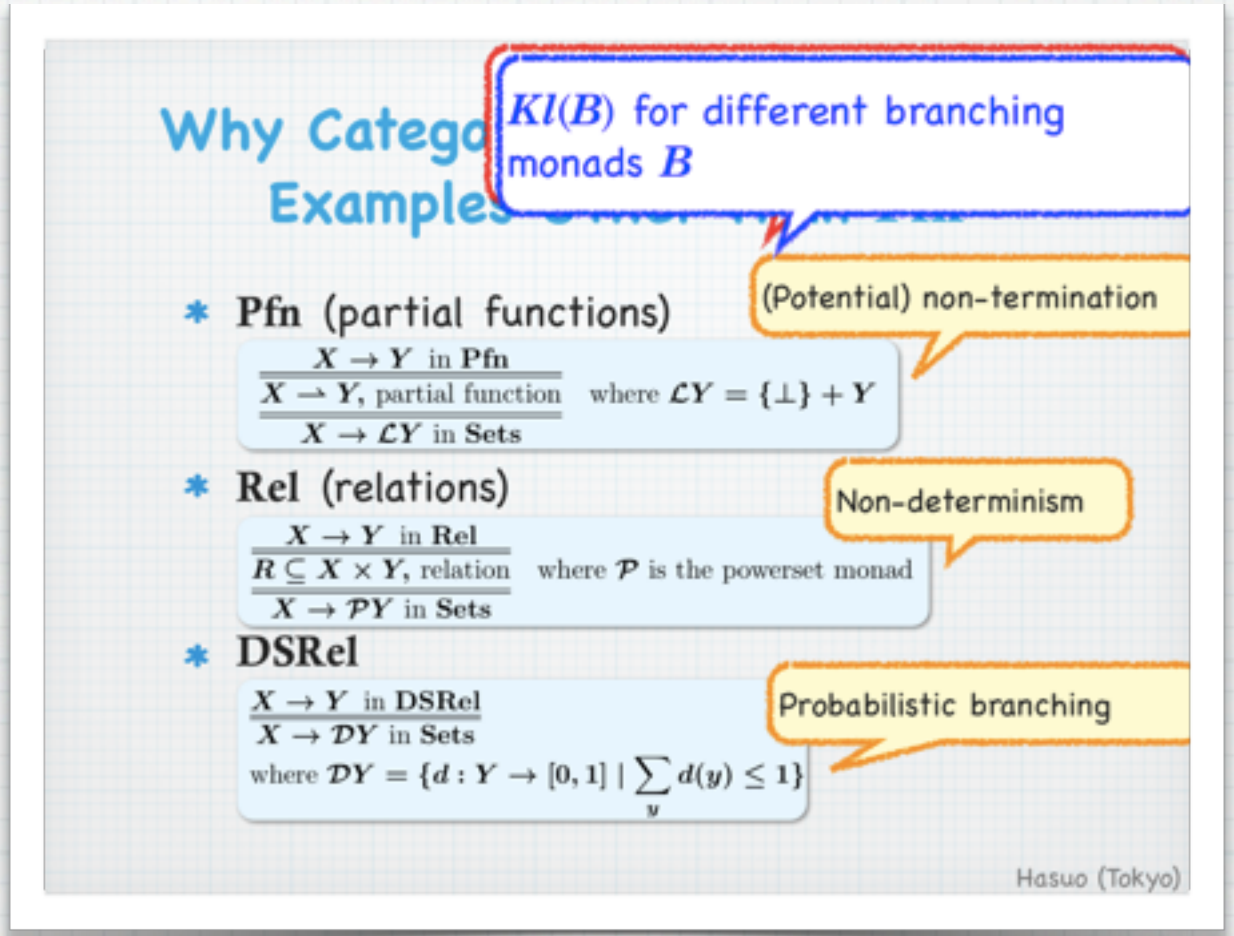
- Separation between B and F

Hasuo (Tokyo)

Examples

- A branching monad B:
  - Lift monad $\mathcal{L} = 1 + (\_)$, powerset monad $\mathcal{P}$, subdistribution monad $\mathcal{D}$
  - Precisely those in



Why Catego

Example

$Kl(B)$ for different branching monads B

- Pfn (partial functions)

$$\begin{array}{c} X \to Y \text{ in } \mathbf{Pfn} \\ \hline\hline X \rightharpoonup Y\text{, partial function} \\ \hline\hline X \to \mathcal{L}Y \text{ in } \mathbf{Sets} \end{array} \quad \text{where } \mathcal{L}Y = \{\bot\} + Y$$

(Potential) non-termination

- Rel (relations)

$$\begin{array}{c} X \to Y \text{ in } \mathbf{Rel} \\ \hline\hline R \subseteq X \times Y\text{, relation} \\ \hline\hline X \to \mathcal{P}Y \text{ in } \mathbf{Sets} \end{array} \quad \text{where } \mathcal{P} \text{ is the powerset monad}$$

Non-determinism

- DSRel

$$\begin{array}{c} X \to Y \text{ in } \mathbf{DSRel} \\ \hline\hline X \to \mathcal{D}Y \text{ in } \mathbf{Sets} \end{array}$$

where $\mathcal{D}Y = \{d : Y \to [0,1] \mid \sum_y d(y) \leq 1\}$

Probabilistic branching

Hasuo (Tokyo)

- A functor F: polynomial functors

From Coalgebraic Trace to Monoidal Trace

Thm. ([Jacobs,CMCS10])
Given a "branching monad" $\boldsymbol{B}$ on **Sets**, the monoidal category

$$(\mathcal{K}\ell(B), +, \mathbf{0})$$

is a traced symmetric monoidal category.

Cor.
$\big(\, (\mathcal{K}\ell(B), +, \mathbf{0}),\ \mathbb{N}\cdot\_\,,\ \mathbb{N}\,\big)$ is a GoI situation.

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Proof. We need

$$\frac{X + Z \xrightarrow{\ f\ } Y + Z \quad \text{in } \mathcal{K}\ell(T)}{X \xrightarrow{\ \mathrm{tr}(f)\ } Y \quad \text{in } \mathcal{K}\ell(T)}$$

- $X + Z \xrightarrow{\ f\ } Y + Z \xrightarrow{\ \kappa\ } Y + (X + Z)$ is a $Y + (\_)$-coalgebra

- $$\begin{tikzcd} Y + \mathbb{N}\cdot Y \arrow[d, "\cong"', "\alpha"] \\ \mathbb{N}\cdot Y \end{tikzcd}$$ is an initial algebra in **Sets**

- Therefore in $\mathcal{K}\ell(T)$:

$$\begin{tikzcd}
Y + (X + Z) \arrow[r, dashed] & Y + \mathbb{N}\cdot Y \\
X + Z \arrow[u, "\kappa \circ f", "|" marking] \arrow[r, dashed, "\mathrm{tr}(c)"', "|" marking] & \mathbb{N}\cdot Y \arrow[u, "\text{final}"', "|" marking] \arrow[d, "\nabla", "|" marking] \\
X \arrow[u, "\kappa_1", "|" marking] & Y
\end{tikzcd}$$

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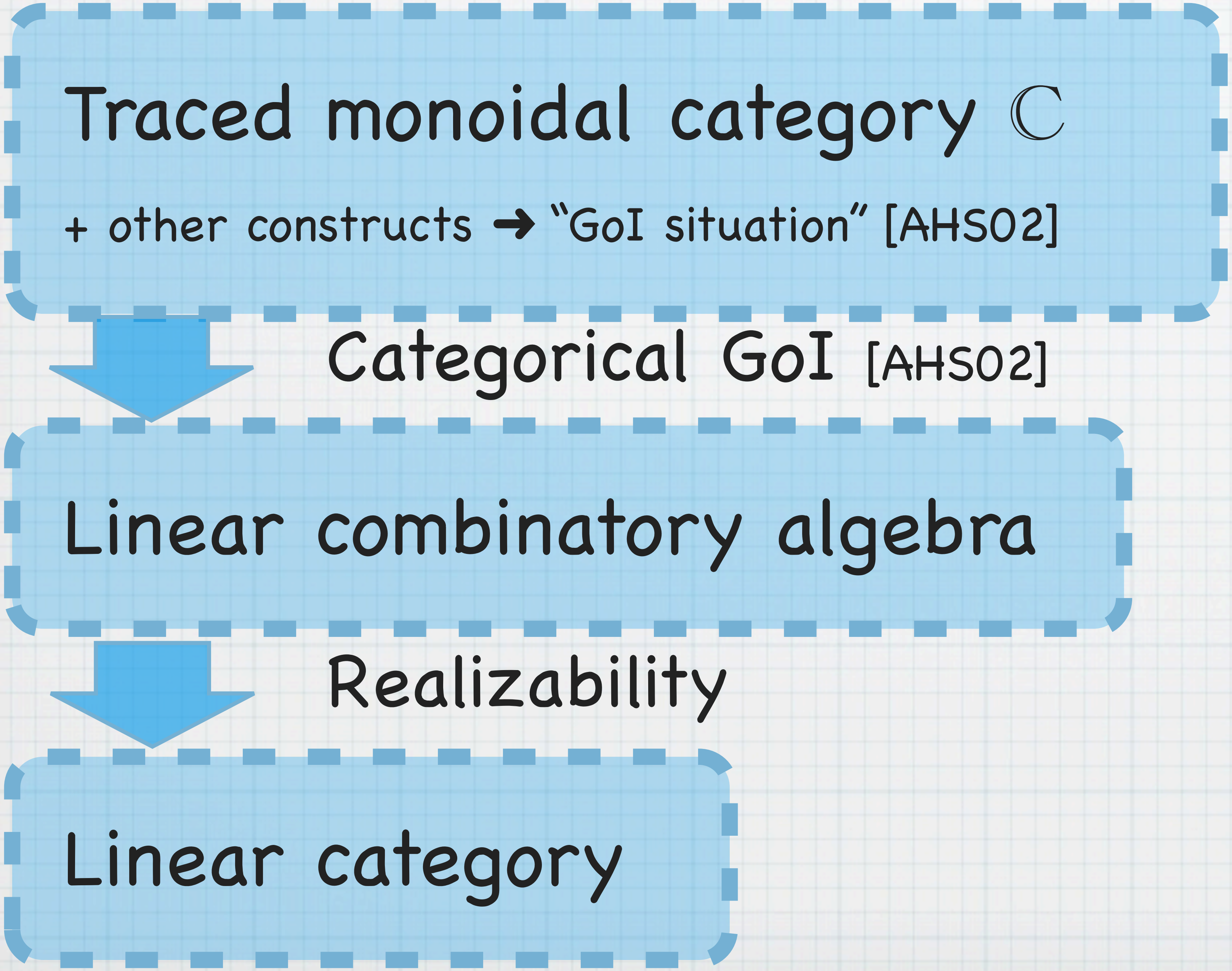
The Categorical GoI Workflow



Traced monoidal category $\mathbb{C}$

+ other constructs ➜ "GoI situation" [AHS02]

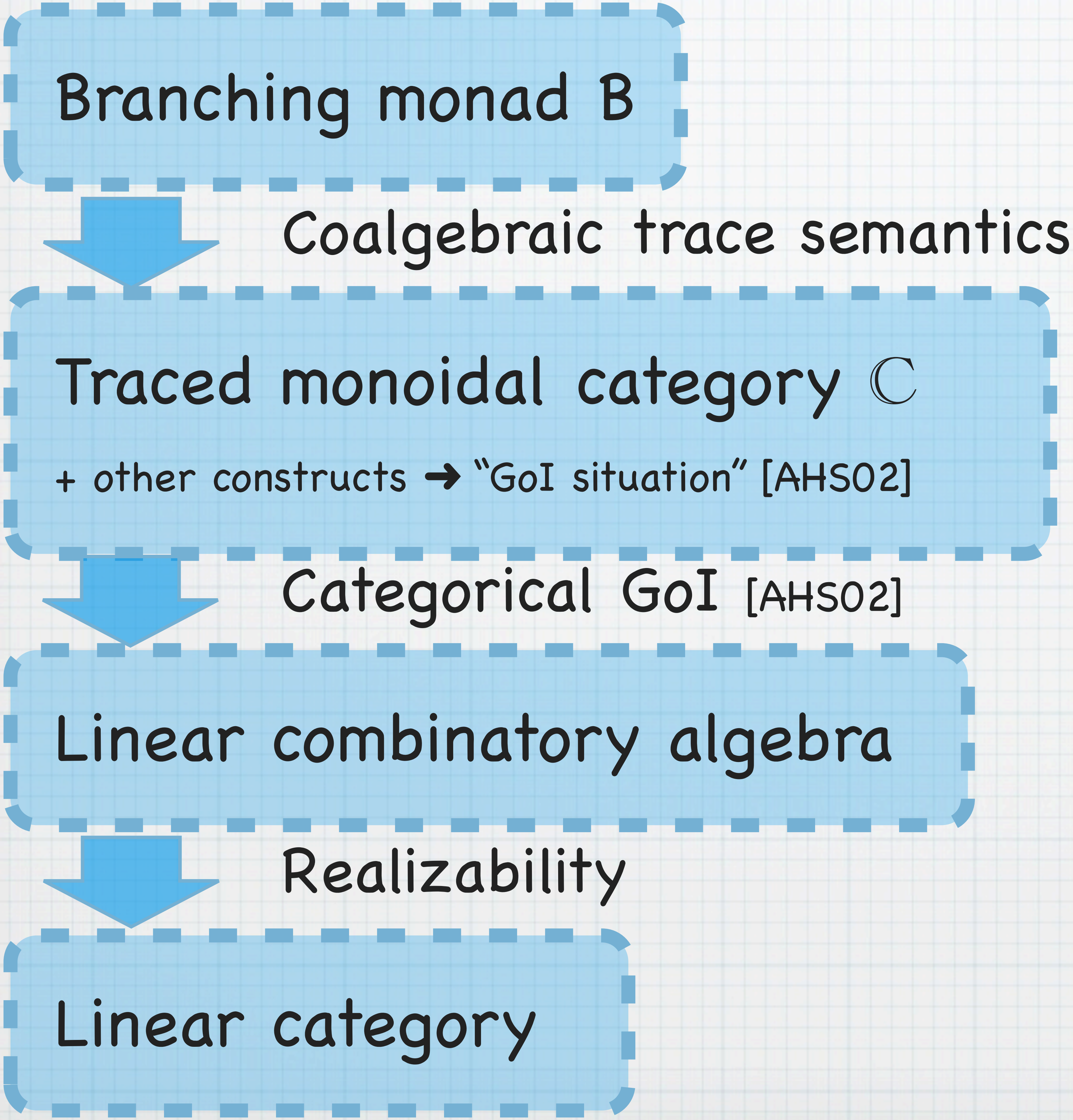
Categorical GoI [AHS02]

Linear combinatory algebra

Realizability

Linear category

The Categorical GoI Workflow



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Coalgebraic trace semantics

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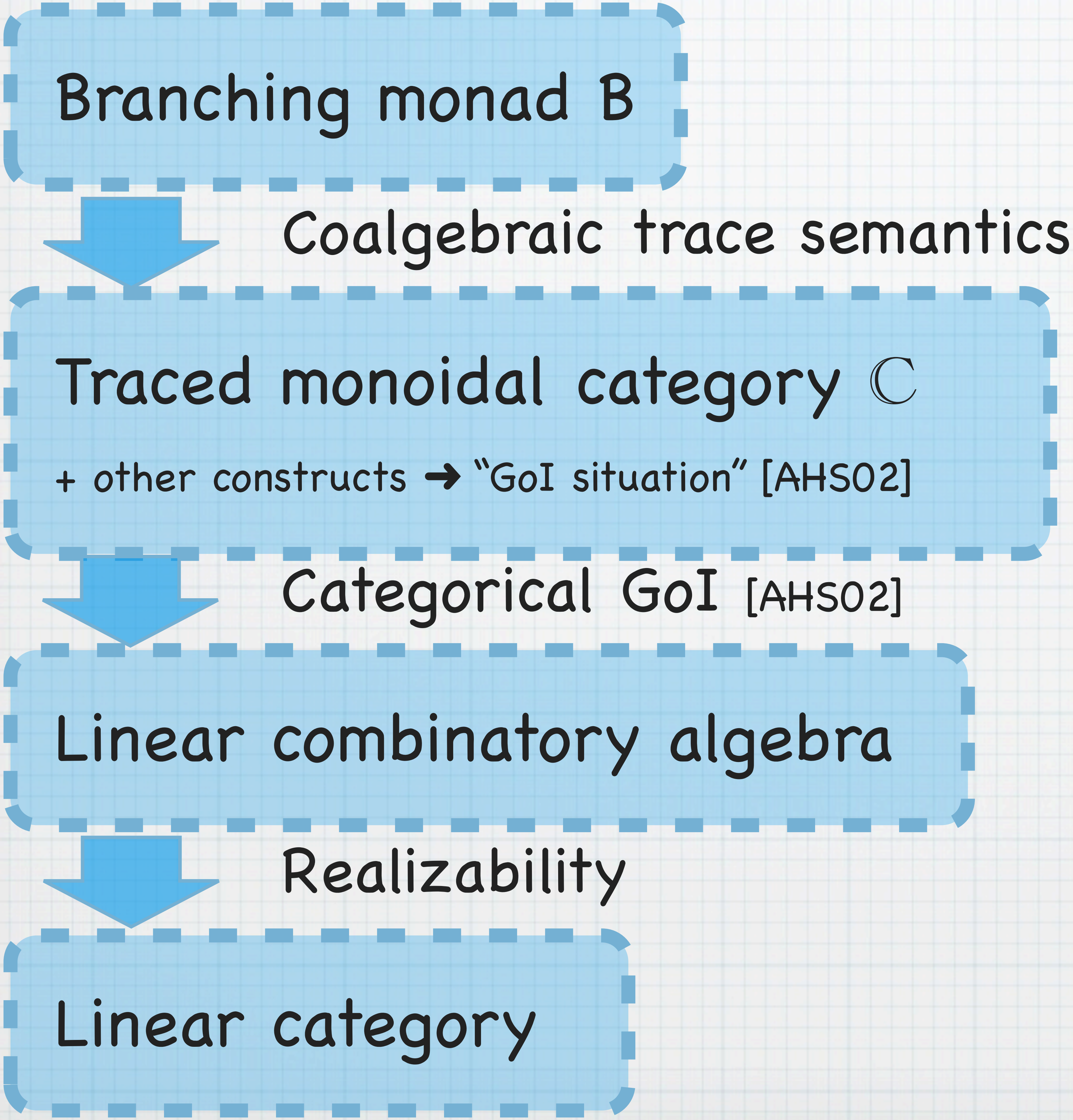

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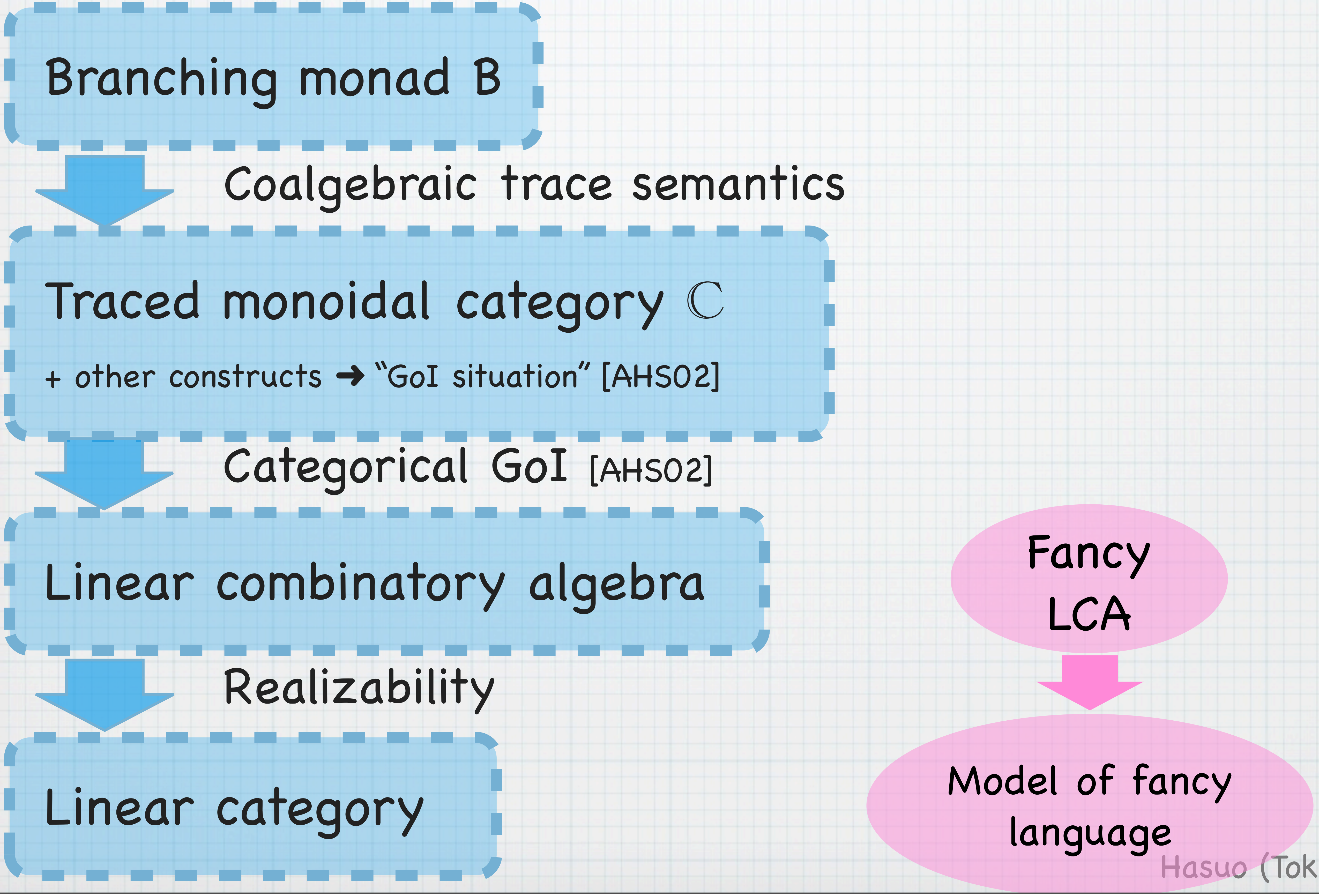

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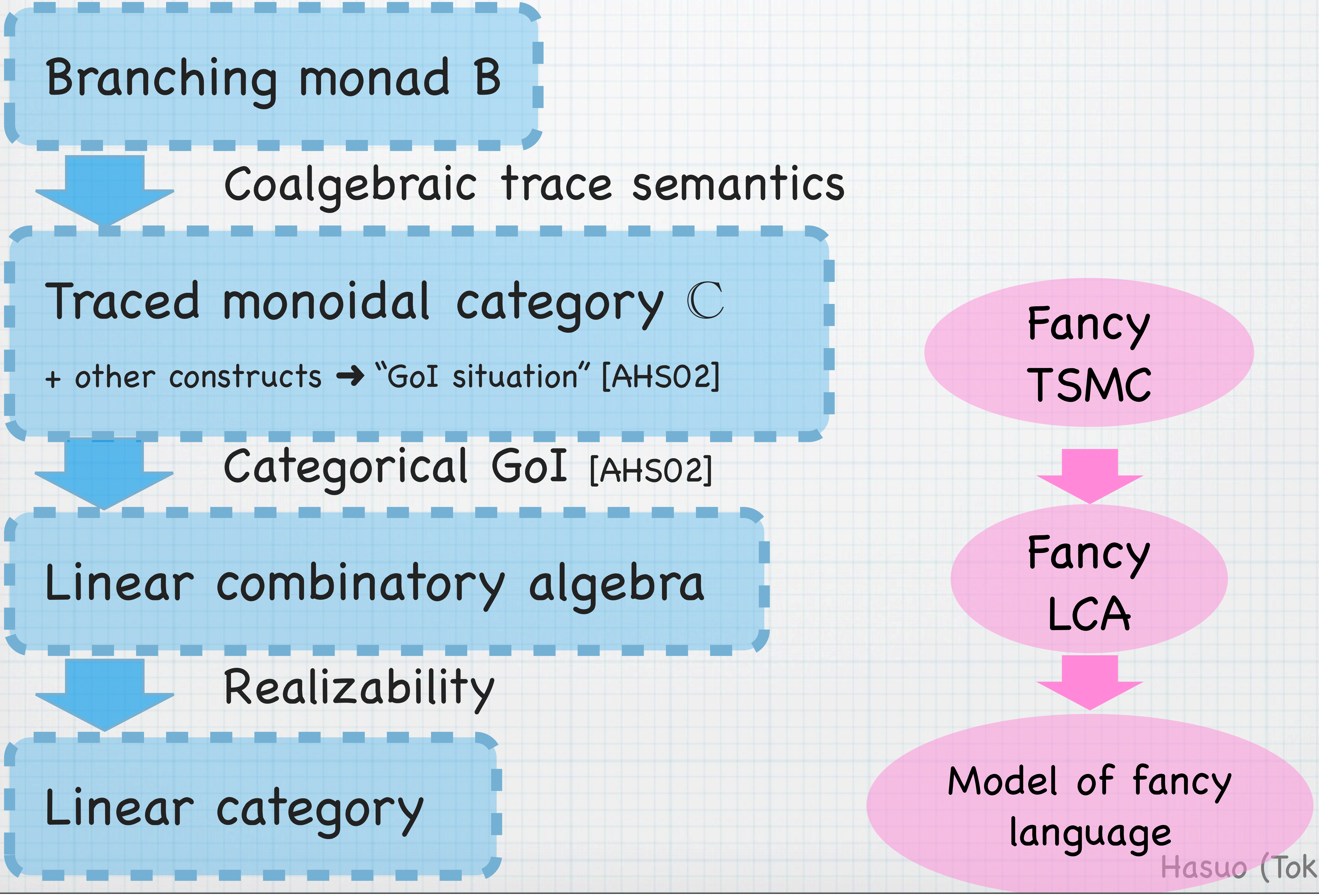

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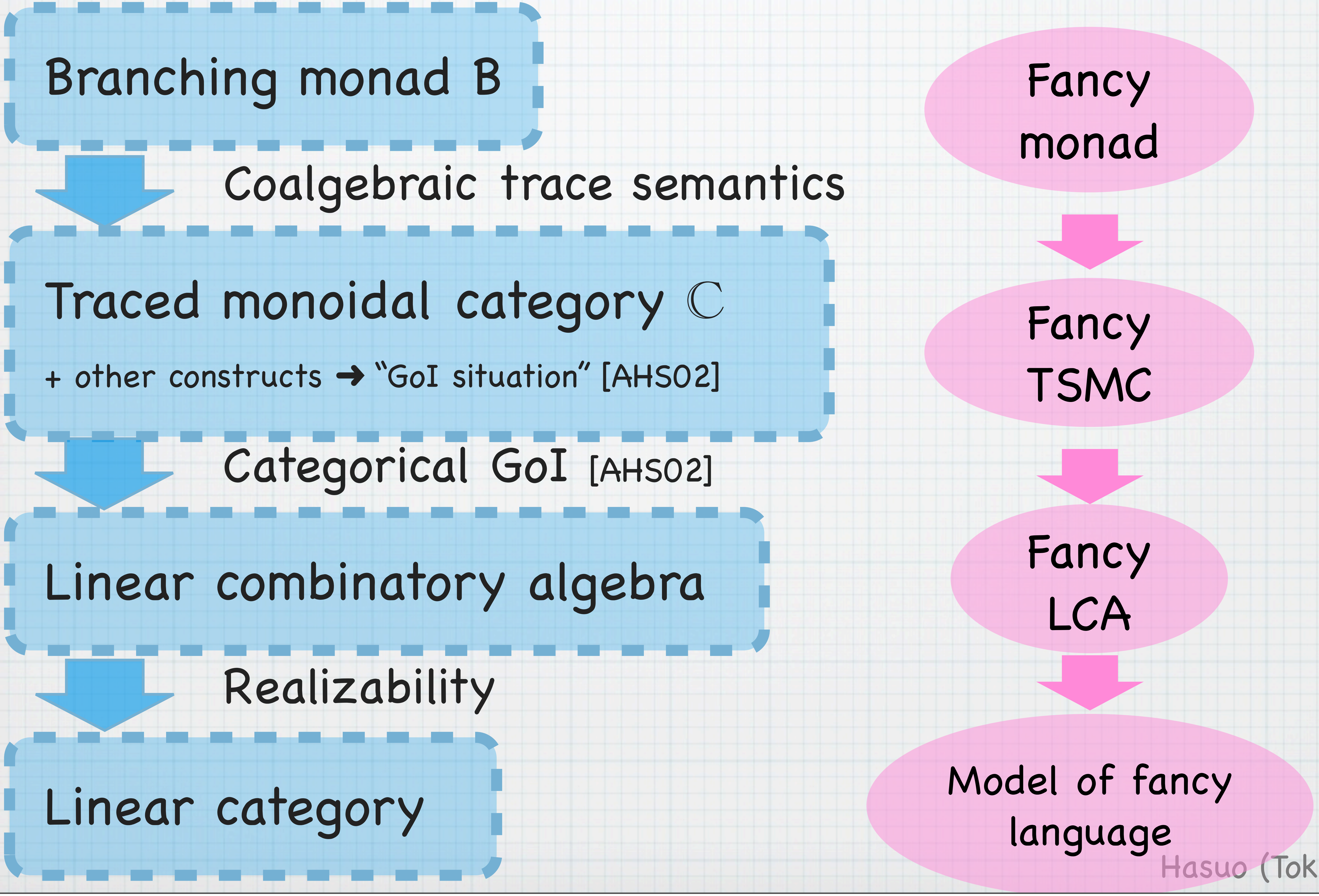

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Linear category

Fancy monad

Fancy TSMC

Fancy LCA

Model of fancy language

What is Fancy, Nowadays?

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- Biology?
- Hybrid systems?
    - Both discrete and continuous data, typically in **cyber-physical systems** (CPS)
    - ➜ Our approach via **non-standard analysis** [Suenaga, IH, ICALP'11] [IH, Suenaga, CAV'12] [Suenaga, Sekine, IH, POPL'13]
- Quantum?
    - Yes this worked!

Part 3

(Optional)

Realizability:

from Untyped to Typed

Realizability

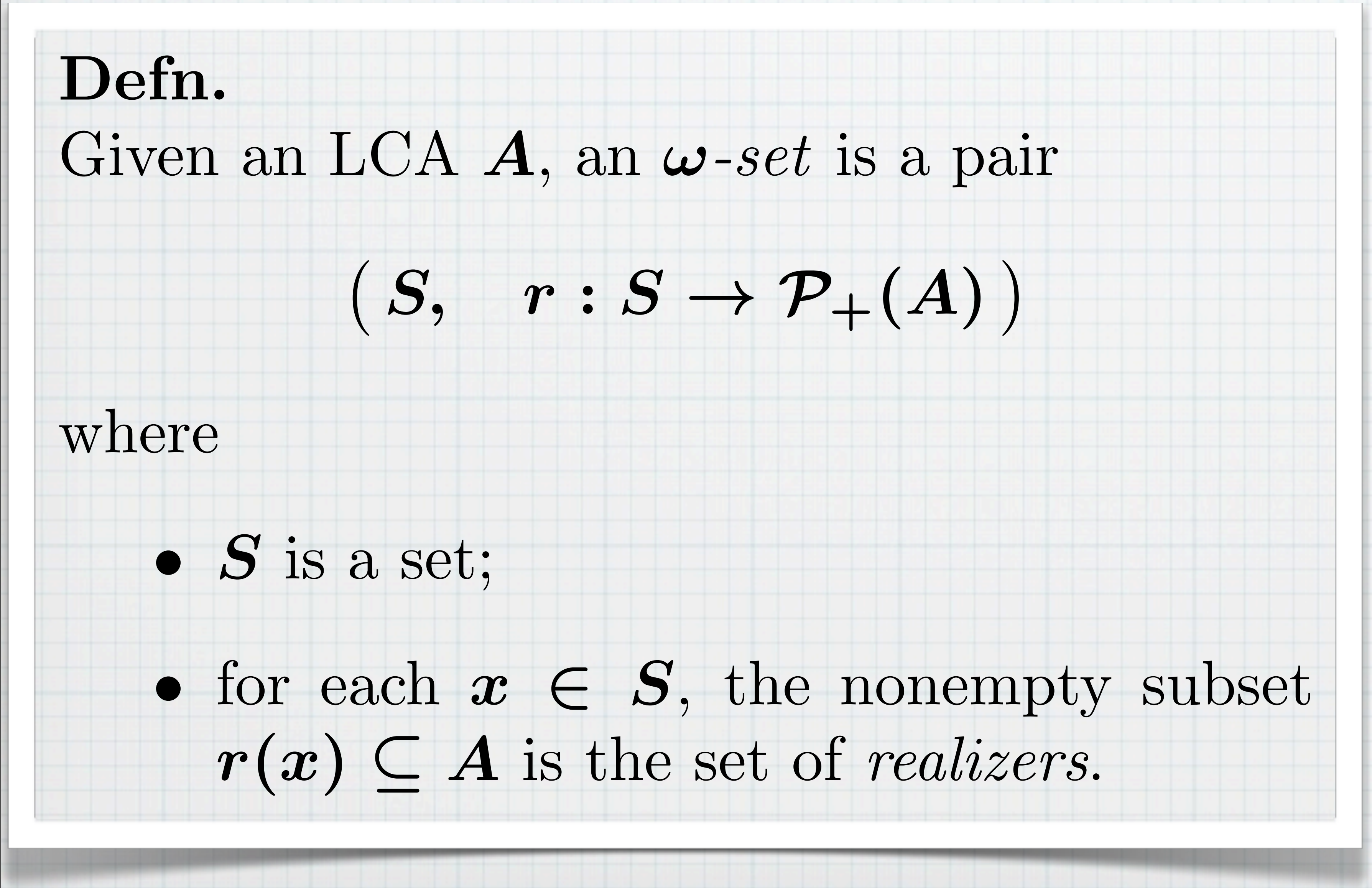
- Dates back to Kleene
- Cf. the Brouwer-Heyting-Kolmogorov (BHK) interpretation
  - A p'f of $A \wedge B$ is a pair: (p'f of A, p'f of B)
  - A p'f of $A \rightarrow B$ is a function carrying (p'f of A) to (p'f of B)
  - Proof = "realizer"

Realizability

- Our technical view on realizability: a construction
  - from a combinatory algebra,
  - of a categorical model of a typed calculus
- Here: construct a linear category from an LCA
- References:
  - [AL05] S. Abramsky and M. Lenisa, "Linear realizability and full completeness for typed lambda-calculi," APAA 2005.
  - [Hos07] N. Hoshino, "Linear realizability," CSL 2007.

Realizability

- Either by **ω-sets** (intuitive) or by **PERs** (tech. convenient)



Defn.
Given an LCA $\boldsymbol{A}$, an $\boldsymbol{\omega}$-*set* is a pair

$$(\, \boldsymbol{S}, \quad \boldsymbol{r} : \boldsymbol{S} \to \boldsymbol{\mathcal{P}}_{+}(\boldsymbol{A}) \,)$$

where

- $\boldsymbol{S}$ is a set;
- for each $\boldsymbol{x} \in \boldsymbol{S}$, the nonempty subset $\boldsymbol{r}(\boldsymbol{x}) \subseteq \boldsymbol{A}$ is the set of *realizers*.

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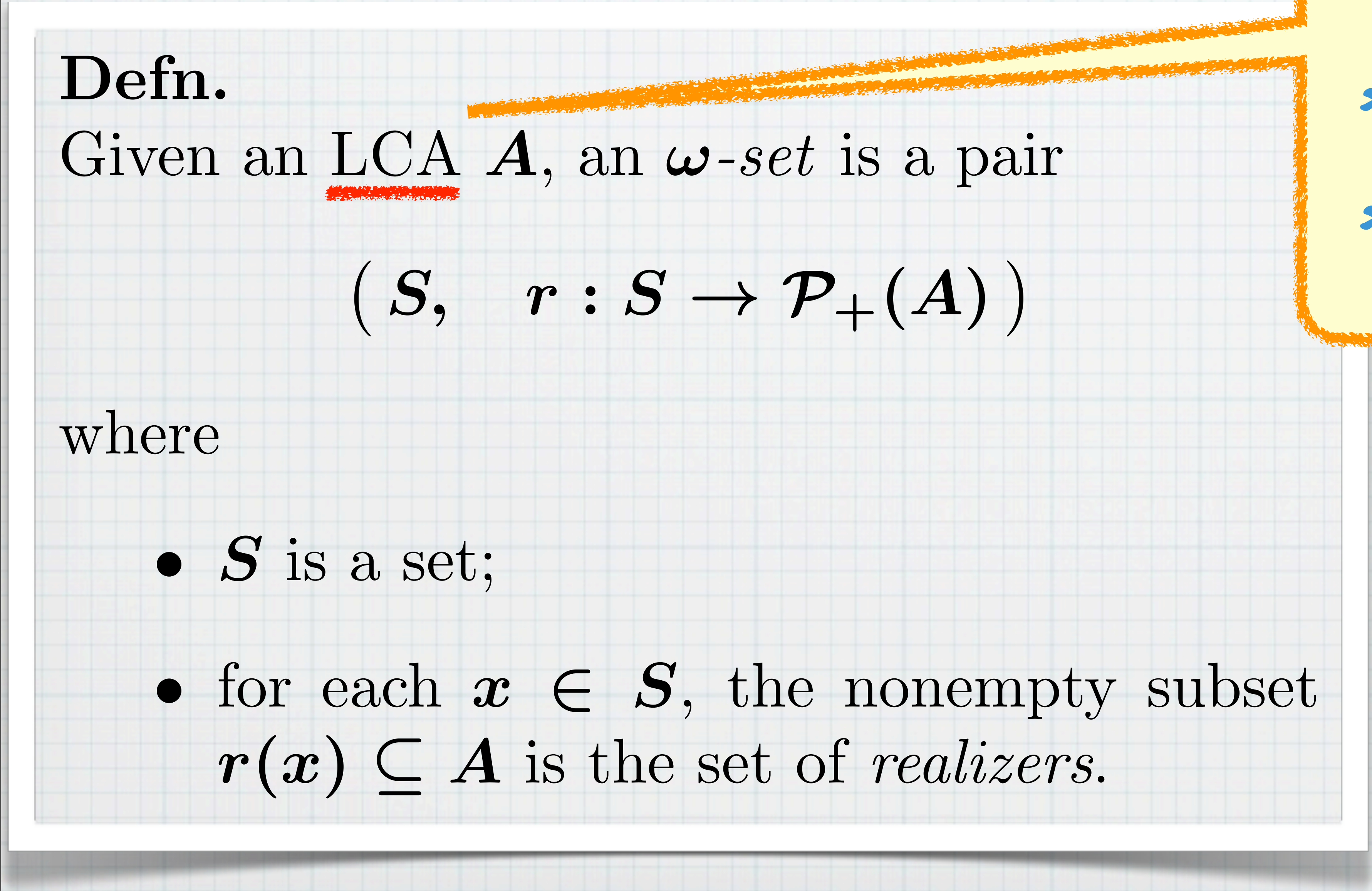

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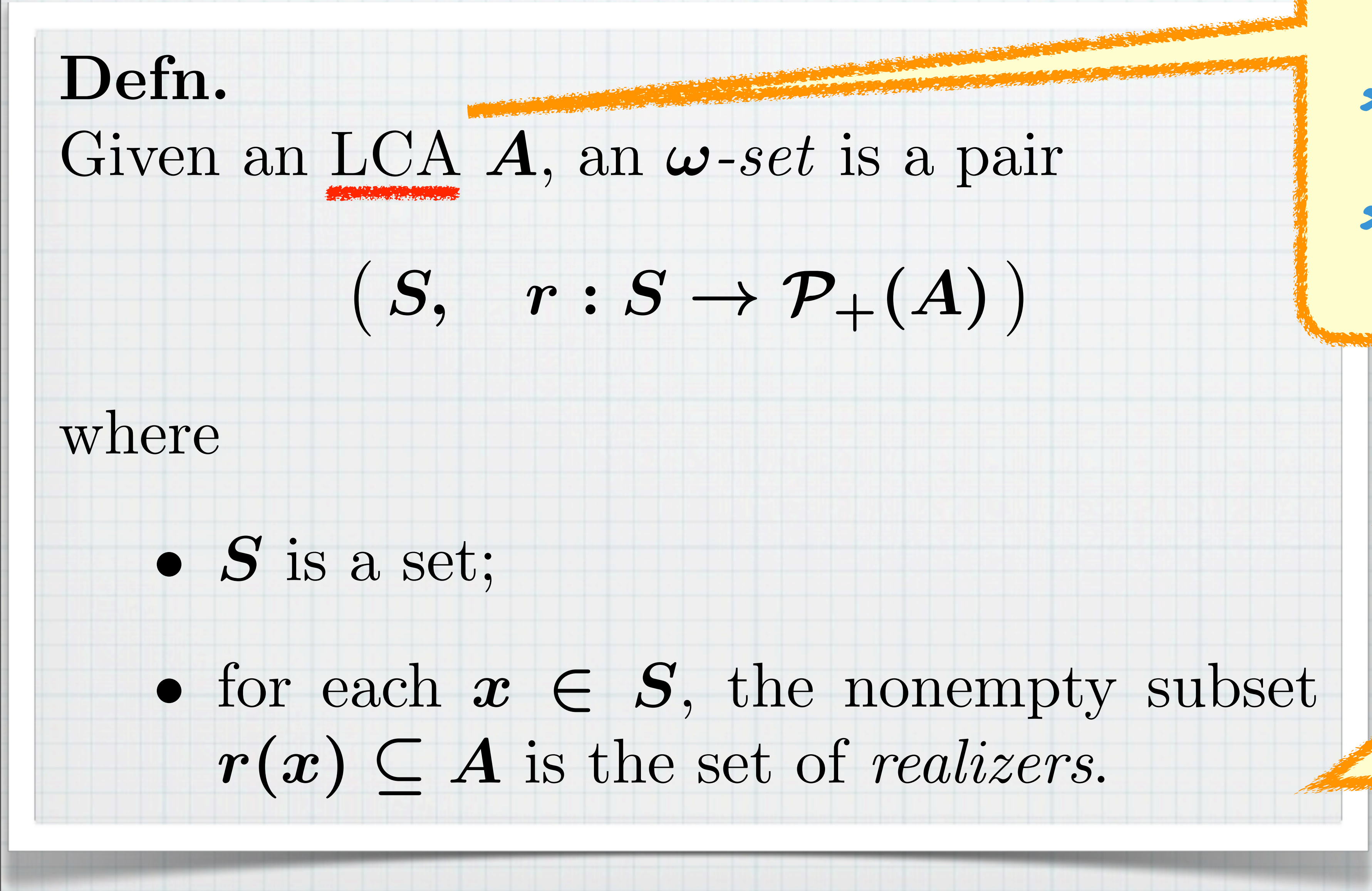

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a ∈ r(x) :

- "realizes" x, or
- "witnesses existence of" x

Realizability

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A *partial equivalence relation (PER)* $\boldsymbol{X}$ is a transitive and symmetric relation on $\boldsymbol{A}$.

$$\begin{aligned} |\boldsymbol{X}| &:= \{\boldsymbol{a} \mid (\boldsymbol{a}, \boldsymbol{a}) \in \boldsymbol{X}\} \\ &= \{\boldsymbol{a} \mid \exists \boldsymbol{b}.\, (\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{X}\} \\ &= \{\boldsymbol{a} \mid \exists \boldsymbol{b}.\, (\boldsymbol{b}, \boldsymbol{a}) \in \boldsymbol{X}\} \end{aligned}$$

is the *domain* of $\boldsymbol{X}$.

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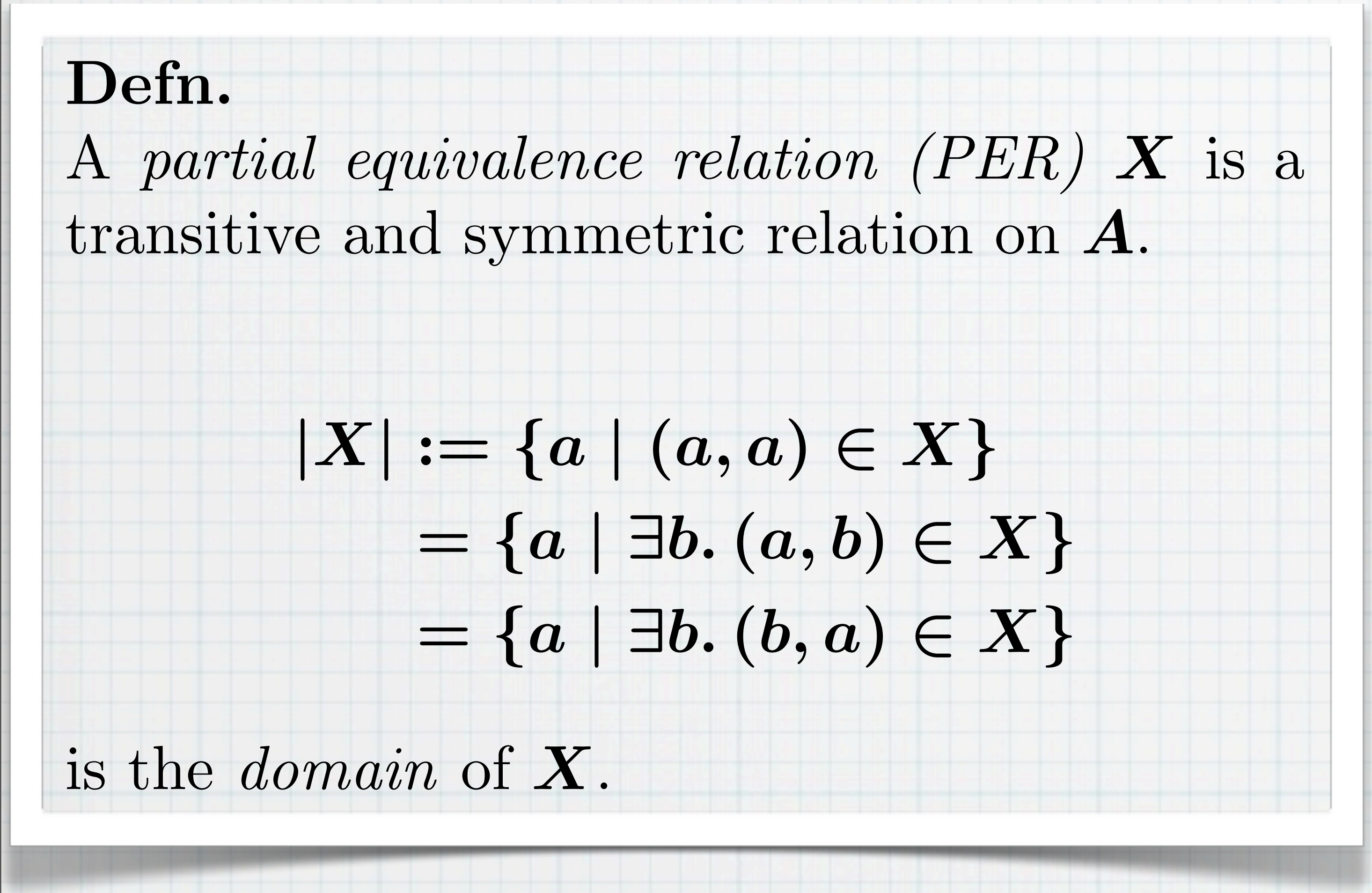

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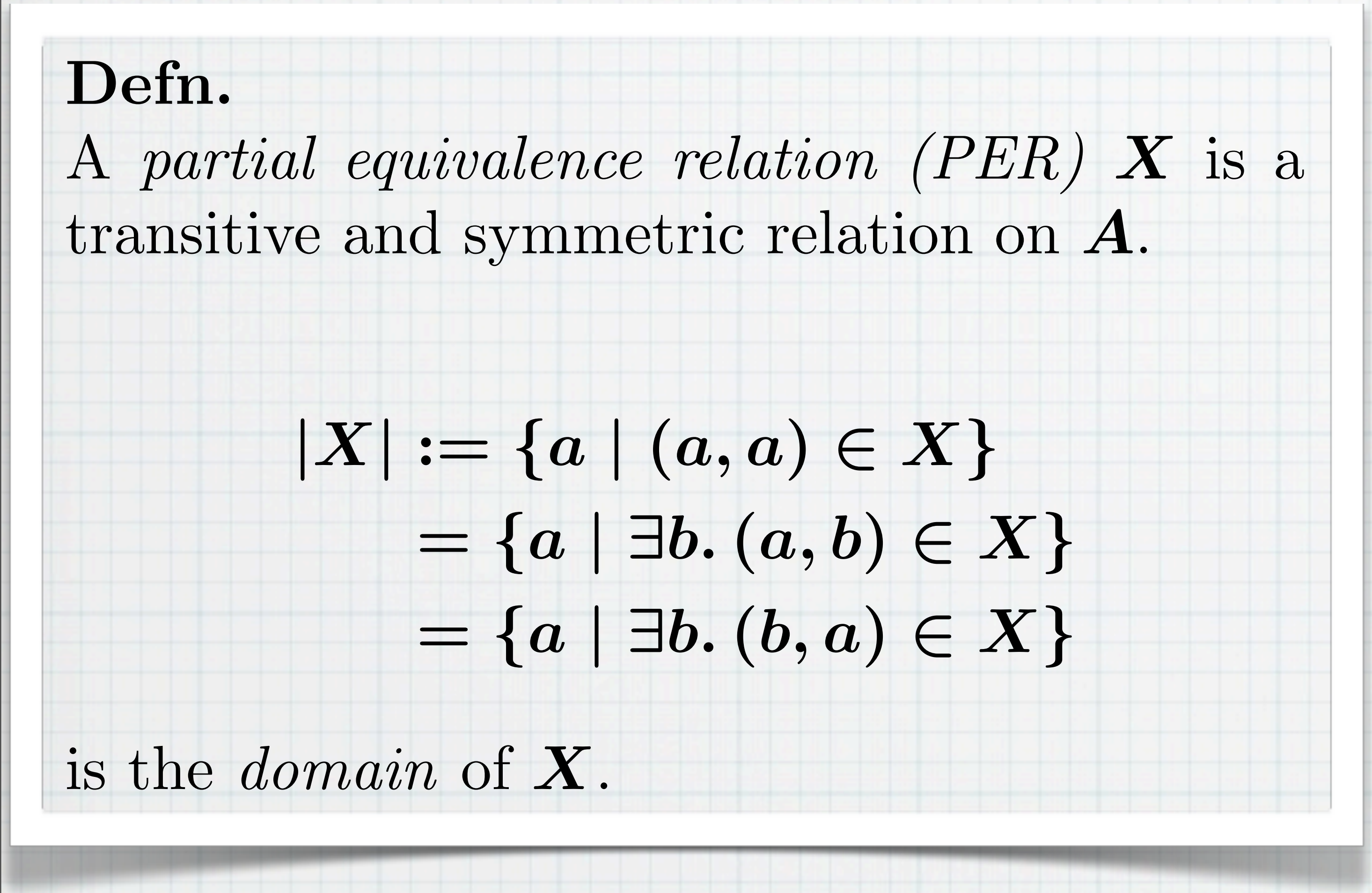

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Hasuo (Tokyo)

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- Also: ω-set to PER

$\mathbf{PER}_A$:
The Category of PERs

- **Obj.** A PER X on A

- **Arr.** The homset is

$$\mathbf{PER}_A(X, Y)$$

$$= \left\{ c \in A \;\middle|\; (x, x') \in X \Longrightarrow (cx, cx') \in Y \right\}$$

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Type Constructors in $\mathbf{PER}_A$

Thm. ([AL05])
If $\boldsymbol{A}$ is an affine LCA, then $\mathbf{PER}_{\boldsymbol{A}}$ is a linear category.
Furthermore, $\mathbf{PER}_{\boldsymbol{A}}$ has finite products and coproducts.

- Linear category [Benton&Wadler,LICS'96][Bierman,TLCA'95]
  - Categorical model of linear logic/linear λ, with
    - Monoidal closed with $\boxtimes, \mathbf{I}, \multimap$
    - Linear exponential comonad $\mathbf{!}$

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Not ⊗,
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Hasuo (Tokyo)

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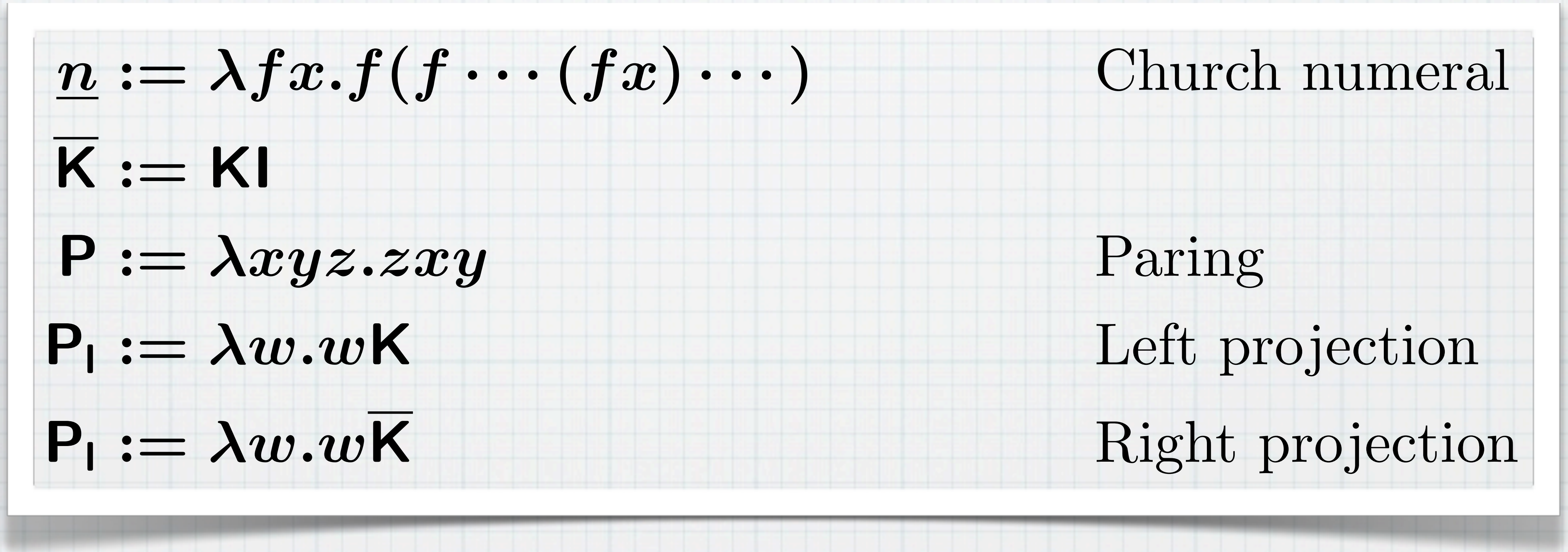

$\underline{n} := \lambda fx.f(f\cdots(fx)\cdots)$	Church numeral
$\overline{\mathsf{K}} := \mathsf{KI}$	
$\mathsf{P} := \lambda xyz.zxy$	Paring
$\mathsf{P}_\mathsf{l} := \lambda w.w\mathsf{K}$	Left projection
$\mathsf{P}_\mathsf{l} := \lambda w.w\overline{\mathsf{K}}$	Right projection

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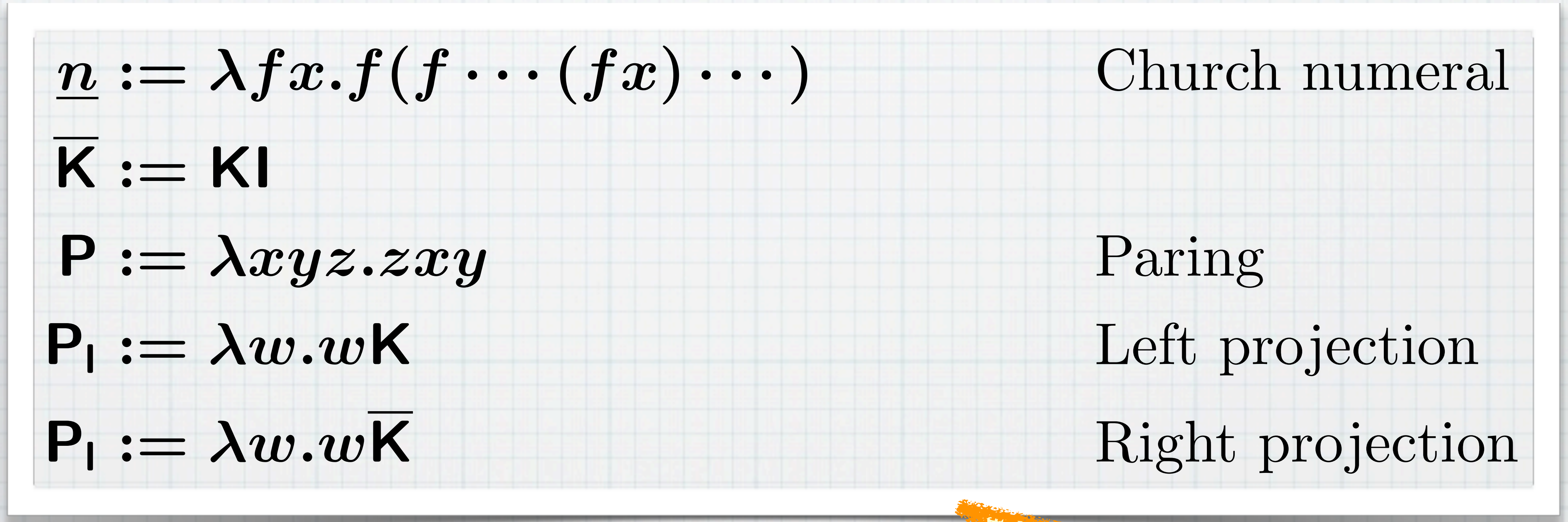

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P_l	$:= \lambda w.w\mathsf{K}$	Left projection
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$$\mathsf{P}_\mathsf{l}(\mathsf{P}xy) = x$$

$$\mathsf{P}_\mathsf{r}(\mathsf{P}xy) = y$$

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$$\mathsf{P}_\mathsf{l}(\mathsf{P}xy) = x$$

$$\mathsf{P}_\mathsf{r}(\mathsf{P}xy) = y$$

  - Cf. Combinaroty completeness

Type Constructors in $\mathbf{PER}_A$

$$\frac{\boldsymbol{X} \in \mathbf{PER}_{\boldsymbol{A}}}{\boldsymbol{X} \subseteq \boldsymbol{A} \times \boldsymbol{A}, \text{ sym., trans.}}$$

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Hasuo (Tokyo)

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additive and

CPS-style. k_1, k_2: "access methods"

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Summary: Realizability

Affine LCA A

$a \cdot b, \quad !\,a, \quad \mathsf{B}, \mathsf{C}, \mathsf{I}, \ldots$

$\longmapsto$

Linear category $\mathbf{PER}_A$

- $$\begin{tikzcd} \boldsymbol{X} \arrow[r, "{[\boldsymbol{c}]}"] & \boldsymbol{Y} \\ {[\boldsymbol{a}]} \arrow[r, mapsto] & {[\boldsymbol{c} \cdot \boldsymbol{a}]} \end{tikzcd}$$ $(a,c \in A)$
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Not ⊗,
for distinction

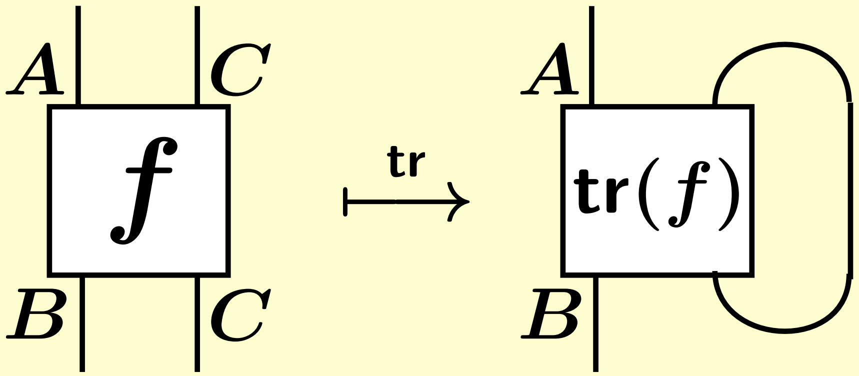
The Categorical GoI Workflow

Branching monad B

Coalgebraic trace semantics

Traced monoidal category $\mathbb{C}$

+ other constructs ➜ "GoI situation" [AHS02]



Categorical GoI [AHS02]

Linear combinatory algebra

- ✱ Applicative str. + combinators
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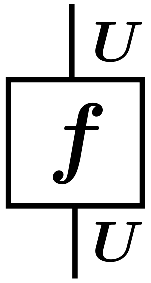
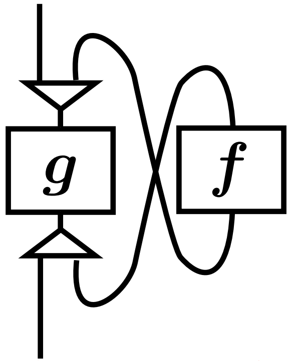


$\in \mathbb{C}(U,U)$

$g \cdot f =$



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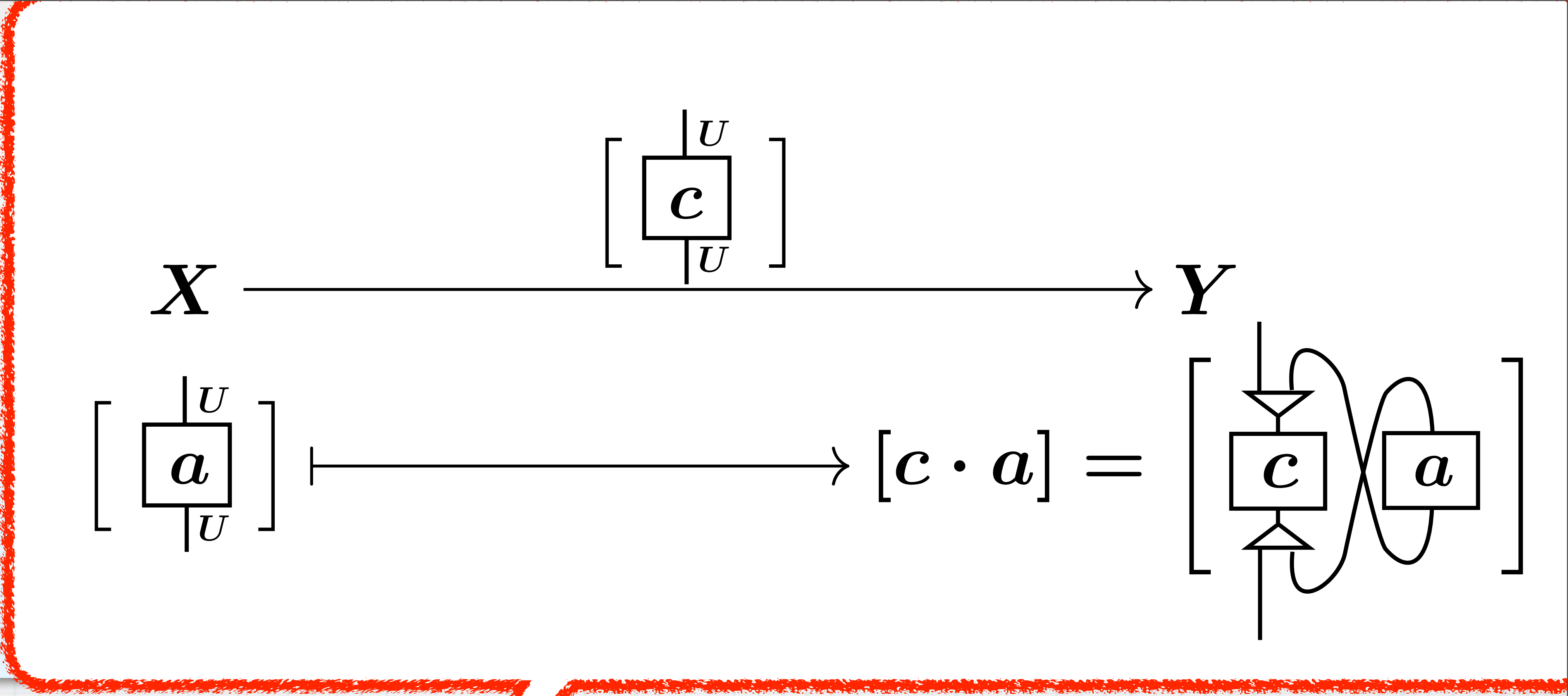

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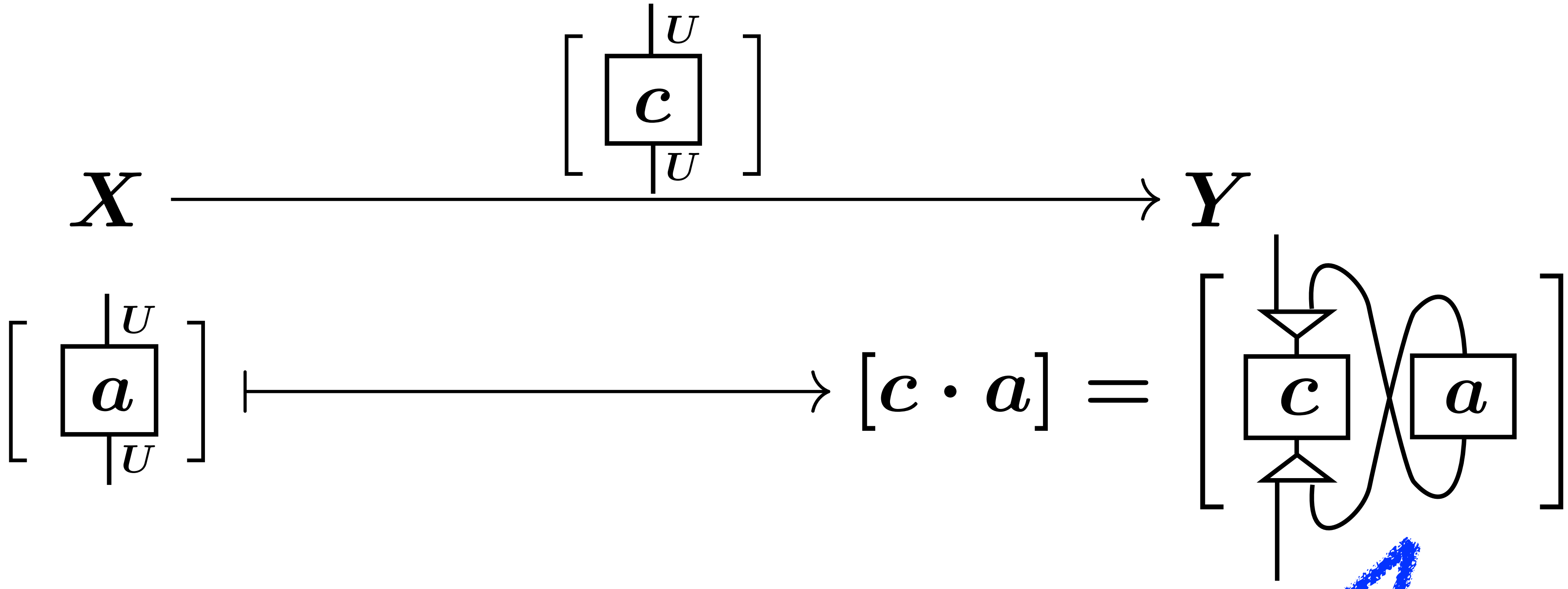

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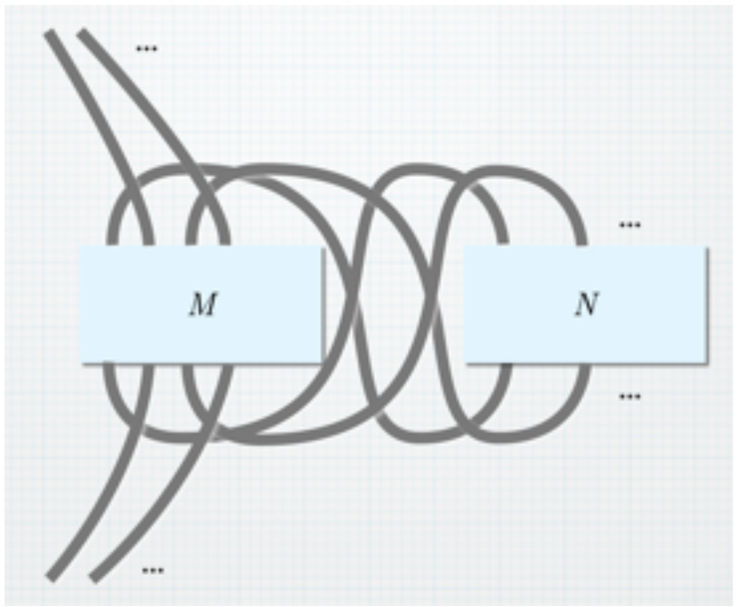

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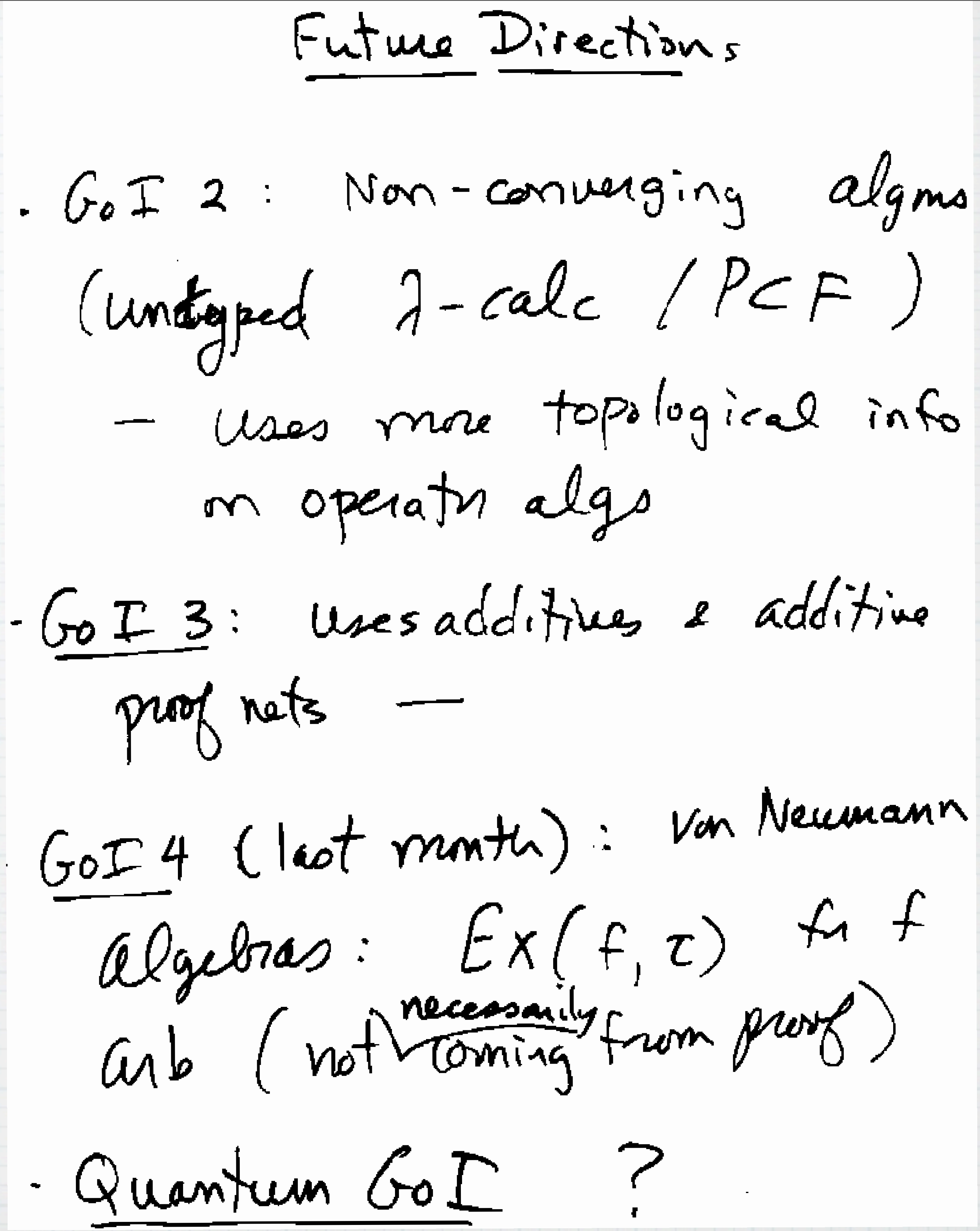
Part 4

Phil Scott.
Tutorial on Geometry of Interaction, FMCS 2004.
Page 47/47



Future Directions

- GoI 2: Non-converging algms (untyped λ-calc / PCF)
  - uses more topological info on operatr algs
- GoI 3: uses additives & additive proof nets —
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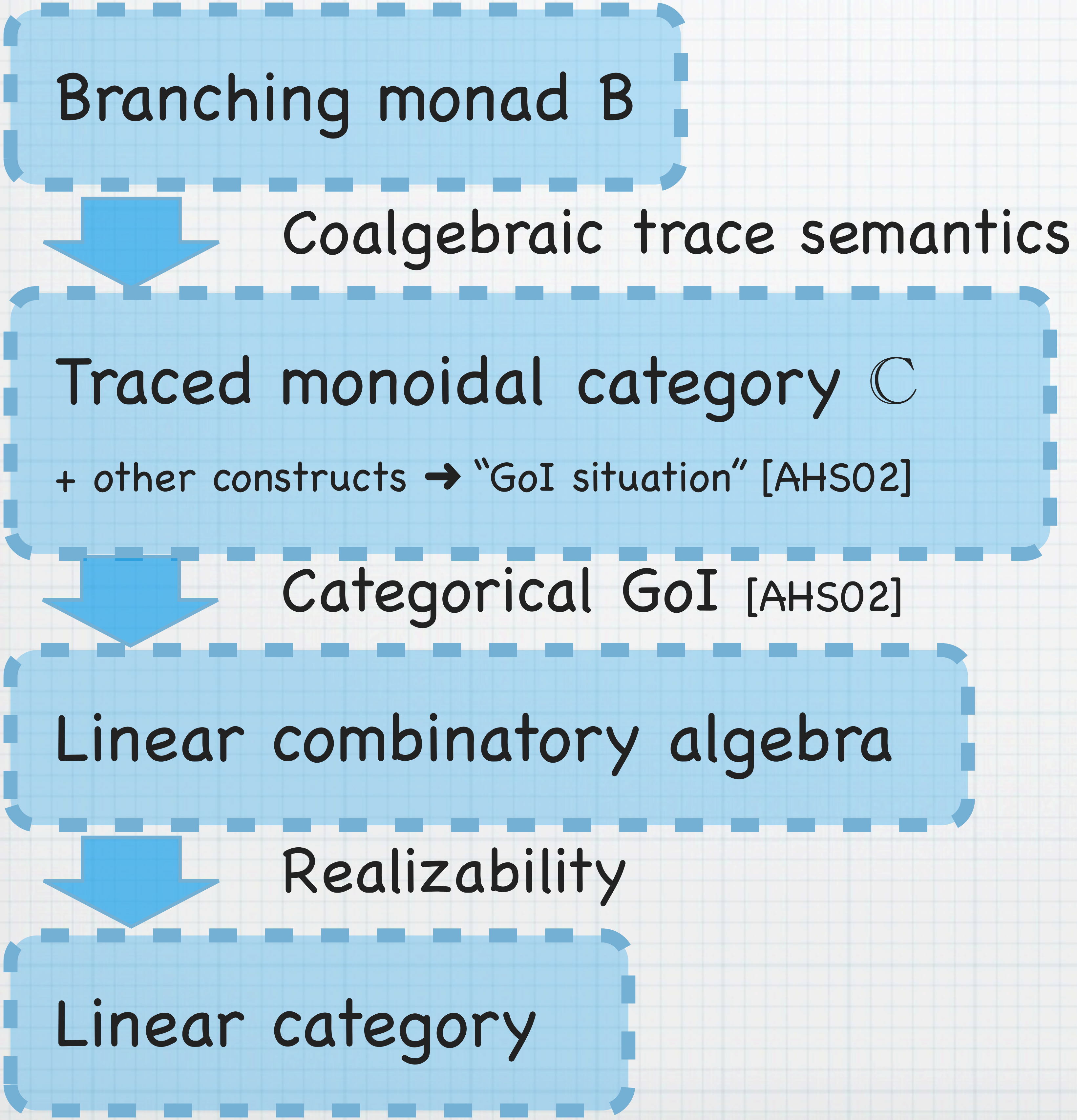
The Categorical GoI Workflow



Branching monad B

Coalgebraic trace semantics

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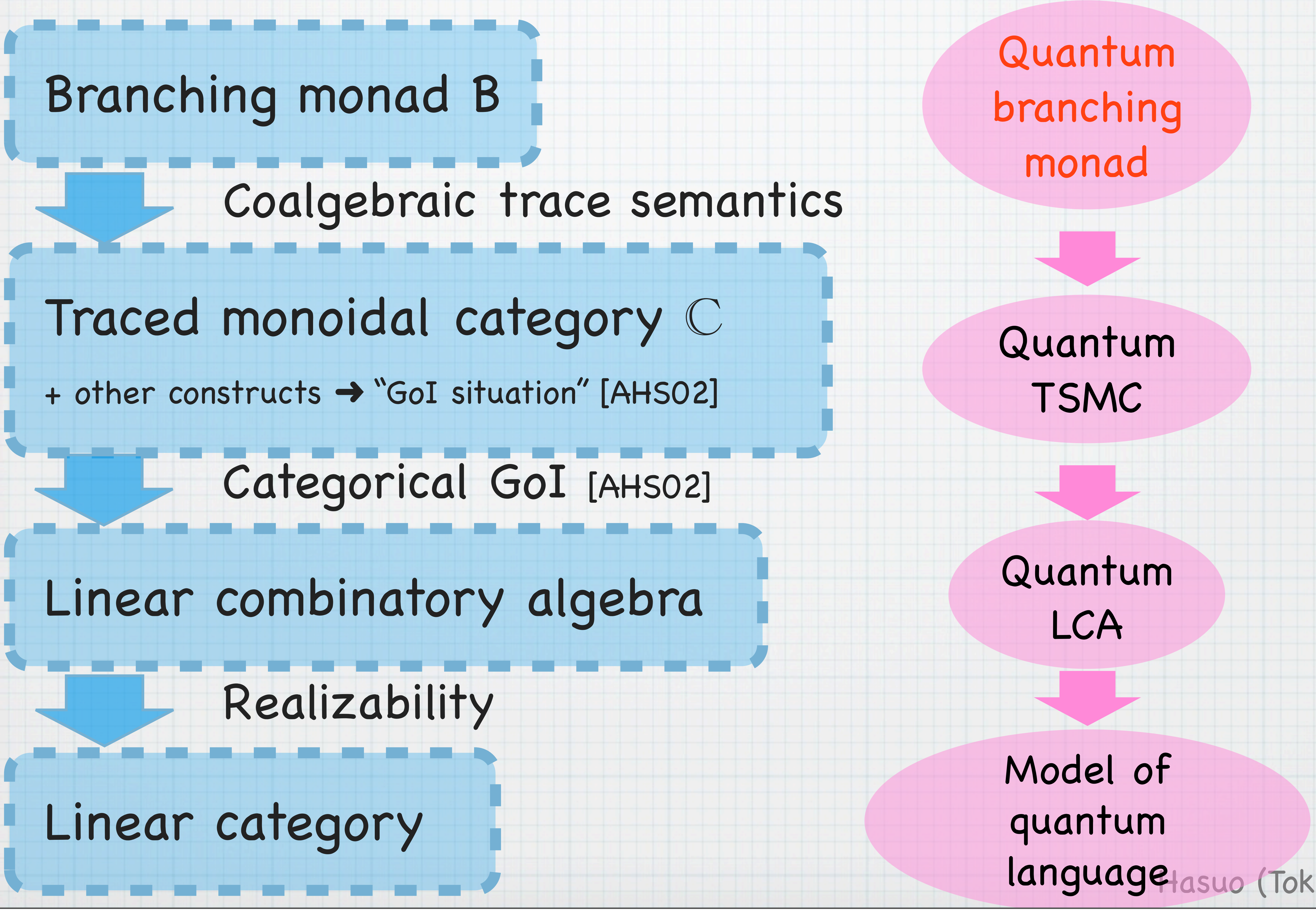

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Quantum branching monad

Quantum TSMC

Quantum LCA

Model of quantum language

The Quantum Branching Monad

$$\boldsymbol{\mathcal{Q}Y} = \left\{ \boldsymbol{c : Y \to} \prod_{\boldsymbol{m,n} \in \mathbb{N}} \mathbf{QO}_{\boldsymbol{m,n}} \;\middle|\; \text{the trace condition} \right\}$$

The Quantum Branching M

$$\mathbf{QO}_{m,n} := \left\{ \begin{array}{l} \text{quantum operations,} \\ \text{from dim. } m \text{ to dim. } n \end{array} \right\}$$

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$$\sum_{\boldsymbol{y}\in\boldsymbol{Y}} \sum_{\boldsymbol{n}\in\mathbb{N}} \mathbf{tr}\bigl[(\boldsymbol{c}(\boldsymbol{y}))_{\boldsymbol{m},\boldsymbol{n}}(\boldsymbol{\rho})\bigr] \leq \mathbf{1} \ ,$$

$$\forall \boldsymbol{m} \in \mathbb{N},\ \forall \boldsymbol{\rho} \in \boldsymbol{D}_{\boldsymbol{m}}.$$

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- Compare with

$$\mathcal{P}Y = \{ c : Y \to 2 \}$$

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$$\begin{array}{c} X \xrightarrow{f} Y \;\text{ in } \mathcal{K}\ell(\mathcal{Q}) \\ \hline\hline X \xrightarrow{f} \mathcal{Q}Y \;\text{ in } \mathbf{Sets} \end{array}$$

- Given $x \in X$, $y \in Y$, $m \in \mathbb{N}$, $n \in \mathbb{N}$ determines a quantum operation

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Any opr. on quantum data:

combination of

- preparation
- unitary transf.
- measurement

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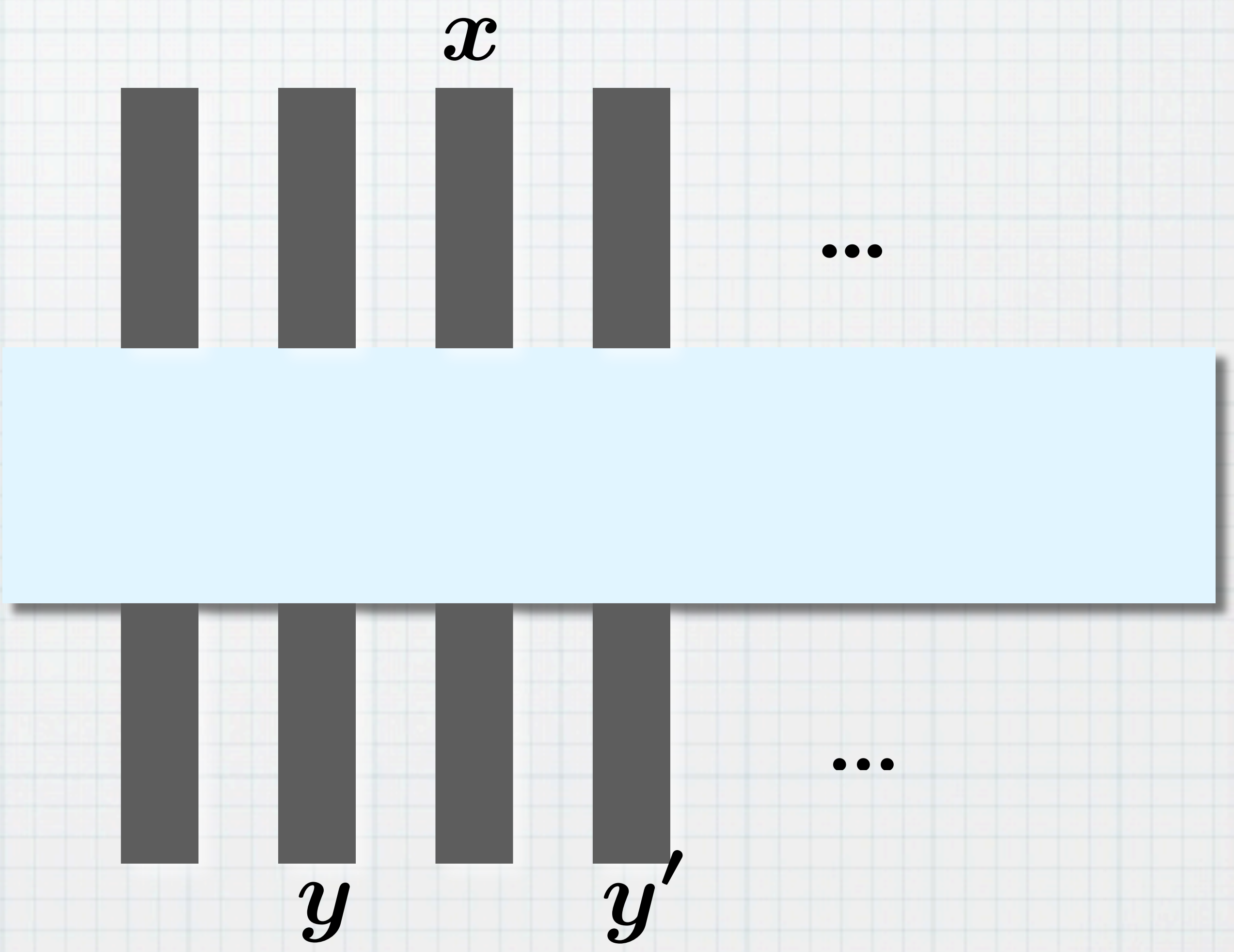

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entrance exit in dim. out dim.

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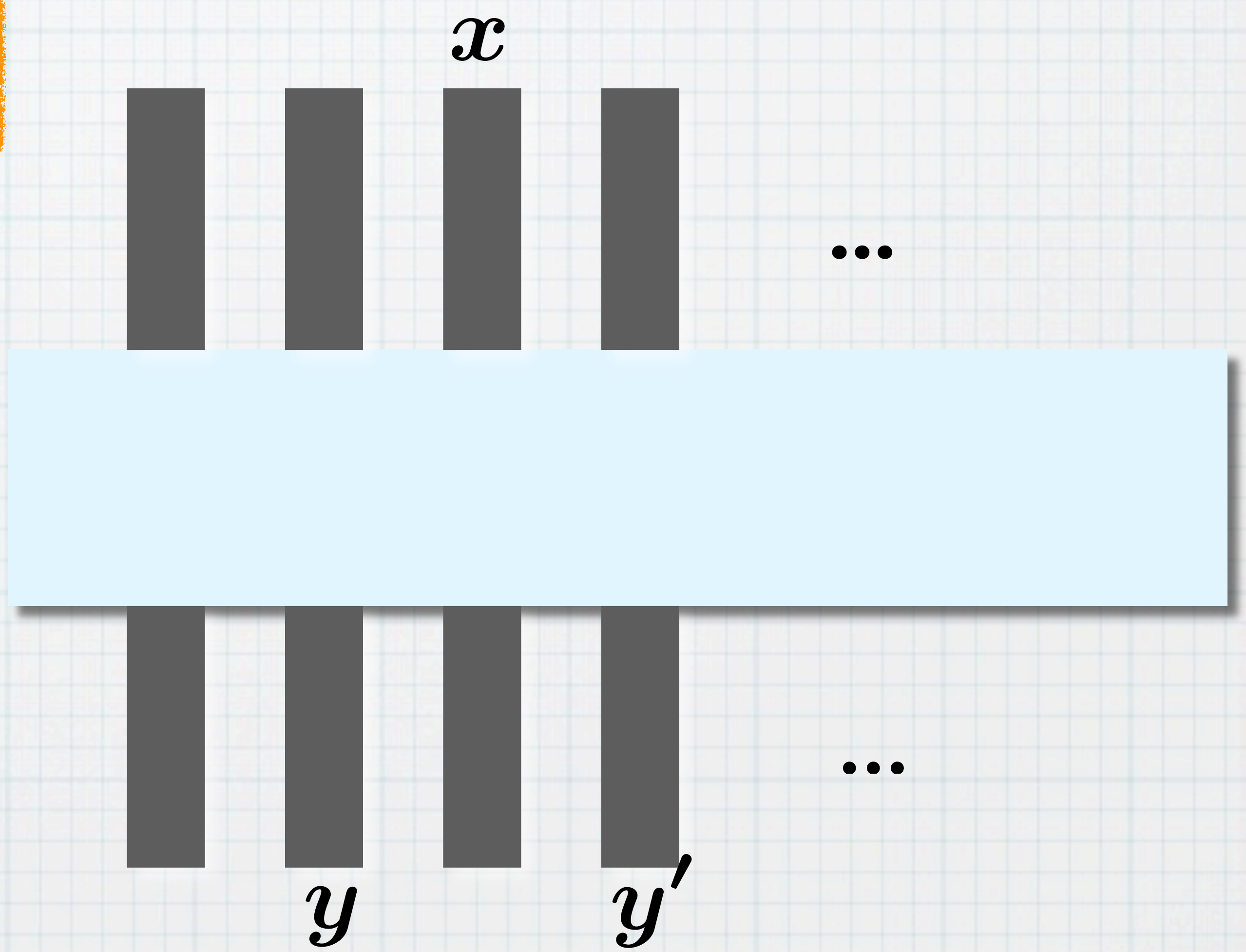

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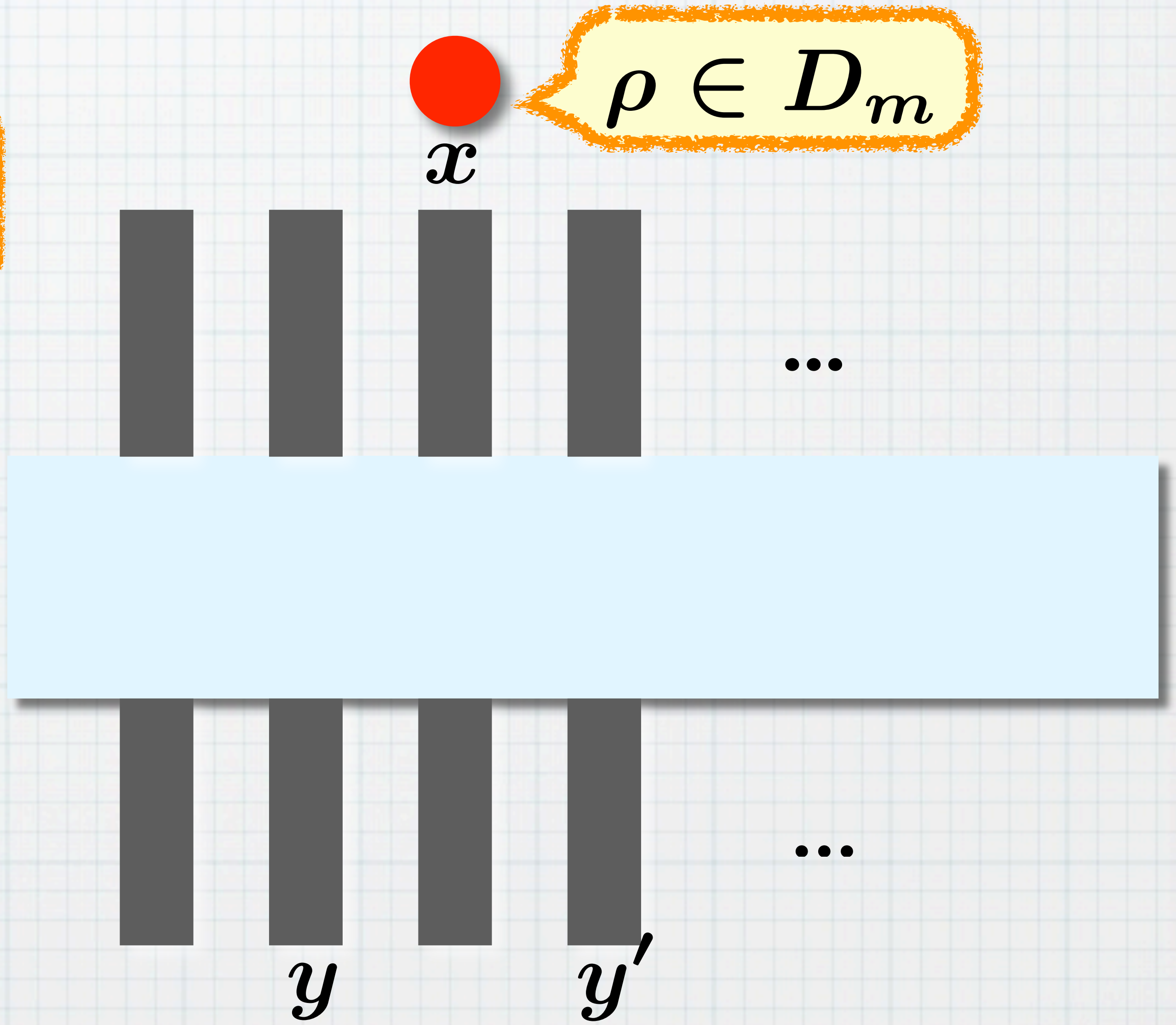

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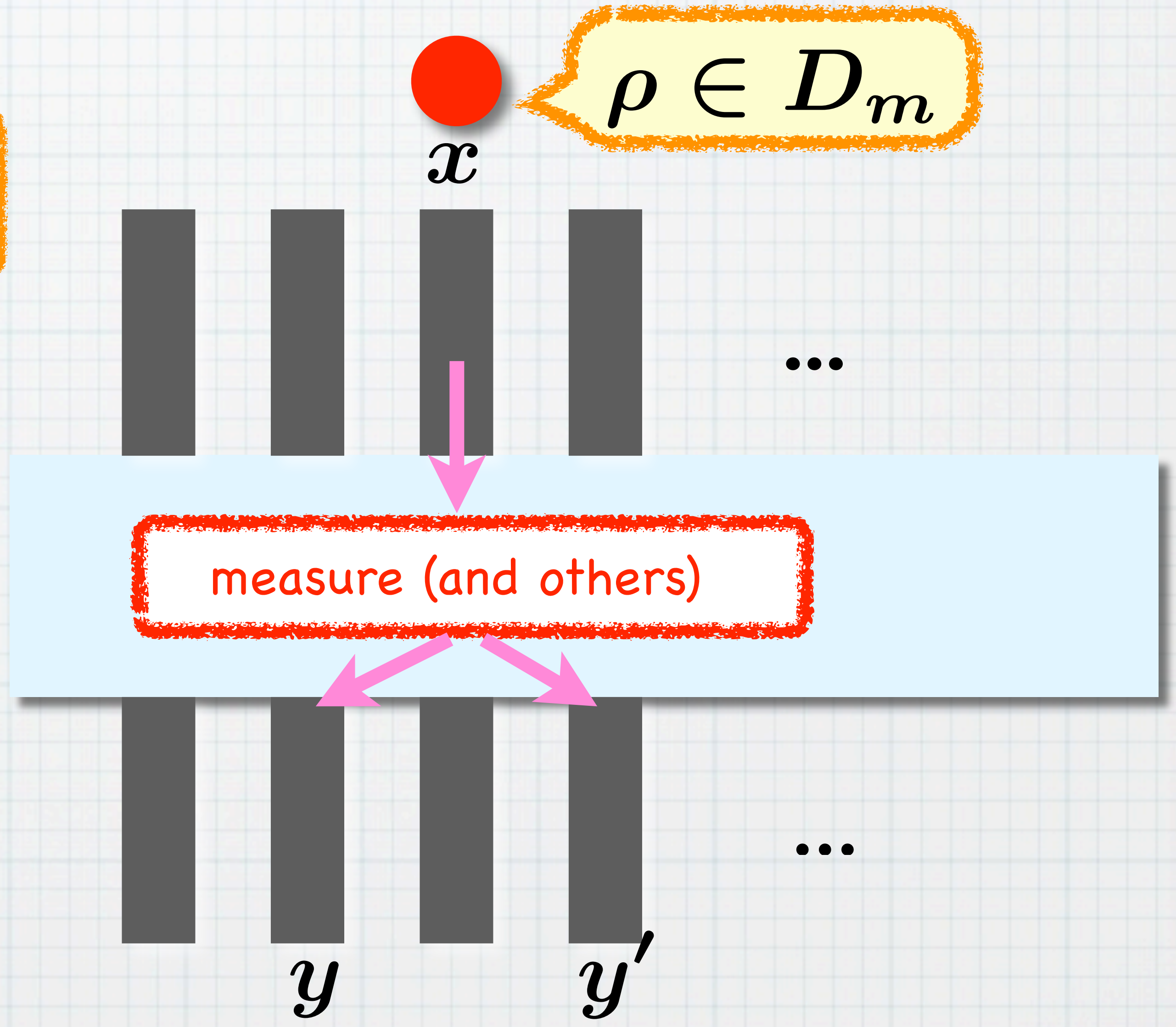

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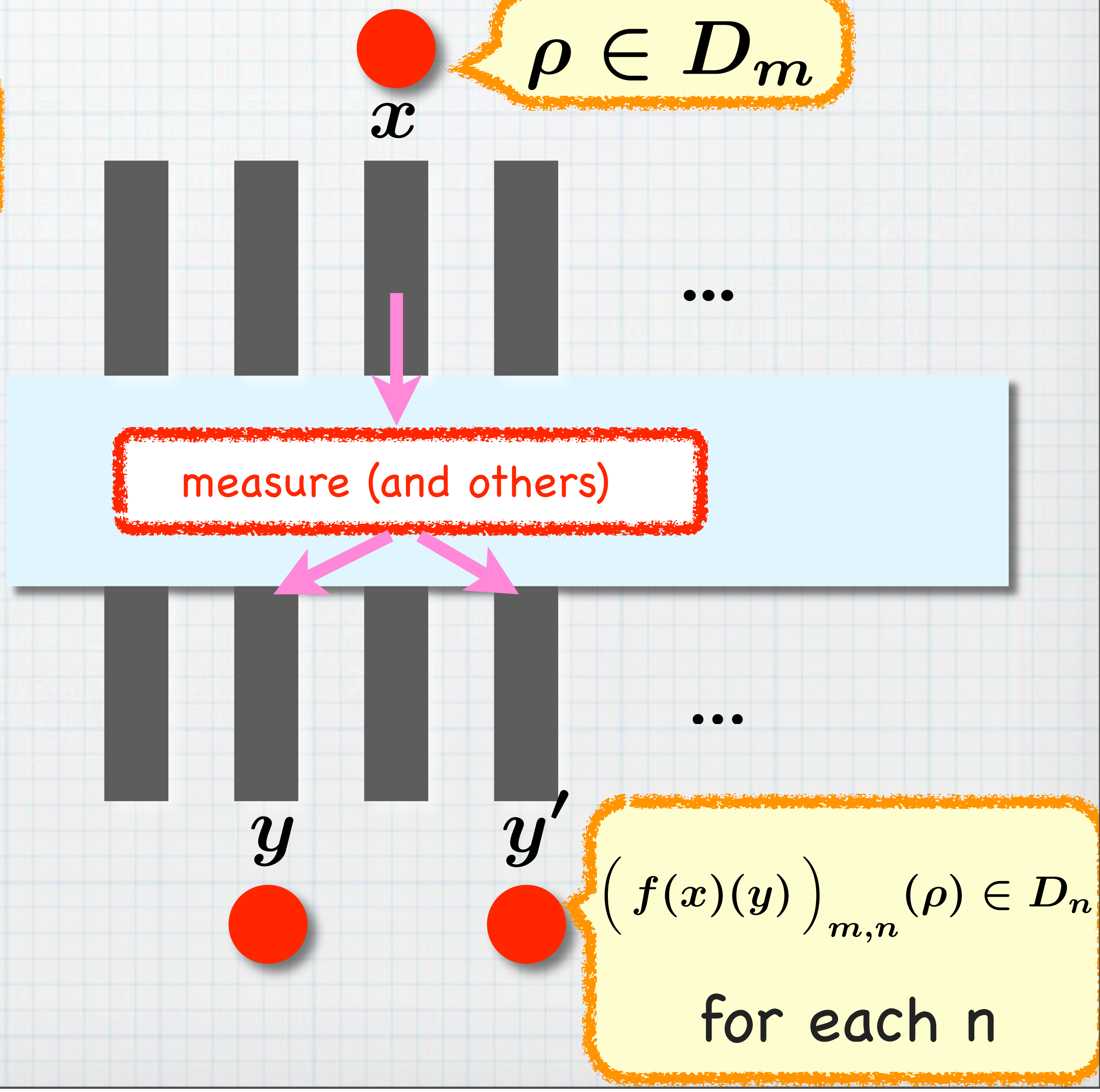

$\rho \in D_m$

$\big(f(x)(y) \big)_{m,n}(\rho) \in D_n$

for each n

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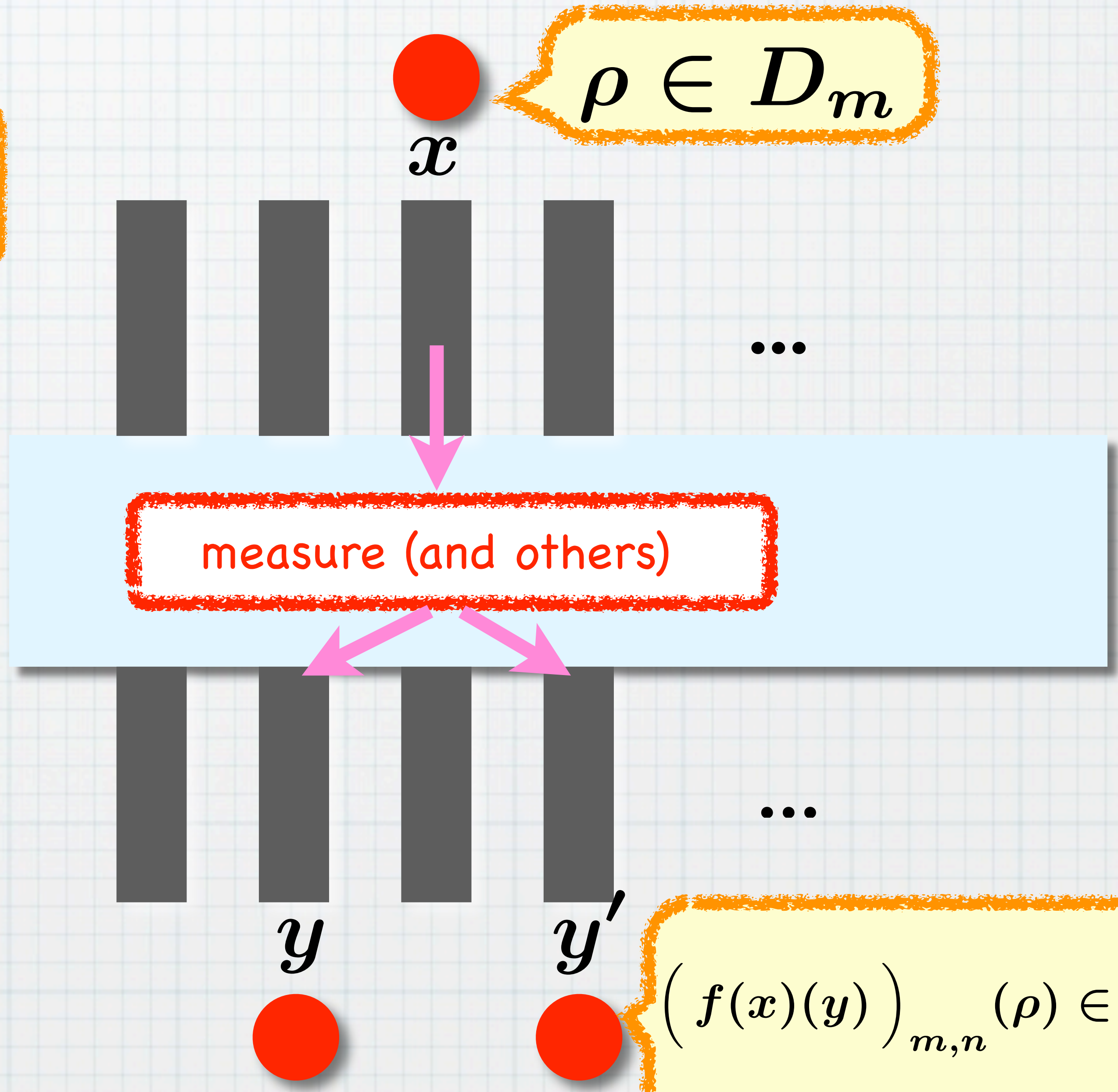
$$\sum_{y\in Y}\sum_{n\in\mathbb{N}} \mathbf{tr}\big[\big(c(y)\big)_{m,n}(\rho)\big] \le 1 \;,\quad \forall m\in\mathbb{N},\ \forall \rho\in D_m .$$

$$\frac{X \xrightarrow{f} Y \ \text{ in } \mathcal{K}\ell(\mathcal{Q})}{X \xrightarrow{f} \mathcal{Q}Y \ \text{ in } \mathbf{Sets}}$$

entrance exit in dim. out dim.

- Given $x\in X$, $y\in Y$, $m\in\mathbb{N}$, $n\in\mathbb{N}$ determines a quantum operation $\big(f(x)(y)\big)_{m,n}$
- **trace cond.:**

$$\sum_{y,n} \Pr\left(\begin{array}{c}\text{Token led}\\ \text{to } y\\ \text{with dim. } n\end{array}\right) \le 1$$



$\rho\in D_m$

$\big(f(x)(y)\big)_{m,n}(\rho) \in D_n$

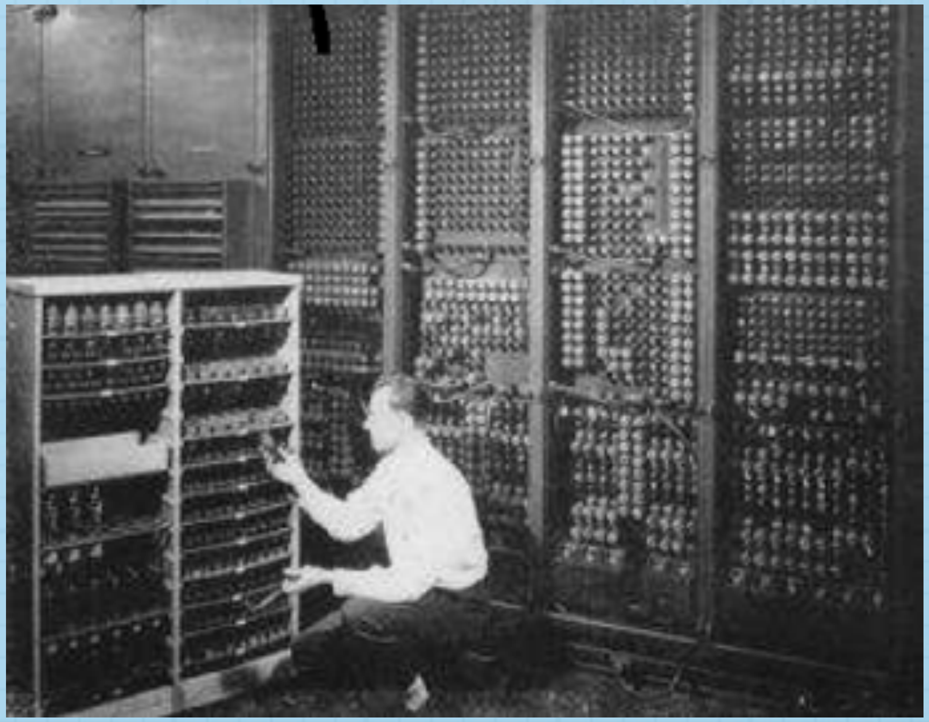
for each n

“Quantum Data, Classical Control”

Illustration by N. Hoshino

Quantum data

Classical control



“Quantum Data, Classical Control”

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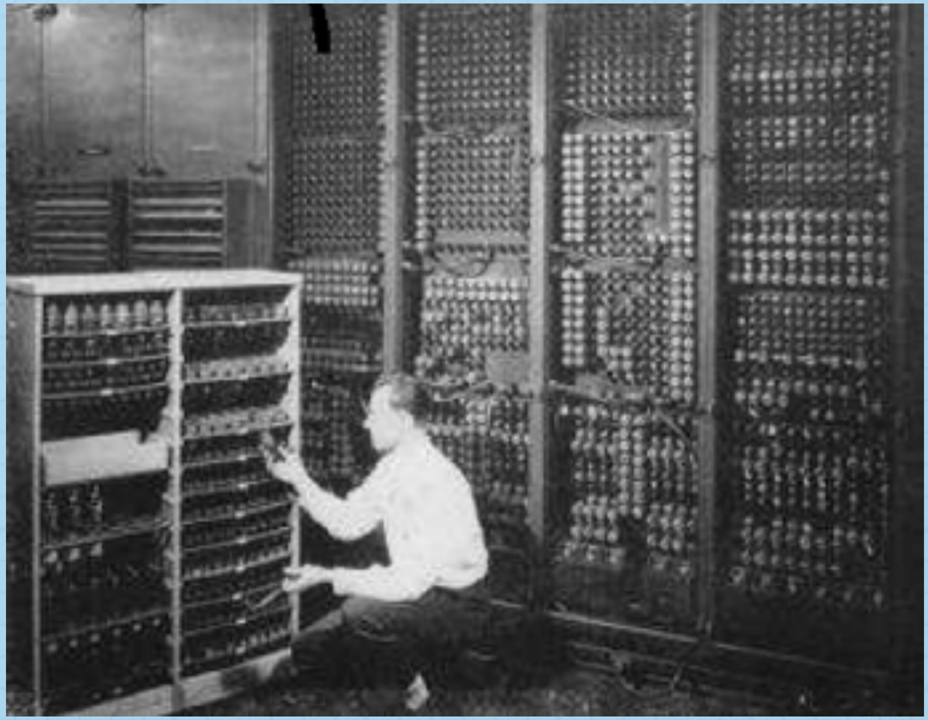
Quantum data

$$\frac{1}{\sqrt{2}}$$

Classical control



“Quantum Data, Classical Control”

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Quantum data

$$\frac{1}{\sqrt{2}}$$

$$+\frac{1}{\sqrt{2}}$$

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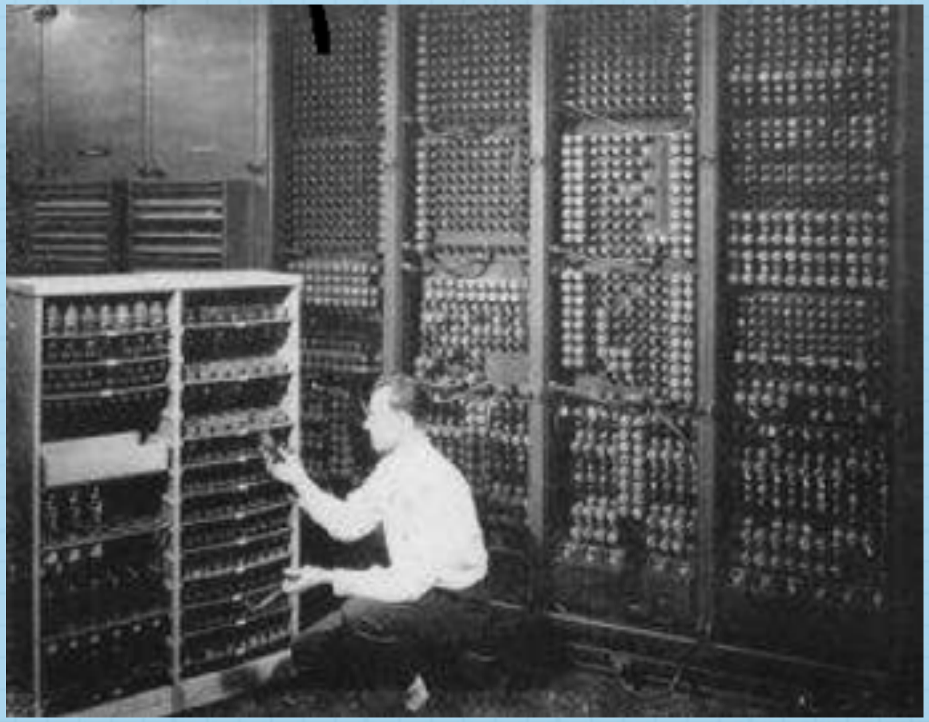
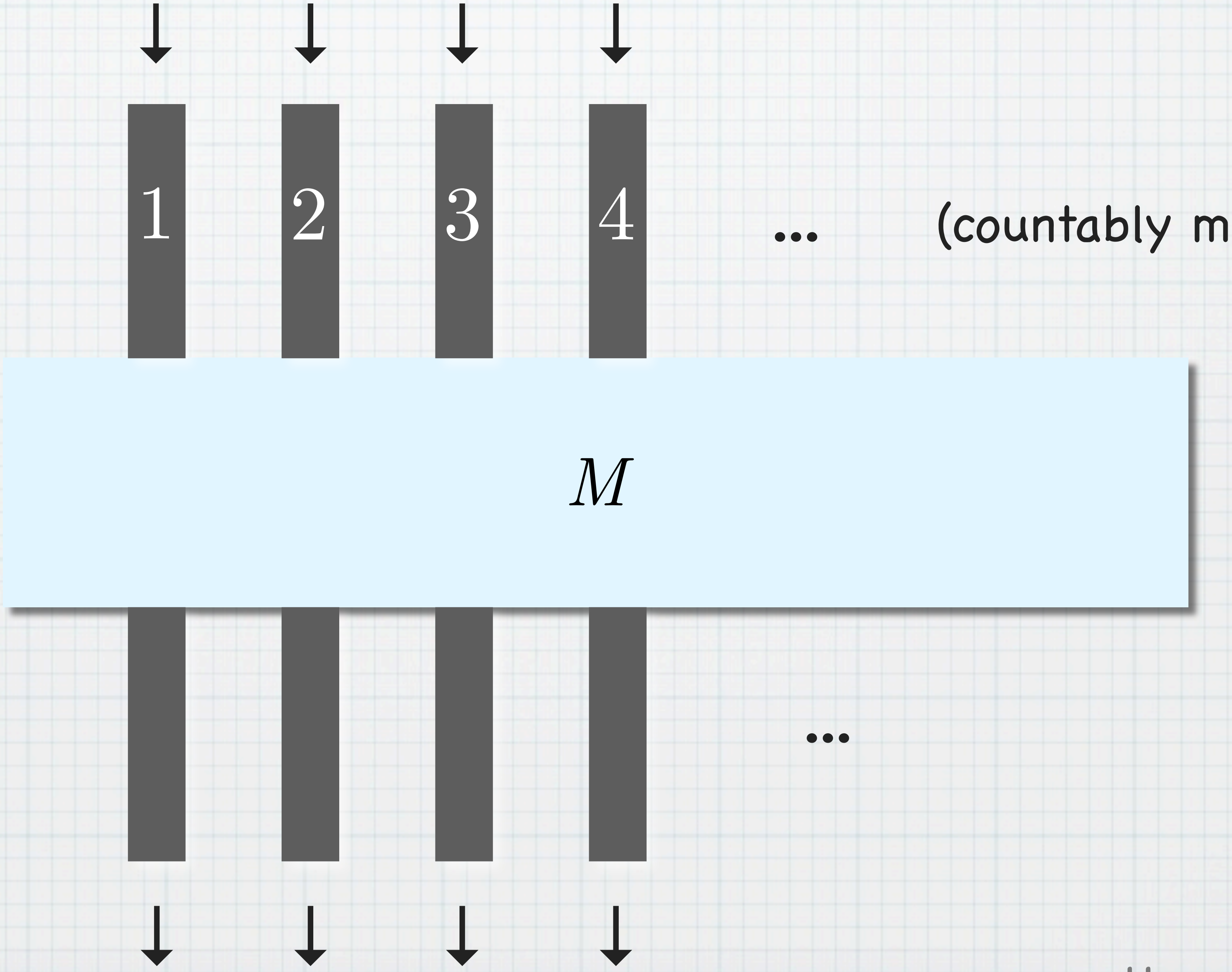


Quantum
Geometry of Interaction

$[\![M]\!] =$



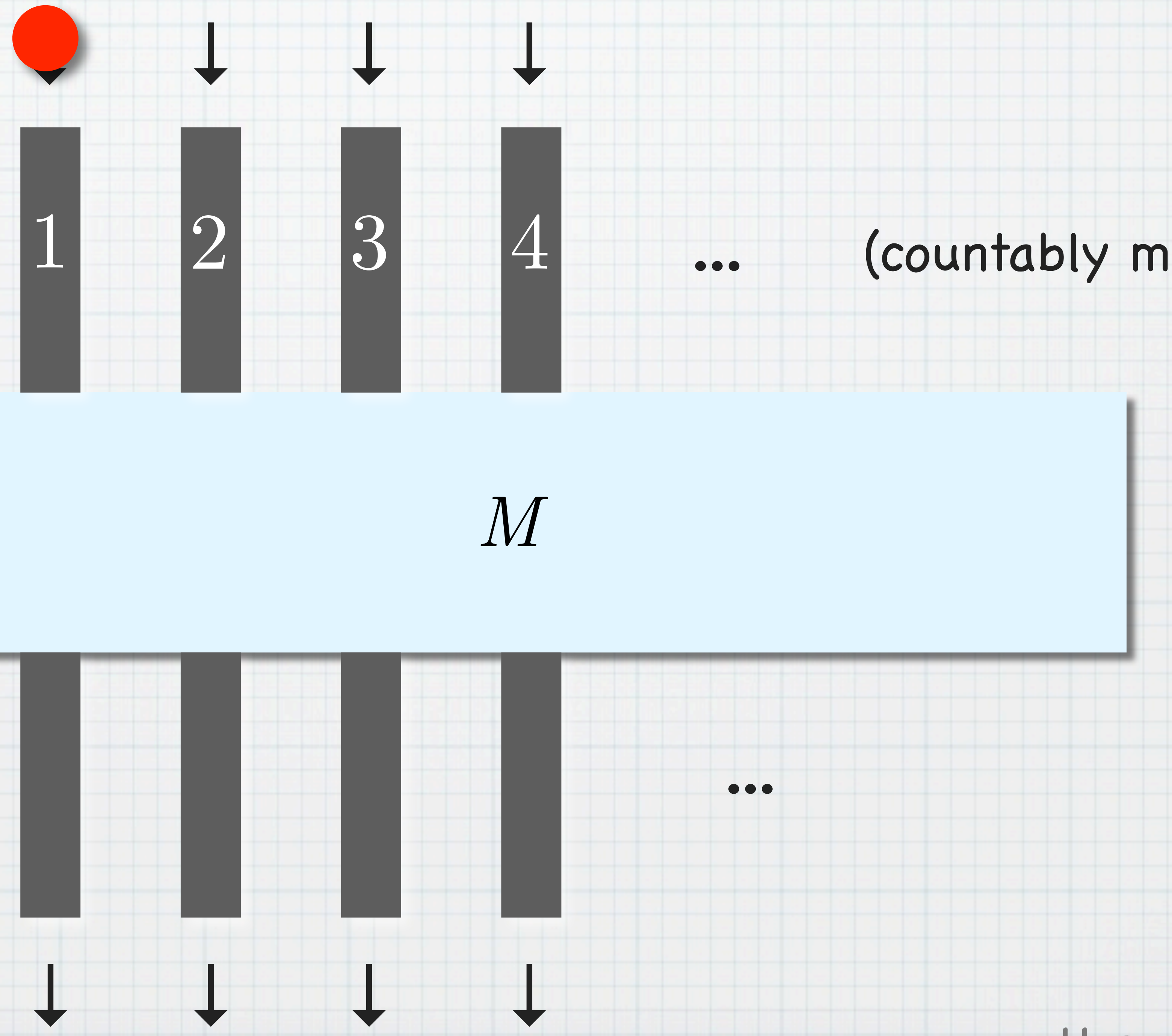
(countably many)

Hasuo (Tokyo)

Quantum
Geometry of Interaction

Not just a token/ particle, but quantum state!

$$[\![M]\!] =$$



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Quantum
Geometry of Interaction

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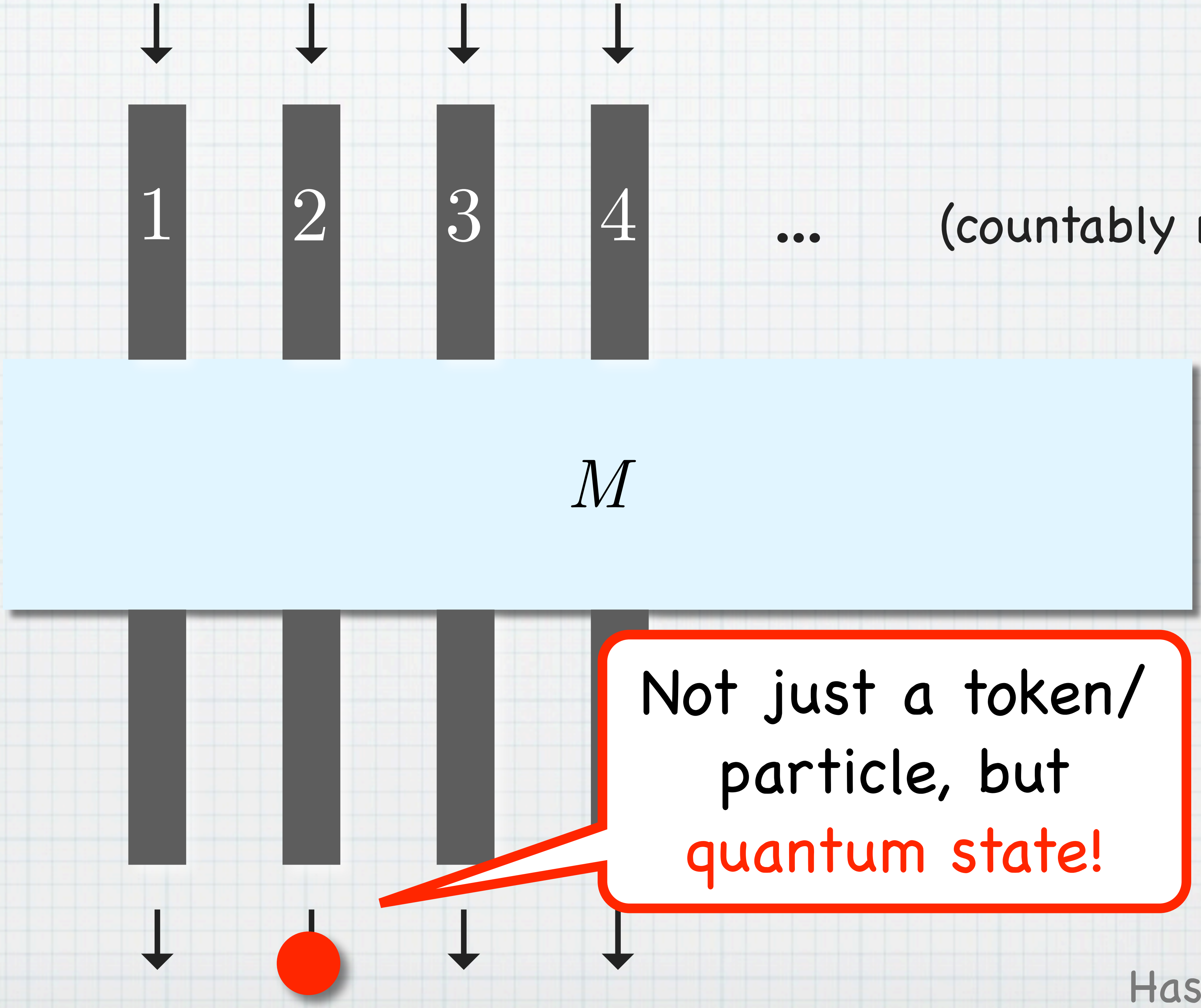

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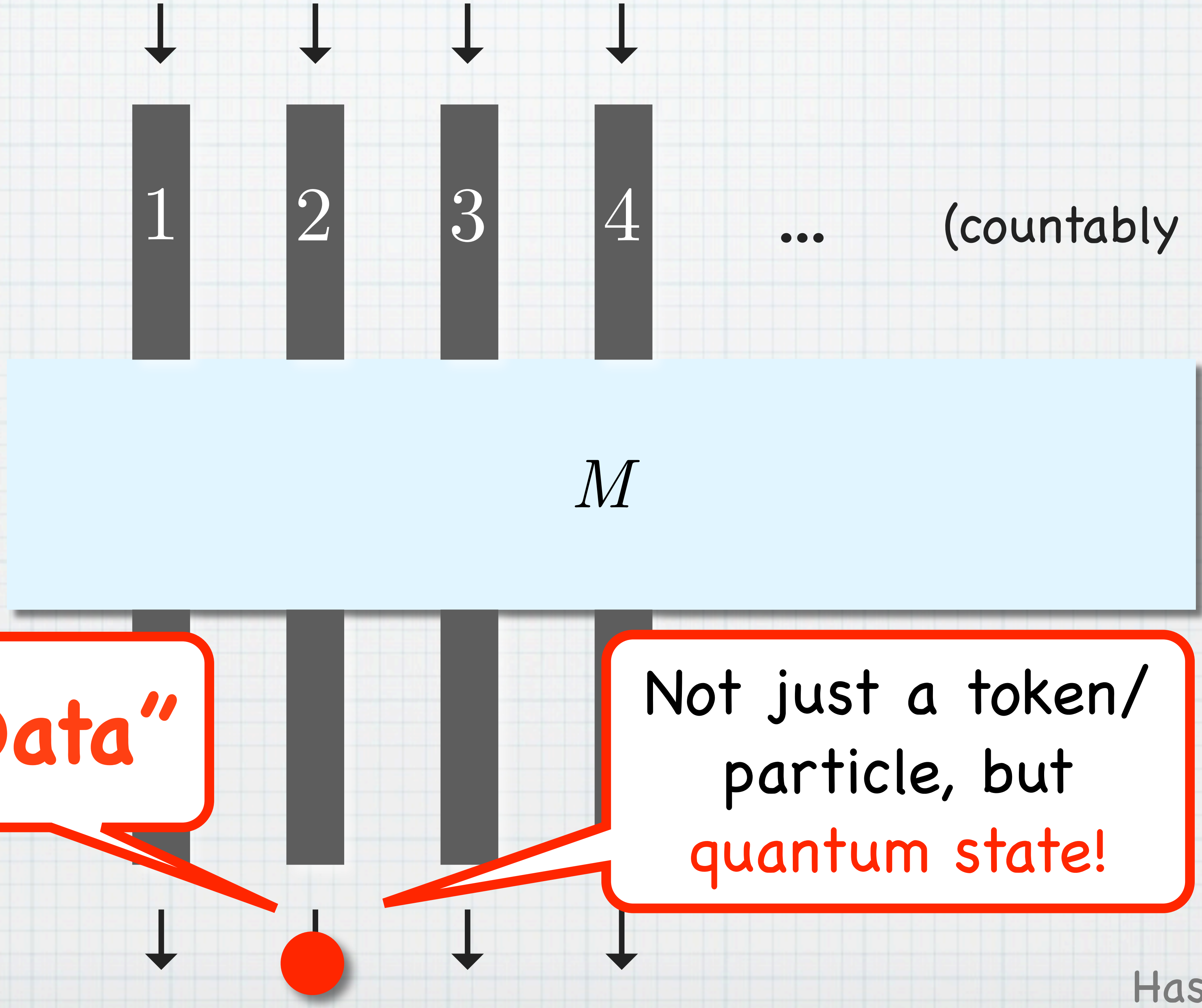

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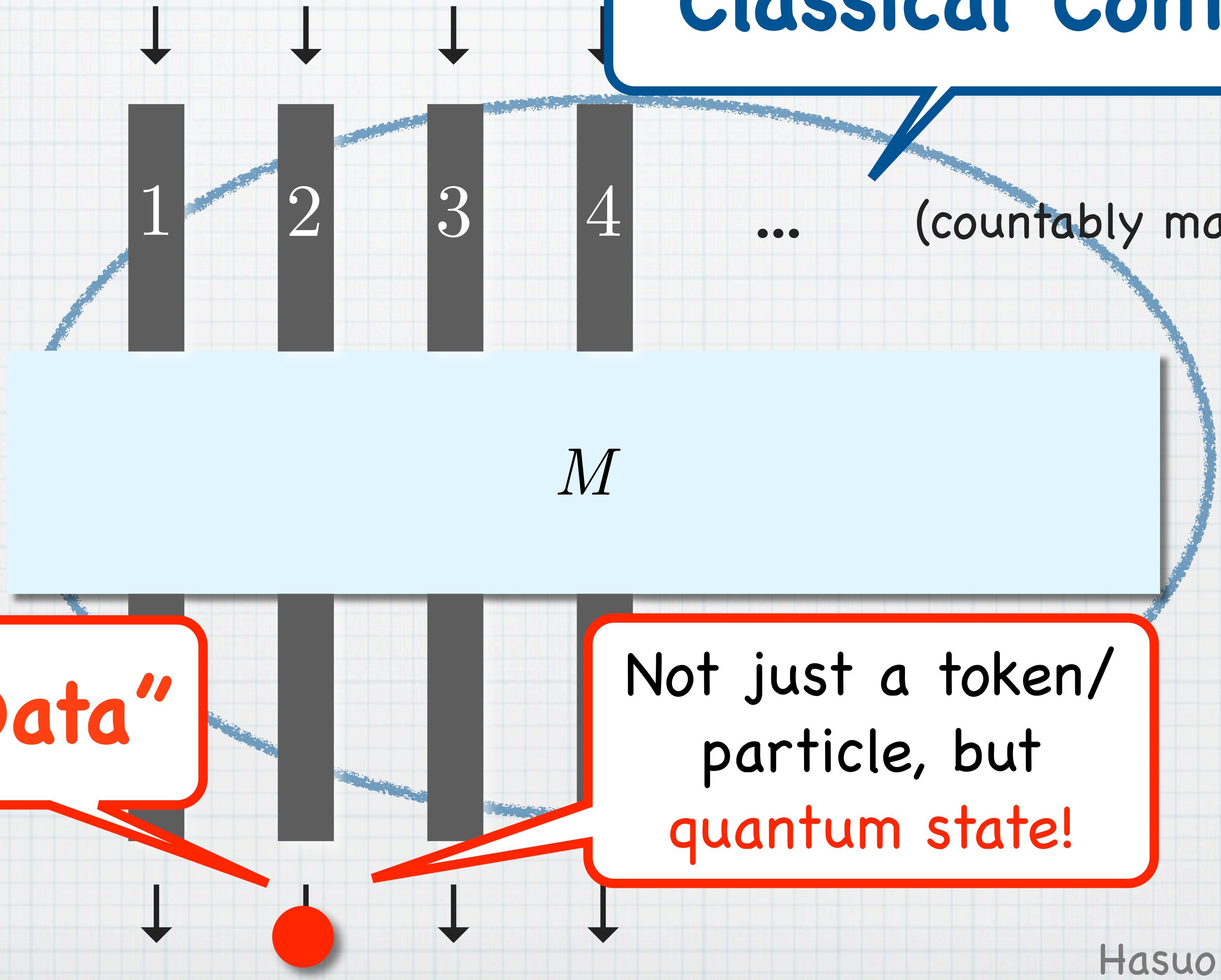


“Classical Control”

“Quantum Data”

Not just a token/ particle, but quantum state!

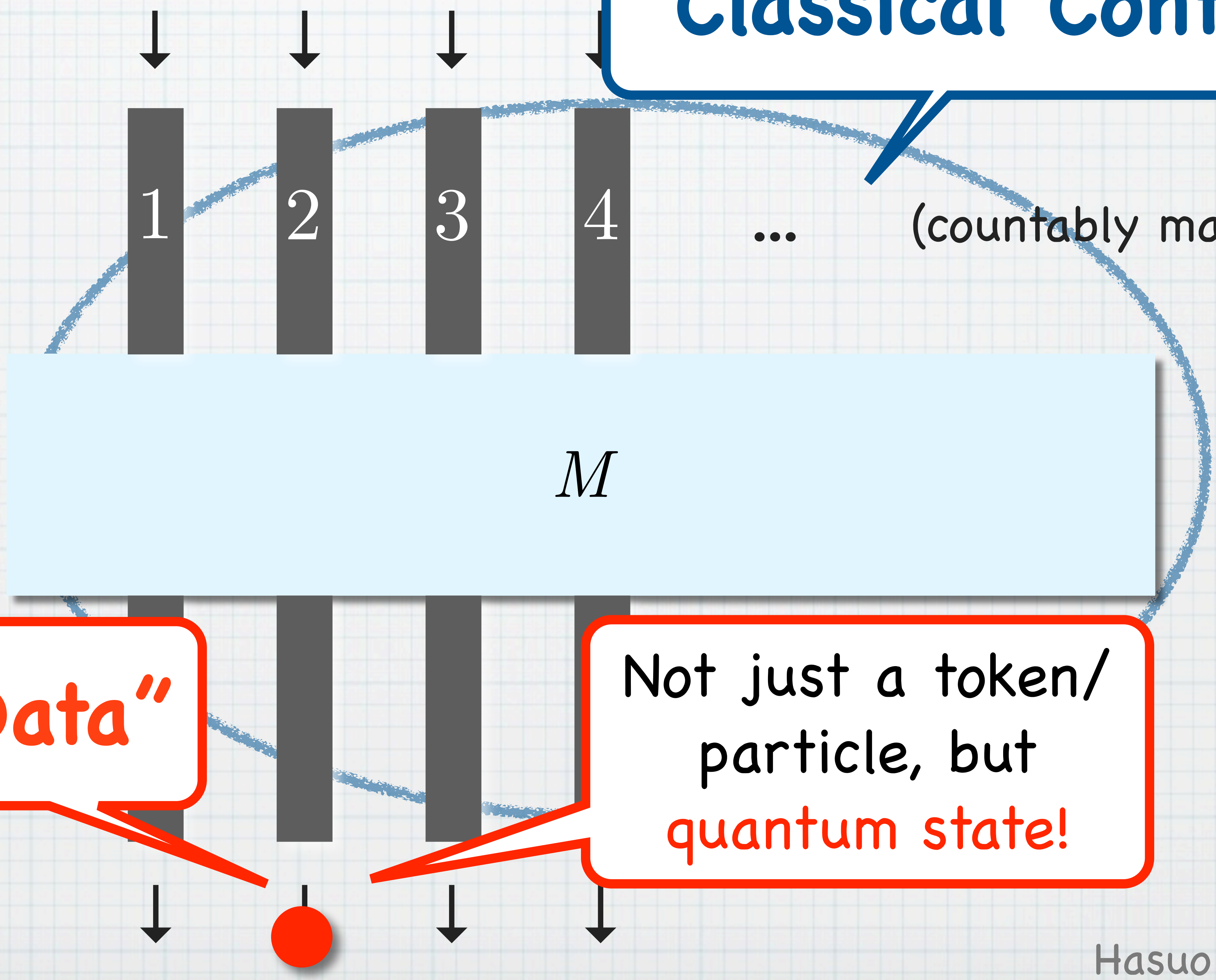
Qua

Geometry o

- "in which pipe"
- (measurement ➜ case-distinction) leads a token to different pipes

"Classical Control"

$$[\![M]\!] =$$



"Quantum Data"

Not just a token/ particle, but quantum state!

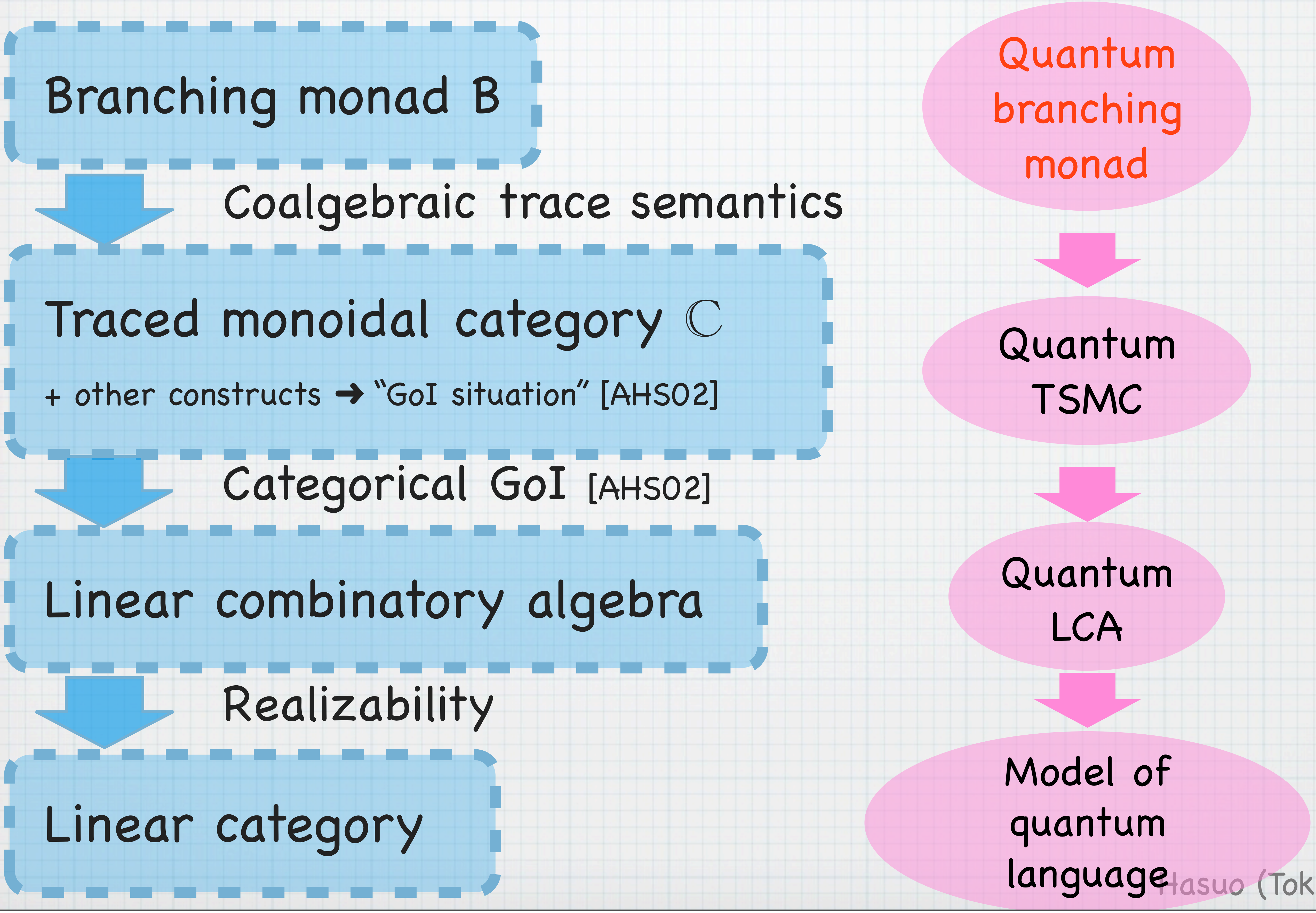
End of the Story?

- No! All the technicalities are yet to come:
  - CPS interpretation (for partial measurement)
  - Result type: a final coalgebra in $\mathbf{PER}_Q$
  - **Admissible PERs** for recursion
  - ...
- On the next occasion :-)

Results

- The monad $\mathcal{Q}$ qualifies as a "branching monad"
- The quantum GoI workflow leads to a linear category $\mathbf{PER}_{\mathcal{Q}}$
- From which we construct an adequate denotational model for a quantum λ-calculus (a variant of Selinger & Valiron's)

Conclusion: the Categorical GoI Workflow



Branching monad B

Coalgebraic trace semantics

Traced monoidal category $\mathbb{C}$

+ other constructs → "GoI situation" [AHS02]

Categorical GoI [AHS02]

Linear combinatory algebra

Realizability

Linear category

Quantum branching monad

Quantum TSMC

Quantum LCA

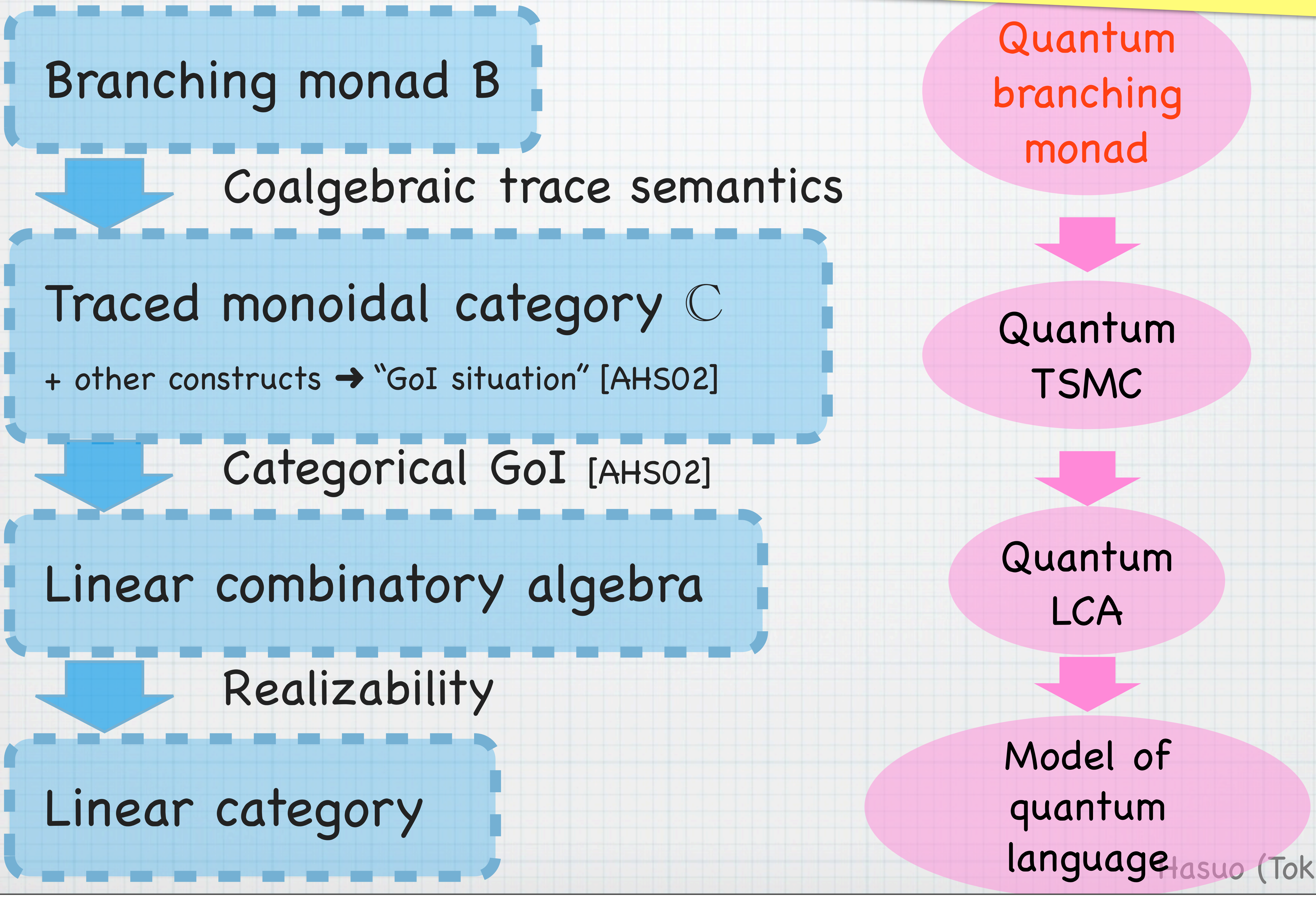
Model of quantum language

Conclusion: the Cat

Thank you for your attention!

Ichiro Hasuo (Dept. CS, U Tokyo)

http://www-mmm.is.s.u-tokyo.ac.jp/~ichiro/



Branching monad B

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Categorical GoI [AHS02]

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Model of quantum language