Coalgebraic Representation Theory of Fractals

Ichiro Hasuo
RIMS, Kyoto Univ., JP
PRESTO Promotion Program, JST, JP

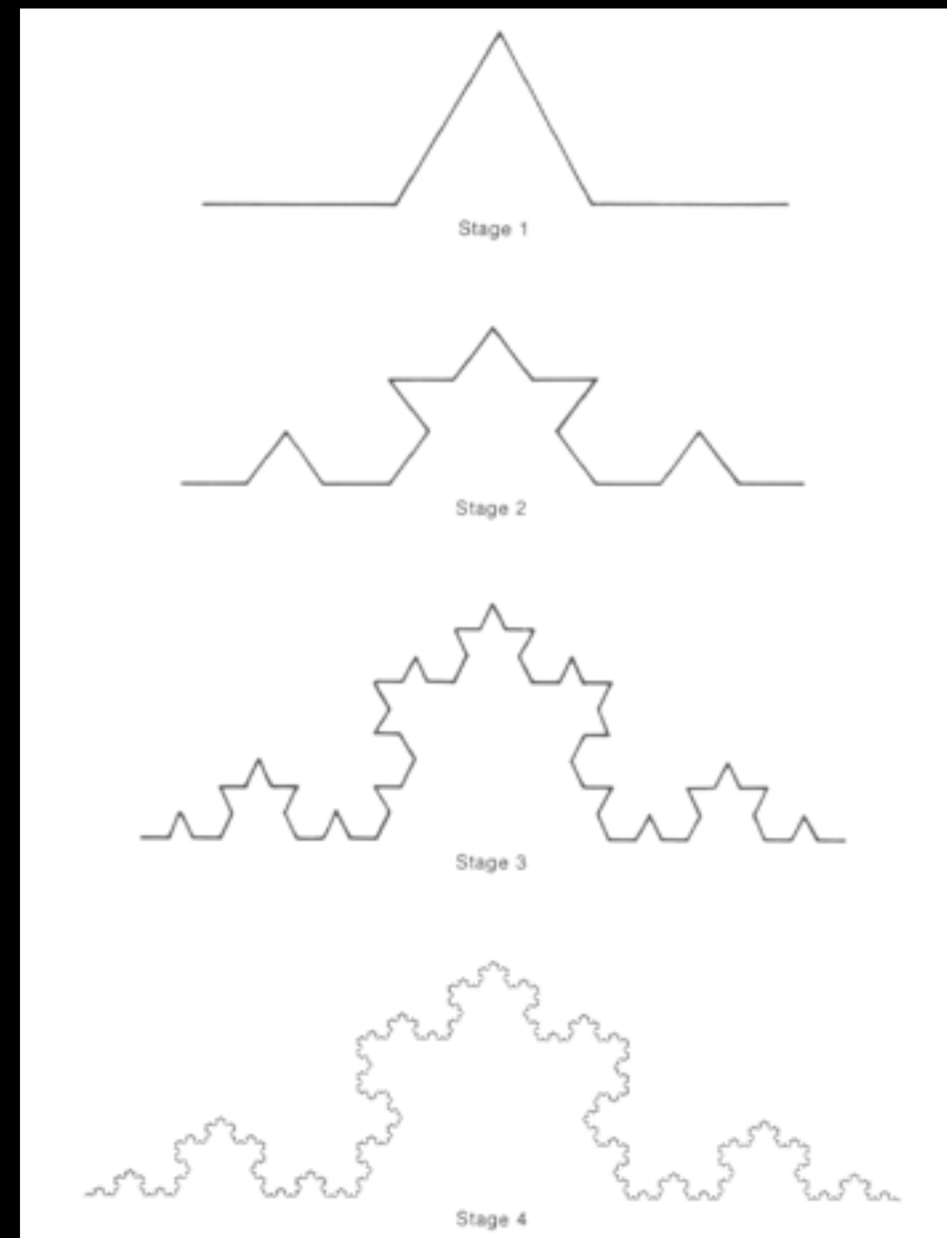
Bart Jacobs
Radboud Univ. Nijmegen, NL

Milad Niqui
CWI, NL

30 min talk, 5 min Q&A

Fractals

the Koch curve



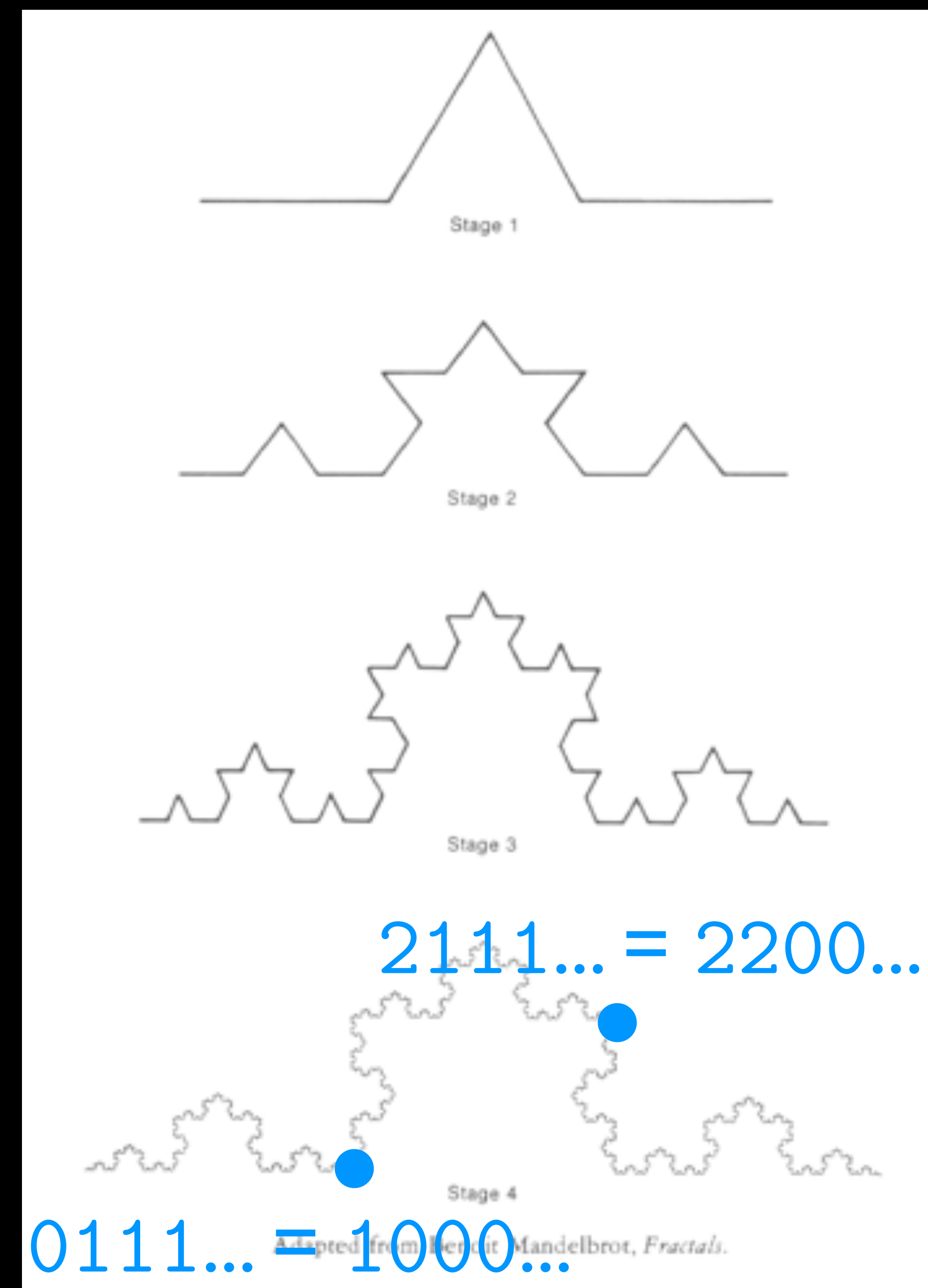
Adapted from Benoit Mandelbrot, *Fractals*.

"A rough or fragmented geometric shape that can be split into parts, each of which (at least approximately) a reduced size copy of the whole" [Mandelbrot]

Fractals

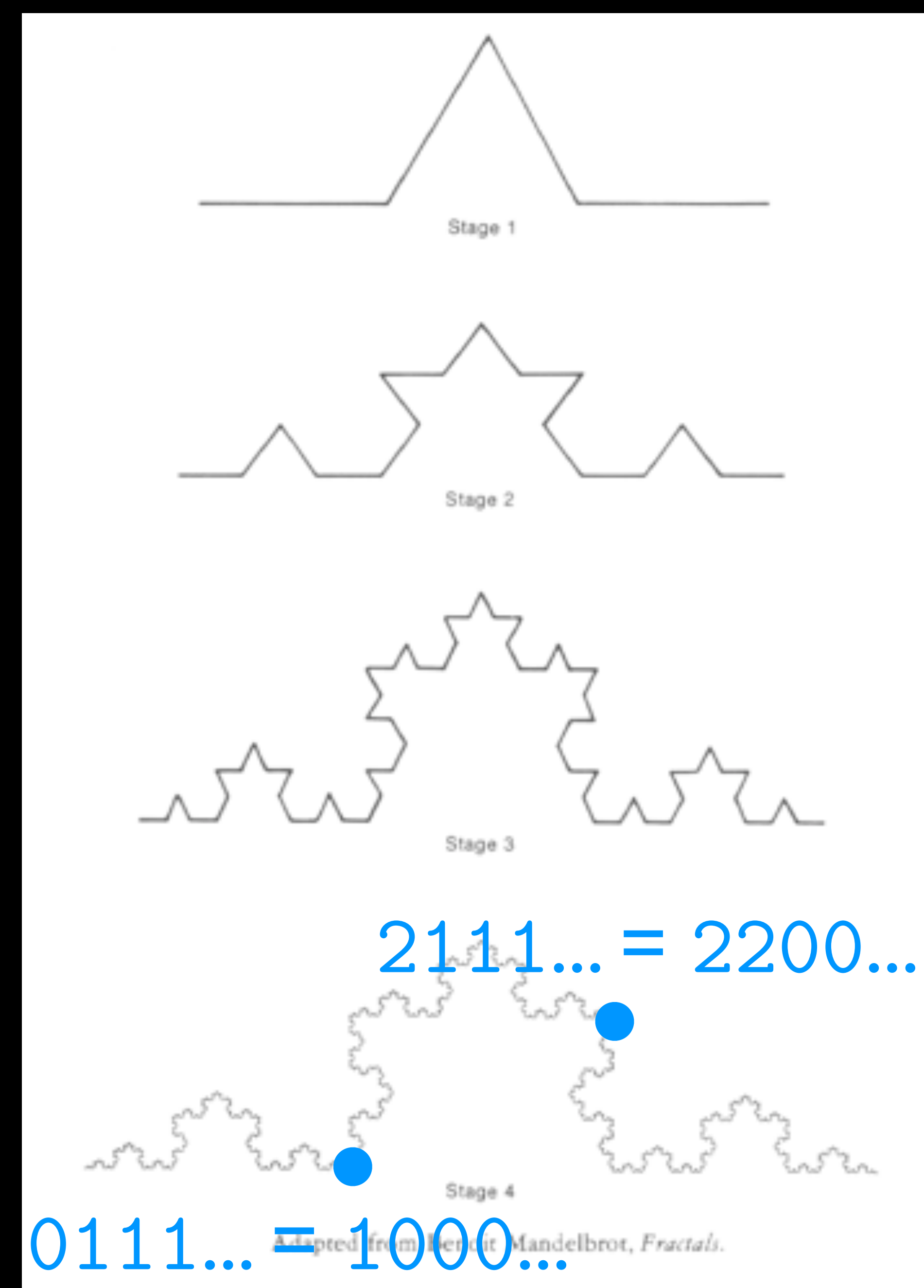
“A rough or fragmented geometric shape that can be split into parts, each of which (at least approximately) a reduced size copy of the whole” [Mandelbrot]

the Koch curve



Fractals

the Koch curve



"A rough or fragmented geometric shape that can be split into parts, each of which (at least approximately) a reduced size copy of the whole" [Mandelbrot]

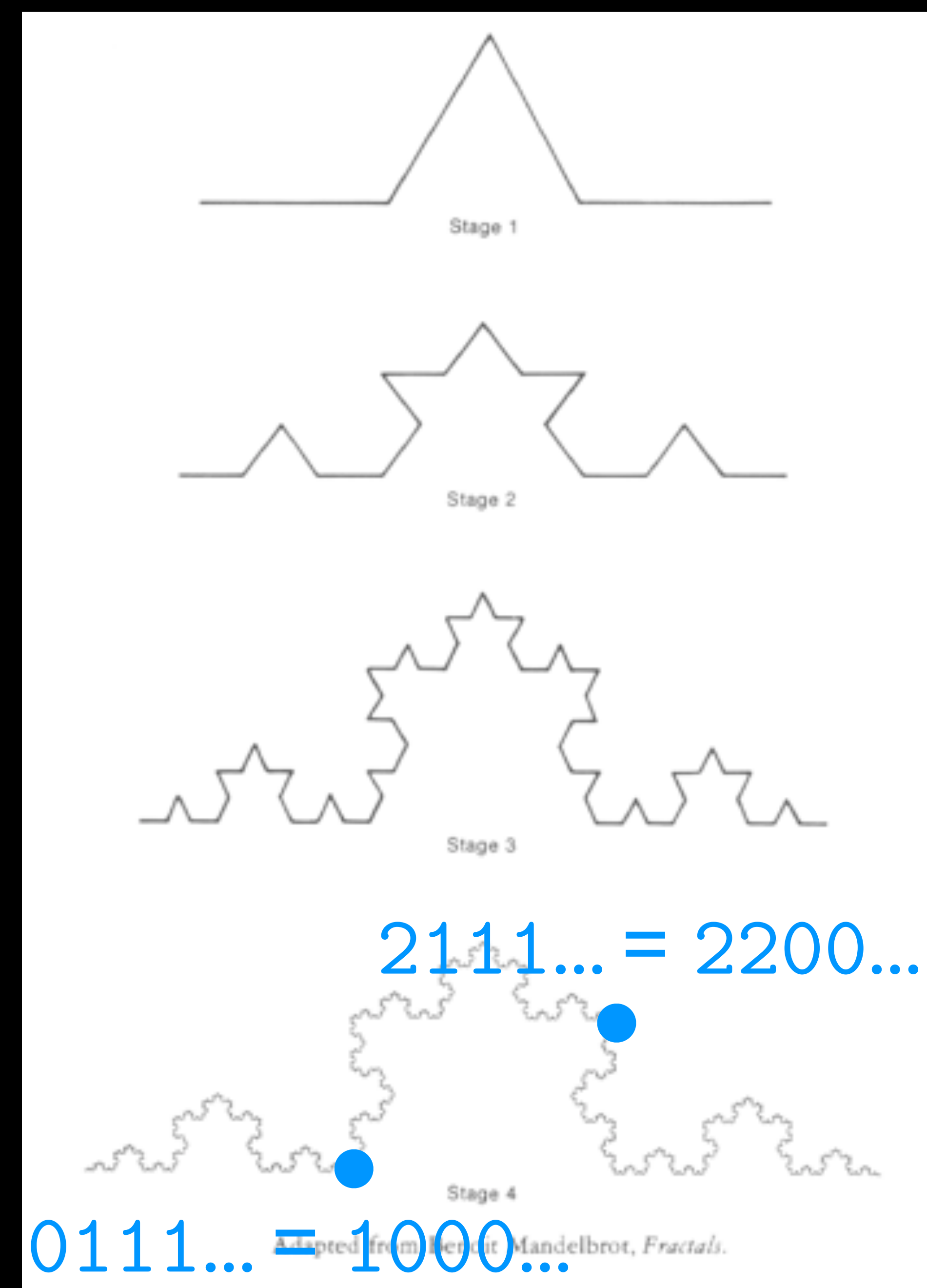
the unit interval $\mathbb{I}$

Fractals

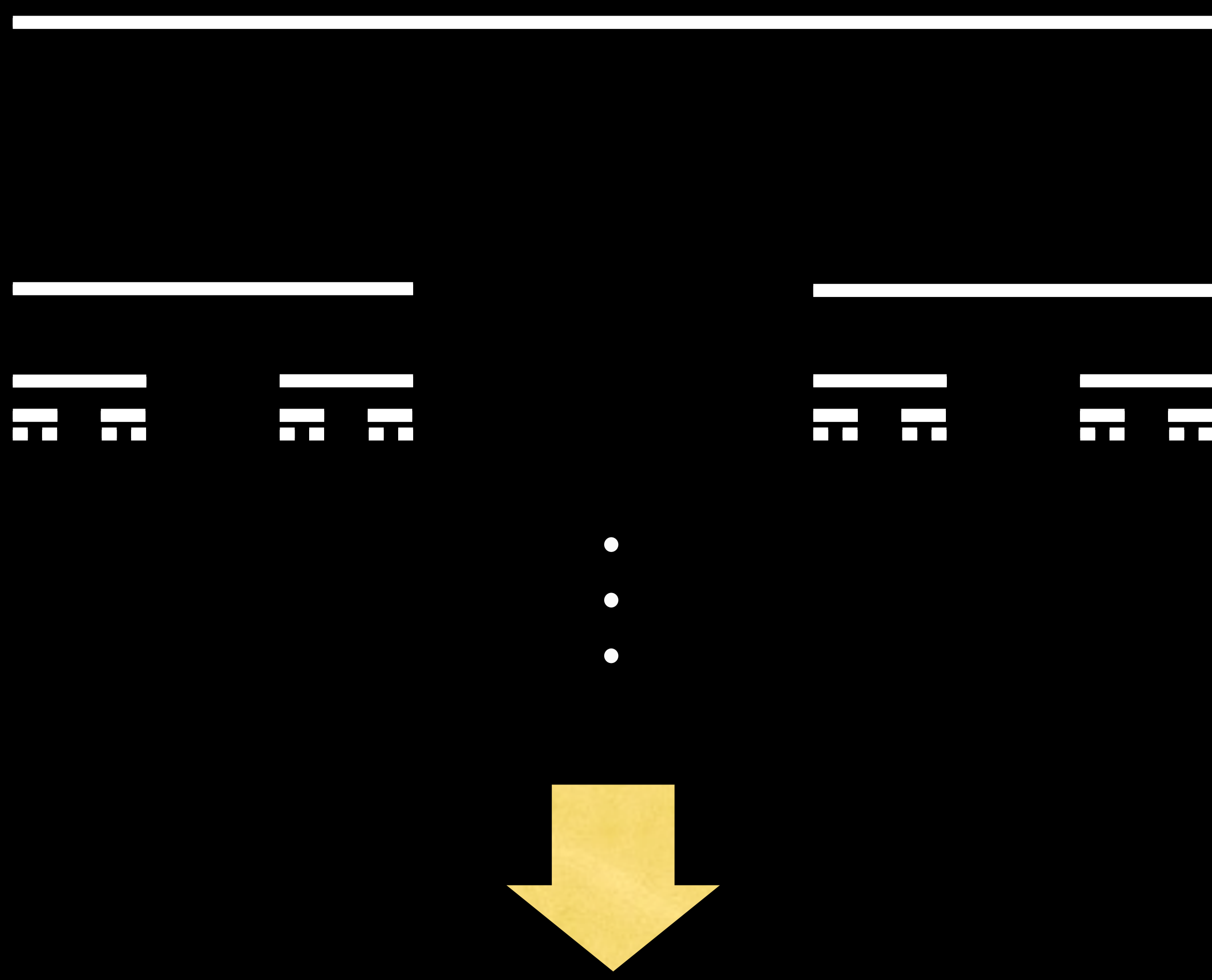
the Koch curve

"A rough or fragmented geometric shape that can be split into parts, each of which (at least approximately) a reduced size copy of the whole" [Mandelbrot]

the unit interval $\mathbb{I}$

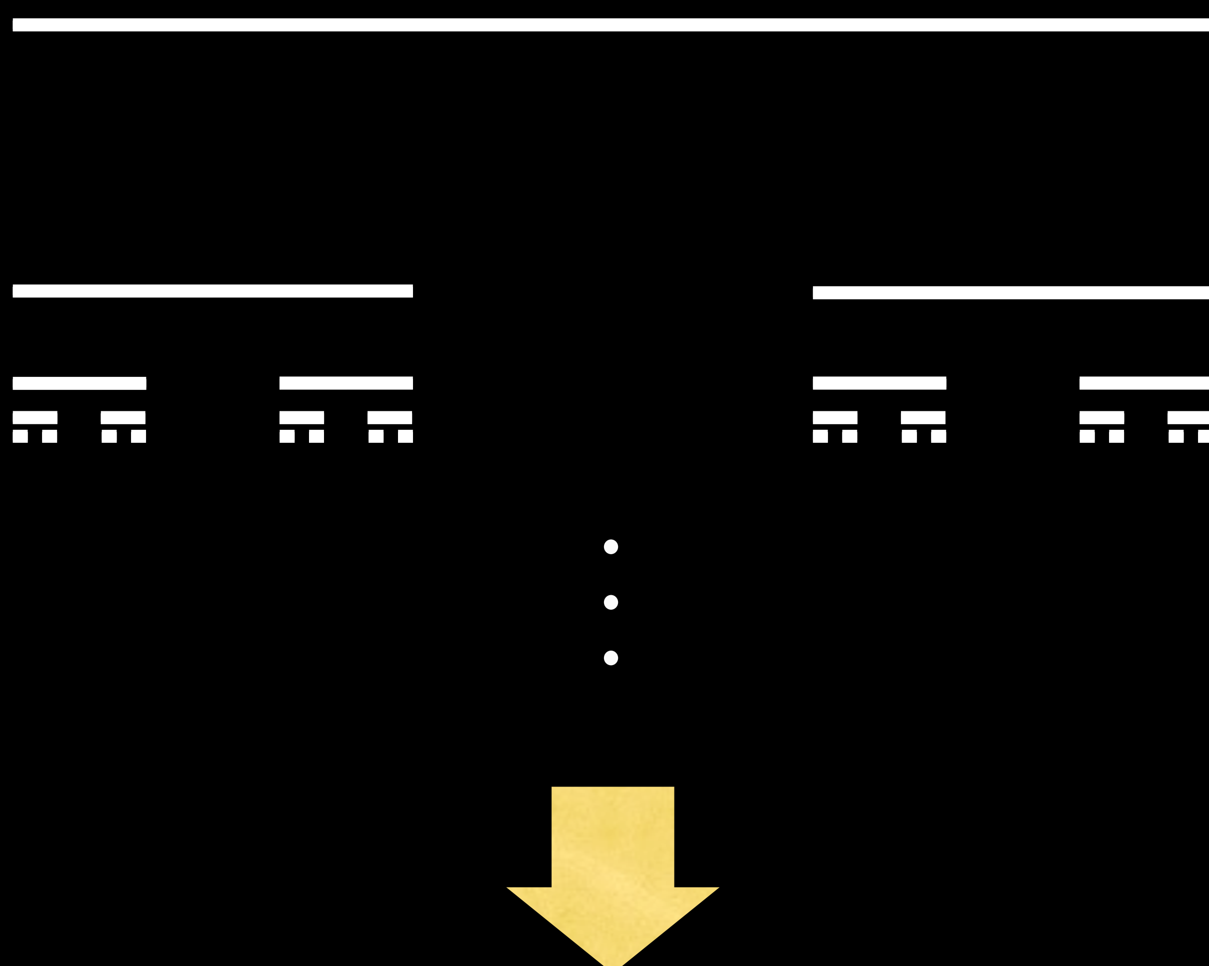


The Cantor Set



the Cantor set C

The Cantor Set



the Cantor set C

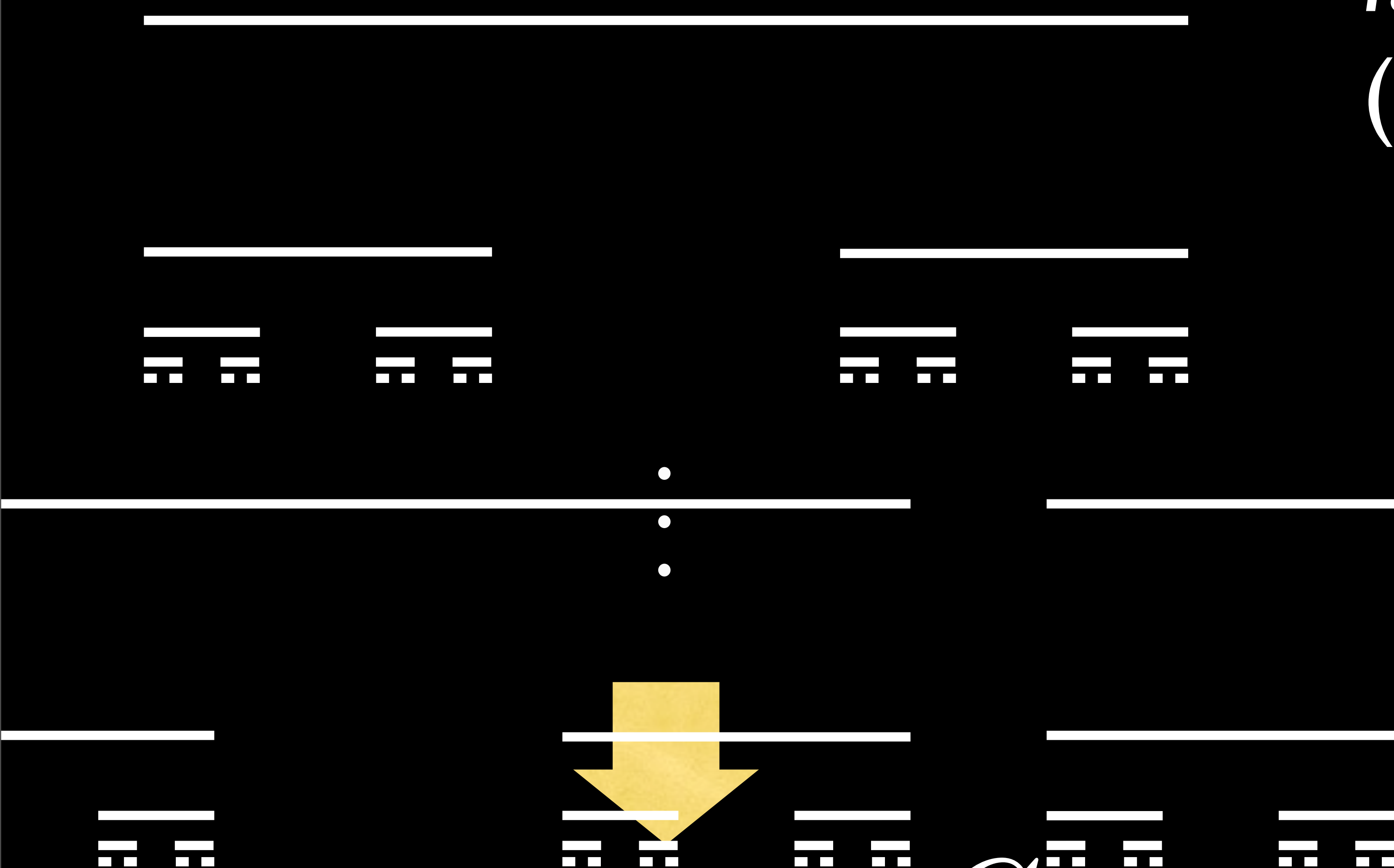
Iterated function system (IFS)

$$\varphi_0, \varphi_1 \; : \mathbb{I} \longrightarrow \mathbb{I} \; ,$$
$$\varphi_0(x) = \frac{x}{3} \; ,$$
$$\varphi_1(x) = \frac{2+x}{3} \; .$$

C as the unique *attractor:*

$$C = \varphi_0(C) \cup \varphi_1(C)$$

The Cantor Set



the Cantor set C

Iterated function system (IFS)

$$\varphi_0, \varphi_1 : \mathbb{I} \longrightarrow \mathbb{I} ,$$

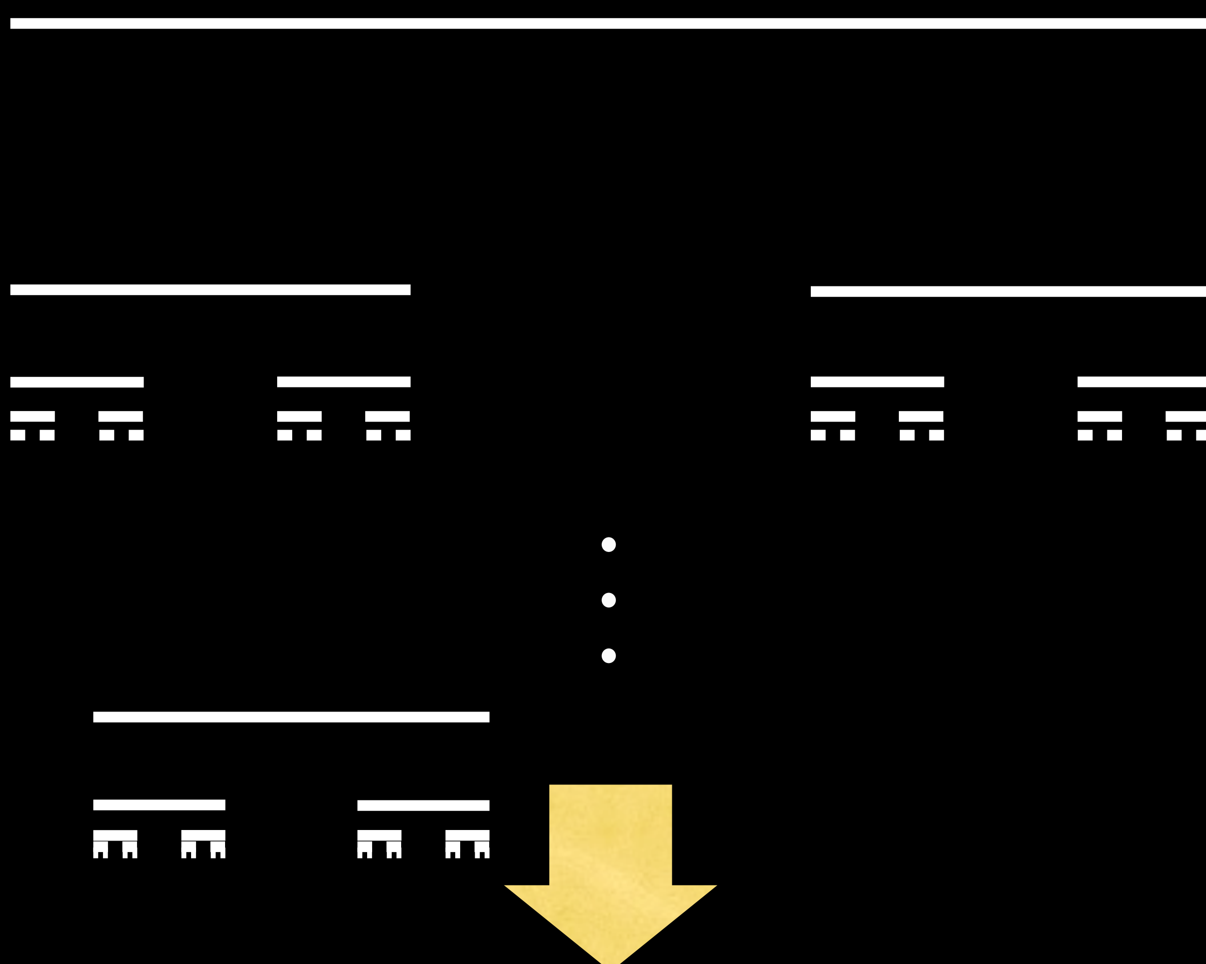
$$\varphi_0(x) = \frac{x}{3} ,$$

$$\varphi_1(x) = \frac{2+x}{3} .$$

C as the unique *attractor:*

$$C = \varphi_0(C) \cup \varphi_1(C)$$

The Cantor Set



the Cantor set C

Iterated function system (IFS)

$$\varphi_0, \varphi_1 \ : \ \mathbb{I} \longrightarrow \mathbb{I} \ ,$$

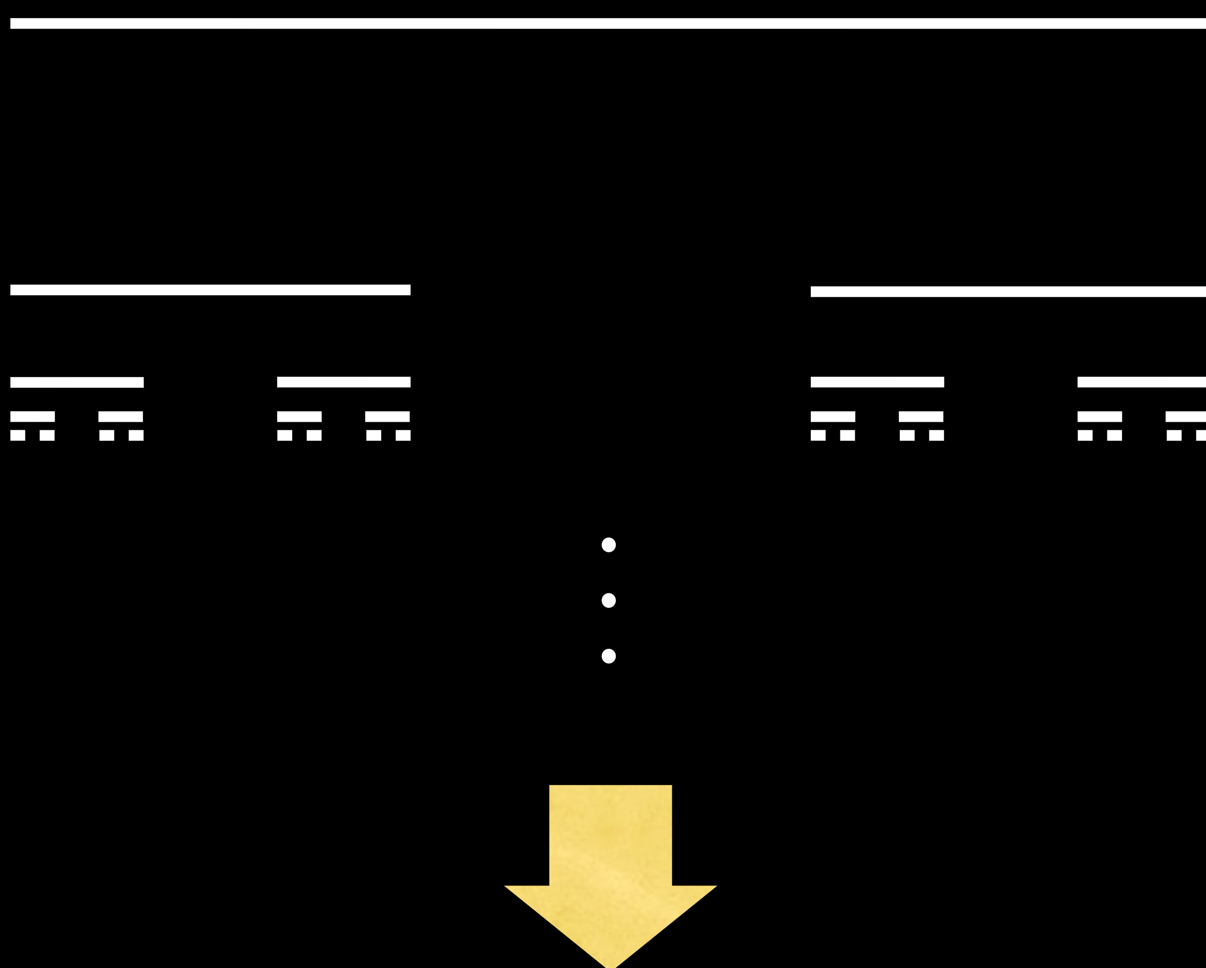
$$\varphi_0(x) = \frac{x}{3} \ ,$$

$$\varphi_1(x) = \frac{2+x}{3} \ .$$

C as the unique *attractor:*

$$C = \varphi_0(C) \cup \varphi_1(C)$$

The Cantor Set



the Cantor set C

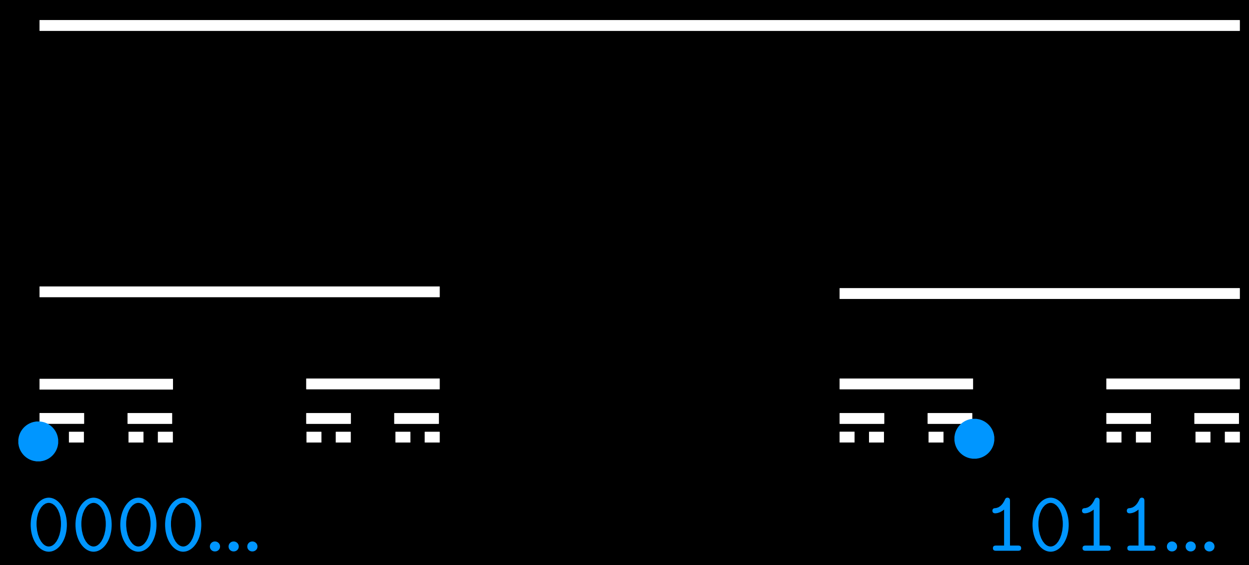
Iterated function system (IFS)

$$\varphi_0, \varphi_1 \; : \; \mathbb{I} \longrightarrow \mathbb{I} \; ,$$
$$\varphi_0(x) = \frac{x}{3} \; ,$$
$$\varphi_1(x) = \frac{2+x}{3} \; .$$

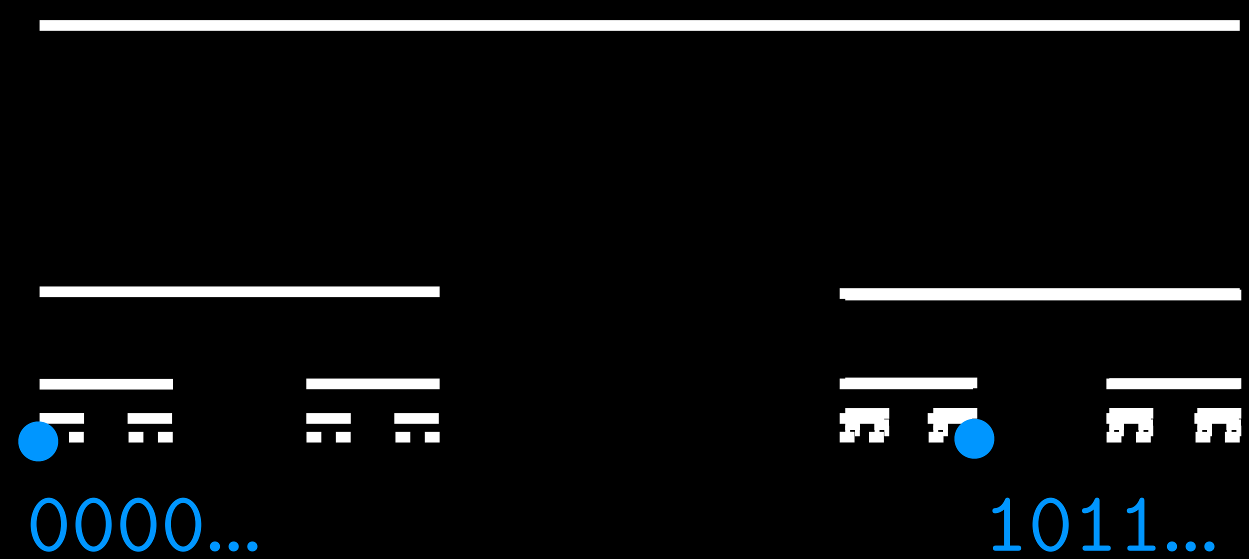
C as the unique *attractor:*

$$C = \varphi_0(C) \cup \varphi_1(C)$$

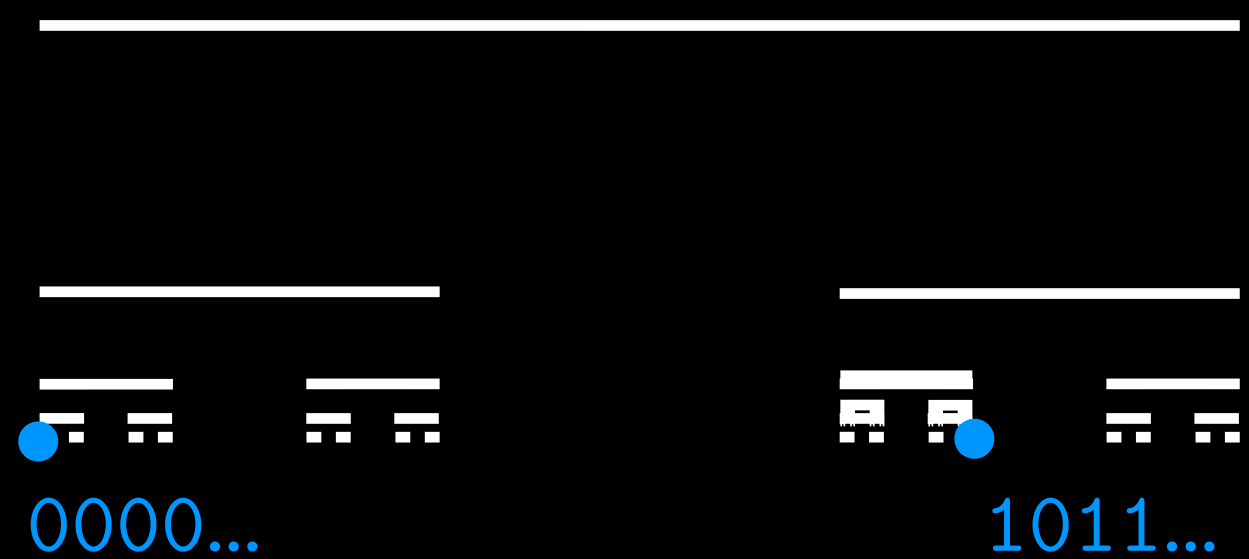
Symbolic Representation



Symbolic Representation

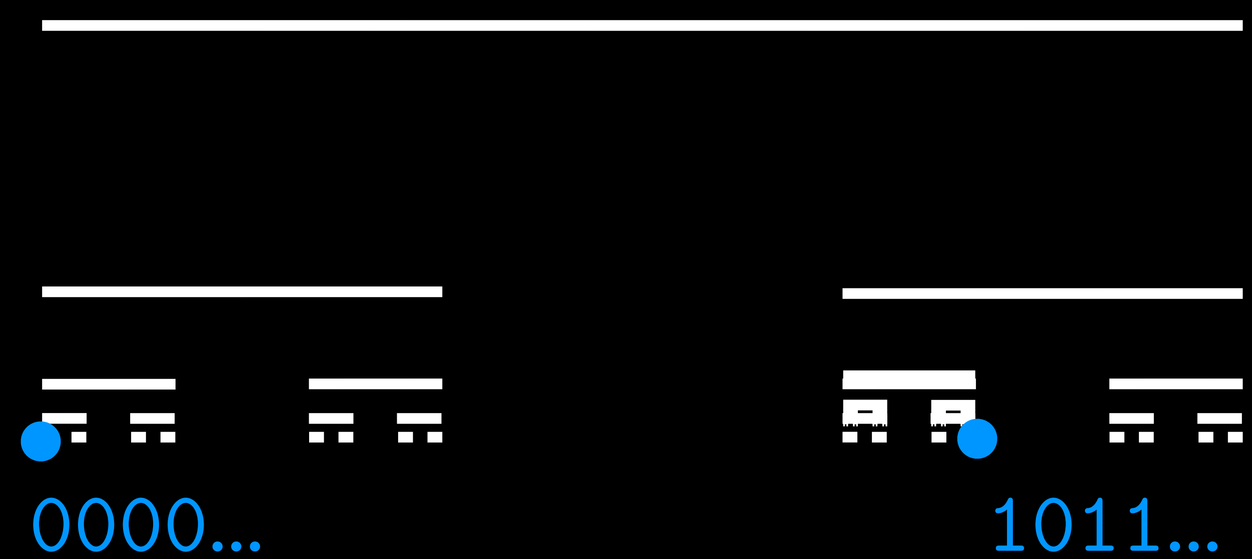


Symbolic Representation



Friday, May 7, 2010

Symbolic Representation



fractal structure → symbolic representation of the shape, by infinite streams

Observation 1

Symb. representatives

$$2^\omega = \{0, 1\}^\omega$$

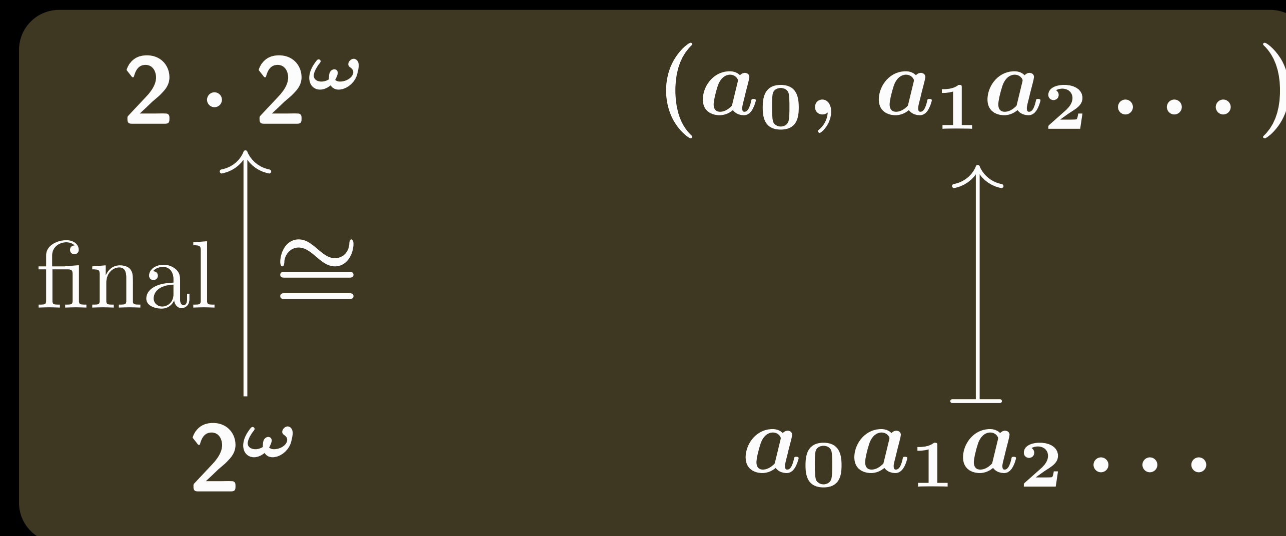
carry a final coalgebra

Observation 1

Symb. representatives

$$2^\omega = \{0,1\}^\omega$$

carry a final coalgebra



Observation 1

Symb. representatives

$$2^\omega = \{0,1\}^\omega$$

carry a final coalgebra

$$\begin{array}{ccc} 2 \cdot 2^\omega & & (a_0,\ a_1 a_2 \ldots) \\ \text{final} \uparrow \cong & & \uparrow \\ 2^\omega & & a_0 a_1 a_2 \ldots \end{array}$$

- final coalgebra as a "fractal"

Observation 1

Symb. representatives

$$2^\omega = \{0,1\}^\omega$$

carry a final coalgebra

$$\begin{array}{ccc} 2 \cdot 2^\omega & & (a_0,\ a_1 a_2 \ldots) \\ \text{final} \uparrow \cong & & \uparrow \\ 2^\omega & & a_0 a_1 a_2 \ldots \end{array}$$

- final coalgebra as a "fractal"
- *combinatorial specification* of the Cantor set:

$$2 \cdot (\_\ ) : \mathrm{Sets} \longrightarrow \mathrm{Sets}$$

Observation 2

$$\varphi_0, \varphi_1 \ : \ \mathbb{I} \longrightarrow \mathbb{I} \ ,$$

$$\varphi_0(x) = \frac{x}{3} \ ,$$

$$\varphi_1(x) = \frac{2+x}{3} \ .$$

Observation 2

$$\varphi_0, \varphi_1 \; : \; \mathbb{I} \longrightarrow \mathbb{I} \; ,$$

$$\varphi_0(x) = \frac{x}{3} \; ,$$

$$\varphi_1(x) = \frac{2+x}{3} \; .$$

$$\chi : 2 \cdot \mathbb{I} \longrightarrow \mathbb{I}$$

$$(0, x) \longmapsto \frac{x}{3} \qquad (1, x) \longmapsto \frac{2+x}{3}$$

Observation 2

$$\varphi_0, \varphi_1 \; : \; \mathbb{I} \longrightarrow \mathbb{I} \; ,$$

$$\varphi_0(x) = \frac{x}{3} \; ,$$

$$\varphi_1(x) = \frac{2+x}{3} \; .$$

$$\chi : 2 \cdot \mathbb{I} \longrightarrow \mathbb{I}$$

$$(0, x) \longmapsto \frac{x}{3} \qquad (1, x) \longmapsto \frac{2+x}{3}$$

An IFS is an algebra

Observation 2

$$\varphi_0, \varphi_1 \ : \ \mathbb{I} \longrightarrow \mathbb{I} \ ,$$

$$\varphi_0(x) = \frac{x}{3} \ ,$$

$$\varphi_1(x) = \frac{2+x}{3} \ .$$

$$\chi : 2 \cdot \mathbb{I} \longrightarrow \mathbb{I}$$

$$(0, x) \longmapsto \frac{x}{3} \qquad (1, x) \longmapsto \frac{2+x}{3}$$

An IFS is an algebra

- for the same functor $2 \cdot (\_\ ) : \mathrm{Sets} \to \mathrm{Sets}$

Observation 3

$$\begin{array}{c} 2 \cdot 2^{\omega} \\ \cong \big\uparrow \text{final} \\ 2^{\omega} \end{array} \qquad\qquad \begin{array}{c} 2 \cdot \mathbb{I} \\ \big\downarrow \chi \\ \mathbb{I} \end{array}$$

Observation 3

$$\begin{array}{ccc} 2 \cdot 2^\omega & & 2 \cdot \mathbb{I} \\ \cong \big\uparrow \text{final} & & \big\downarrow \chi \\ 2^\omega & & \mathbb{I} \end{array}$$

{symb. representatives}

IFS

$$\varphi_0, \varphi_1 \ : \mathbb{I} \longrightarrow \mathbb{I} \ ,$$
$$\varphi_0(x) = \frac{x}{3} \ ,$$
$$\varphi_1(x) = \frac{2+x}{3} \ .$$

Observation 3

$$
\begin{array}{ccc}
\mathbf{2}\cdot\mathbf{2}^{\omega} & \xrightarrow{\ \mathbf{2}\cdot[\![\_]\!]_{\chi}\ } & \mathbf{2}\cdot\mathbb{I} \\
\cong\ \big\uparrow\ \text{final} & & \big\downarrow\ \chi \\
\mathbf{2}^{\omega} & \xrightarrow[\ [\![\_]\!]_{\chi}\ ]{} & \mathbb{I}
\end{array}
$$

{symb. representatives}

IFS

$$\varphi_0, \varphi_1 \ : \ \mathbb{I} \longrightarrow \mathbb{I} \ ,$$
$$\varphi_0(x) = \frac{x}{3} \ ,$$
$$\varphi_1(x) = \frac{2+x}{3} \ .$$

Observation 3

$$\begin{array}{ccc} 2 \cdot 2^{\omega} & \xrightarrow{\;2 \cdot [\![\_]\!]_{\chi}\;} & 2 \cdot \mathbb{I} \\ \cong \uparrow \text{final} & & \downarrow \chi \\ 2^{\omega} & \xrightarrow[\;[\![\_]\!]_{\chi}\;]{} & \mathbb{I} \end{array}$$

{symb. representatives}

IFS

$$\varphi_0, \varphi_1 \;:\; \mathbb{I} \longrightarrow \mathbb{I}\ ,$$
$$\varphi_0(x) = \frac{x}{3}\ ,$$
$$\varphi_1(x) = \frac{2+x}{3}\ .$$

0000... $\longmapsto$

1011... $\longmapsto$

Observation 3

$$\begin{array}{ccc} 2 \cdot 2^\omega & \xrightarrow{\;2 \cdot [\![\text{-}]\!]_\chi\;} & 2 \cdot \mathbb{I} \\ \cong \Big\uparrow \text{final} & & \Big\downarrow \chi \\ 2^\omega & \xrightarrow[\;[\![\text{-}]\!]_\chi\;]{} & \mathbb{I} \end{array}$$

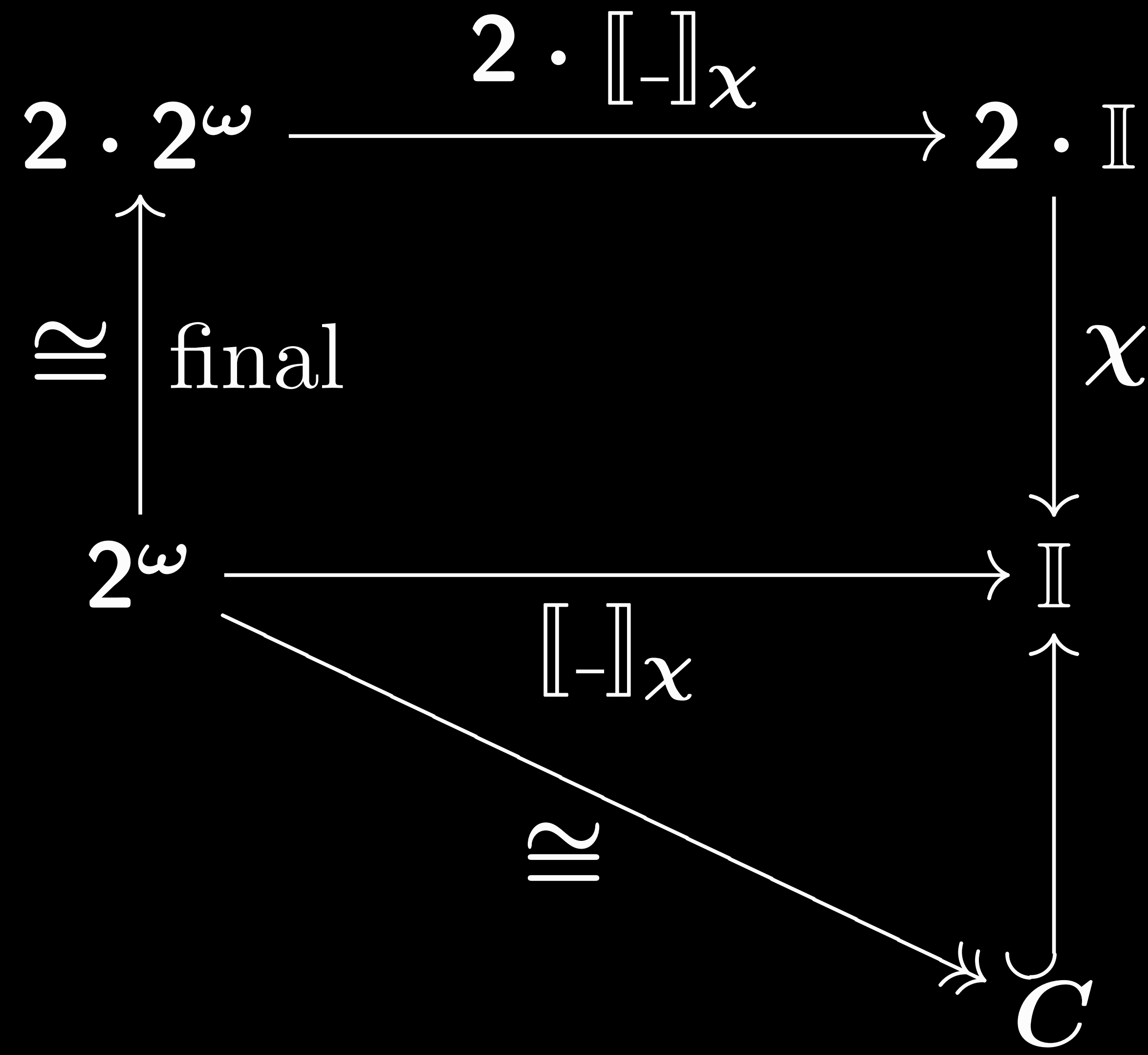
- Such $[\![\text{-}]\!]_\chi$ uniquely exists.

Observation 3

$$
\begin{array}{ccc}
2 \cdot 2^{\omega} & \xrightarrow{\;2 \cdot [\![\_]\!]_{\chi}\;} & 2 \cdot \mathbb{I} \\
{\scriptstyle \cong}\big\uparrow{\scriptstyle \text{final}} & & \big\downarrow{\scriptstyle \chi} \\
2^{\omega} & \xrightarrow[\;[\![\_]\!]_{\chi}\;]{} & \mathbb{I}
\end{array}
$$

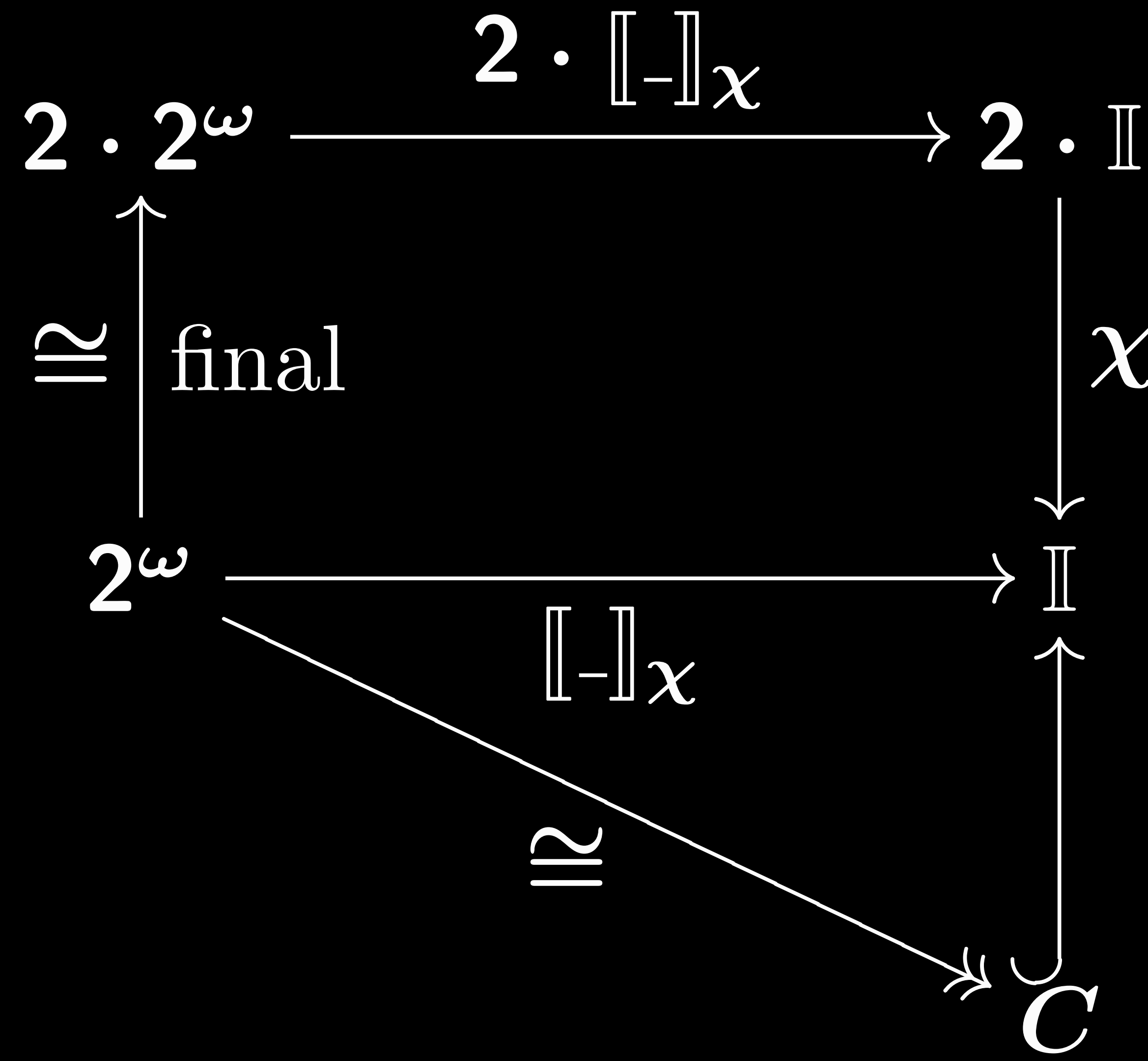
- $[\![\_]\!]_{\chi}$ is injective.

Observation 3

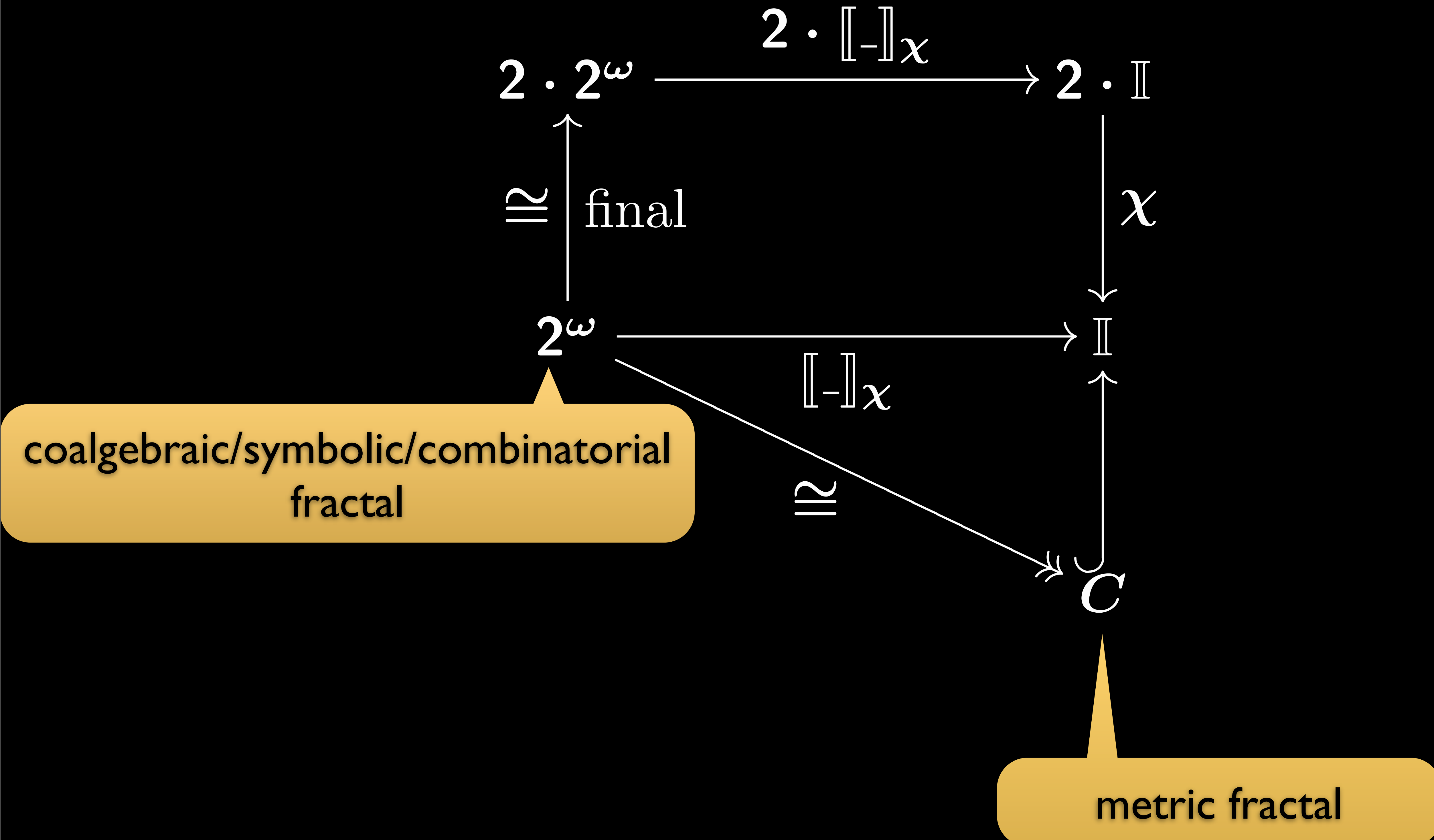


- $[\![-]\!]_\chi$ is injective.

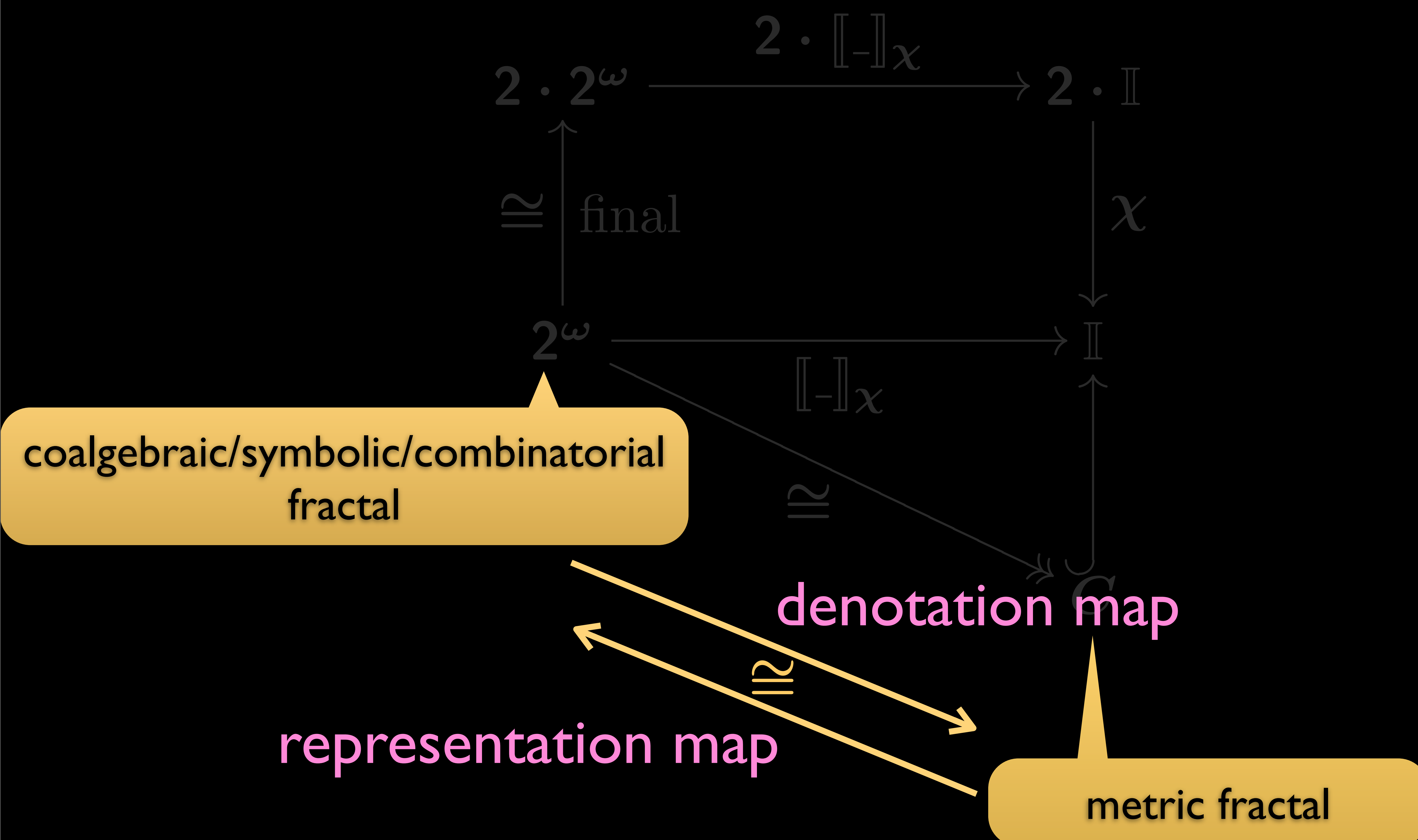
“Representation Theory”



“Representation Theory”

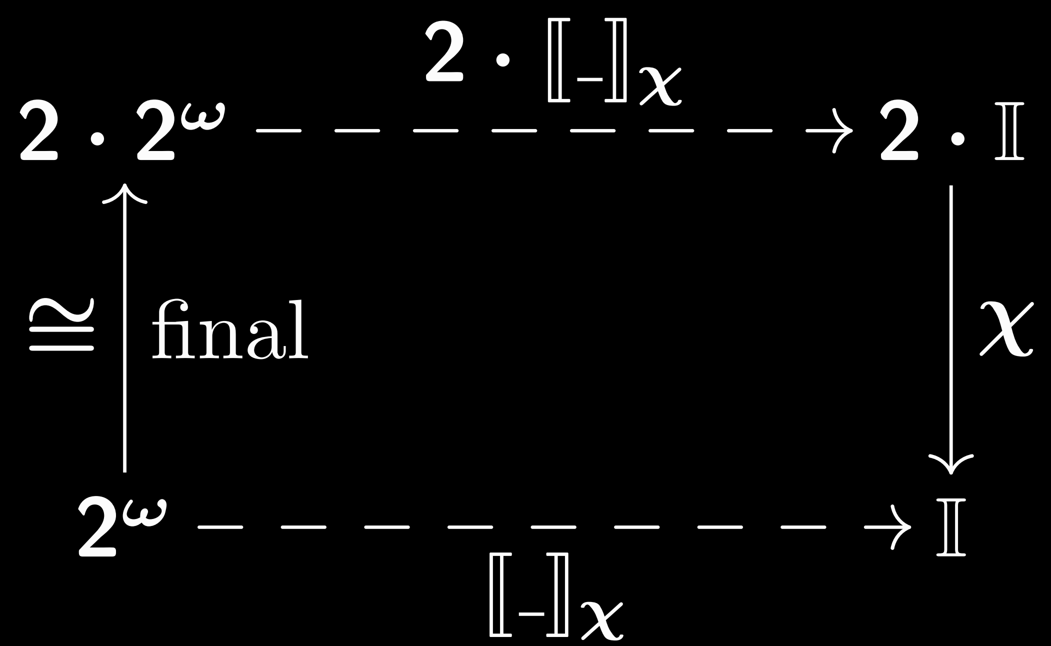


“Representation Theory”



Axiomatic Domain Theory

- **Thm.** $\exists! \, [\![\_]\!]_\chi$ s.t.



Proof

- $\mathrm{Sets}(2^\omega, \mathbb{I})$ is a complete metric space (CMS).
- $\Phi : \quad \mathrm{Sets}(2^\omega, \mathbb{I}) \longrightarrow \mathrm{Sets}(2^\omega, \mathbb{I})$

is a contracting map.

- Use the Banach fixed pt. thm.

Axiomatic Domain Theory

- **Thm.** $\exists!\ [\![\_]\!]_\mathcal{X}$ s.t.

$$\begin{array}{ccc}
2 \cdot 2^\omega & \xrightarrow{\ 2 \cdot [\![_]\!]_\mathcal{X}\ } & 2 \cdot \mathbb{I} \\
\cong \uparrow \text{final} & & \downarrow \mathcal{X} \\
2^\omega & \xrightarrow[\ [\![_]\!]_\mathcal{X}\ ]{} & \mathbb{I}
\end{array}$$

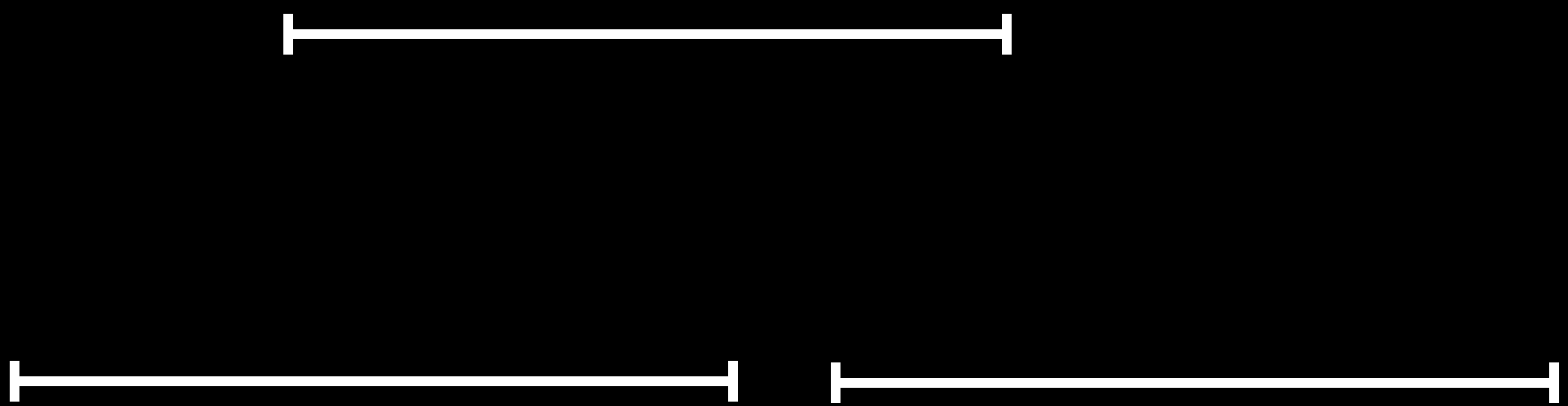
cf. *initial algebra-final coalgebra coincidence*

- solving domain equation [Freyd]
  $$X = (X \Rightarrow X)$$
- coalgebraic trace semantics [IH-Jacobs-Sokolova]
- *corecursive algebra* [Capretta-Uustalu-Vene]

- typical in enrichment with "approximation structure"
  - of *infinitary* data, by *finitary* ones
  - order/CPO [Smyth-Plotkin], complete metric [America-Rutten]

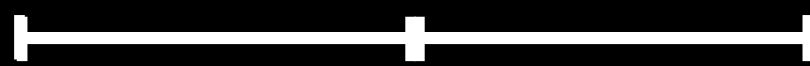
Unit interval $\mathbb{I}$

Unit interval $\mathbb{I}$

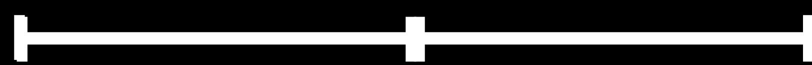


Unit interval $\mathbb{I}$

Unit interval $\mathbb{I}$



Unit interval $\mathbb{I}$

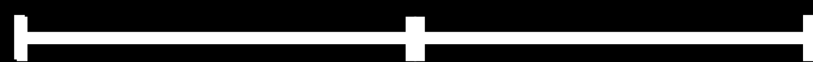


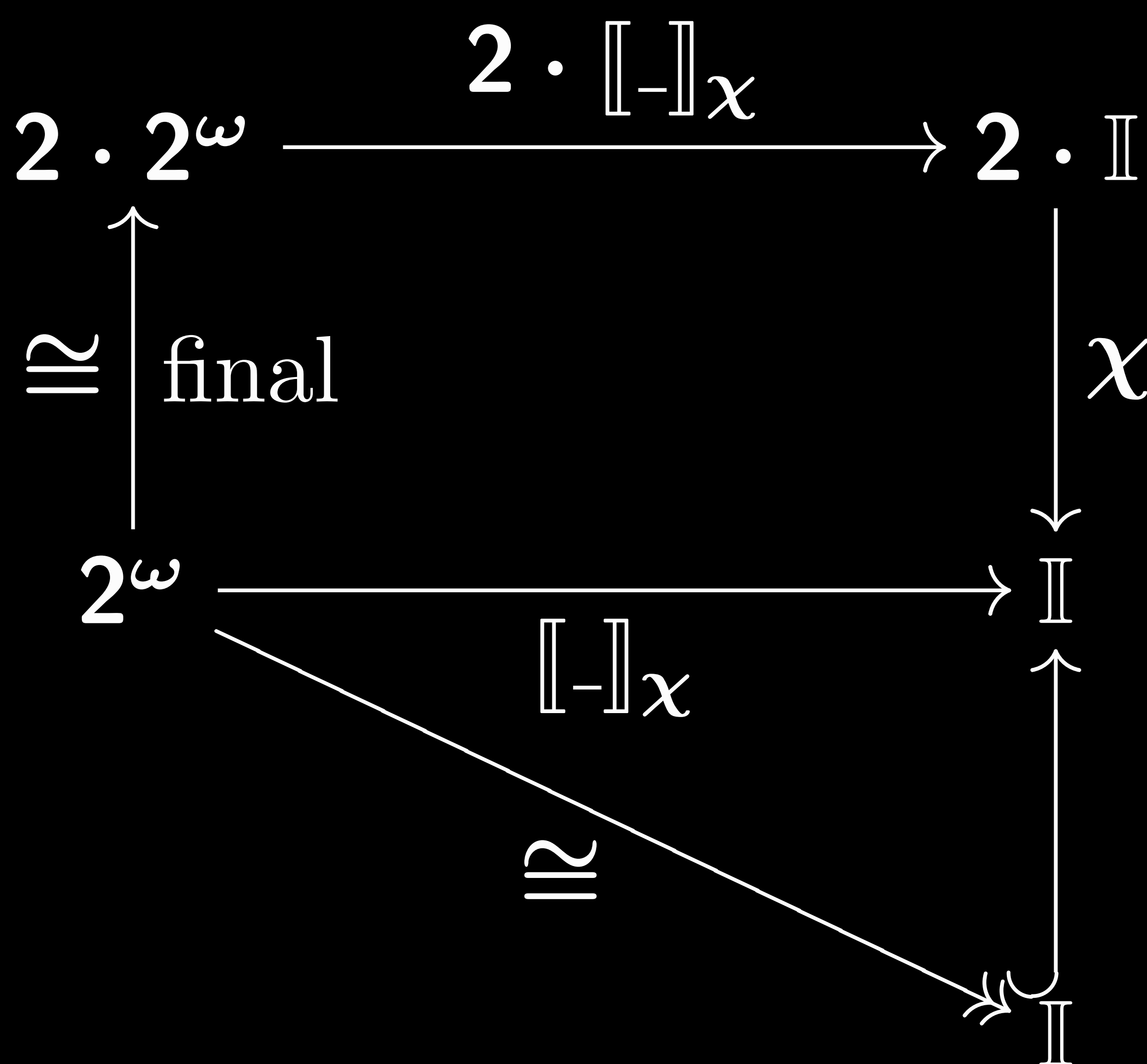
$$\varphi_0, \varphi_1 \;:\; \mathbb{C} \longrightarrow \mathbb{C} \;,$$

$$\varphi_0(x) = \frac{x}{2} \;,$$

$$\varphi_1(x) = \frac{1+x}{2} \;.$$

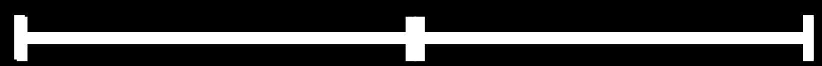
Unit interval $\mathbb{I}$

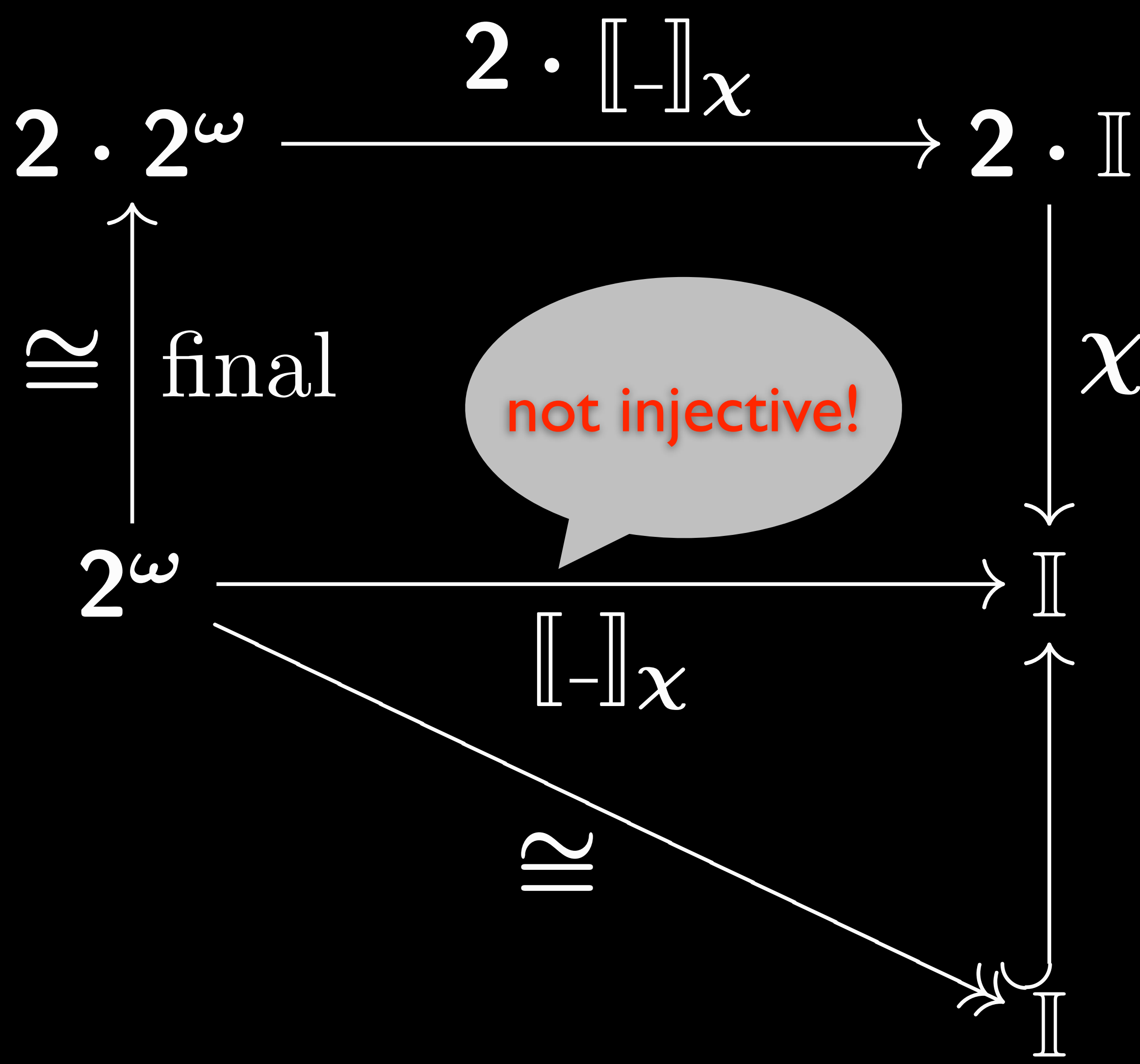




$$\varphi_0, \varphi_1 \ : \ \mathbb{C} \longrightarrow \mathbb{C} \ ,$$
$$\varphi_0(x) = \frac{x}{2} \ ,$$
$$\varphi_1(x) = \frac{1+x}{2} \ .$$

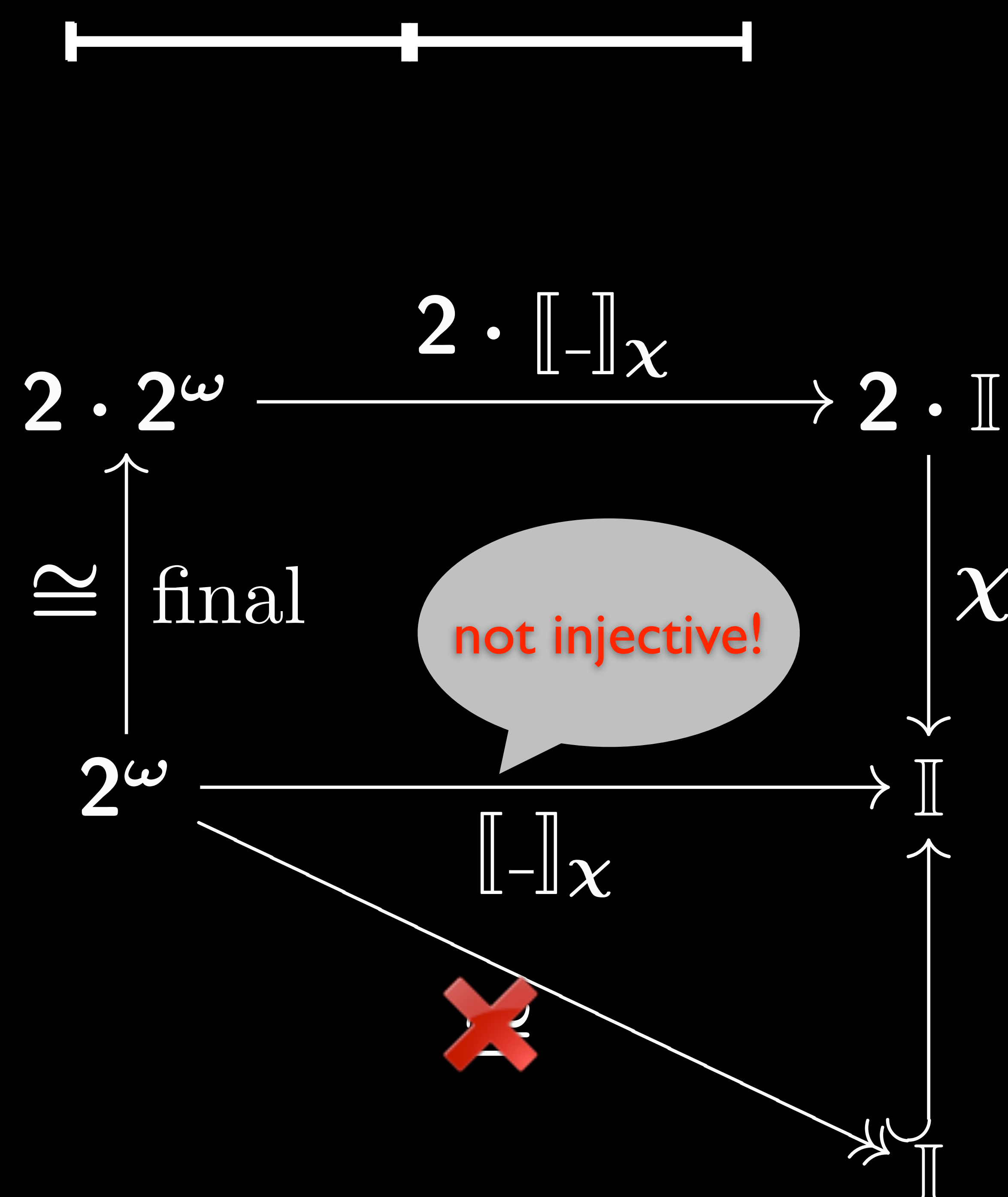
Unit interval $\mathbb{I}$





$$\varphi_0, \varphi_1 \; : \; \mathbb{C} \longrightarrow \mathbb{C} \; ,$$
$$\varphi_0(x) = \frac{x}{2} \; ,$$
$$\varphi_1(x) = \frac{1+x}{2} \; .$$

Unit interval $\mathbb{I}$

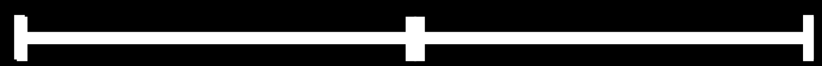


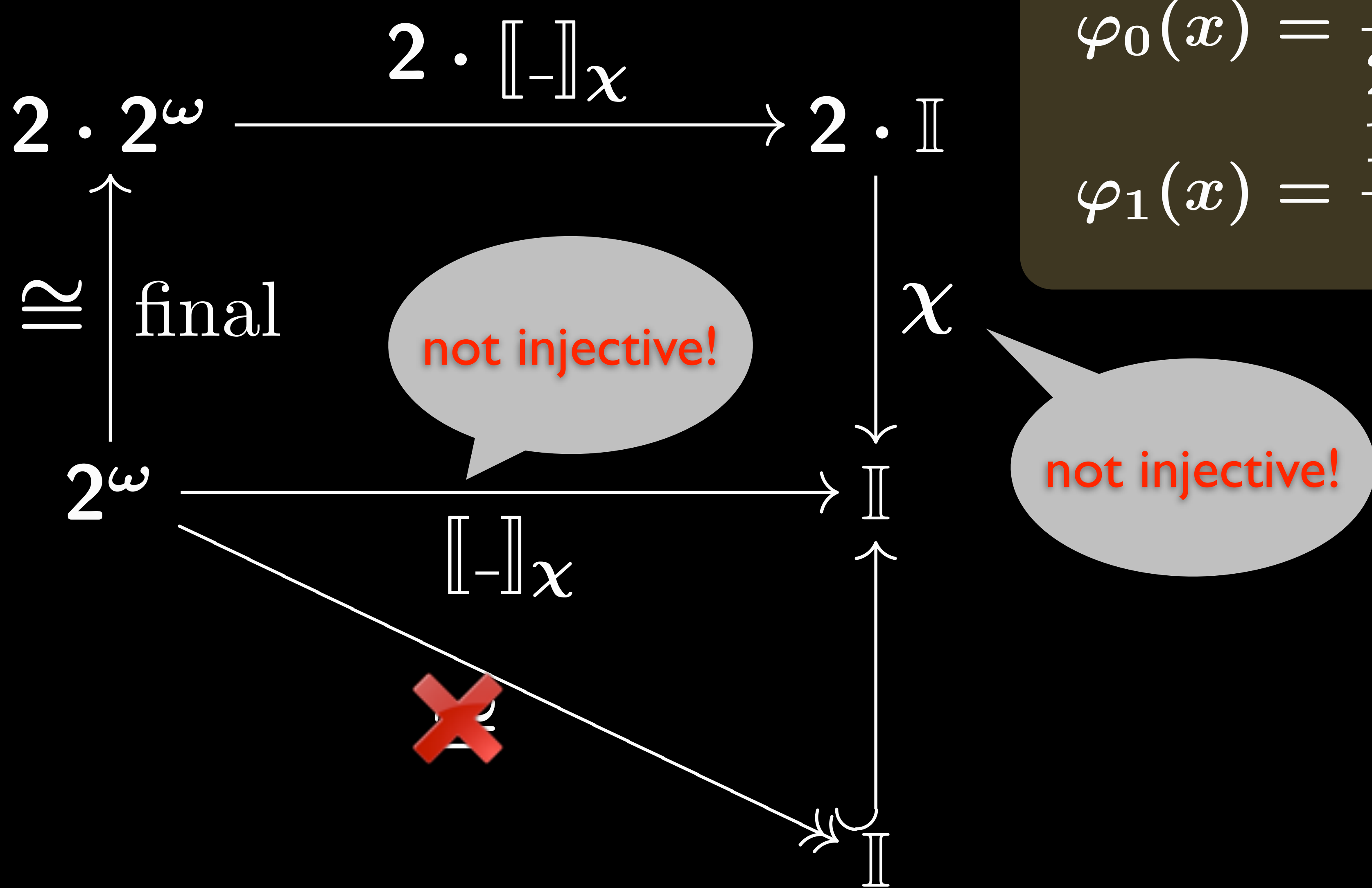
$$\varphi_0, \varphi_1 \ : \ \mathbb{C} \longrightarrow \mathbb{C} \ ,$$

$$\varphi_0(x) = \frac{x}{2} \ ,$$

$$\varphi_1(x) = \frac{1+x}{2} \ .$$

Unit interval $\mathbb{I}$





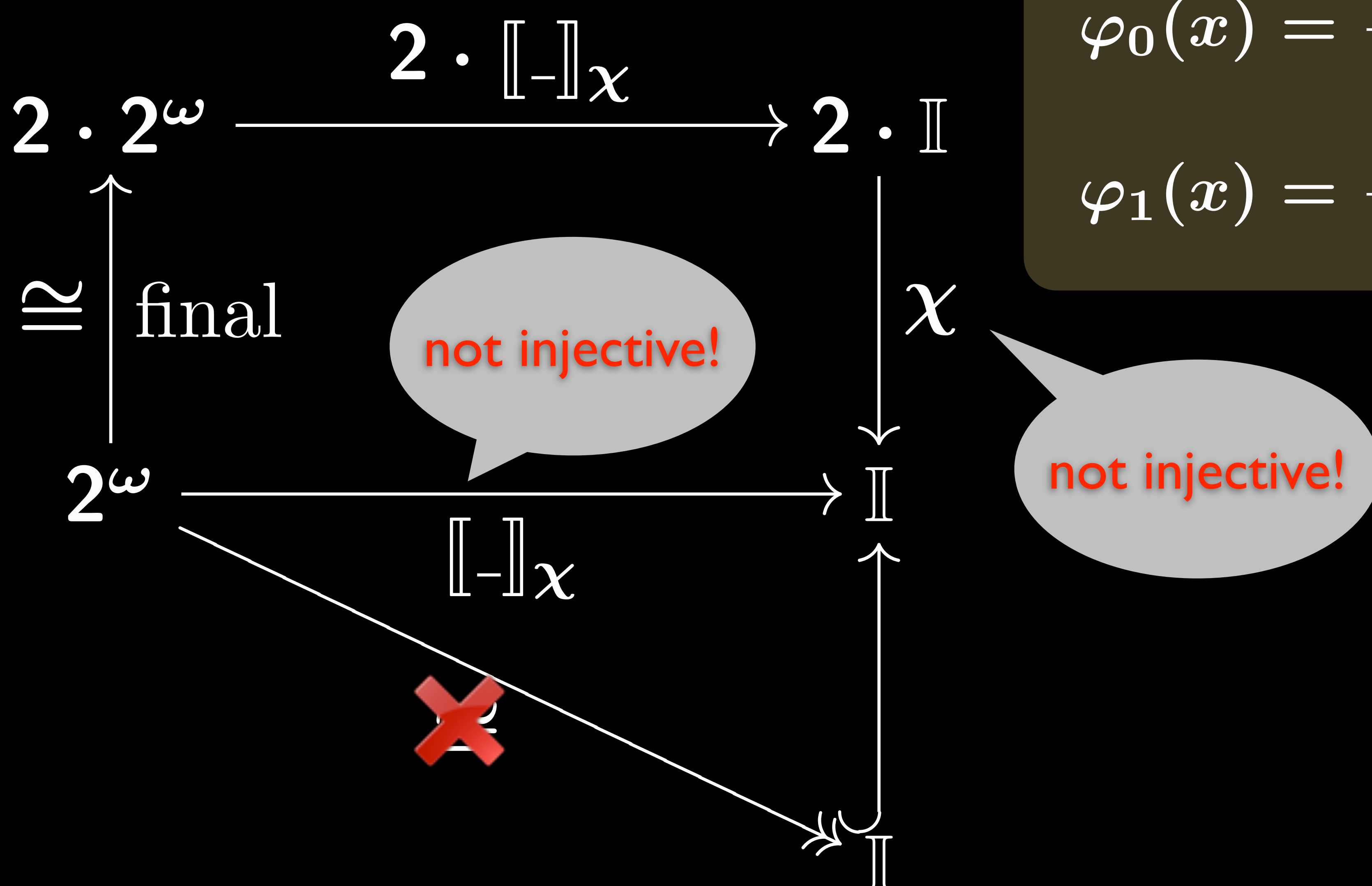
$$\varphi_0, \varphi_1 \;:\; \mathbb{C} \longrightarrow \mathbb{C}\ ,$$

$$\varphi_0(x) = \frac{x}{2}\ ,$$

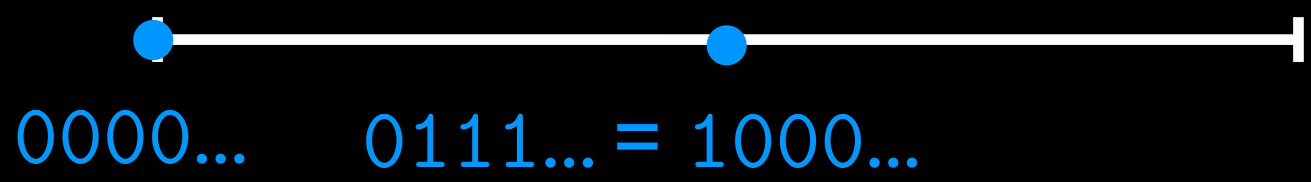
$$\varphi_1(x) = \frac{1+x}{2}\ .$$

Unit interval $\mathbb{I}$

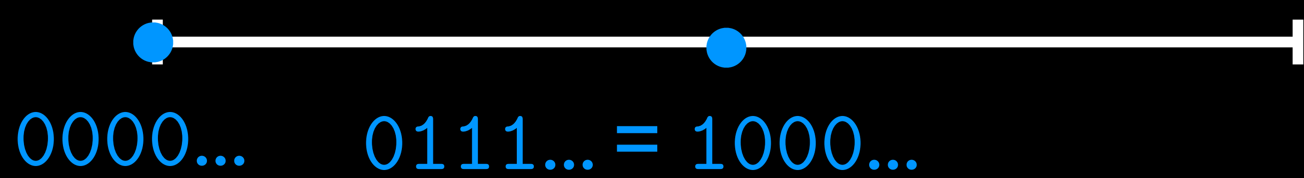
$$\varphi_0, \varphi_1 \; : \; \mathbb{C} \longrightarrow \mathbb{C} \; ,$$
$$\varphi_0(x) = \frac{x}{2} \; ,$$
$$\varphi_1(x) = \frac{1+x}{2} \; .$$



Unit interval $\mathbb{I}$

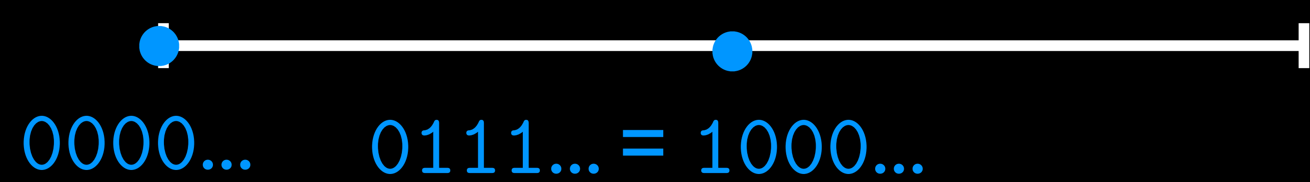


Unit interval $\mathbb{I}$



{ symbolic representatives } = $2^\omega / \sim$

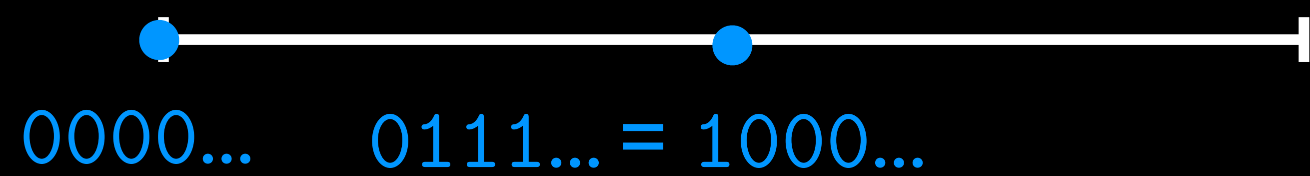
Unit interval $\mathbb{I}$



categorically?

{ symbolic representatives } = $2^{\omega} / \sim$

Unit interval $\mathbb{I}$



categorically?

{ symbolic representatives } = $2^\omega / \sim$

- Use of *presheaves* and *modules/distributors/profunctors*
  - Freyd's observation
  - Tom Leinster. A general theory of self-similarity I, II. In arXiv.

Related Work

- Real number computation, exact arithmetic
  [Edalat-Heckmann, Kreitz-Weihrauch, Weihrauch, ...]
  - "representation": $$\begin{array}{ccc} \mathbb{N}^\omega & & \\ \uparrow & & \\ \boldsymbol{A} & \twoheadrightarrow & \mathbb{I} \end{array}$$
  - ours: $$\begin{pmatrix}\text{coalgebraic/symbolic} \\ \text{fractal}\end{pmatrix} \underset{\text{representation}}{\overset{\text{denotation}}{\rightleftarrows}} \begin{pmatrix}\text{metric} \\ \text{fractal}\end{pmatrix} \quad (\cong)$$

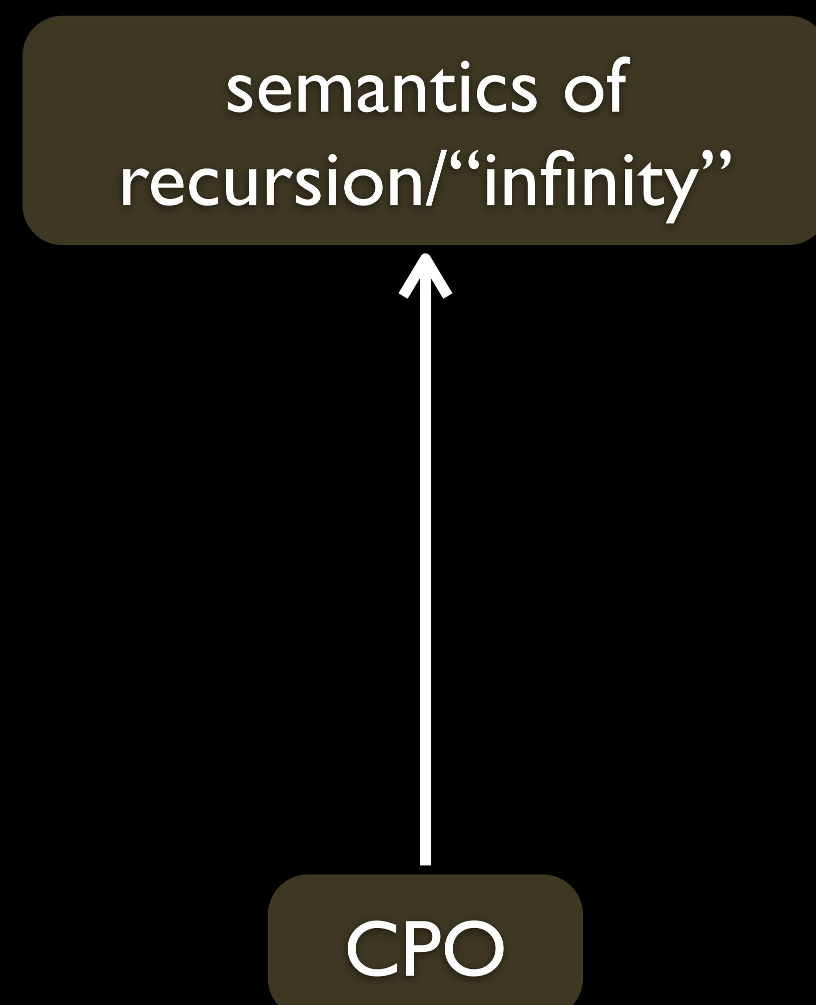
Related Work

- Real number computation, exact arithmetic
  [Edalat-Heckmann, Kreitz-Weihrauch, Weihrauch, ...]
  - "representation": $\mathbb{N}^\omega \hookleftarrow \boldsymbol{A} \twoheadrightarrow \mathbb{I}$
  - ours: $$\begin{pmatrix}\text{coalgebraic/symbolic}\\ \text{fractal}\end{pmatrix} \underset{\text{representation}}{\overset{\text{denotation}}{\rightleftarrows}} \begin{pmatrix}\text{metric}\\ \text{fractal}\end{pmatrix} \quad (\cong)$$
- Real lines $\mathbb{R}, \mathbb{I}, [0,1), (0,1), \ldots$ categorically/coalgebraically [Escardo-Simpson, Pavlovic-Pratt, ...]
  - introducing the *fractal* point of view

Related Work

- Domain-theoretic approach to fractals
  [Hayashi, Edalat, Scriven, Coquand, ...]

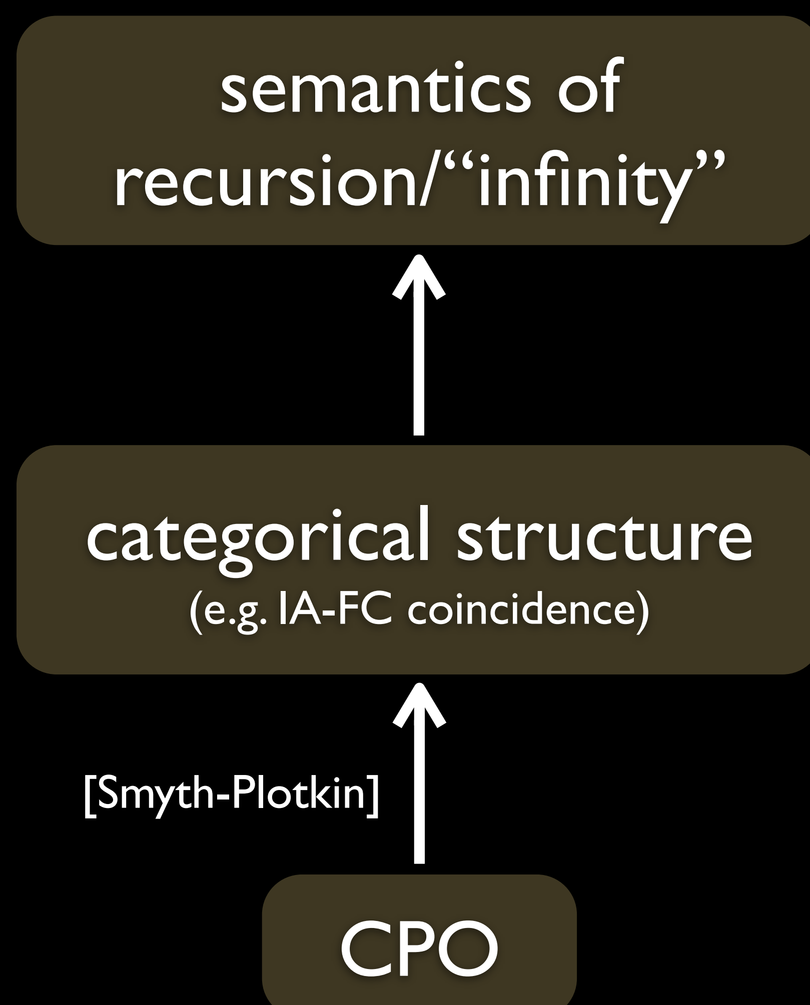
domain theory



Related Work

- Domain-theoretic approach to fractals
  [Hayashi, Edalat, Scriven, Coquand, ...]

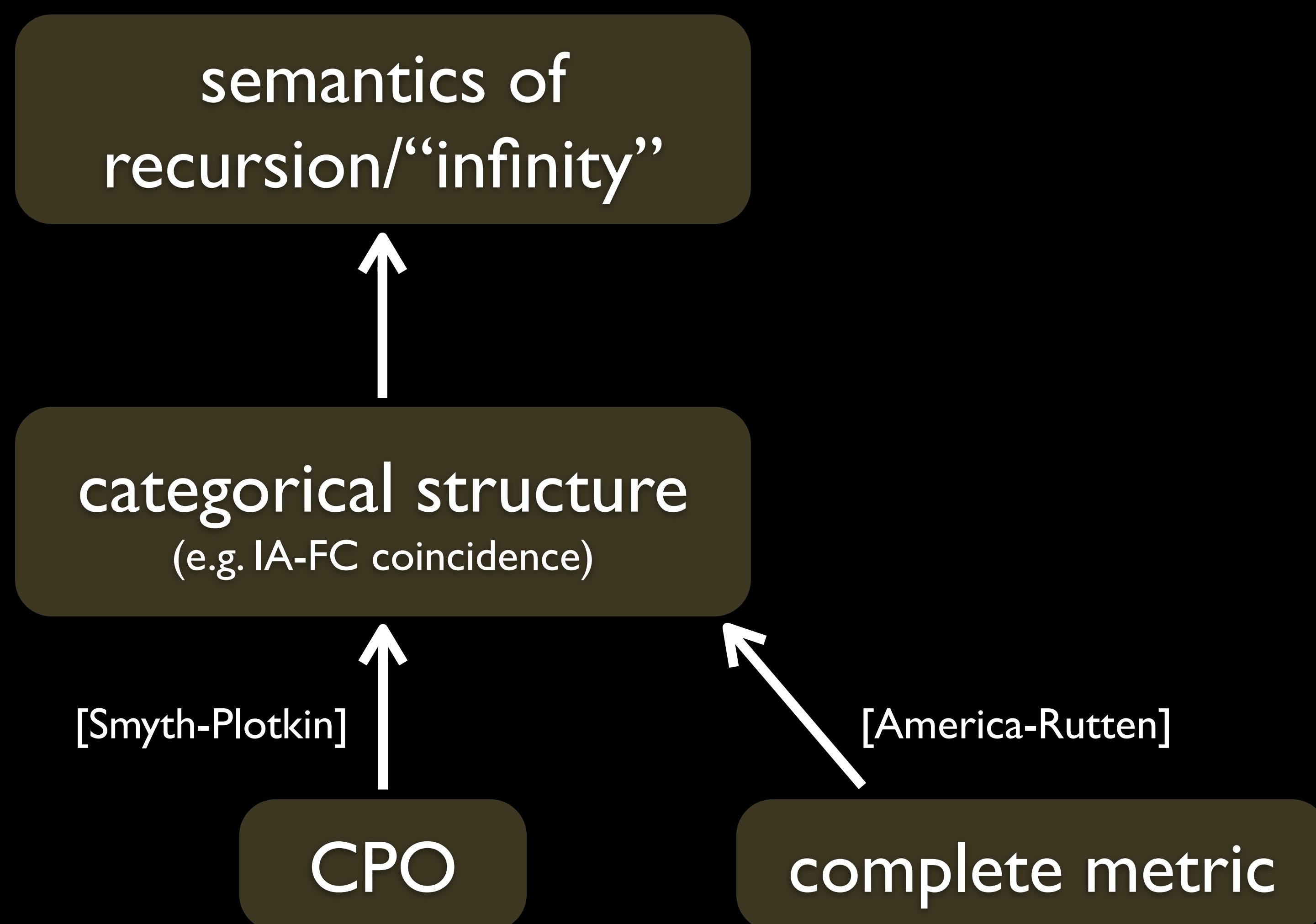
axiomatic domain theory
[Freyd, Fiore, ...]



Related Work

- Domain-theoretic approach to fractals
  [Hayashi, Edalat, Scriven, Coquand, ...]

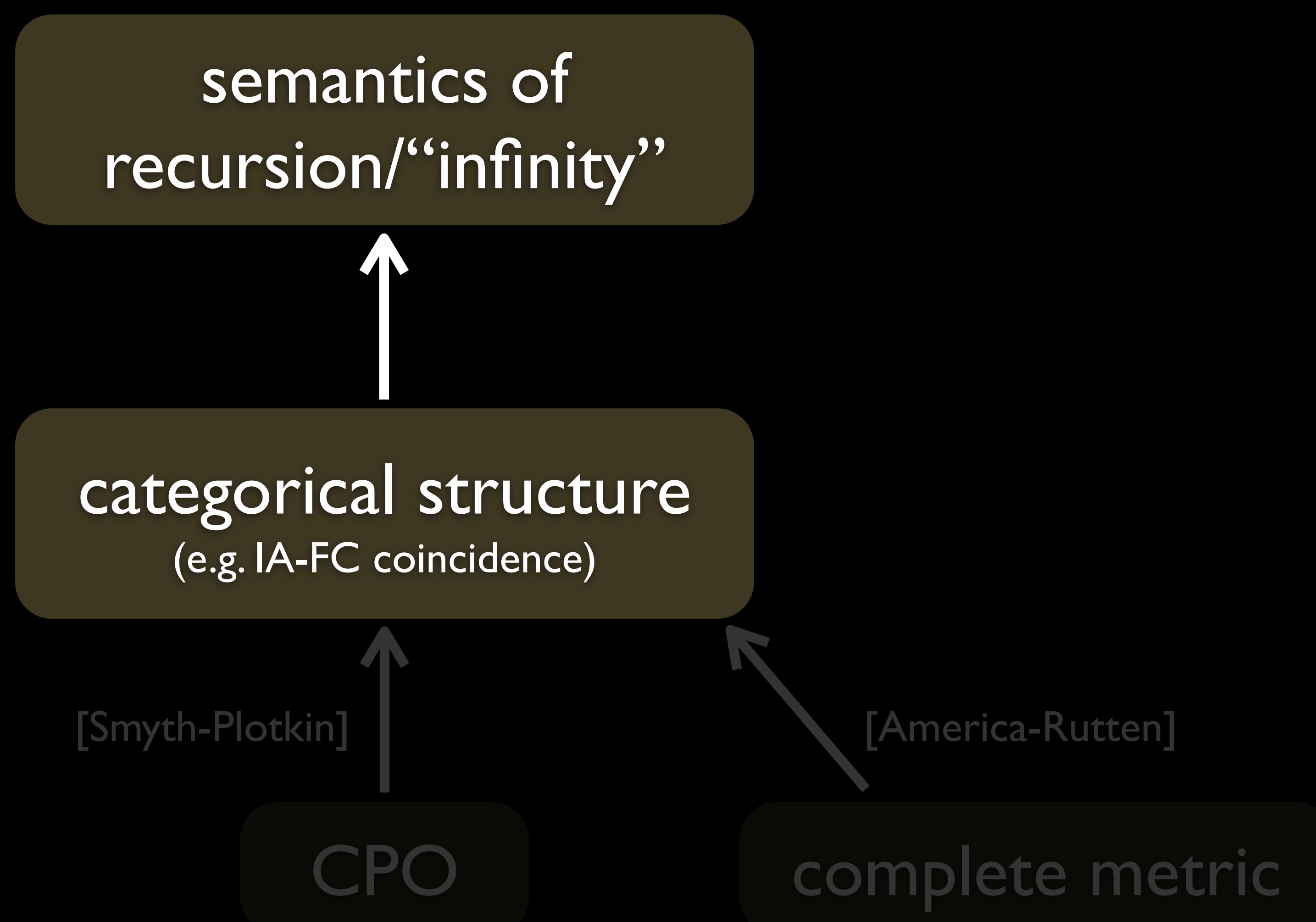
axiomatic
domain theory
[Freyd, Fiore, ...]



Related Work

- Domain-theoretic approach to fractals
  [Hayashi, Edalat, Scriven, Coquand, ...]

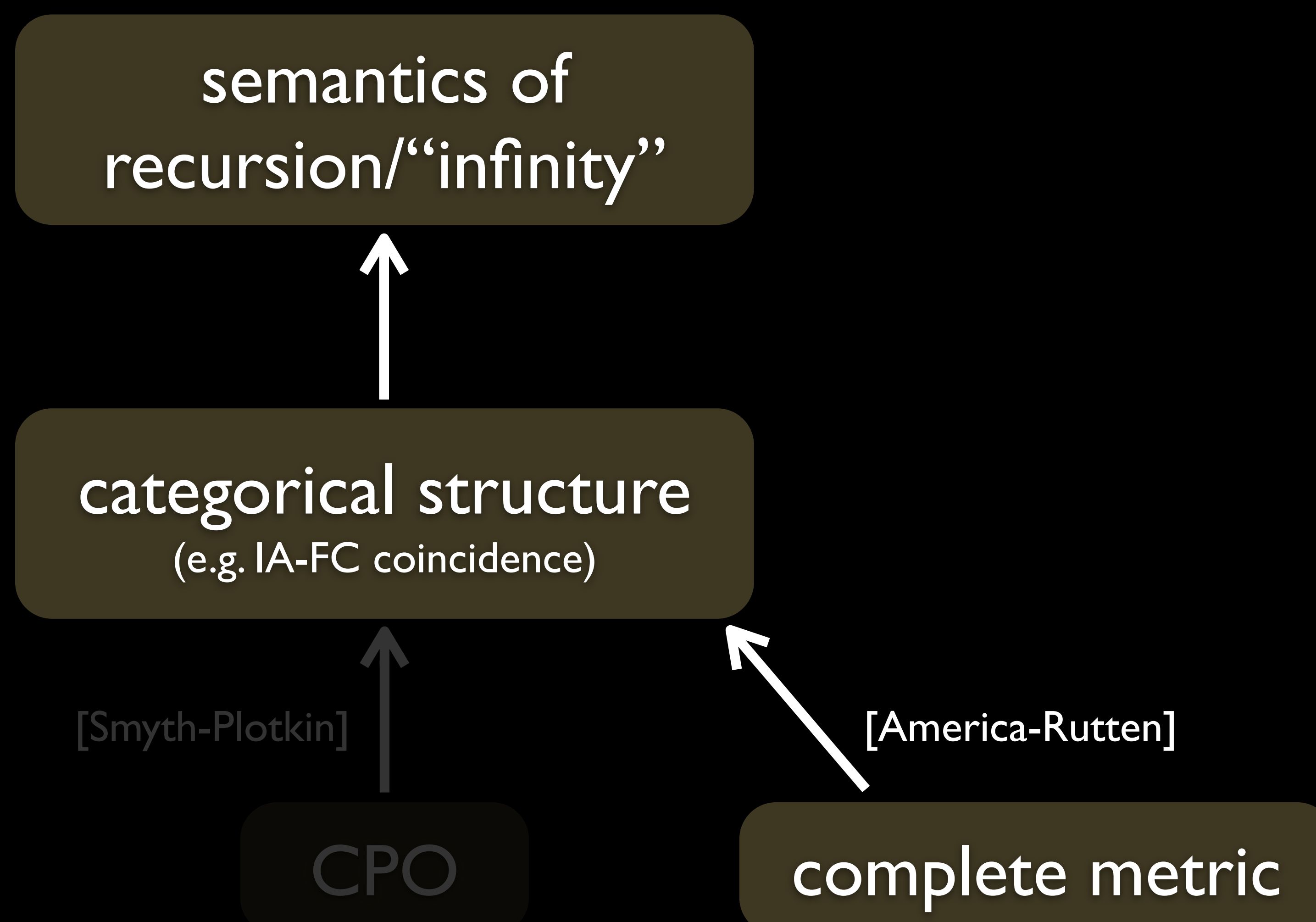
coalgebraic modeling
[Rutten, Jacobs, ...]



Related Work

- Domain-theoretic approach to fractals
  [Hayashi, Edalat, Scriven, Coquand, ...]

current work



Related Work

- Tom Leinster.
  A general theory of self-similarity I, II.

The Scenario

1. combinatorial spec.

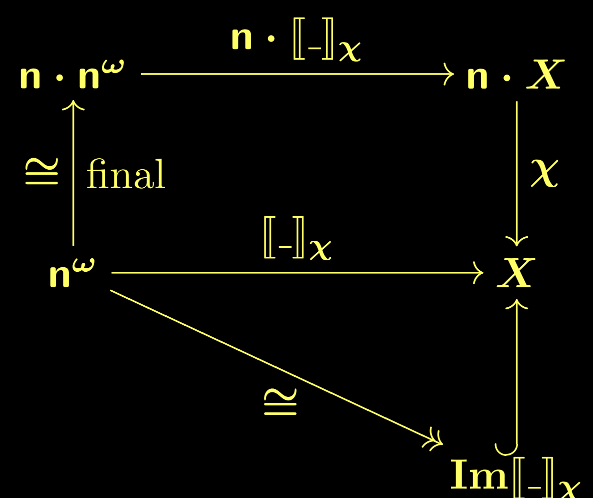
   $\mathbf{n} \cdot (\_) : \mathbf{Sets} \to \mathbf{Sets}$

2. symbolic representatives

   $$\mathbf{n}^\omega \xrightarrow[\cong]{\text{final}} \mathbf{n} \cdot \mathbf{n}^\omega$$

3. IFS

   $$\mathbf{n} \cdot \boldsymbol{X} \xrightarrow{\mathcal{X}} \boldsymbol{X}$$

4. 

with gluing

The Scenario

1. combinatorial spec.

   $$\mathbf{n}\cdot(\_\,):\mathbf{Sets}\to\mathbf{Sets}$$

2. symbolic representatives

   $$\begin{array}{c}\mathbf{n}\cdot\mathbf{n}^{\omega}\\ \text{final}\uparrow\cong\\ \mathbf{n}^{\omega}\end{array}$$

3. IFS

   $$\begin{array}{c}\mathbf{n}\cdot\boldsymbol{X}\\ \chi\downarrow\\ \boldsymbol{X}\end{array}$$

4. $$\begin{array}{ccc}\mathbf{n}\cdot\mathbf{n}^{\omega} & \xrightarrow{\mathbf{n}\cdot[\![\text{-}]\!]_{\chi}} & \mathbf{n}\cdot\boldsymbol{X}\\ \cong\uparrow\text{final} & & \downarrow\chi\\ \mathbf{n}^{\omega} & \xrightarrow{[\![\text{-}]\!]_{\chi}} & \boldsymbol{X}\\ & \searrow\cong & \uparrow\\ & & \mathbf{Im}[\![\text{-}]\!]_{\chi}\end{array}$$

---

1. combinatorial spec.
2. symbolic representatives
3. IFS
4.

Fractal as Presheaf Coalgebra [Leinster]

- **Aim** Turn

$$\varphi_0, \varphi_1 \;:\; \mathbb{C} \longrightarrow \mathbb{C} \;,$$
$$\varphi_0(x) = \frac{x}{2} \;,$$
$$\varphi_1(x) = \frac{1+x}{2} \;.$$

  into an *injective* X

Fractal as Presheaf Coalgebra [Leinster]

- **Aim** Turn

$$\varphi_0, \varphi_1 \; : \; \mathbb{C} \longrightarrow \mathbb{C} \; ,$$
$$\varphi_0(x) = \frac{x}{2} \; ,$$
$$\varphi_1(x) = \frac{1+x}{2} \; .$$

into an *injective* X

⇨ "mod out" via *presheaves* and *modules*

Fractal as Presheaf Coalgebra [Leinster]

- **Aim** Turn

$$\varphi_0, \varphi_1 : \mathbb{C} \longrightarrow \mathbb{C} \ ,$$
$$\varphi_0(x) = \frac{x}{2} \ ,$$
$$\varphi_1(x) = \frac{1+x}{2} \ .$$

into an *injective* X

In this talk:

- presheaf $P : \mathbb{A} \to \mathbf{Sets}$
- module $\dfrac{M : \mathbb{A} \longrightarrow\!\!\!\!\!\!+\ \mathbb{B}}{M : \mathbb{A}^{\mathrm{op}} \times \mathbb{B} \longrightarrow \mathbf{Sets}}$

⇒ "mod out" via *presheaves* and *modules*

Module

(bimodule/distributor/profunctor)

$$\frac{\boldsymbol{M} : \mathbb{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{A}\ , \quad \text{a module}}{\boldsymbol{M} : \mathbb{A}^{\mathbf{op}} \times \mathbb{A} \longrightarrow \mathbf{Sets}\ , \quad \text{a functor}}$$

Module

(bimodule/distributor/profunctor)

$$\frac{\boldsymbol{M} : \mathbb{A} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathbb{A} \;\;, \quad \text{a module}}{\boldsymbol{M} : \mathbb{A}^{\mathbf{op}} \times \mathbb{A} \longrightarrow \mathbf{Sets} \;\;, \quad \text{a functor}}$$

- $\dfrac{\text{relation}}{\text{function}} = \dfrac{\text{module}}{\text{functor}}$

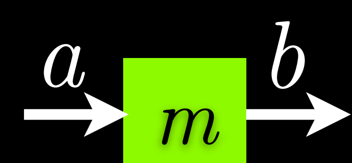
Module

(bimodule/distributor/profunctor)

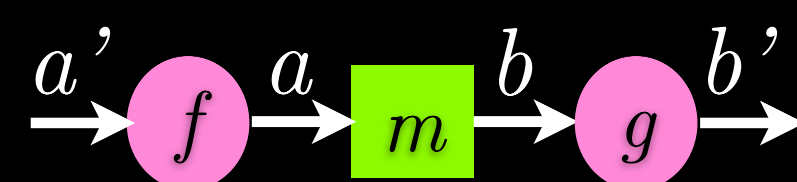
$$\frac{M : \mathbb{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{A} \ , \quad \text{a module}}{M : \mathbb{A}^{\mathbf{op}} \times \mathbb{A} \longrightarrow \mathbf{Sets} \ , \quad \text{a functor}}$$

- $\dfrac{\text{relation}}{\text{function}} = \dfrac{\text{module}}{\text{functor}}$

- *sets with left and right* $\mathbb{A}$*-action*

$M(a,b) = \{$ 

$\}$

$M(f,g)m = g \cdot m \cdot f =$ 

Module

(bimodule/distributor/profunctor)

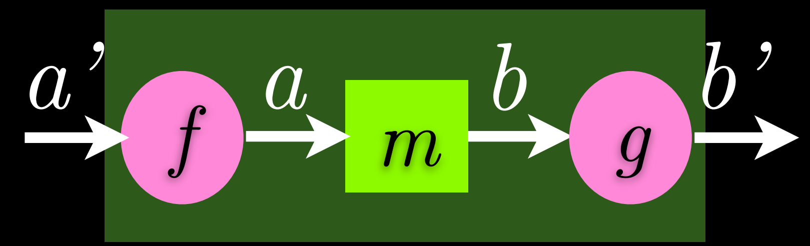
$$\frac{M : \mathbb{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{A} \;\;, \qquad \text{a module}}{M : \mathbb{A}^{\mathbf{op}} \times \mathbb{A} \longrightarrow \mathbf{Sets} \;\;, \qquad \text{a functor}}$$

- $\dfrac{\text{relation}}{\text{function}} = \dfrac{\text{module}}{\text{functor}}$

- *sets with left and right* $\mathbb{A}$*-action*

$$M(a,b) = \{ \xrightarrow{a} \boxed{m} \xrightarrow{b} \}$$

$$M(f,g)m = g \cdot m \cdot f =$$



The Base Category $\mathbb{A}_{\mathbb{I}}$

$$\mathbb{A}_{\mathbb{I}} \quad = \quad \left(\mathbf{0} \underset{r}{\overset{l}{\rightrightarrows}} \mathbf{1} \right)$$

The Base Category $\mathbb{A}_{\mathbb{I}}$

$$\mathbb{A}_{\mathbb{I}} \quad = \quad \left(\mathbf{0} \overset{l}{\underset{r}{\rightrightarrows}} \mathbf{1} \right)$$

$$\begin{array}{llcl} P_{\mathbb{I}} : & \mathbb{A}_{\mathbb{I}} & \longrightarrow & \mathbf{Sets} \\ & \mathbf{0} \\ & l \downarrow\downarrow r & \longmapsto & \{*\} \\ & \mathbf{1} & & \mathbf{0} \downarrow\downarrow \mathbf{1} \\ & & & \mathbb{I} \end{array}$$

The Base Category $\mathbb{A}_{\mathbb{I}}$

$$\mathbb{A}_{\mathbb{I}} \quad = \quad \left(\mathbf{0} \overset{l}{\underset{r}{\rightrightarrows}} \mathbf{1} \right)$$

$$\begin{array}{cccc} P_{\mathbb{I}} : & \mathbb{A}_{\mathbb{I}} & \longrightarrow & \mathbf{Sets} \\ & \begin{array}{c} \mathbf{0} \\ l \downarrow\downarrow r \\ \mathbf{1} \end{array} & \longmapsto & \begin{array}{c} \{*\} \\ 0 \downarrow\downarrow 1 \\ \mathbb{I} \end{array} \end{array}$$

Combinatorial Specification as a Module

$$\frac{M_{\mathbb{I}} : \mathbb{A}_{\mathbb{I}} \longrightarrow\!\!\!\!\!\!+\;\; \mathbb{A}_{\mathbb{I}}}{M_{\mathbb{I}} : \mathbb{A}_{\mathbb{I}}^{\mathbf{op}} \times \mathbb{A}_{\mathbb{I}} \longrightarrow \mathbf{Sets}}$$

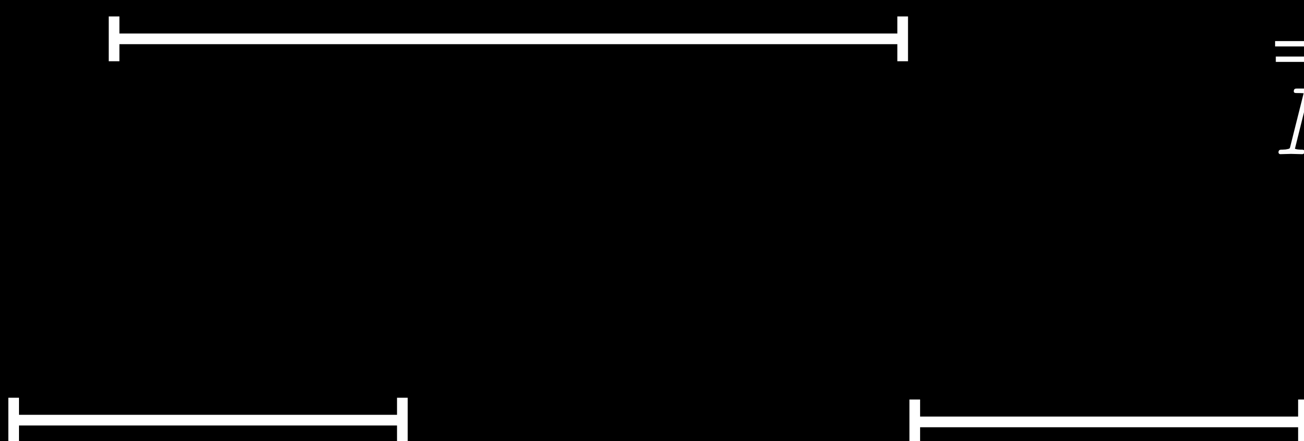
gluing, indeed!

Combinatorial Specification as a Module

$$\frac{M_{\mathbb{I}} : \mathbb{A}_{\mathbb{I}} \longrightarrow\!\!\!\!\!+\; \mathbb{A}_{\mathbb{I}}}{M_{\mathbb{I}} : \mathbb{A}_{\mathbb{I}}^{\mathbf{op}} \times \mathbb{A}_{\mathbb{I}} \longrightarrow \mathbf{Sets}}$$

gluing, indeed!

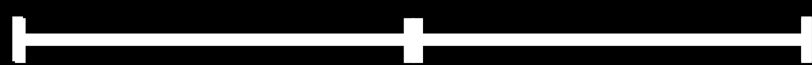
Combinatorial Specification as a Module



$$\frac{M_{\mathbb{I}} : \mathbb{A}_{\mathbb{I}} \longrightarrow\!\!\!\!\!\!+\;\; \mathbb{A}_{\mathbb{I}}}{M_{\mathbb{I}} : \mathbb{A}_{\mathbb{I}}^{\mathbf{op}} \times \mathbb{A}_{\mathbb{I}} \longrightarrow \mathbf{Sets}}$$

gluing, indeed!

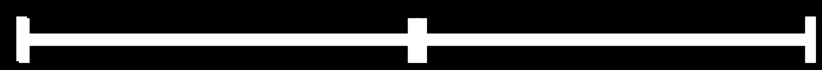
Combinatorial Specification as a Module



$$\frac{M_{\mathbb{I}} : \mathbb{A}_{\mathbb{I}} \longrightarrow\!\!\!\!\!\!\!+\;\;\; \mathbb{A}_{\mathbb{I}}}{M_{\mathbb{I}} : \mathbb{A}_{\mathbb{I}}^{\mathbf{op}} \times \mathbb{A}_{\mathbb{I}} \longrightarrow \mathbf{Sets}}$$

gluing, indeed!

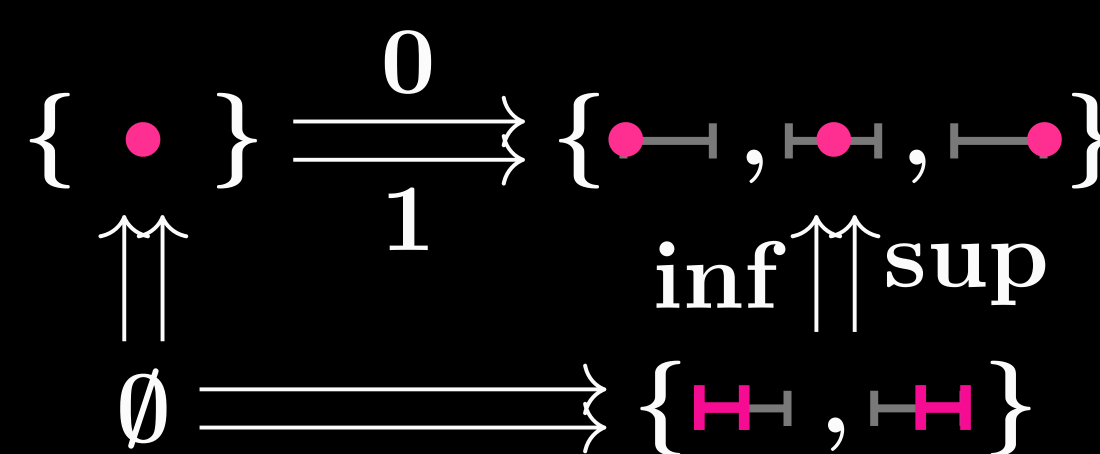
Combinatorial Specification as a Module



$$\frac{M_{\mathbb{I}} : \mathbb{A}_{\mathbb{I}} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{A}_{\mathbb{I}}}{M_{\mathbb{I}} : \mathbb{A}_{\mathbb{I}}^{\mathrm{op}} \times \mathbb{A}_{\mathbb{I}} \longrightarrow \mathbf{Sets}}$$

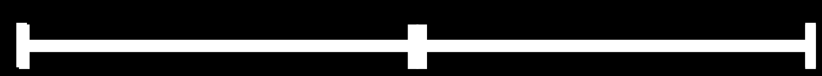
$$\begin{array}{ccc} M_{\mathbb{I}}(0,0) & \overset{l\cdot\_}{\underset{r\cdot\_}{\rightrightarrows}} & M_{\mathbb{I}}(0,1) \\ {\scriptstyle \_\cdot l}\upuparrows{\scriptstyle \_\cdot r} & & \upuparrows \\ M_{\mathbb{I}}(1,0) & \rightrightarrows & M_{\mathbb{I}}(1,1) \end{array}$$

$=$



gluing, indeed!

Combinatorial Specification as a Module

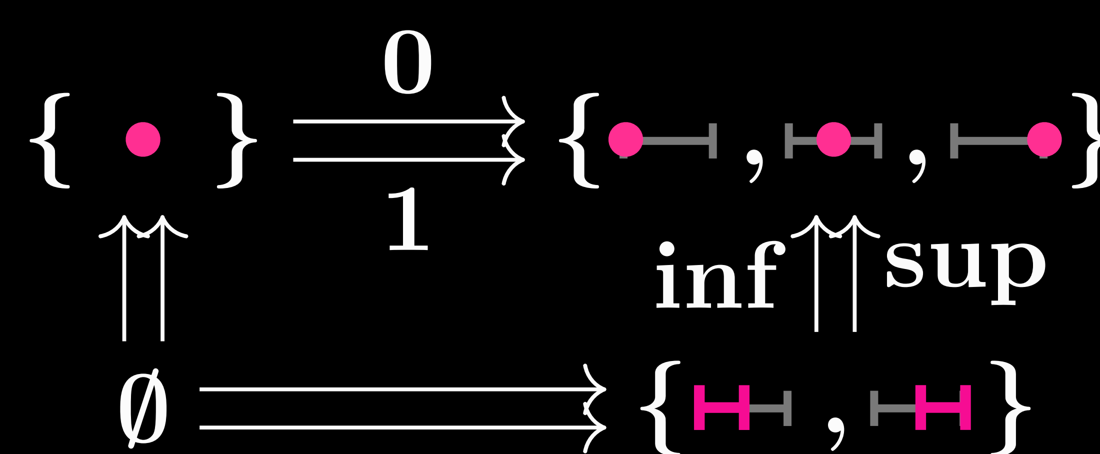


$$M_{\mathbb{I}} : \mathbb{A}_{\mathbb{I}} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{A}_{\mathbb{I}}$$

$$M(a,b) = \{ \xrightarrow{a} \boxed{m} \xrightarrow{b} \}$$

how many *ingredients* (a-shapes) are used in the *outcome* (b-shape)

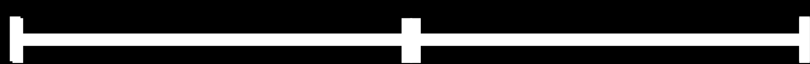
$$\begin{array}{ccc} M_{\mathbb{I}}(0,0) & \overset{l\cdot\_}{\underset{r\cdot\_}{\rightrightarrows}} & M_{\mathbb{I}}(0,1) \\ {\scriptstyle \_\cdot l}\upuparrows{\scriptstyle \_\cdot r} & & \upuparrows \\ M_{\mathbb{I}}(1,0) & \Longrightarrow & M_{\mathbb{I}}(1,1) \end{array} =$$



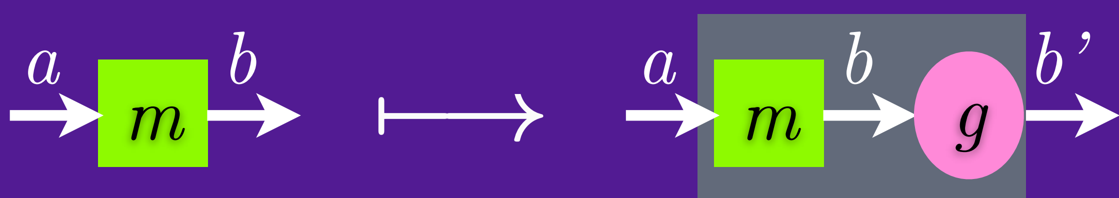
gluing, indeed!

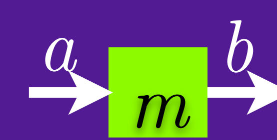
Combinatorial Specification as a Module

$$M_{\mathbb{I}} : \mathbb{A}_{\mathbb{I}} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathbb{A}_{\mathbb{I}}$$



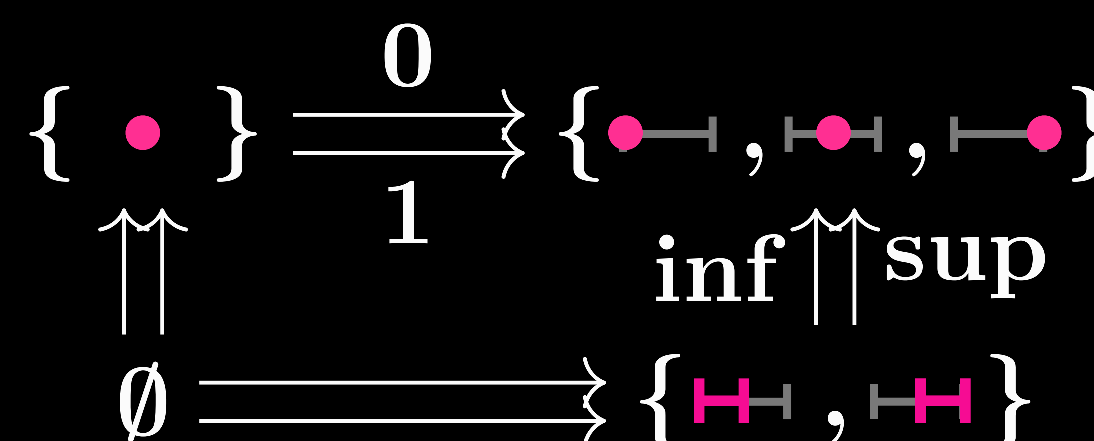
$M(\mathrm{id}, g) = g \cdot \_ \;:$



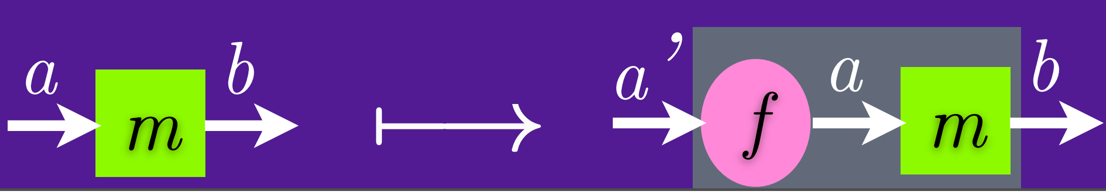
$M(a, b) = \{$  $\}$

how many *ingredients* (a-shapes) are used in the *outcome* (b-shape)

$$\begin{array}{ccc} M_{\mathbb{I}}(0,0) & \overset{l\cdot\_}{\underset{r\cdot\_}{\rightrightarrows}} & M_{\mathbb{I}}(0,1) \\ {\scriptstyle \_\cdot l}\upuparrows{\scriptstyle \_\cdot r} & & \upuparrows \\ M_{\mathbb{I}}(1,0) & \Longrightarrow & M_{\mathbb{I}}(1,1) \end{array} \quad = $$



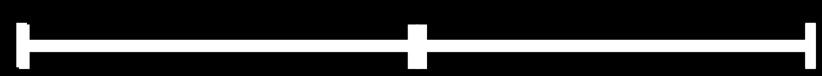
$M(f, \mathrm{id}) = \_ \cdot f \;:$



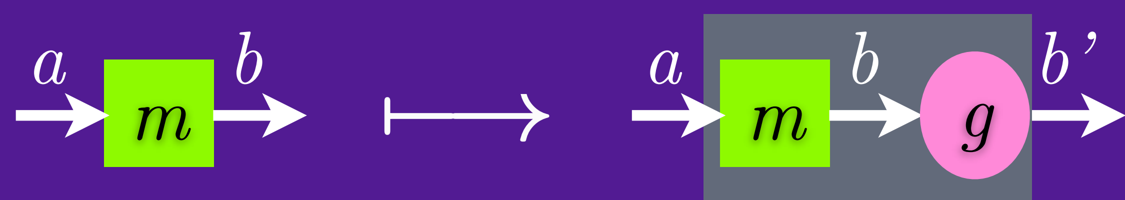
gluing, indeed!

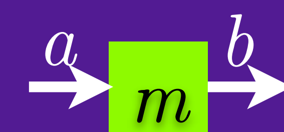
Combinatorial Specification as a Module

$$M_{\mathbb{I}} : \mathbb{A}_{\mathbb{I}} \longrightarrow\!\!\!\!\!\!\!+\ \ \mathbb{A}_{\mathbb{I}}$$



$M(\mathrm{id}, g) = g \cdot \_$:



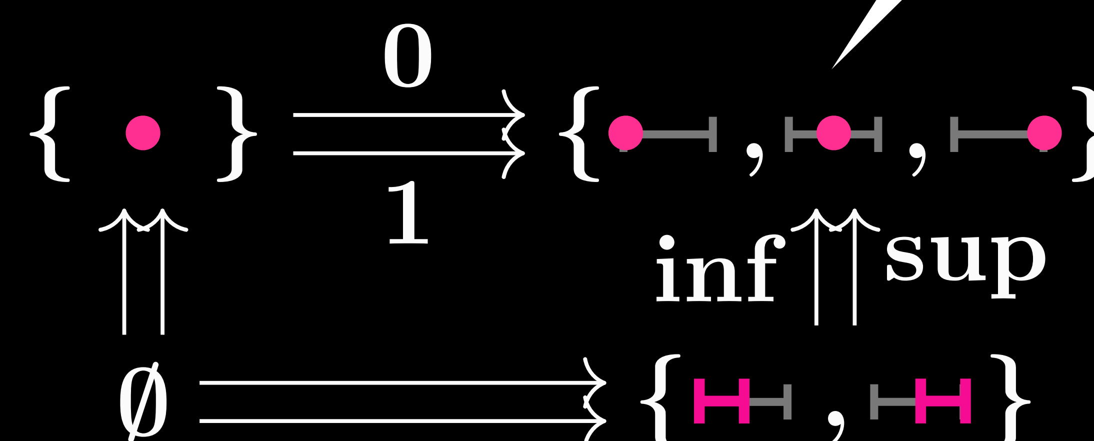
$M(a,b) = \{$  $\}$

how many *ingredients* (a-shapes) are used in the *outcome* (b-shape)

gluing

$$\begin{array}{ccc}
M_{\mathbb{I}}(0,0) & \overset{l\cdot_}{\underset{r\cdot_}{\rightrightarrows}} & M_{\mathbb{I}}(0,1) \\
{\scriptstyle _\cdot l}\upuparrows{\scriptstyle _\cdot r} & & \upuparrows \\
M_{\mathbb{I}}(1,0) & \rightrightarrows & M_{\mathbb{I}}(1,1)
\end{array}$$

$=$



$M(f, \mathrm{id}) = \_ \cdot f$:

gluing, indeed!

with gluing

The Scenario

1. combinatorial spec.

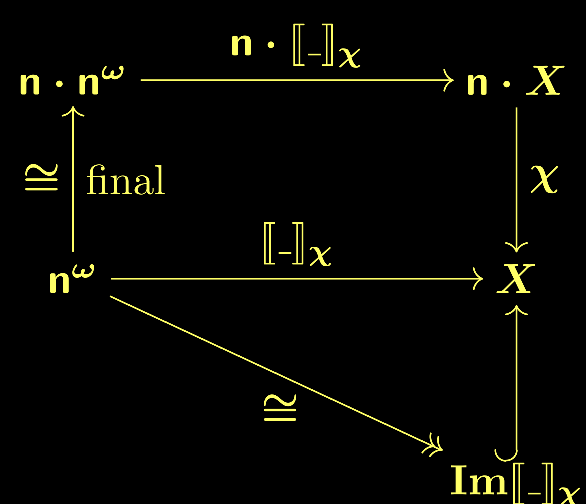
   $\mathbf{n} \cdot (\_\,) : \mathbf{Sets} \to \mathbf{Sets}$

2. symbolic representatives

   $\mathbf{n}^\omega \xrightarrow[\text{final}]{\cong} \mathbf{n} \cdot \mathbf{n}^\omega$

3. IFS

   $\mathbf{n} \cdot X \xrightarrow{\chi} X$

4. 

1. combinatorial spec.

   $(\mathbb{A}\,,\ M : \mathbb{A} \nrightarrow \mathbb{A})$

2. symbolic representatives

3. IFS

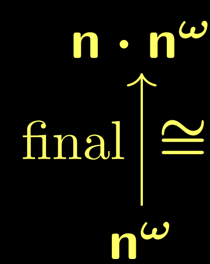
4.

The Scenario

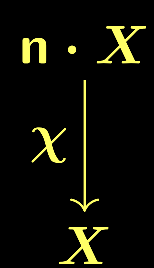
1. combinatorial spec.

   $\mathbf{n} \cdot (\_) : \mathbf{Sets} \to \mathbf{Sets}$

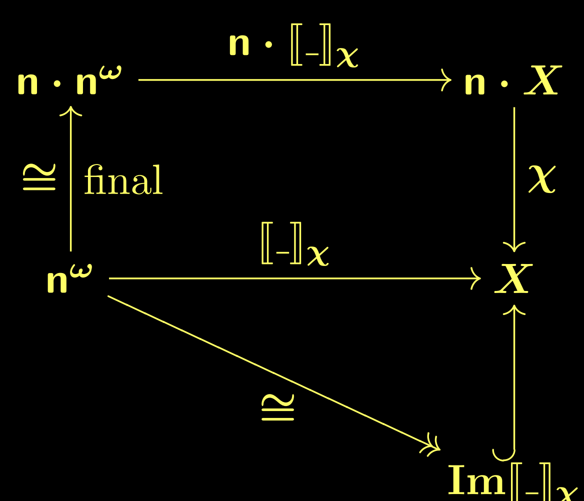
2. symbolic representatives



3. IFS



4.



1. combinatorial spec.

   $(\mathbb{A}\,,\ M : \mathbb{A} \nrightarrow \mathbb{A})$

2. symbolic representatives

3. IFS

4.

Tensor Product

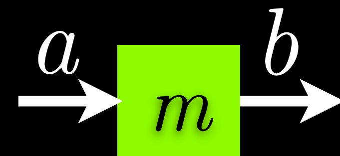
$$M \otimes (\_\,) : \quad \mathbf{Sets}^{\mathbb{A}} \longrightarrow \mathbf{Sets}^{\mathbb{A}}$$

Tensor Product

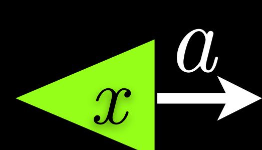
$$\begin{array}{rccc} M \otimes (\_\,) : & \mathbf{Sets}^{\mathbb{A}} & \longrightarrow & \mathbf{Sets}^{\mathbb{A}} \\ & P & \longmapsto & M \otimes P \end{array}$$

Tensor Product

$$M \otimes (\_\,) : \quad \mathbf{Sets}^{\mathbb{A}} \longrightarrow \mathbf{Sets}^{\mathbb{A}}$$
$$P \longmapsto M \otimes P$$

$M(a,b) = \{$ 

$\}$

$P(a) = \{$ 

$\}$

Tensor Product

$$
\begin{array}{rccc}
M \otimes (\_) : & \mathbf{Sets}^{\mathbb{A}} & \longrightarrow & \mathbf{Sets}^{\mathbb{A}} \\
 & P & \longmapsto & M \otimes P
\end{array}
$$

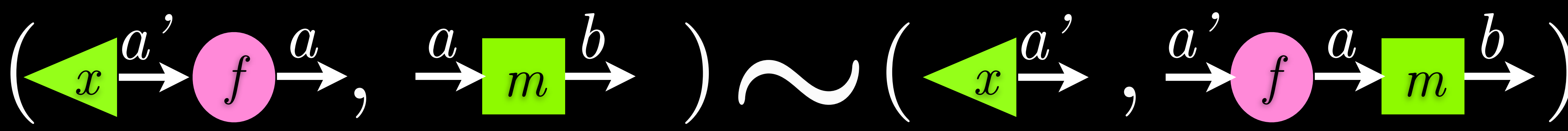
$$M(a,b) = \{ \xrightarrow{a} \boxed{m} \xrightarrow{b} \}$$

$$P(a) = \{ \triangleleft x \xrightarrow{a} \}$$

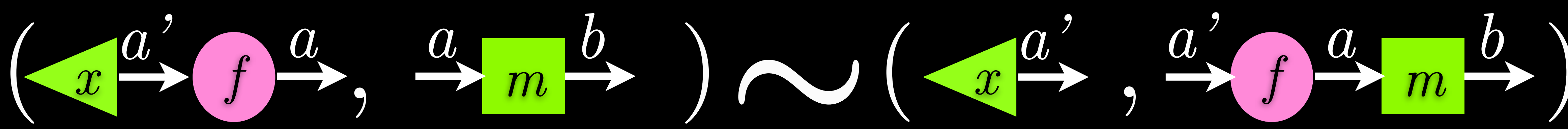
$$(M \otimes P)b = \Big(\coprod_{a \in \mathbb{A}} P(a) \times M(a,b) \Big) / \sim$$

$$= \Big\{ \big(\triangleleft x \xrightarrow{a} \, , \, \xrightarrow{a} \boxed{m} \xrightarrow{b} \big) \,\Big|\, a \in \mathbb{A} \Big\} / \sim$$

Tensor Product



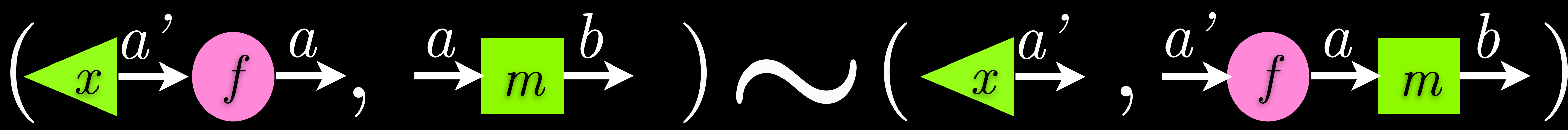
Tensor Product



- cf. tensor product of vector spaces

$$cx \otimes y = x \otimes cy$$

Tensor Product

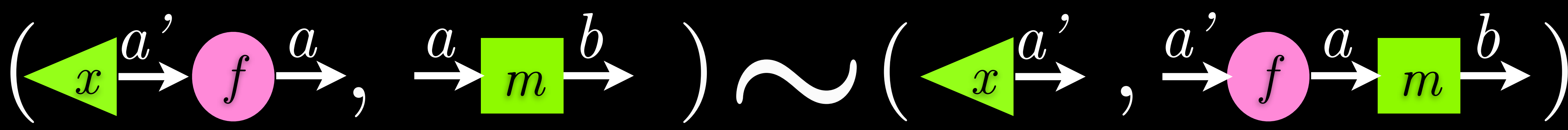


- cf. tensor product of vector spaces

$$cx \otimes y = x \otimes cy$$

- def. by coend: $(M \otimes P)b = \int^{a \in \mathbb{A}} P(a) \times M(a,b)$

Tensor Product



- cf. tensor product of vectors spaces

$$cx \otimes y = x \otimes cy$$

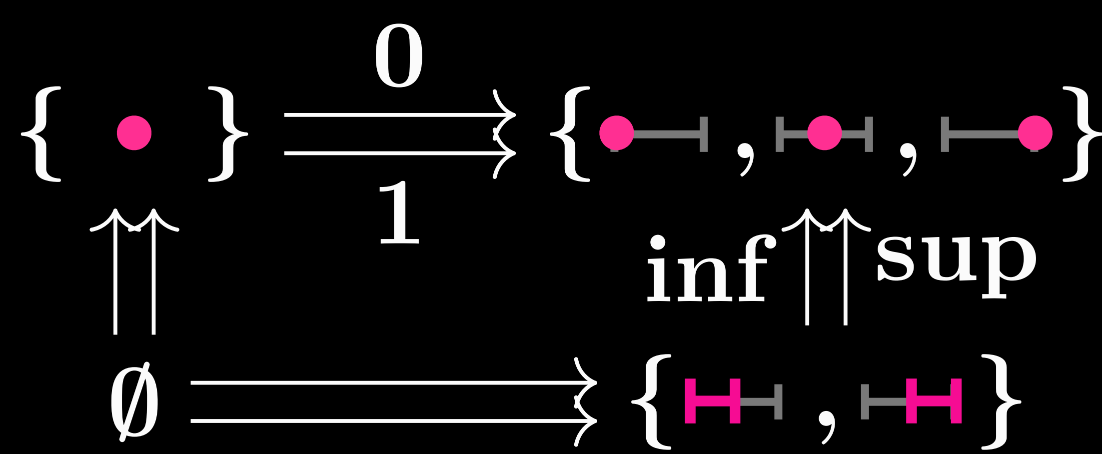
- def. by coend: $(M \otimes P)b = \int^{a \in \mathbb{A}} P(a) \times M(a,b)$

- In the bicategory $\mathbf{Prof}/\mathbf{Dist}$:

$$\frac{1 \overset{P}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} \mathbb{A} \qquad \mathbb{A} \overset{M}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} \mathbb{A}}{M \otimes P : \; 1 \overset{P}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} \mathbb{A} \overset{M}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} \mathbb{A}}$$

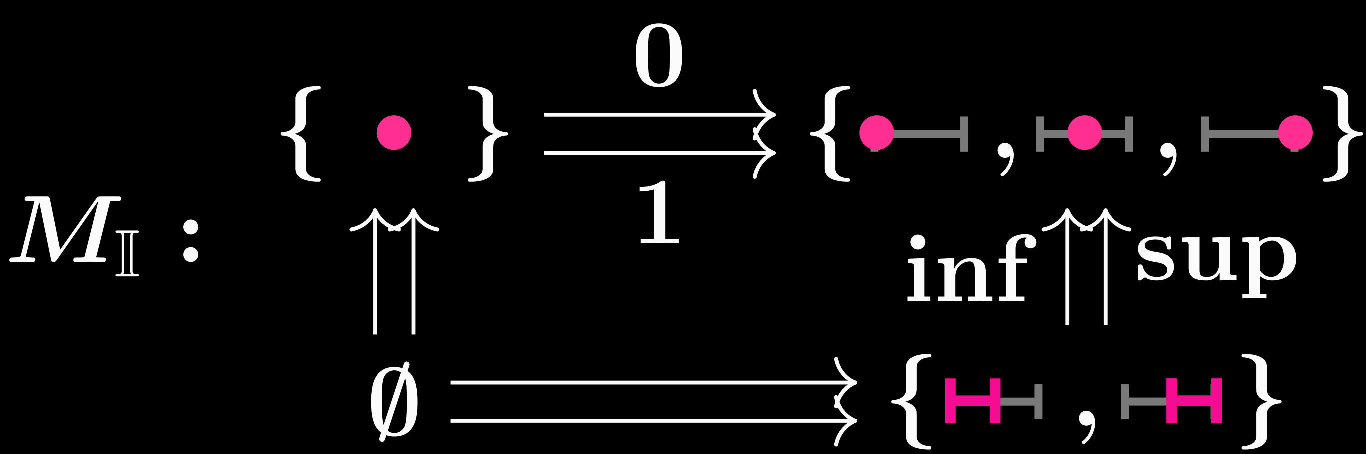
Tensor Product

$M_{\mathbb{I}}$:



$P_{\mathbb{I}} : \{*\} \rightrightarrows \mathbb{I}$

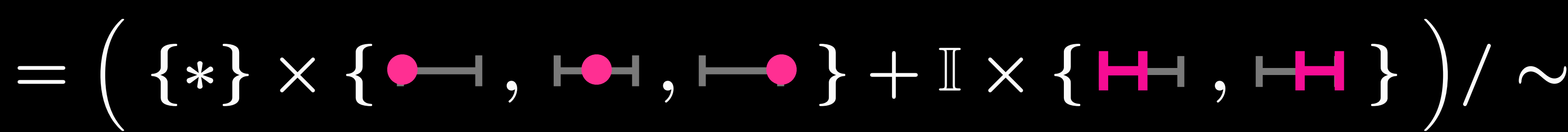
Tensor Product



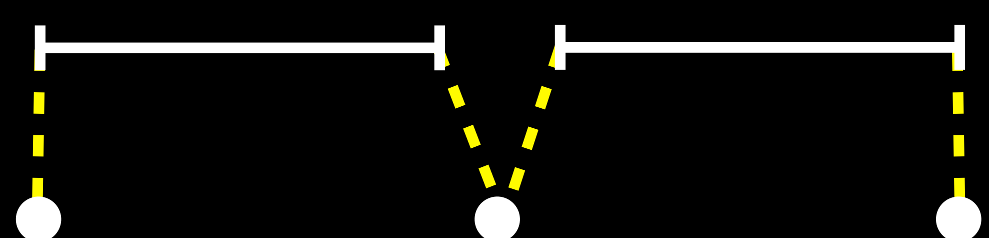
$P_{\mathbb{I}} : \{*\} \rightrightarrows \mathbb{I}$

$(M_{\mathbb{I}} \otimes P_{\mathbb{I}})_1$

$= \Big(P_{\mathbb{I}}(0) \times M_{\mathbb{I}}(0,1) + P_{\mathbb{I}}(1) \times M_{\mathbb{I}}(1,1) \Big) / \sim$



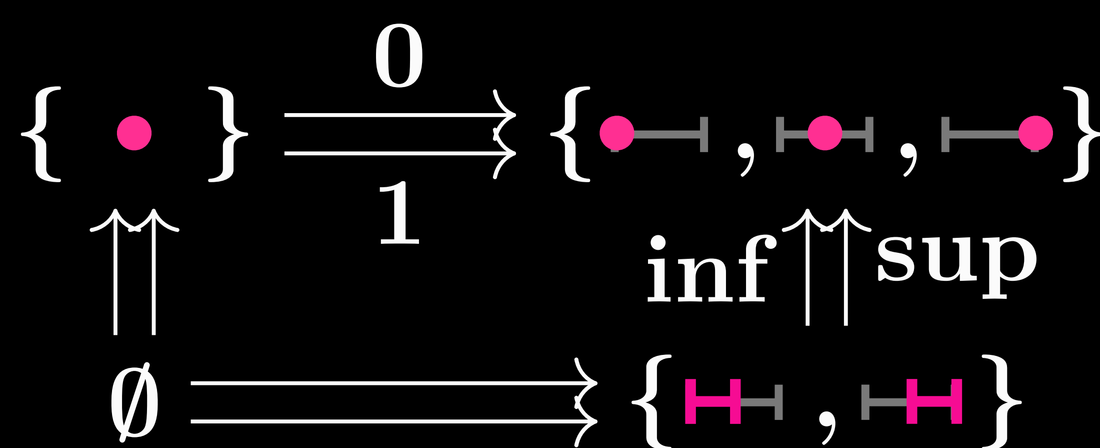
=



$\cong \mathbb{I}$

Tensor Product

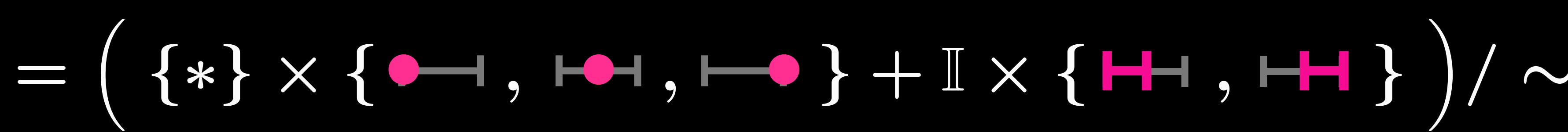
$M_{\mathbb{I}}$:



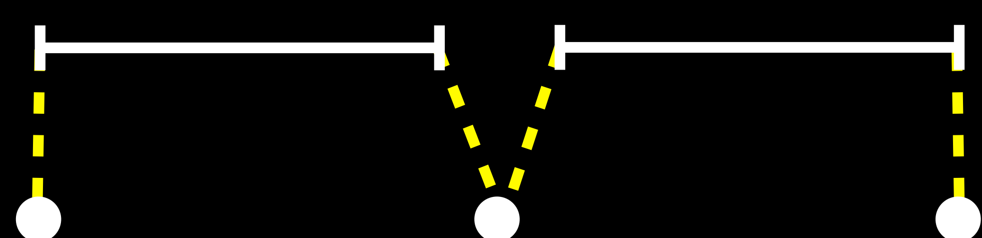
$P_{\mathbb{I}} : \{*\} \rightrightarrows \mathbb{I}$

$(M_{\mathbb{I}} \otimes P_{\mathbb{I}})_1$

$= \Big(P_{\mathbb{I}}(0) \times M_{\mathbb{I}}(0,1) + P_{\mathbb{I}}(1) \times M_{\mathbb{I}}(1,1) \Big) / \sim$



$=$



$\therefore M_{\mathbb{I}} \otimes P_{\mathbb{I}} \cong P_{\mathbb{I}}$

$\cong \mathbb{I}$

with gluing

The Scenario

1. combinatorial spec.

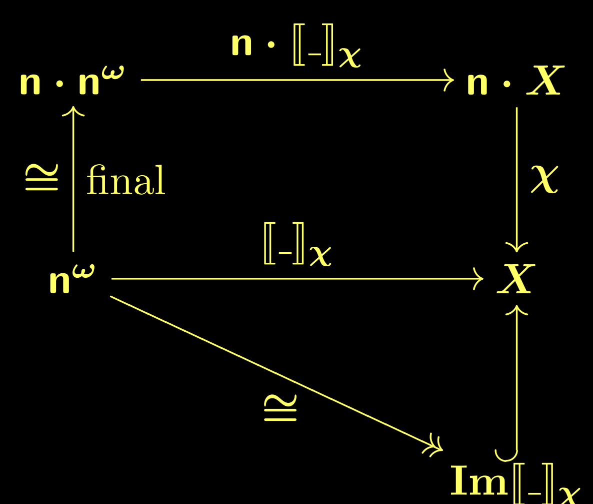
   $\mathbf{n}\cdot(\_) : \mathbf{Sets} \to \mathbf{Sets}$

2. symbolic representatives

   $$\mathbf{n}^\omega \xrightarrow[\cong]{\text{final}} \mathbf{n}\cdot\mathbf{n}^\omega$$

3. IFS

   $$\mathbf{n}\cdot X \xrightarrow{\chi} X$$

4. 

1. combinatorial spec.

   $(\mathbb{A},\ M : \mathbb{A} \nrightarrow \mathbb{A})$

2. symbolic representatives

3. IFS

4.

with gluing

The Scenario

1. combinatorial spec.

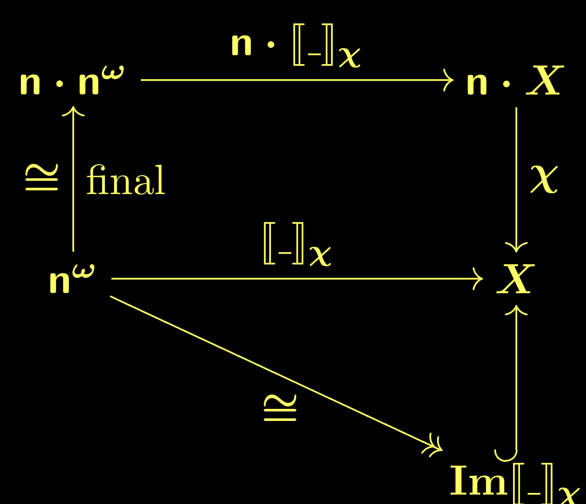
   $\mathbf{n} \cdot (\_) : \mathbf{Sets} \to \mathbf{Sets}$

2. symbolic representatives

   $$\begin{array}{c} \mathbf{n} \cdot \mathbf{n}^{\omega} \\ \text{final} \uparrow \cong \\ \mathbf{n}^{\omega} \end{array}$$

3. IFS

   $$\begin{array}{c} \mathbf{n} \cdot X \\ \chi \downarrow \\ X \end{array}$$

4. 

1. combinatorial spec.

   $M \otimes (\_) : \mathbf{Sets}^{\mathbb{A}} \to \mathbf{Sets}^{\mathbb{A}}$

2. symbolic representatives

3. IFS

4.

with gluing

The Scenario

1. combinatorial spec.

   $\mathbf{n} \cdot (\_) : \mathbf{Sets} \to \mathbf{Sets}$

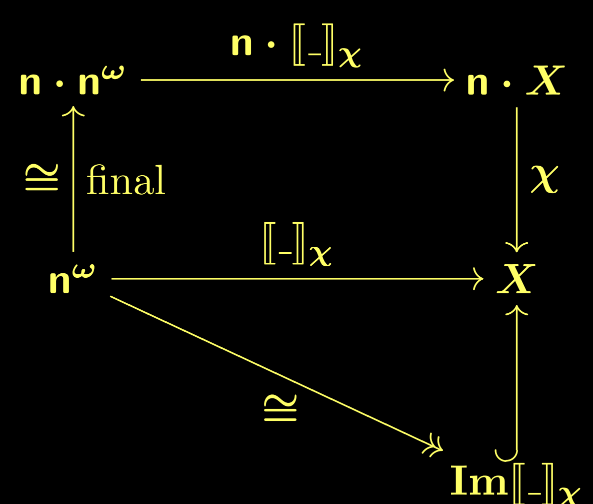
2. symbolic representatives

   $\mathbf{n}^\omega \xrightarrow[\cong]{\text{final}} \mathbf{n} \cdot \mathbf{n}^\omega$

3. IFS

   $\mathbf{n} \cdot \boldsymbol{X} \xrightarrow{\chi} \boldsymbol{X}$

4.



1. combinatorial spec.

   $M \otimes (\_) : \mathbf{Sets}^{\mathbb{A}} \to \mathbf{Sets}^{\mathbb{A}}$

2. symbolic representatives

   $Z \xrightarrow[\cong]{\text{final}} M \otimes Z$

3. IFS

4.

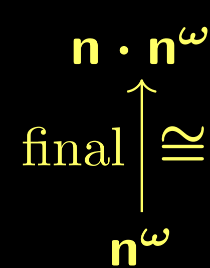
with gluing

The Scenario

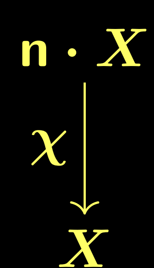
1. combinatorial spec.

   $\mathbf{n} \cdot (\_) : \mathbf{Sets} \to \mathbf{Sets}$

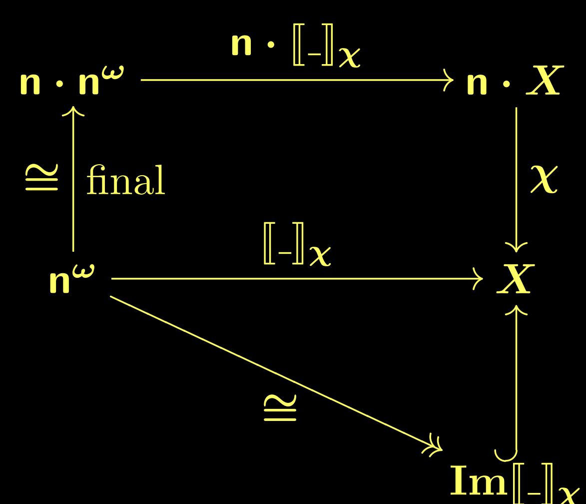
2. symbolic representatives



3. IFS



4.



1. combinatorial spec.

   $M \otimes (\_) : \mathbf{Sets}^{\mathbb{A}} \to \mathbf{Sets}^{\mathbb{A}}$

2. symbolic representatives

3. IFS

4.

Non-Degeneracy: “Only Forced Equalities”

$$M_{\mathbb{I}} \otimes P_{\mathbf{deg}}$$

$$\text{final} \Big\uparrow \cong$$

$$P_{\mathbf{deg}}$$

$$P_{\mathbf{deg}} : \quad \mathbb{A}_{\mathbb{I}} \longrightarrow \mathbf{Sets}$$

$$\begin{matrix} \mathbf{0} \\ l\downarrow\downarrow r \\ \mathbf{1} \end{matrix} \quad \longmapsto \quad \begin{matrix} \{*\} \\ !\downarrow\downarrow ! \\ \{*\} \end{matrix}$$

Non-Degeneracy: "Only Forced Equalities"

$$M_{\mathbb{I}} \otimes P_{\mathbf{deg}} \overset{\text{final}}{\underset{\cong}{\longleftarrow}} P_{\mathbf{deg}}$$

$$\begin{array}{cccc} P_{\mathbf{deg}} : & \mathbb{A}_{\mathbb{I}} & \longrightarrow & \mathbf{Sets} \\ & \mathbf{0} & & \{*\} \\ & l\downarrow\downarrow r & \longmapsto & !\downarrow\downarrow ! \\ & \mathbf{1} & & \{*\} \end{array}$$

- **Def.** $P : \mathbb{A} \to \text{Sets}$ is *non-degenerate* if, in $\text{el}(P)$

(ND1)

$$\begin{array}{ccc} (a,x) & & (a',x') \\ & {}_{f}\searrow \quad \swarrow_{f'} & \\ & (b,y) & \end{array} \Longrightarrow \begin{array}{ccc} & \exists(c,z) & \\ & {}^{\exists g}\swarrow \quad \searrow^{\exists g'} & \\ (a,x) & & (a',x') \\ & {}_{f}\searrow \quad \swarrow_{f'} & \\ & (b,y) & \end{array}$$

(ND2)

$$\begin{array}{c} (a,x) \\ f'\downarrow\downarrow f \\ (b,y) \end{array} \Longrightarrow \begin{array}{c} \exists(c,z) \\ \exists g\downarrow \\ (a,x) \\ f'\downarrow\downarrow f \\ (b,y) \end{array}$$

Non-Degeneracy: "Only Forced Equalities"

- **Prop.** $P : \quad \mathbb{A}_{\mathbb{I}} \longrightarrow \mathbf{Sets}$

$$\begin{array}{c} \mathbf{0} \\ l\downarrow\downarrow r \\ \mathbf{1} \end{array} \longmapsto \begin{array}{c} P(\mathbf{0}) \\ P(l)\downarrow\downarrow P(r) \\ P(\mathbf{1}) \end{array}$$

  is non-degenerate iff

  - $P(l)$ and $P(r)$ are injective
  - their images are disjoint

Final *Non-Degenerate* Coalgebra

- **<u>Prop.</u>** [Freyd]

$$\begin{array}{c} M_{\mathbb{I}} \otimes P_{\mathbb{I}} \\ \uparrow \cong \\ P_{\mathbb{I}} \end{array}$$

is a final *non-degenerate* coalgebra

of symboic nature

Final *Non-Degenerate* Coalgebra

- **Prop.** [Freyd]

$$\begin{array}{c} M_{\mathbb{I}} \otimes P_{\mathbb{I}} \\ \uparrow \cong \\ P_{\mathbb{I}} \end{array}$$

is a final *non-degenerate* coalgebra

- **Thm.** (Leinster's construction)

$$\begin{array}{c} M \otimes I \\ \text{final ND} \uparrow \cong \\ I \end{array}$$

$$I(a) = \{ \cdots \xrightarrow{a_3} m_3 \xrightarrow{a_2} m_2 \xrightarrow{a_1} m_1 \xrightarrow{a} \} / \sim$$

of symboic nature

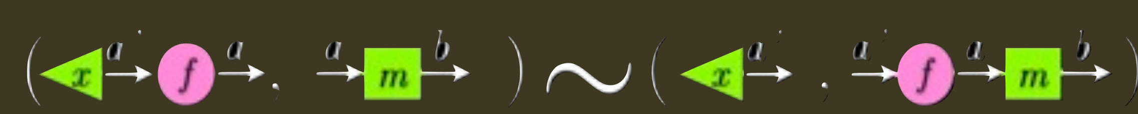
Final *Non-Degenerate* Coalgebra

- **Prop.** [Freyd]

$$\begin{array}{c} M_{\mathbb{I}} \otimes P_{\mathbb{I}} \\ \uparrow \cong \\ P_{\mathbb{I}} \end{array}$$

is a final *non-degenerate* coalgebra

cf. in $M \otimes P$



- **Thm.** (Leinster's construction)

$$\begin{array}{c} M \otimes I \\ \text{final ND} \uparrow \cong \\ I \end{array}$$

$$I(a) = \{ \cdots \xrightarrow{a_3} m_3 \xrightarrow{a_2} m_2 \xrightarrow{a_1} m_1 \xrightarrow{a} \} / \sim$$

of symboic nature

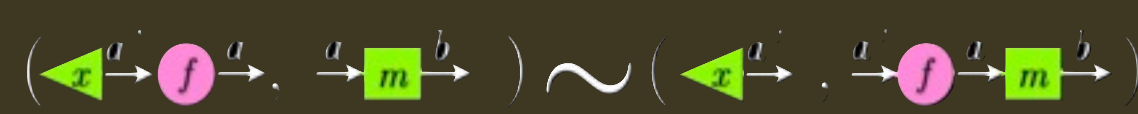
Final *Non-Degenerate* Coalgebra

- **Prop.** [Freyd]

$$\begin{array}{c} M_{\mathbb{I}} \otimes P_{\mathbb{I}} \\ \uparrow \cong \\ P_{\mathbb{I}} \end{array}$$

is a final *non-degenerate* coalgebra

cf. in $M \otimes P$



- **Thm.** (Leinster's construction)

$$\begin{array}{c} M \otimes I \\ \text{final ND} \uparrow \cong \\ I \end{array}$$

$$I(a) = \{ \cdots \xrightarrow{a_3} \boxed{m_3} \xrightarrow{a_2} \boxed{m_2} \xrightarrow{a_1} \boxed{m_1} \xrightarrow{a} \} / \sim$$

symbolic!

of symboic nature

with gluing

The Scenario

1. combinatorial spec.

   $\mathbf{n} \cdot (\_) : \mathbf{Sets} \to \mathbf{Sets}$

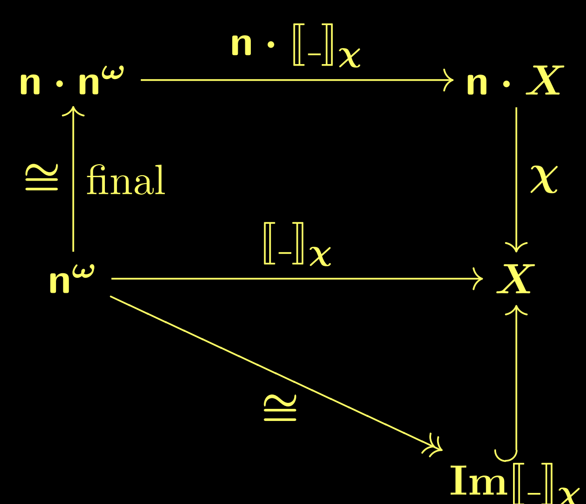
2. symbolic representatives

   $$\begin{array}{c} \mathbf{n} \cdot \mathbf{n}^\omega \\ \text{final} \uparrow \cong \\ \mathbf{n}^\omega \end{array}$$

3. IFS

   $$\begin{array}{c} \mathbf{n} \cdot X \\ \chi \downarrow \\ X \end{array}$$

4.



1. combinatorial spec.

   $M \otimes (\_) : \mathbf{Sets}^{\mathbb{A}} \to \mathbf{Sets}^{\mathbb{A}}$

2. symbolic representatives

   $$\begin{array}{c} M \otimes Z \\ \text{final} \uparrow \cong \\ Z \end{array}$$

3. IFS

4.

with gluing

The Scenario

1. combinatorial spec.

   $\mathbf{n}\cdot(\_) : \mathbf{Sets} \to \mathbf{Sets}$

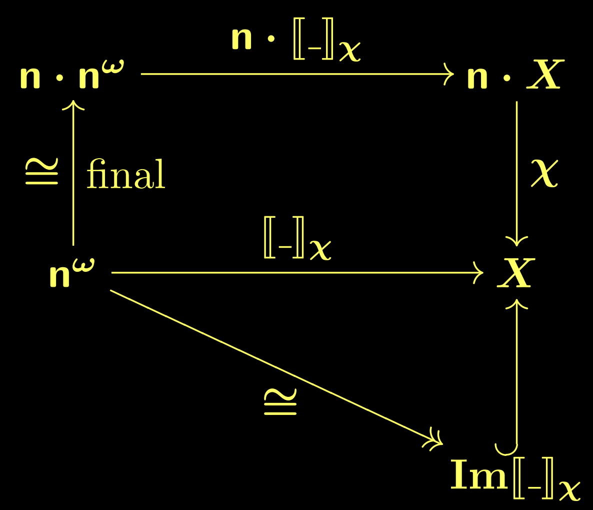
2. symbolic representatives

   $\mathbf{n}^\omega \xrightarrow[\cong]{\text{final}} \mathbf{n}\cdot\mathbf{n}^\omega$

3. IFS

   $\boldsymbol{X} \xleftarrow{\chi} \mathbf{n}\cdot\boldsymbol{X}$

4.



1. combinatorial spec.

   $M\otimes(\_) : \mathbf{Sets}^{\mathbb{A}} \to \mathbf{Sets}^{\mathbb{A}}$

2. symbolic representatives

   $I \xrightarrow[\cong]{\text{final ND}} M\otimes I$

3. IFS

4.

with gluing

The Scenario

1. combinatorial spec.

   $\mathbf{n} \cdot (\_) : \mathbf{Sets} \to \mathbf{Sets}$

2. symbolic representatives

   $$\begin{array}{c} \mathbf{n} \cdot \mathbf{n}^{\omega} \\ \text{final} \uparrow \cong \\ \mathbf{n}^{\omega} \end{array}$$

3. IFS

   $$\begin{array}{c} \mathbf{n} \cdot X \\ \chi \downarrow \\ X \end{array}$$

4.

   $$\begin{array}{ccc} \mathbf{n} \cdot \mathbf{n}^{\omega} & \xrightarrow{\mathbf{n} \cdot [\![\text{-}]\!]_{\chi}} & \mathbf{n} \cdot X \\ \cong \uparrow \text{final} & & \downarrow \chi \\ \mathbf{n}^{\omega} & \xrightarrow{[\![\text{-}]\!]_{\chi}} & X \\ & \searrow^{\cong} & \uparrow \\ & & \mathbf{Im}[\![\text{-}]\!]_{\chi} \end{array}$$

1. combinatorial spec.

   $M \otimes (\_) : \mathbf{Sets}^{\mathbb{A}} \to \mathbf{Sets}^{\mathbb{A}}$

2. symbolic representatives

   $$\begin{array}{c} M \otimes I \\ \text{final ND} \uparrow \cong \\ I \end{array}$$

3. IFS

4.

IIFS = Injective IFS

$$\varphi_0, \varphi_1 \ : \ \mathbb{C} \longrightarrow \mathbb{C} \ ,$$

$$\varphi_0(x) = \frac{x}{2} \ ,$$

$$\varphi_1(x) = \frac{1+x}{2} \ .$$

IIFS = Injective IFS

$$\varphi_0, \varphi_1 \;:\; \mathbb{C} \longrightarrow \mathbb{C} \;,$$

$$\varphi_0(x) = \frac{x}{2} \;,$$

$$\varphi_1(x) = \frac{1+x}{2} \;.$$

$$\begin{array}{c} 2 \cdot \mathbb{I} \\ \chi \big\downarrow \\ \mathbb{I} \end{array}$$

not injective!

IIFS = Injective IFS

$$\varphi_0, \varphi_1 \; : \; \mathbb{C} \longrightarrow \mathbb{C} \; ,$$
$$\varphi_0(x) = \frac{x}{2} \; ,$$
$$\varphi_1(x) = \frac{1+x}{2} \; .$$

$$2 \cdot \mathbb{I} \xrightarrow{\;\chi\;} \mathbb{I}$$

not injective!

$$M_{\mathbb{I}} \otimes P_{\mathbb{I}} \xrightarrow{\;\chi\;} P_{\mathbb{I}}$$

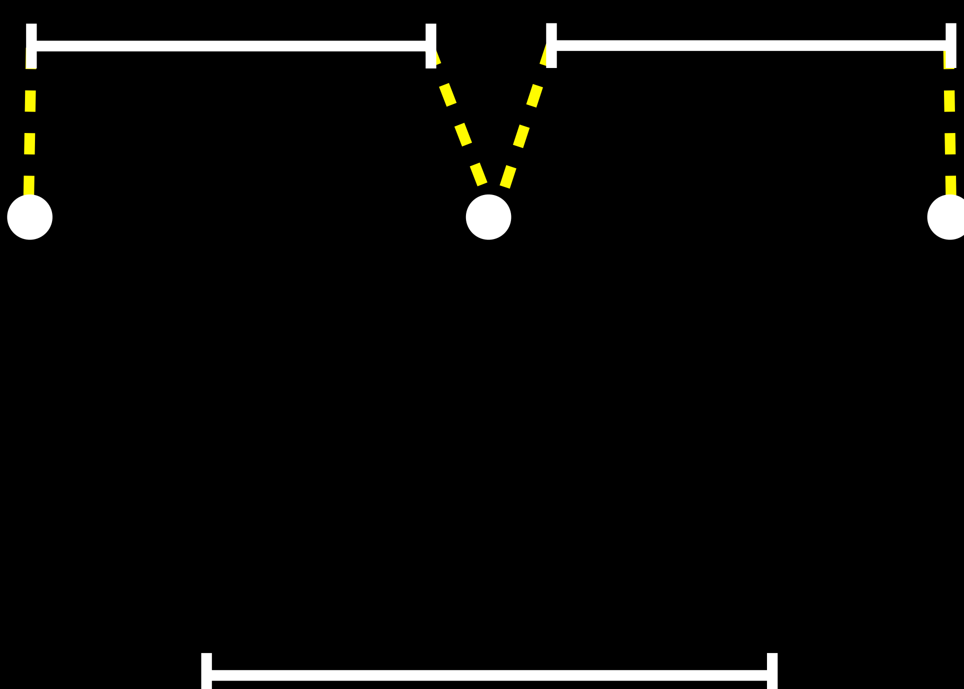
IIFS = Injective IFS

$$\varphi_0, \varphi_1 \; : \; \mathbb{C} \longrightarrow \mathbb{C} \; ,$$
$$\varphi_0(x) = \frac{x}{2} \; ,$$
$$\varphi_1(x) = \frac{1+x}{2} \; .$$

$$2 \cdot \mathbb{I} \xrightarrow{\;\chi\;} \mathbb{I}$$

not injective!

$$M_{\mathbb{I}} \otimes P_{\mathbb{I}} \xrightarrow{\;\chi\;} P_{\mathbb{I}}$$



$$M \otimes X \xrightarrow{\chi} X$$

Injective IFS ?

1. IFS + explicit gluing structure

Injective IFS ?

$$M \otimes X \overset{\chi}{\rightarrowtail} X$$

1. IFS + explicit gluing structure
   - currently: IFS $\not\mapsto$ / $\not\leftarrow$ IIFS

$$M \otimes X \overset{\chi}{\rightarrowtail} X$$

Injective IFS ?

1. IFS + explicit gluing structure
   - currently: IFS $\underset{\not\longleftarrow}{\not\longmapsto}$ IIFS

2. Comb. spec. $(\mathbb{A}, M)$
   + "how to *metrically realize* it in a CMS X"

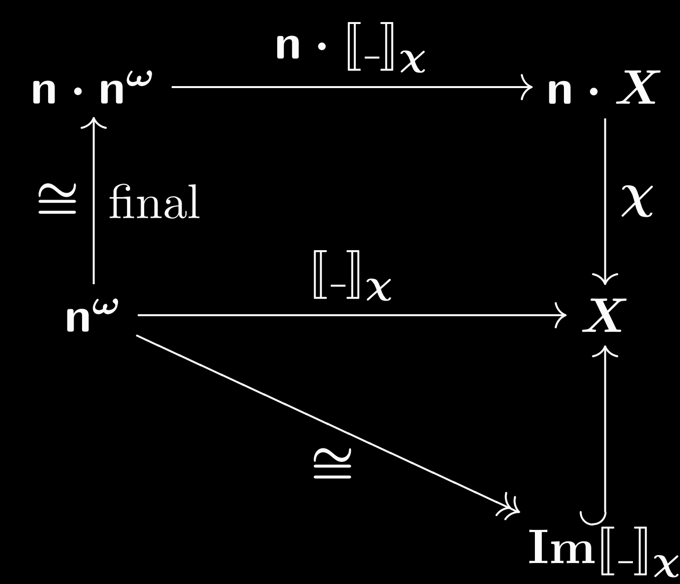
$$M \otimes X \overset{\chi}{\rightarrowtail} X$$

Injective IFS ?

1. IFS + explicit gluing structure
   - currently: IFS $\overset{\not\mapsto}{\underset{\not\mapsfrom}{}}$ IIFS

2. Comb. spec. $(\mathbb{A}, M)$
   + "how to *metrically realize* it in a CMS X"

3. "Sanity check" for $(\mathbb{A}, M)$, if works



with gluing

The Scenario

1. combinatorial spec.

   $\mathbf{n} \cdot (\_) : \mathbf{Sets} \to \mathbf{Sets}$

2. symbolic representatives

   $$\begin{array}{c} \mathbf{n} \cdot \mathbf{n}^{\omega} \\ \text{final} \uparrow \cong \\ \mathbf{n}^{\omega} \end{array}$$

3. IFS

   $$\begin{array}{c} \mathbf{n} \cdot \boldsymbol{X} \\ \chi \downarrow \\ \boldsymbol{X} \end{array}$$

4. $$\begin{array}{ccc} \mathbf{n} \cdot \mathbf{n}^{\omega} & \xrightarrow{\mathbf{n} \cdot [\![\text{-}]\!]_{\chi}} & \mathbf{n} \cdot \boldsymbol{X} \\ \cong \uparrow \text{final} & & \downarrow \chi \\ \mathbf{n}^{\omega} & \xrightarrow{[\![\text{-}]\!]_{\chi}} & \boldsymbol{X} \\ & \searrow^{\cong} & \uparrow \\ & & \mathbf{Im}[\![\text{-}]\!]_{\chi} \end{array}$$

1. combinatorial spec.

   $M \otimes (\_) : \mathbf{Sets}^{\mathbb{A}} \to \mathbf{Sets}^{\mathbb{A}}$

2. symbolic representatives

   $$\begin{array}{c} M \otimes I \\ \text{final ND} \uparrow \cong \\ I \end{array}$$

3. Injective IFS

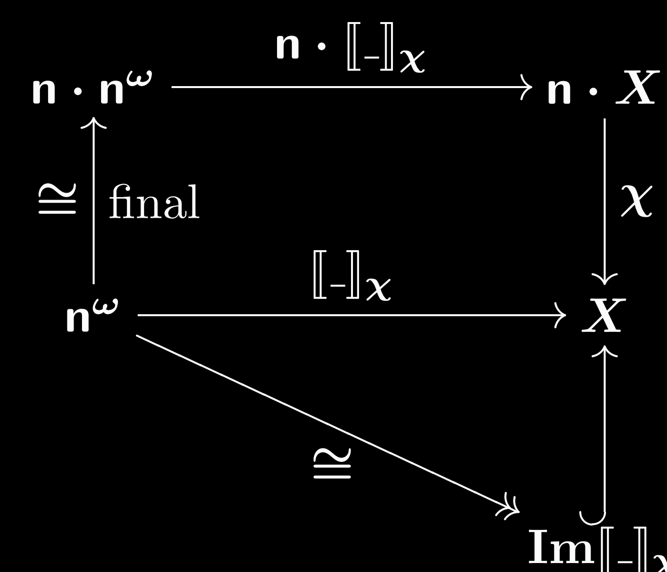
   $$\begin{array}{c} M \otimes \delta X \\ \chi \downarrow \\ X \end{array}$$

4. $$\begin{array}{ccc} M \otimes I & \xrightarrow{M \otimes [\![\text{-}]\!]_{\chi}} & M \otimes X \\ \text{final ND} \uparrow \cong & & \downarrow \chi \\ I & \xrightarrow{[\![\text{-}]\!]_{\chi}} & X \\ & \searrow^{\cong} & \uparrow \\ & & \mathbf{Im}\, [\![\text{-}]\!]_{\chi} \end{array}$$

Injective IFS ?

$$M \otimes X \xrightarrow{\ \chi\ } X$$

1. IFS + explicit gluing structure
   - currently: IFS $\not\mapsto$ / $\not\mapsfrom$ IIFS
2. Comb. spec. $(\mathbb{A}, M)$
   + "how to *metrically realize* it in a CMS X"
3. "Sanity check" for $(\mathbb{A}, M)$, if works



Injective IFS

- **Def.** An *injective IFS* over $(\mathbb{A}, M)$ and $\delta \in [0,1)$ is
  - $\boldsymbol{X} : \mathbb{A} \longrightarrow \mathbf{CMet}_{\mathbf{1}}^{\mathrm{TB}}$, non-degenerate
  - $$\begin{array}{c} M \otimes \delta \boldsymbol{X} \\ \chi \big\downarrow \\ \boldsymbol{X} \end{array}$$

subject to

1. $\chi_{\boldsymbol{a}}$ is injective, $\forall \boldsymbol{a} \in \mathbb{A}$
2. $\boldsymbol{X}\boldsymbol{a} \neq \emptyset$ and $\boldsymbol{I}\boldsymbol{a} \neq \emptyset$, $\forall \boldsymbol{a} \in \mathbb{A}$
3. for $\boldsymbol{f} : \boldsymbol{b} \to \boldsymbol{a}$ in $\mathbb{A}$, $(\boldsymbol{X}\boldsymbol{f})^{-1}(\mathbf{Im}\,\chi_{\boldsymbol{a}}) \subseteq \mathbf{Im}\,\chi_{\boldsymbol{b}}$

Injective IFS

- **Def.** An *injective IFS* over $(\mathbb{A}, M)$ and $\delta \in [0,1)$ is
  - $X : \mathbb{A} \longrightarrow \mathbf{CMet}_1^{\mathrm{TB}}$, non-degenerate (one-bounded, totally bounded)
  - $$\begin{array}{c} M \otimes \delta X \\ \chi \big\uparrow\!\!\downarrow \\ X \end{array}$$

subject to

1. χ_a is injective, $\forall a \in \mathbb{A}$
2. $Xa \neq \emptyset$ and $Ia \neq \emptyset$, $\forall a \in \mathbb{A}$
3. for $f : b \to a$ in $\mathbb{A}$, $(Xf)^{-1}(\mathrm{Im}\,\chi_a) \subseteq \mathrm{Im}\,\chi_b$

Injective IFS

- **Def.** An *injective IFS* over $(\mathbb{A}, M)$ and $\delta \in [0,1)$ is
  - $X : \mathbb{A} \longrightarrow \mathbf{CMet}_1^{\mathrm{TB}}$, non-degenerate
  - $$\begin{array}{c} M \otimes \delta X \\ \chi \big\downarrow \\ X \end{array}$$

one-bounded, totally bounded

coproduct metric + quotient pseudometric

subject to

1. χ_a is injective, $\forall a \in \mathbb{A}$
2. $Xa \neq \emptyset$ and $Ia \neq \emptyset$, $\forall a \in \mathbb{A}$
3. for $f : b \to a$ in $\mathbb{A}$, $(Xf)^{-1}(\mathrm{Im}\,\chi_a) \subseteq \mathrm{Im}\,\chi_b$

Injective IFS

- **Def.** An *injective IFS* over $(\mathbb{A}, M)$ and $\delta \in [0,1)$ is
  - $X : \mathbb{A} \longrightarrow \mathbf{CMet}_1^{\mathrm{TB}}$, non-degenerate
  - $M \otimes \delta X \xrightarrow{\chi} X$

one-bounded, totally bounded

non-expansive

coproduct metric + quotient pseudometric

subject to

1. χ_a is injective, $\forall a \in \mathbb{A}$
2. $Xa \neq \emptyset$ and $Ia \neq \emptyset$, $\forall a \in \mathbb{A}$
3. for $f : b \to a$ in $\mathbb{A}$, $(Xf)^{-1}(\mathrm{Im}\,\chi_a) \subseteq \mathrm{Im}\,\chi_b$

Main Result I

- **Thm.** $\begin{array}{c} M \otimes \delta X \\ {\scriptstyle \chi}\downarrow \\ X \end{array}$: IIFS

1. $\exists!\ [\![\_]\!]_{\chi}$
2. $[\![\_]\!]_{\chi}$ is monic

$$\begin{array}{ccc} M \otimes I & \xrightarrow{M \otimes [\![\_]\!]_{\chi}} & M \otimes X \\ {\scriptstyle \text{final ND}}\uparrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \chi} \\ I & \overset{[\![\_]\!]_{\chi}}{\rightarrowtail} & X \\ & {\scriptstyle \cong}\searrow & \uparrow \\ & & \mathbf{Im}\,[\![\_]\!]_{\chi} \end{array}$$

with gluing

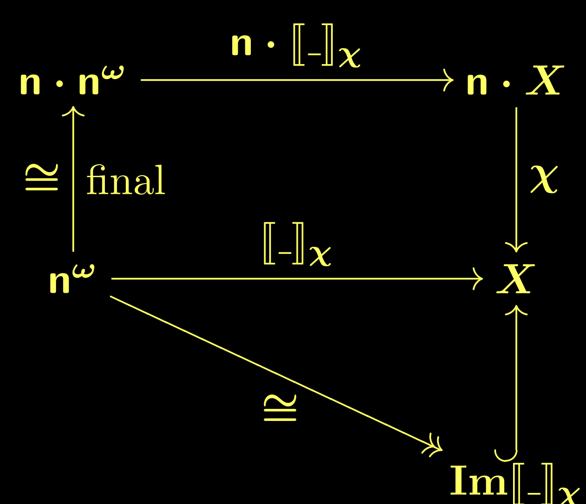
The Scenario

1. combinatorial spec.

   $\mathbf{n} \cdot (\_) : \mathbf{Sets} \to \mathbf{Sets}$

2. symbolic representatives $\quad \mathbf{n}^\omega \xrightarrow[\cong]{\text{final}} \mathbf{n} \cdot \mathbf{n}^\omega$

3. IFS $\quad \mathbf{n} \cdot X \xrightarrow{\chi} X$

4. 

1. combinatorial spec.

   $M \otimes (\_) : \mathbf{Sets}^{\mathbb{A}} \to \mathbf{Sets}^{\mathbb{A}}$

2. symbolic representatives $\quad I \xrightarrow[\cong]{\text{final ND}} M \otimes I$

3. Injective IFS

4.

with gluing

The Scenario

1. combinatorial spec.

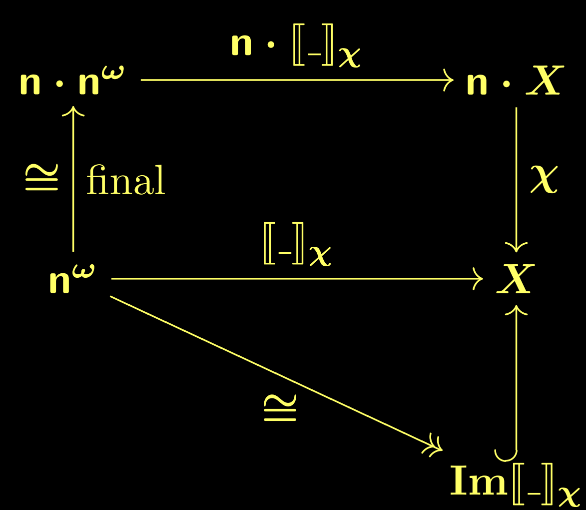
   $\mathbf{n} \cdot (\_) : \mathbf{Sets} \to \mathbf{Sets}$

2. symbolic representatives

   $\mathbf{n}^\omega \xrightarrow[\text{final}]{\cong} \mathbf{n} \cdot \mathbf{n}^\omega$

3. IFS

   $\chi : \mathbf{n} \cdot \boldsymbol{X} \to \boldsymbol{X}$

4. 

1. combinatorial spec.

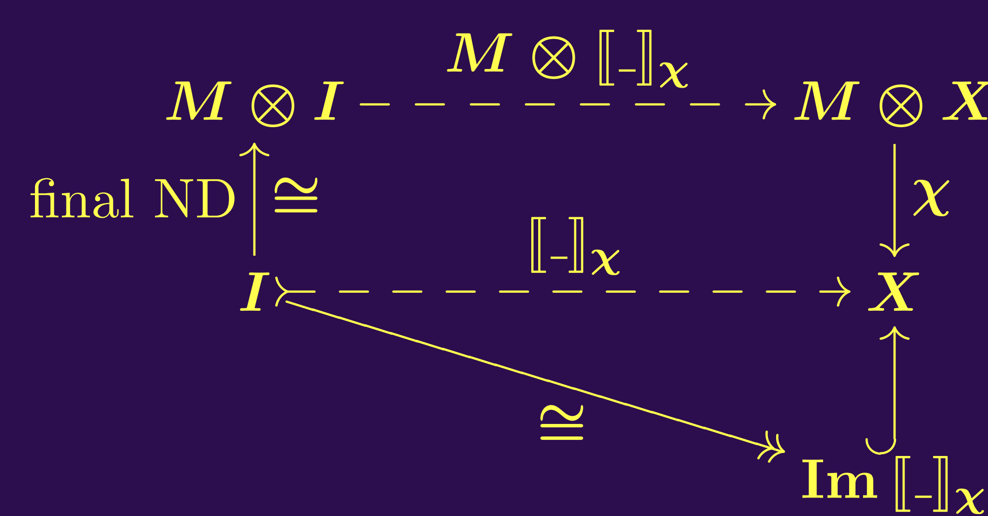
   $M \otimes (\_) : \mathbf{Sets}^{\mathbb{A}} \to \mathbf{Sets}^{\mathbb{A}}$

2. symbolic representatives

   $\boldsymbol{I} \xrightarrow[\text{final ND}]{\cong} M \otimes \boldsymbol{I}$

3. Injective IFS

   $\chi : M \otimes \delta \boldsymbol{X} \to \boldsymbol{X}$

4. 

Main Result II

- **Def.** $\begin{array}{c} M \otimes \delta X \\ \chi \big\downarrow \\ X \end{array}$: IIFS. An *attractor* is $S \overset{\varepsilon}{\hookrightarrow} X$

  such that

  1. ε_a is injective, $\forall a \in \mathbb{A}$
  2. $Sa \neq \emptyset$, $\forall a \in \mathbb{A}$
  3. $\exists \sigma$, an *isomorphism*, s.t.

$$\begin{array}{ccc} M \otimes S & \xrightarrow{M \otimes \varepsilon} & M \otimes X \\ \sigma \big\uparrow \cong & & \big\downarrow \chi \\ S & \xrightarrow[\varepsilon]{} & X \end{array}$$

Main Result II

- **Def.** $\begin{array}{c} M \otimes \delta X \\ \chi \big\downarrow \\ X \end{array}$: IIFS. An *attractor* is $S \overset{\varepsilon}{\hookrightarrow} X$ such that

  1. ε_a is injective, $\forall a \in \mathbb{A}$
  2. $Sa \neq \emptyset$, $\forall a \in \mathbb{A}$
  3. $\exists \sigma$, an *isomorphism*, s.t.

  $$\begin{array}{ccc} M \otimes S & \xrightarrow{M \otimes \varepsilon} & M \otimes X \\ \sigma \big\uparrow \cong & & \big\downarrow \chi \\ S & \xrightarrow[\varepsilon]{} & X \end{array}$$

- **Thm.** $\begin{array}{c} M \otimes I \\ \text{final ND} \big\uparrow \cong \\ I \end{array}$ is the unique attractor.

Related Work

- Tom Leinster.
  A general theory of self-similarity I, II.

$$\begin{array}{c} M \otimes I \\ \text{final ND} \uparrow \cong \\ I \end{array}$$

Related Work

- Tom Leinster.
  A general theory of self-similarity I, II.

$$\begin{array}{c} M \otimes I \\ \text{final ND} \uparrow \cong \\ I \end{array}$$

topological →

- canonical topology over I
  cf. [Barr, Adamek]
- *recognition theorem:* every compact metrizable space is a "fractal"

Related Work

- Tom Leinster.
  A general theory of self-similarity I, II.

$$\text{final ND} \quad \begin{array}{c} M \otimes I \\ \uparrow \cong \\ I \end{array}$$

topological

- canonical topology over I
  cf. [Barr, Adamek]
- *recognition theorem:* every compact metrizable space is a "fractal"

metric

- injective IFS
- symbolic representation

current work

Conclusion

fractal structure ➡ symbolic representation of the shape, by infinite streams

- (co)algebra, IA-FC coincidence (no CPO)
- bijective representation
- presheaves & modules for *gluing* [Leinster]
- future work: implement *computation*

Conclusion

fractal structure ⇒ symbolic representation of the shape, by infinite streams

- (co)algebra, IA-FC coincidence (no CPO)
- bijective representation
- presheaves & modules for *gluing* [Leinster]
- future work: implement *computation*

Thank you for your attention!

Ichiro Hasuo (RIMS, Kyoto U.)
http://www.kurims.kyoto-u.ac.jp/~ichiro