Codensity Lifting

A Unified Frmwk for Observations in Process Theory

A few open positions

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Relevance to IFIP WG 1.3

$$\begin{tikzcd} FX \\ X \arrow[u] \end{tikzcd}$$

- State-based systems as coalgebras
- Bisimilarity-like notions via fibrational coinduction (incl. bisim. metric)

$$\begin{tikzcd} \mathbb{E} \arrow[loop right, "\overline{F}"] \arrow[d] \\ \mathbb{C} \arrow[loop right, "F"] \end{tikzcd}$$

- Specification in modal logics,
  - firstly by dual adjunctions
  - secondly by fibration

$$\begin{tikzcd} \mathbb{C} \arrow[loop left, "F"] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbb{D}^{op} \arrow[l, shift left] \arrow[loop below, "G"] \end{tikzcd} \quad L^{op}$$

Outline

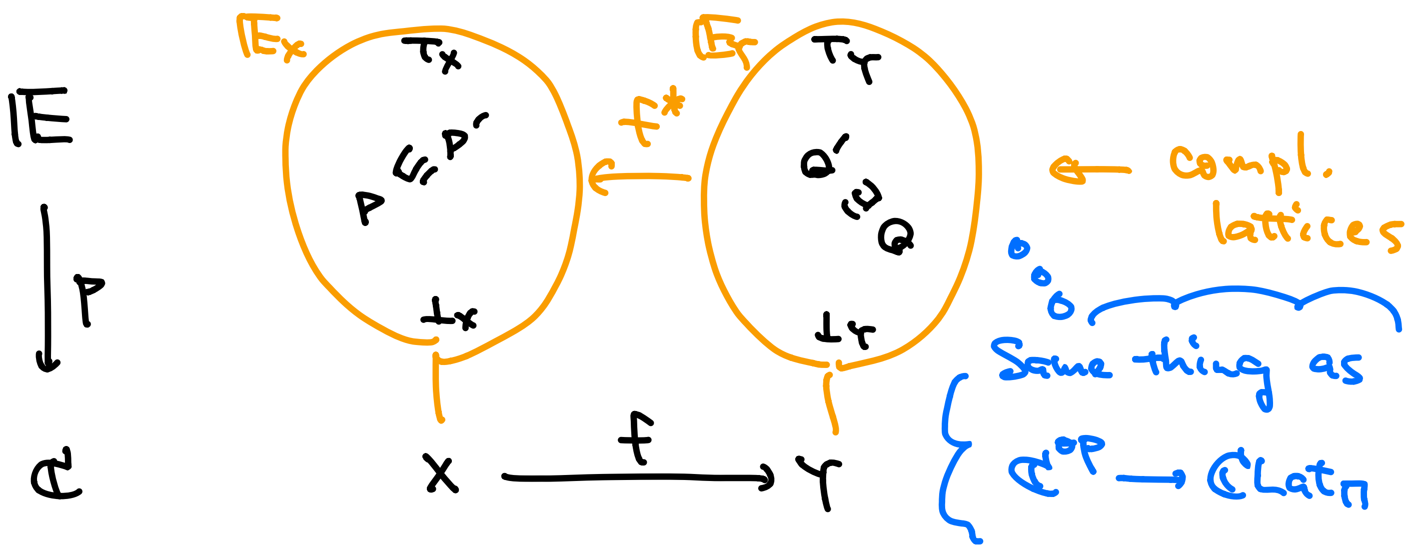
- Fibration
  esp. $\mathbf{CLat}_{\sqcap}$-fibr. for observability structures
- Fibrational Coinduction
  for bisim.-like notions
- Codensity Lifting
  "Canonical functor lifting based on observations"
- Usage 1: Codensity Bisimilarity Games
  Komorida+, lics19
- Usage 2: Expressivity of Modal Logics
  Komorida+, preprint

References

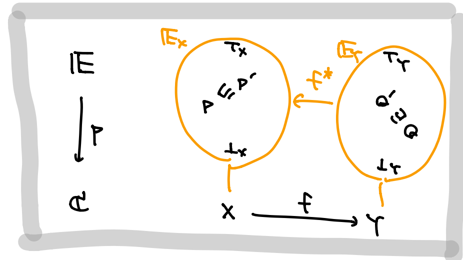
- [Katsumata, Sato, CALCO'15]
- [Sprunger, Katsumata, Dubut, Hasuo, CMCS'18]
- [Komorida, Katsumata, Hu, Klin, Hasuo, LICS'19]
- [Komorida, Katsumata, Kupke, Rot, Jacobs, Hasuo, preprint]

Fibration

- Ref.: Categorical Logic & Type Theory, Bart Jacobs, 1999, Elsevier
- We focus on $\mathbf{CLat}_\sqcap$-fibrations:



Fibration



<u>**Def.**</u> A functor $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{C}\end{array}$ is a **fibration** if

$$
\begin{tikzcd}
\mathbb{E} \arrow[d, "p"] & & Q & & & f^*Q \arrow[r, "\tilde{f}_Q"] & Q \\
\mathbb{C} & X \arrow[r, "f"] & Y & \Longrightarrow & & X \arrow[r, "f"] & Y
\end{tikzcd}
$$

w/ universality ...

<u>Cartesian lifting</u>

$$
\begin{tikzcd}
\mathbb{E} \arrow[d, "p"] & & Q & & \forall R \arrow[dr, dashed] \arrow[rr, bend left, "\forall u"] & & Q \\
& & & \Longrightarrow & & f^*Q \arrow[ur, "\tilde{f}_Q"] & \\
\mathbb{C} & X \arrow[r, "f"] & Y & & pR \arrow[dr, "\exists g"'] \arrow[rr, "pu"] & & Y \\
& & & & & X \arrow[ur, "f"'] \arrow[u, phantom, "\parallel"] &
\end{tikzcd}
$$

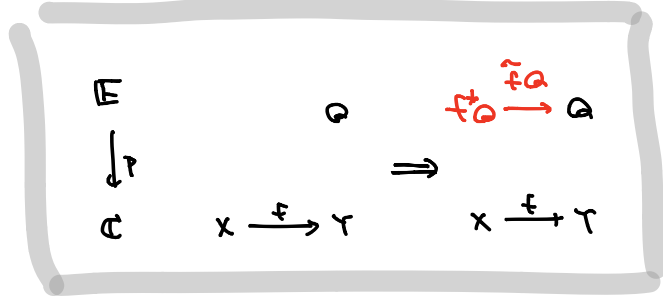
Prop. Cartesian liftings induce functors

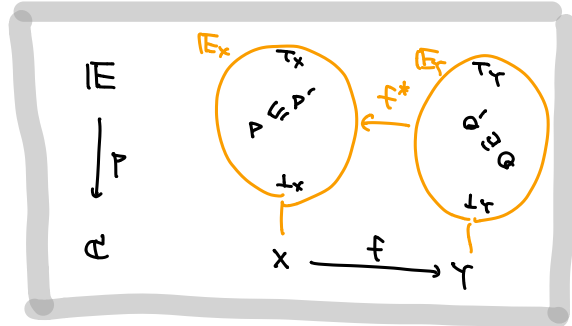
$$f^*: \mathbb{E}_Y \longrightarrow \mathbb{E}_X$$

The fiber over X

obj. P s.t. $pP = X$

arr. u s.t. $pu = \mathrm{id}_X$





Def. $\begin{matrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{matrix}$ is a $\mathbf{CLat}_\sqcap$-fib. if

- each fiber $\mathbb{E}_X$ is a compl. lattice
- each reindexing f^* preserve $\sqcap$ (meet)

Cartn-Fib. Examples

Pred
$\downarrow$
Sets

obj. $(X, P \subseteq X)$

arr. pred.-preserving maps

$f : (X, P) \to (Y, Q)$

$\iff \forall x \in P.\ f(x) \in Q$

$\iff P \subseteq f^{-1}(Q)$

ERel
$\downarrow$
Sets

obj. $(X, R \subseteq X \times X)$

arr. endorelation-preserving maps

$$\begin{pmatrix} X, \\ f^{-1}(Q) \subseteq X \end{pmatrix} \longrightarrow \begin{pmatrix} Y, \\ Q \subseteq Y \end{pmatrix}$$

$$\begin{pmatrix} X, \\ (f \times f)^{-1}(S) \\ \subseteq X \times X \end{pmatrix} \longrightarrow \begin{pmatrix} Y, \\ S \subseteq Y \times Y \end{pmatrix}$$

$$X \xrightarrow{f} Y$$

$$X \xrightarrow{f} Y$$

$\mathbb{C}\mathbf{Lat}_\sqcap$-Fib. for Observability Structures

$$\begin{array}{c} \mathbb{E} \\ \downarrow{\scriptstyle p} \\ \mathbb{C} \end{array}$$

$\mathbb{E}_X$

$\top_X$

P'

$\sqcup\!\!|$ (i.e. $P \sqsubseteq P'$)

P

$\bot_X$

X

coarse ↑ ↓ fine

observability str. over X

P is more fine-grained, discriminating.

$$\begin{array}{c} \mathbf{ERel} \\ \downarrow \\ \mathbf{Sets} \end{array}$$

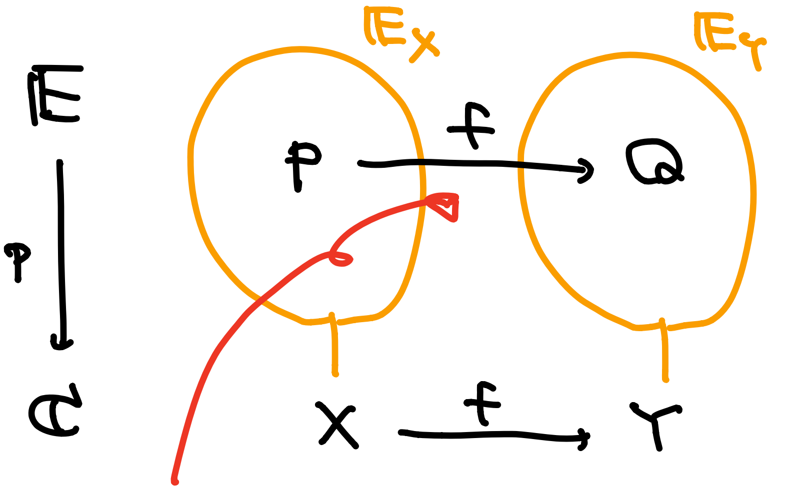
obj. $(X,\ R \subseteq X \times X)$

arr. endorelation-preserving maps

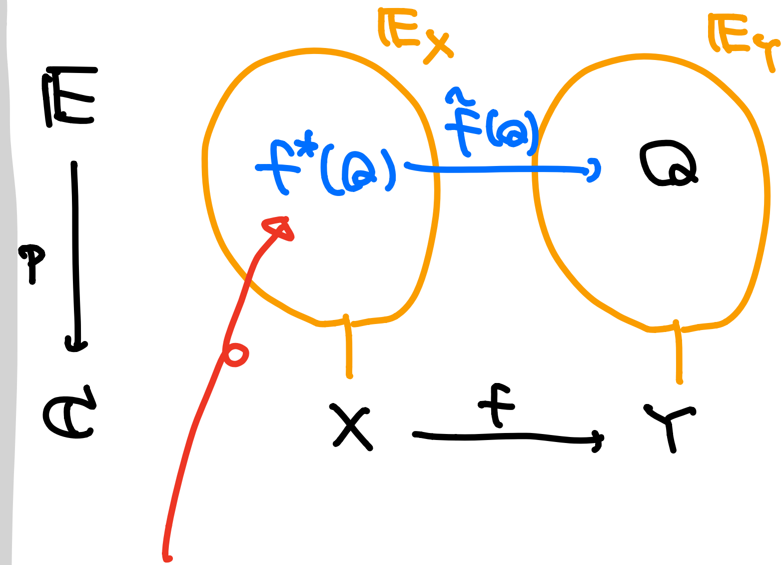
$$\left(X,\ (f\times f)^{-1}(S) \subseteq X \times X \right) \longrightarrow \left(Y,\ S \subseteq Y \times Y \right)$$

$$X \xrightarrow{f} Y$$

$\mathbb{C}$Latn-Fib. for Observability Structures



- f respects observability:
  x, x' are indistinguishable
  $\Longrightarrow$ so are $f(x), f(x')$



- The observability structure induced by $(Y, Q), f$

$\mathbf{CLat}_\sqcap$-Fib. Examples

$\mathbf{PMet}_1$
$\downarrow$
$\mathbf{Sets}$

<u>obj.</u> (X, d)
$\uparrow$ 1-b'dd pseudometric

<u>arr.</u> Nonexpansive maps

$f : (X, d) \to (Y, e)$

$\Leftrightarrow e(fx, fx') \leq d(x, x')$

- $(\mathbf{PMet}_1)_X$ collects all 1-b'dd pseudometrics over X

- $$\begin{array}{ccc} (X, d) & & d(x, x') \\ \sqsubseteq & \Longleftrightarrow & \geq \\ (X, d') & & d'(x, x') \end{array}$$

coarse

fine, discriminating

$\mathbf{Top}$
$\downarrow$
$\mathbf{Sets}$

<u>obj.</u> topol. spaces

<u>arr.</u> conti. maps

- $\mathbf{Top}_X$ collects topologies:

$\top_X = \{\emptyset, X\}$ (attached)

$\mathbf{Top}_X$

$$\begin{array}{c} \mathcal{O}' \\ \sqsubseteq \\ \mathcal{O} \end{array} \quad \text{if } \mathcal{O} \supseteq \mathcal{O}'$$

$\bot_X = \mathcal{P}X$ (discr.)

X

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Fibrational Coinduction

[Hermida, Jacobs, I&C'98]
[Bonchi, Petrisan, Pous, Rot, Acta Inf.'17]
[Hasuo, Kataoka, Cho, MSCS'18]
[Komorida, Katsumata, Hu, Klin, Hasuo, LICS'19]
[Kupke, Rot, CSL'20]

- "The observability defined by coalgebraic behaviors"

- ... with respect to how we observe,

  specified by

$$\begin{array}{c} \overline{F} \circlearrowright \mathbb{E} \\ \big\downarrow p \\ F \circlearrowright \mathbb{C} \end{array}$$

w/ $p\overline{F} = Fp$

- Ex.
  - (Coalg.) bisimilarity
  - bisim. metric [Baldan, Bonchi, Kerstan, Koenig, LMCS'18]
  - "bisim. topology"
  - ...

Fibrational Coinduction

- Predicate Transformer $c^*\overline{F}$

$$
\begin{tikzcd}
\mathbb{E} \arrow[loop left, "\overline{F}"] \arrow[d, "p"'] & c^*(\overline{F}P) \arrow[r] & \overline{F}P \\
 & P & \\
\mathbb{C} \arrow[loop left, "F"] & X \arrow[r, "c"'] & FX
\end{tikzcd}
$$

③ $c^*(\overline{F}P)$ ② $\overline{F}P$ ① P

... that is,

$$
\begin{tikzcd}
\mathbb{E}_X \arrow[r, "\overline{F}_X"] \arrow[rr, bend right, "c^*\overline{F}"'] & \mathbb{E}_{FX} \arrow[r, "c^*"] & \mathbb{E}_X
\end{tikzcd}
$$

- $(c^*\overline{F})P$: "observability induced by P one step ahead"

Fibrational Coinduction

$$\begin{tikzcd} \overline{F} \circlearrowright \mathbb{E} \arrow[d, "p"] \\ F \circlearrowright \mathbb{C} \end{tikzcd} \qquad X \xrightarrow{c} FX$$

- ... which suggests

$P \sqsubseteq (c^* \overline{F})\, P$ ... "P is a bisimulation"

$\nu(c^* \overline{F}) \in \mathbb{E}_X$ ... "bisimilarity"

- Indeed,

$$\begin{tikzcd} \overline{F} \circlearrowright \mathbf{ERel} \arrow[d] \\ F \circlearrowright \mathbf{Sets} \end{tikzcd}$$

for many coalg. bisimilarity,

suitable

$$\begin{tikzcd} \overline{\mathcal{D}} \circlearrowright \mathbf{PMet}_1 \arrow[d] \\ \mathcal{D} \circlearrowright \mathbf{Sets} \end{tikzcd}$$

for bisimulation metric,

...

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Codensity Lifting

- History

[Katsumata & Sato, CALCO'15]:
introduction
[Sprunger, Katsumata, Dubut, Hasuo, CMCS'18]:
elaboration
[Komorida, Katsumata, Hu, Klin, Hasuo, LICS'19]:
intuitions and combination with games

- Related topics

* $\top\top$-lifting [Abadi '00] ← fib. models for λ-calculi
  top top
* Kantorovich lifting [Baldan, Bonchi, Kerstan, König, LMCS'18]
  ⇒ Special case for $\mathbf{PMet}_1 \downarrow \mathbf{Sets}$

Codensity Lifting

Given a $\mathbf{CLat}_\sqcap$-fib. $\begin{array}{c} \mathbb{E} \\ \downarrow p \\ F \circlearrowright \mathbb{C} \end{array}$ with $\begin{array}{c} \Omega \\ \\ F\Omega \xrightarrow{\tau} \Omega \end{array}$,

define $\mathbb{E} \xrightarrow{F^{\Omega,\tau}} \mathbb{E}$ by

$$F^{\Omega,\tau}(P) = \bigsqcap_{\substack{f : P \to \Omega \\ \text{in } \mathbb{E}}} (\tau \circ F(pf))^*(\Omega)$$

Codensity Lifting

Given a $\mathbf{CLat}_\sqcap$-fib. $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ F \circlearrowright \mathbb{C}\end{array}$ with $\begin{array}{c}\Omega\\ \\ F\Omega \xrightarrow{\tau} \Omega\end{array}$,

observability str. over Ω

modality

observ. domain

define $\mathbb{E} \xrightarrow{F^{\Omega,\tau}} \mathbb{E}$ by

$$F^{\Omega,\tau}(P) = \bigsqcap_{\substack{f : P \to \Omega\\ \text{in } \mathbb{E}}} (\tau \circ F(pf))^*(\Omega)$$

the coarsest obs. str. that makes all the observations (via f) decent

$$\begin{array}{ccc} \mathbb{E} & & \dashrightarrow \Omega \\ \downarrow p & & \\ \mathbb{C} & & F(pP) \xrightarrow{F(pf)} F\Omega \xrightarrow{\tau} \Omega \end{array}$$

Codensity Lifting

- Example: "language topology"
  + Setting: acceptance is semi-observable
    (Acc: observable, Rej.: unobservable)
  * Param:

Sierpinski

$$\begin{pmatrix} \Omega \\ F\Omega \to \Omega \end{pmatrix} = \begin{pmatrix} (2, \{\emptyset, \{\top\}, 2\}) \\ 2 \times 2^A \xrightarrow[\text{acc?},\ \langle a \rangle\ (a \in A)]{} 2 \end{pmatrix}$$

  *

$$\begin{array}{l} \mathbf{Top} \circlearrowleft F^{\Omega,\tau} \\ \downarrow p \\ \mathbf{Sets} \circlearrowleft F = 2 \times (-)^A \end{array}$$

$U(c^* F^{\Omega,\tau})$ is gen. by

$$\left\{ \left\{ x \;\middle|\; \begin{array}{l} w \text{ is accepted} \\ \text{from } x \end{array} \right\} \;\middle|\; w \in \Sigma^* \right\}$$

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Bisimilarity Games

- Conventionally:

pos	player	possible moves
(x, y)	Spoiler	(z, w, a, z') where $\{z, w\} = \{x, y\}$ and $z \xrightarrow{a} z'$
(z, w, a, z')	Duplicator	(z', w') where $w \xrightarrow{a} w'$

claiming $x \sim y$

Bisimilarity Games

- More recently:
  for probabil. bisimilarity in [Clerc, Fijalkow, Klin, Panangaden, I&C'19]

  [$x \sim y \Rightarrow Pr(x \to Z) = Pr(y \to Z)$ for all $\sim$-closed Z [Larsen, Skou, '91]]

TABLE I
THE GAME FOR PROBABILISTIC BISIMILARITY FROM [13]

position	player	possible moves
$(x, y) \in X^2$	S	$Z \subseteq X$ s.t. $c(x)(Z) \neq c(y)(Z)$
$Z \subseteq X$	D	$(x', y') \in X^2$ s.t. $x' \in Z \wedge y' \notin Z$

Codensity Bisimilarity Games

Def. In

$$\begin{tikzcd} \mathbb{E} \arrow[d] \\ F \circlearrowright \mathbb{C} \end{tikzcd} \qquad \Omega \qquad F\Omega \xrightarrow{\tau} \Omega$$

where

$$\left(\begin{tikzcd} F^{\Omega,\tau} P \arrow[rr, dashed] & & \Omega \\ FX \arrow[r, "F(pf)"] & F\Omega \arrow[r, "\tau"] & \Omega \end{tikzcd}\right)$$

with $f : P \to \Omega$

position	pl.	possible moves
$P \in \mathbb{E}_X$	S	$k \in \mathbb{C}(X, \Omega)$ s.t. $\tau \circ Fk \circ c : (X, P) \dot{\nrightarrow} (\Omega, \boldsymbol{\Omega})$
$k \in \mathbb{C}(X, \Omega)$	D	$P' \in \mathbb{E}_X$ s.t. $k : (X, P') \dot{\nrightarrow} (\Omega, \boldsymbol{\Omega})$

Thm. TFAE:

- $P \in \mathbb{E}_X$ is a winning position for Duplicator
- $P \sqsubseteq \nu(c^* F^{\Omega,\tau})$

Trimming Codensity Games

Def. A generating set $\mathcal{G} \subseteq \mathbb{E}_X$ is s.t.

$$\forall P \in \mathbb{E}_X.\ \exists \mathcal{A} \subseteq \mathcal{G}.\ P = \bigsqcup \mathcal{A}.$$

Def. (e.g. in [HasuoKC'18])

$$\begin{array}{c} \mathbb{E} \\ \downarrow \\ \mathbb{C} \ni S \end{array}$$

S is a fibered separator if

$\forall Y \in \mathbb{C}.\ \forall P, Q \in \mathbb{E}_Y.$

$$\left[P \neq Q \Rightarrow \begin{array}{ccc} f^* P & & P \\ \neq & & \neq \\ f^* Q & & Q \\ & S \xrightarrow{\exists f} Y & \end{array} \right]$$

Prop.

$$\left.\begin{array}{l} S : \text{fibered separator} \\ \mathcal{G} \subseteq \mathbb{E}_S,\ \text{generating set} \end{array}\right] \Rightarrow \left\{ f_* P \;\middle|\; \begin{array}{l} f : S \to X \\ P \in \mathcal{G} \end{array} \right\}$$

$S \xrightarrow{f} {}^{\forall}X$

a generating set of $\mathbb{E}_X$

Codensity Bisimilarity Games, Trimmed

Def. In
$$\begin{tikzcd} \mathbb{E} \arrow[d] & & \Omega \\ F \circlearrowright \mathbb{C} & & F\Omega \arrow[r] & \Omega \end{tikzcd}$$

where
$$\left(\begin{tikzcd} F^{\Omega,\tau} P \arrow[rr, dashed] & & \Omega \\ FX \arrow[r, "F(pf)"] & F\Omega \arrow[r, "\tau"] & \Omega \end{tikzcd}\right)$$
with $f : P \to \Omega$

position	pl.	possible moves
$P \in \mathbb{E}_X$	S	$k \in \mathbb{C}(X, \Omega)$ s.t. $\tau \circ Fk \circ c : (X, P) \dot{\not\to} (\Omega, \mathbf{\Omega})$
$k \in \mathbb{C}(X, \Omega)$	D	$P' \in \mathbb{E}_X$ s.t. $k : (X, P') \dot{\not\to} (\Omega, \mathbf{\Omega})$

P can be chosen exclusively from a generated set *G*

Thm. TFAE:

- $P \in \mathbb{E}_X$ is a winning position for Duplicator
- $P \sqsubseteq \nu(c^* F^{\Omega,\tau})$

Examples

TABLE II
THE GAME FOR (PROBABILISTIC) BISIMULATION METRIC, ADAPTING [13]

position	P	possible moves
$(x, y, \varepsilon) \in X^2 \times [0,1]$	S	$f\colon X \to [0,1]$ such that $\left\lvert E_{c(x)}[f] - E_{c(y)}[f] \right\rvert > \varepsilon$
$f\colon X \to [0,1]$	D	$(x', y', \varepsilon') \in X^2 \times [0,1]$ such that $\left\lvert f(x') - f(y') \right\rvert > \varepsilon'$

Examples

TABLE XI

CODENSITY BISIMILARITY GAME FOR CONVENTIONAL BISIMILARITY

position	pl.	possible moves
$(x, y) \in X \times X$	S	$k \in \mathbf{Set}(X, 2)$ s.t. $\exists x' \in c(x).\ k(x') = \top$ $\not\Leftrightarrow \exists y' \in c(y).\ k(y') = \top$
$k \in \mathbf{Set}(X, 2)$	D	(x'', y'') s.t. $k(x'') \neq k(y'')$

TABLE XII

CODENSITY BISIMILARITY GAME FOR DETERMINISTIC AUTOMATA AND THEIR LANGUAGE EQUIVALENCE

position	pl.	possible moves
$(x, y) \in X \times X$	S	If $\pi_1(x) \neq \pi_1(y)$ then S wins If $\pi_1(x) = \pi_1(y)$ then $a \in \Sigma$ and $k \in \mathbf{Set}(X, 2)$ s.t. $k(\pi_2(x)(a)) \neq k(\pi_2(y)(a))$
$a \in \Sigma$ and $k \in \mathbf{Set}(X, 2)$	D	$(x'', y'') \in X \times X$ s.t. $k(x'') \neq k(y'')$

Examples

TABLE XIII
CODENSITY BISIMILARITY GAME FOR DETERMINISTIC AUTOMATA AND BISIMULATION TOPOLOGY

position	pl.	possible moves
$\mathcal{O} \in \mathbf{Top}_X$	S	$a \in \{\varepsilon\} \cup \Sigma$ and $k \in \mathbf{Set}(X, 2)$ s.t. $\tau_a \circ (A_\Sigma k) \circ c : (X, \mathcal{O}) \dot{\nrightarrow} (2, \mathbf{\Omega}_a)$
$a \in \{\varepsilon\} \cup \Sigma$ and $k \in \mathbf{Set}(X, 2)$	D	$\mathcal{O}' \in \mathbf{Top}_X$ s.t. $k : (X, \mathcal{O}') \dot{\nrightarrow} (2, \mathbf{\Omega}_a)$

TABLE XIV
CODENSITY BISIMILARITY GAME FOR NONDETERMINISTIC AUTOMATA AND THEIR BISIMILARITY

position	pl.	possible moves
$(x, y) \in X \times X$	S	If $\pi_1(x) \neq \pi_1(y)$ then S wins If $\pi_1(x) = \pi_1(y)$ then $a \in \Sigma$ and $k \in \mathbf{Set}(X, 2)$ s.t. $\exists x' \in \pi_2(x)(a).\ k(x') = \top$ $\not\Leftrightarrow \exists y' \in \pi_2(y)(a).\ k(y') = \top$
$a \in \Sigma$ and $k \in \mathbf{Set}(X, 2)$	D	$(x'', y'') \in X \times X$ s.t. $k(x'') \neq k(y'')$

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Expressivity

- $x \sim y \quad \overset{\text{adequacy}}{\underset{\text{expressivity}}{\rightleftarrows}} \quad th(x) = th(y)$

- For bisim. metric,

$$d^{\text{bisim.}}(x, y) \;\overset{\text{adequacy}}{\underset{\text{expressivity}}{\gtreqless}}\; d^{\text{logic}}(x, y)$$

- Refs.

[Koenig, Mika-Michalski, CONCUR'18]
[Wild, Schröder, Pattinson, König, LICS'18 & IJCAI'19]
[Clerc, Fijalkow, Klin, Panangaden, I&C'19]

(Technical details omitted... Stay tuned for our forthcoming paper!)

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Other Recent Lines of Work

- Search-based testing by stochastic optimization and logic
  [EMSOFT'18, CAV'19]

- Martingales and automated analysis of probabilistic programs [TACAS'19]

- Monitoring by (parametric) timed automata
  [EMSOFT'18, ICECCS'18, CAV'19]

- Relational extension of differential dynamic logic
  [TACAS'20]

- Weighted-automata extraction from recurrent neural networks
  [AAAI'20]