PROGRAMMING WITH INFINITESIMALS

A WHILE-LANGUAGE FOR HYBRID SYSTEM MODELING

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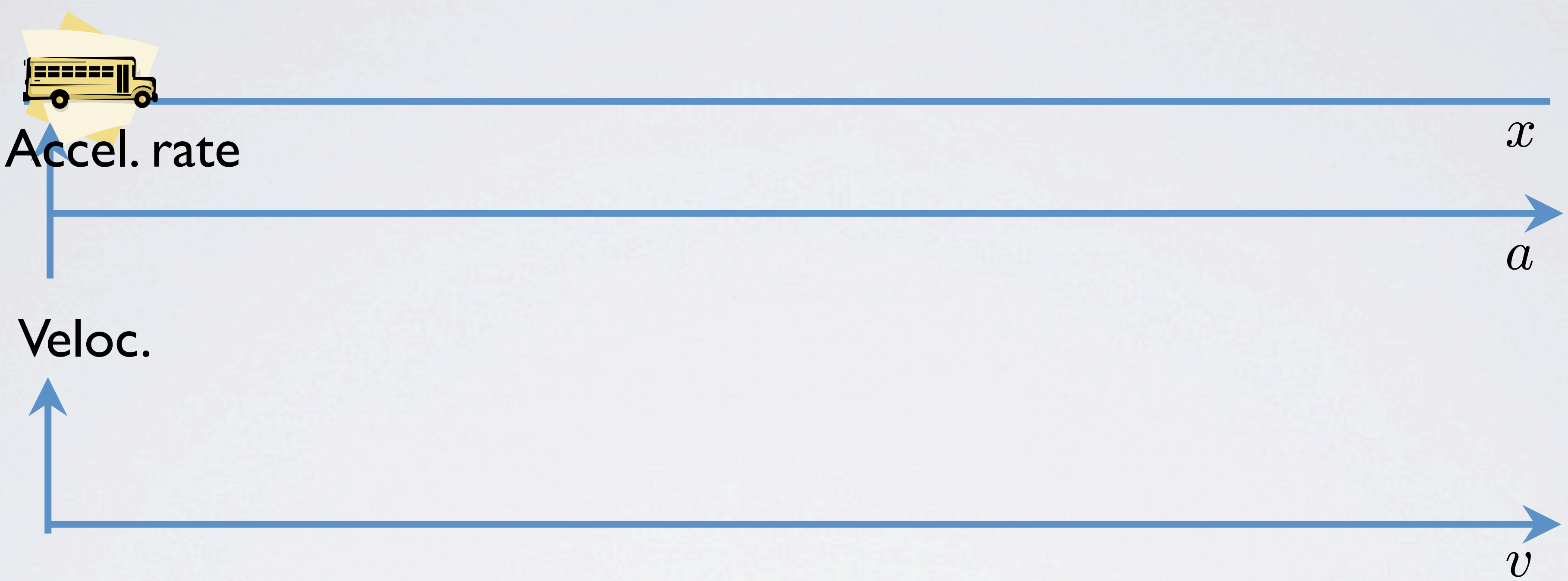
Hybrid System



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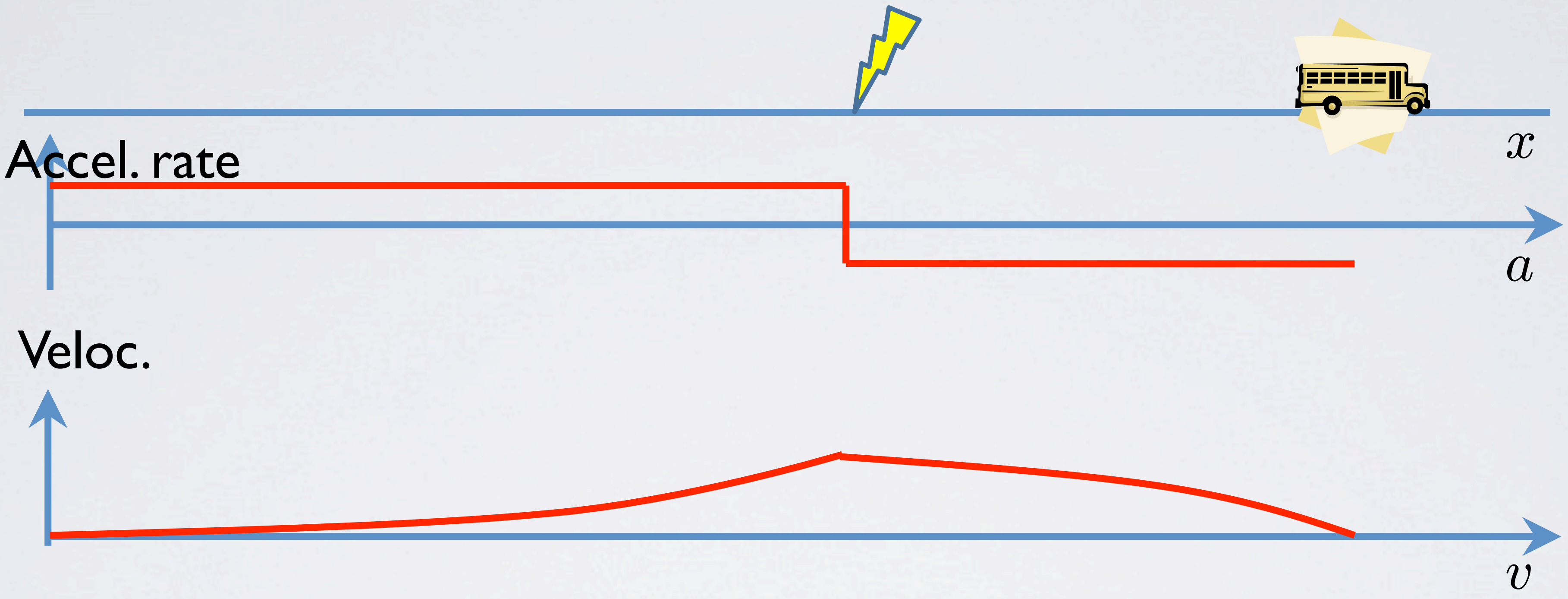

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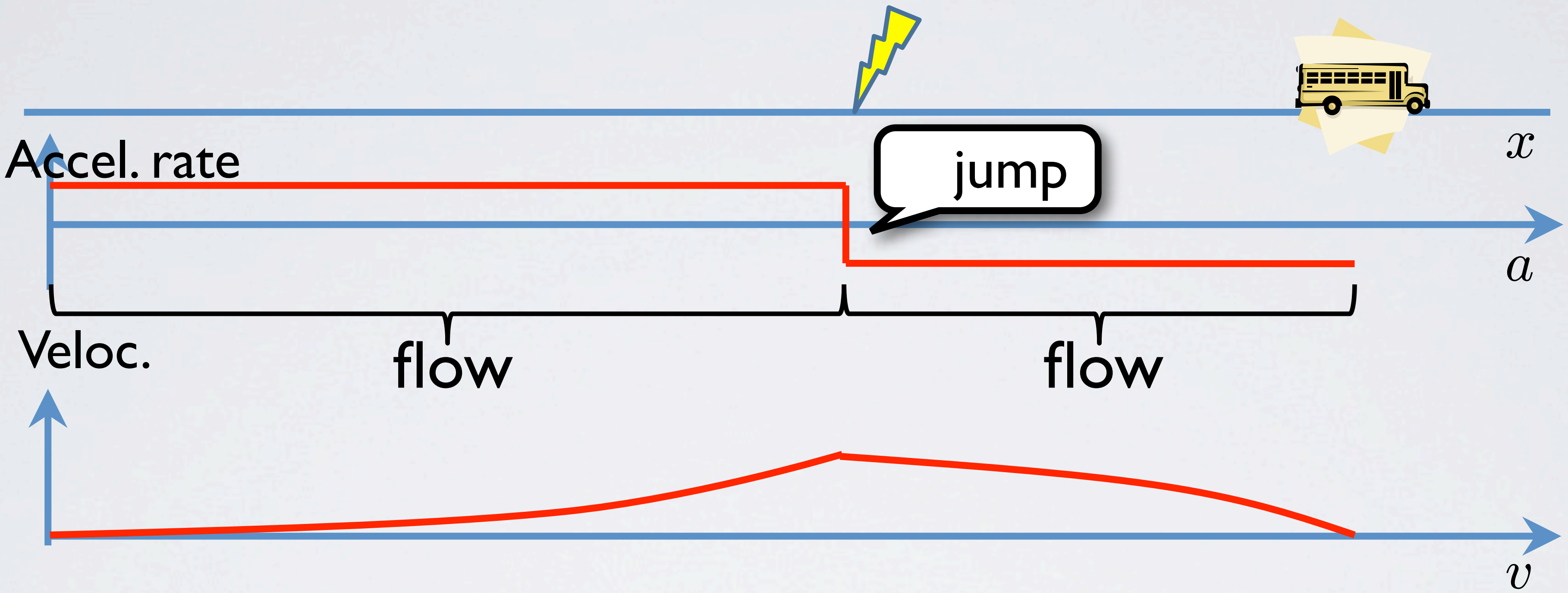
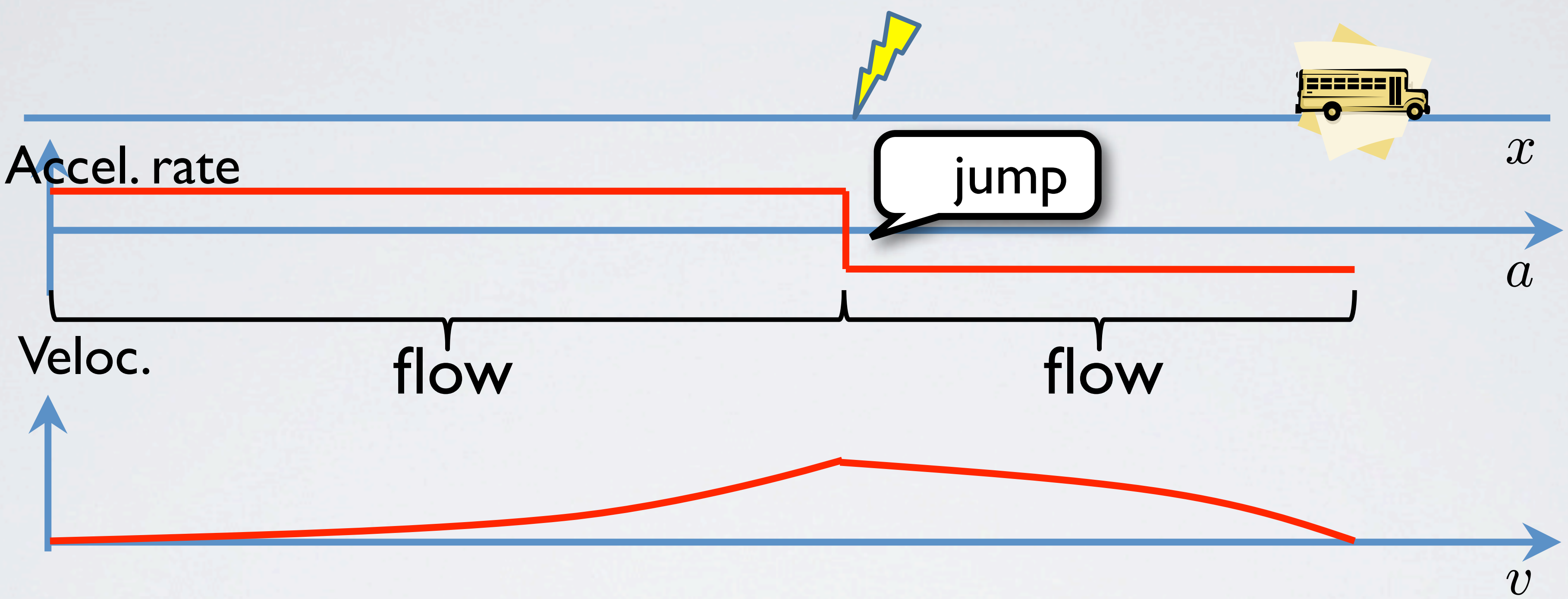


Hybrid System



- Flow & jump
  - Digital control in a physical environment
  - Component of cyber-physical systems

Hybrid System

Discrete
"jump"

and

Continuous
"flow"

Hybrid System

Discrete
"jump"

and

Continuous
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Hybrid System

Formal verification
(computer science)

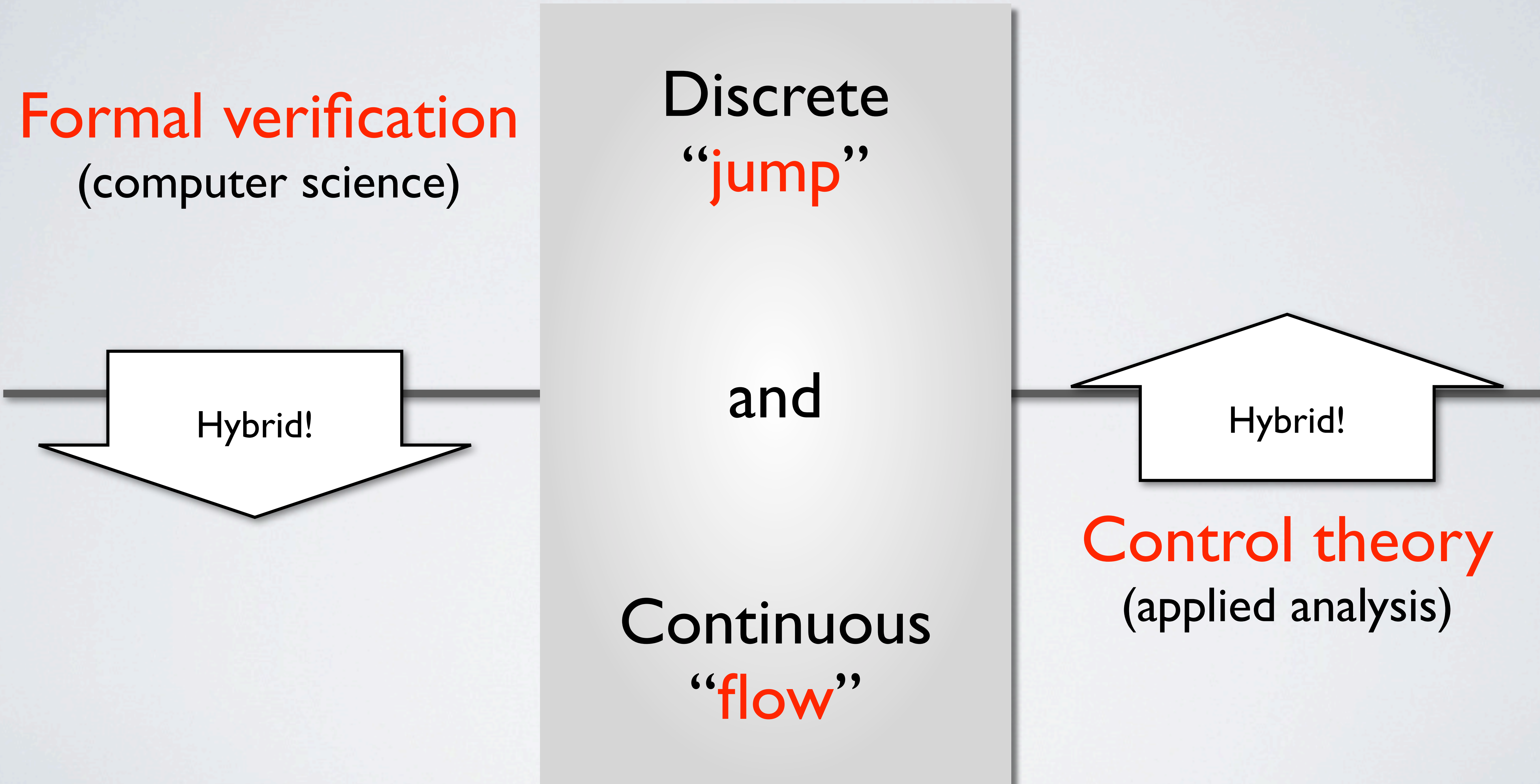
Discrete
“jump”

and

Continuous
“flow”

Control theory
(applied analysis)

Hybrid System



Hybrid System

Formal verification
(computer science)

Hybrid!

- Flow?
- With minimal cost?

Discrete
"jump"

and

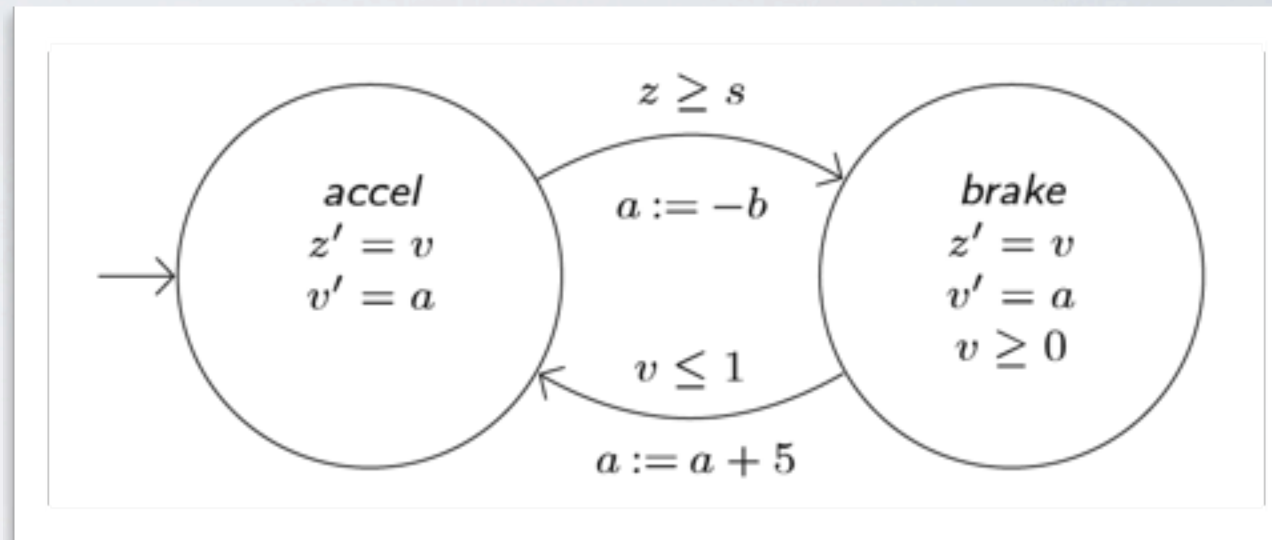
Continuous
"flow"

Hybrid!

Control theory
(applied analysis)

Formal Verification Approaches

- Hybrid automata [Alur & others, ’90s-]



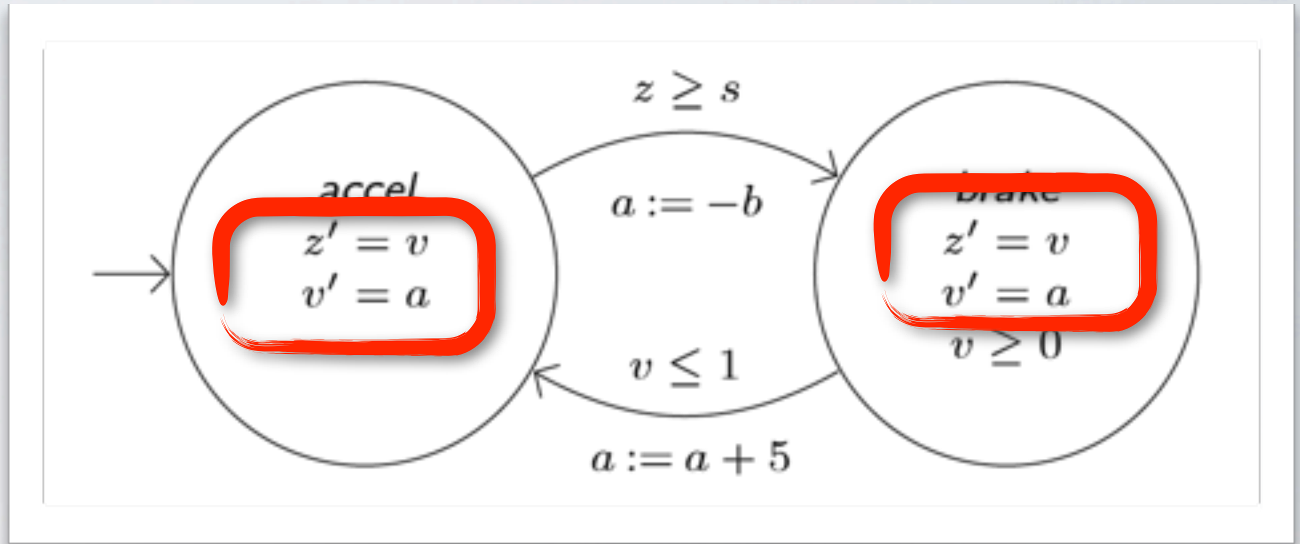
- Differential dynamic logic [Platzer & others, ’07-]

$$[\dot{x} = 1 \text{ while } x \leq 3]\varphi$$

- Differential equations, explicitly ➜ distinction jump vs. flow

Formal Verification Approaches

- Hybrid automata [Alur & others, ’90s-]



- Differential dynamic logic [Platzer & others, ’07-]

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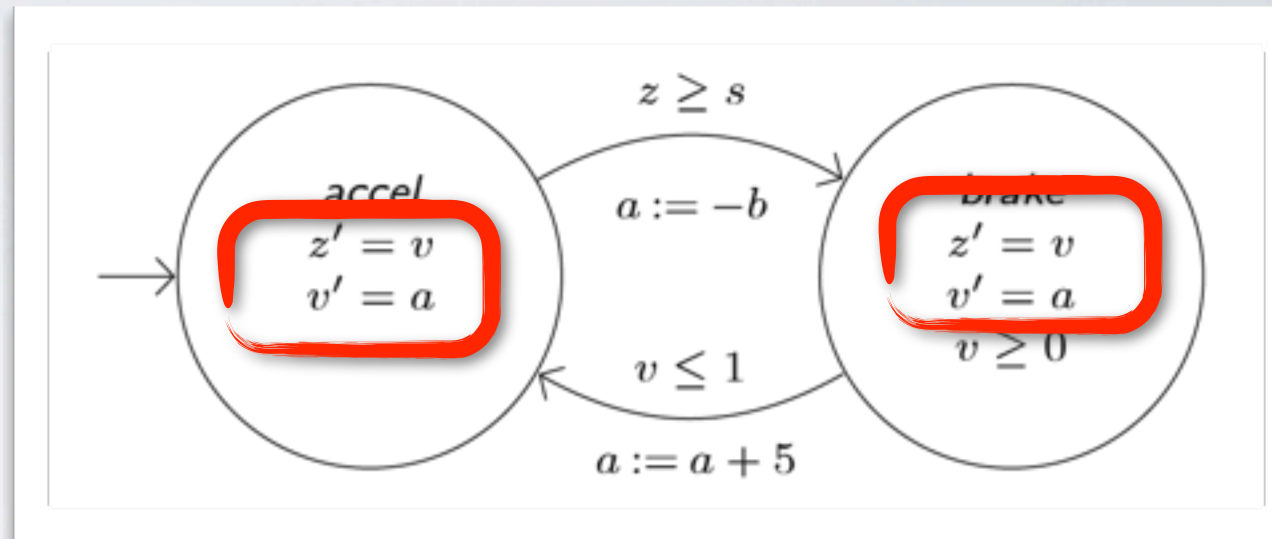

- Differential dynamic logic [Platzer & others, '07-]

$$[\dot{\boldsymbol{x}} = \mathbf{1}\ \text{while}\ \boldsymbol{x} \leq \mathbf{3}]\varphi$$

- Differential equations, explicitly ➜ distinction jump vs. flow

"Turn Flow into Jump"

$$
\begin{array}{l}
\boldsymbol{t} := \mathbf{0}\ ; \\
\texttt{while}\ (\boldsymbol{t} \leq \mathbf{1})\ \texttt{do}\ \{ \\
\quad \boldsymbol{t} := \boldsymbol{t} + \texttt{dt} \\
\}
\end{array}
$$

“Turn Flow into Jump”

$$
\begin{array}{l}
t := 0\ ; \\
\texttt{while}\ (t \le 1)\ \texttt{do}\ \{ \\
\quad t := t + \texttt{dt} \\
\}
\end{array}
$$

- Infinitesimal number $\texttt{dt}$
  - “Infinitely small” : $0 < \texttt{dt} < r$ for any positive real r

"Turn Flow into Jump"

$$
\begin{array}{l}
\boldsymbol{t} := \mathbf{0}\ ; \\
\texttt{while}\ (t \le 1)\ \texttt{do}\ \{ \\
\quad t := t + \mathrm{dt} \\
\}
\end{array}
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- Infinitesimal number dt
  - "Infinitely small" :
    0 < dt < r for any positive real r
- t = 1 after the execution?

"Turn Flow into Jump"

$$
\begin{aligned}
&t := 0\ ;\\
&\texttt{while}\ (t \leq 1)\ \texttt{do}\ \{\\
&\quad t := t + \mathrm{dt}\\
&\}
\end{aligned}
$$

- Infinitesimal number dt
  - "Infinitely small" :
    $0 < \mathrm{dt} < r$ for any positive real r
- $t = 1$ after the execution?
- Non-standard analysis!
  [Robinson '60s]

Contribution

Contribution

The standard textbook [Winskel]

Contribution

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While

Programming lang.

```
while (t<a) do {
  t:=t+1;
  if ...
}
```

Contribution

The standard textbook [Winskel]

While

Programming lang.

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while (t<a) do {
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Assn

First-order assertion lang.

$\exists z(x=2*z \wedge y=3*z)$

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The standard textbook [Winskel]

While

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First-order assertion lang.

$\exists$z(x=2*z $\wedge$ y=3*z)

Hoare

Hoare-style program logic

$$\frac{\{A \wedge b\}\, c\, \{A\}}{\{A\}\ \texttt{while}\ b\ \texttt{do}\ c\ \{A \wedge \neg b\}}$$

Contribution

The standard textbook [Winskel]

While^{dt}

Programming lang.

```
while (t<a) do {
  t:=t+1;
  if ...
}
```

Assn^{dt}

First-order assertion lang.

```
∃z(x=2*z ∧ y=3*z)
```

Hoare^{dt}

Hoare-style program logic

$$\frac{\{A \wedge b\}\; c\; \{A\}}{\{A\}\ \texttt{while}\ b\ \texttt{do}\ c\ \{A \wedge \neg b\}}$$

Contribution

The standard textbook [Winskel]

While$^{\mathrm{dt}}$

Programming lang.

```
while (t<a) do {
  t:=t+1;
  if ...
}
```

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First-order assertion lang.

$\exists z(x=2*z \land y=3*z)$

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Hoare-style program logic

$$\frac{\{A \land b\}\, c\, \{A\}}{\{A\}\ \texttt{while}\ b\ \texttt{do}\ c\ \{A \land \neg b\}}$$

Rigorous semantics by non-standard analysis

Contribution

The standard textbook [Winskel]

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Programming lang.

```
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```
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Rigorous semantics by non-standard analysis

- **Hoare**$^{\mathrm{dt}}$: sound and relatively complete

Contribution

The standard textbook [Winskel]

While$^{\mathrm{dt}}$

Programming lang.

```
while (t<a) do {
  t:=t+1;
  if ...
}
```

Assn$^{\mathrm{dt}}$

First-order assertion lang.

$\exists$z(x=2*z $\wedge$ y=3*z)

Hoare$^{\mathrm{dt}}$

Hoare-style program logic

$$\frac{\{A \wedge b\}\, c\, \{A\}}{\{A\}\, \texttt{while}\ b\ \texttt{do}\ c\, \{A \wedge \neg b\}}$$

Rigorous semantics by non-standard analysis

- **Hoare**$^{\mathrm{dt}}$: sound and relatively complete
- Program verification/static analysis of hybrid systems

Contribution

The standard textbook [Winskel]

While$^{\text{dt}}$

Programming lang.

```
while (t<a) do {
  t:=t+1;
  if ...
}
```

Assn$^{\text{dt}}$

First-order assertion lang.

$\exists z(x=2*z \wedge y=3*z)$

Hoare$^{\text{dt}}$

Hoare-style program logic

$$\frac{\{A \wedge b\}\, c\, \{A\}}{\{A\}\, \texttt{while}\ b\ \texttt{do}\ c\, \{A \wedge \neg b\}}$$

Rigorous semantics by non-standard analysis

- **Hoare**$^{\text{dt}}$: sound and relatively complete
- Program verification/static analysis of hybrid systems
- Actual verification with NSA

Contribution

Assn$^{\mathtt{dt}}$

First-order assertion lang.

`∃z(x=2*z ∧ y=3*z)`

Contribution

First-order language
for hyperreals

```
∃z(x=2*z ∧ y=3*z)
```

+

semantics via
ultraproducts

Contribution

First-order language
for reals

$\exists z(x=2*z \land y=3*z)$

+

usual semantics

First-order language
for hyperreals

$\exists z(x=2*z \land y=3*z)$

+

semantics via
ultraproducts

Contribution

First-order language for reals

∃z(x=2*z ∧ y=3*z)

+

usual semantics

Transfer Principle

[Robinson] (Łos' Theorem)

First-order language for hyperreals

∃z(x=2*z ∧ y=3*z)

+

semantics via ultraproducts

Contribution

First-order language
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∃z(x=2*z ∧ y=3*z)

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[Robinson] (Łos' Theorem)

First-order language
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∃z(x=2*z ∧ y=3*z)

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semantics via
ultraproducts

Computational
aspects

- While-language
- Hoare logic

Contribution

First-order language for reals

∃z(x=2*z ∧ y=3*z)

\+

usual semantics

Transfer Principle

[Robinson] (Łos' Theorem)

First-order language for hyperreals

∃z(x=2*z ∧ y=3*z)

\+

semantics via ultraproducts

Computational aspects

- While-language
- Hoare logic

Computational aspects

- While-language
- Hoare logic

Semantics: Challenge

$$
\begin{array}{l}
\boldsymbol{t} := \mathbf{0}\ ; \\
\texttt{while}\ (\boldsymbol{t} \leq \mathbf{1})\ \texttt{do}\ \{ \\
\quad \boldsymbol{t} := \boldsymbol{t} + \texttt{dt} \\
\}
\end{array}
$$

Semantics: Challenge

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$$
\begin{array}{l}
t := \mathbf{0}\ ; \\
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Semantics: Challenge

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\}
\end{array}
$$

$$\boldsymbol{t} = \mathbf{1} + \texttt{dt}$$

$$
\begin{array}{l}
\boldsymbol{t} := \mathbf{0}\ ; \\
\texttt{while}\ (\texttt{true})\ \texttt{do}\ \{ \\
\quad \boldsymbol{t} := \boldsymbol{t} + \texttt{dt} \\
\}
\end{array}
$$

$$\perp \text{ (divergence)}$$

Semantics: Challenge

$$\begin{array}{l} t := 0\ ; \\ \texttt{while}\ (t \leq 1)\ \texttt{do}\ \{ \\ \quad t := t + \mathtt{dt} \\ \} \end{array}$$

$$\begin{array}{l} t := 0\ ; \\ \texttt{while}\ (\texttt{true})\ \texttt{do}\ \{ \\ \quad t := t + \mathtt{dt} \\ \} \end{array}$$

$$t = 1 + \mathtt{dt}$$

$$\perp \text{ (divergence)}$$

- Semantics by “sectionwise execution”

Semantics: Challenge

$$
\begin{array}{l}
t := 0\ ; \\
\texttt{while}\ (t \leq 1)\ \texttt{do}\ \{ \\
\quad t := t + \texttt{dt} \\
\}
\end{array}
\qquad\qquad
\begin{array}{l}
t := 0\ ; \\
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\quad t := t + \texttt{dt} \\
\}
\end{array}
$$

$$t = 1 + \texttt{dt} \qquad\qquad \bot\ \text{(divergence)}$$

- Semantics by "sectionwise execution"
- Sectionwise execution/satisfaction ➜ much like Łos' thm.

I

$\textbf{While}^{\mathrm{dt}}$

SYNTAX

$\mathbf{While}^{\mathsf{dt}}$: Syntax

$$
\begin{array}{llll}
\mathbf{AExp} \ni & a & ::= & x \mid \mathtt{c}_r \mid a_1 \ \mathtt{aop} \ a_2 \mid \mathtt{dt} \mid \infty \\
& & & \text{where } x \in \mathbf{Var},\ \mathtt{c}_r \text{ is a constant for } r \in \mathbb{R}, \text{ and } \mathtt{aop} \in \{+, -, \cdot, \hat{}\,\} \\
\mathbf{BExp} \ni & b & ::= & \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 \\
\mathbf{Cmd} \ni & c & ::= & \mathtt{skip} \mid x := a \mid c_1; c_2 \mid \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \mid \mathtt{while}\ b\ \mathtt{do}\ c
\end{array}
$$

$\mathbf{While}^{\mathrm{dt}}$: Syntax

const. for reals

const. for an infinitesimal & an infinite

$$\begin{array}{llll}
\mathbf{AExp} \ni & a & ::= & x \mid \mathtt{c}_r \mid a_1 \ \mathtt{aop} \ a_2 \mid \mathtt{dt} \mid \infty \\
& & & \text{where } x \in \mathbf{Var},\ \mathtt{c}_r \text{ is a constant for } r \in \mathbb{R}, \text{ and } \mathtt{aop} \in \{+, -, \cdot, ^\wedge\} \\
\mathbf{BExp} \ni & b & ::= & \mathtt{true} \mid \mathtt{false} \mid b_1 \wedge b_2 \mid \neg b \mid a_1 < a_2 \\
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\end{array}$$

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const. for an infinitesimal & an infinite

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\end{array}$$

- **While** + reals + `dt`

While$^{\text{dt}}$: Syntax

const. for reals

const. for an infinitesimal & an infinite

$$\begin{array}{llll}
\textbf{AExp} \ni & a & ::= & x \mid \mathtt{c}_r \mid a_1 \ \mathtt{aop} \ a_2 \mid \mathtt{dt} \mid \infty \\
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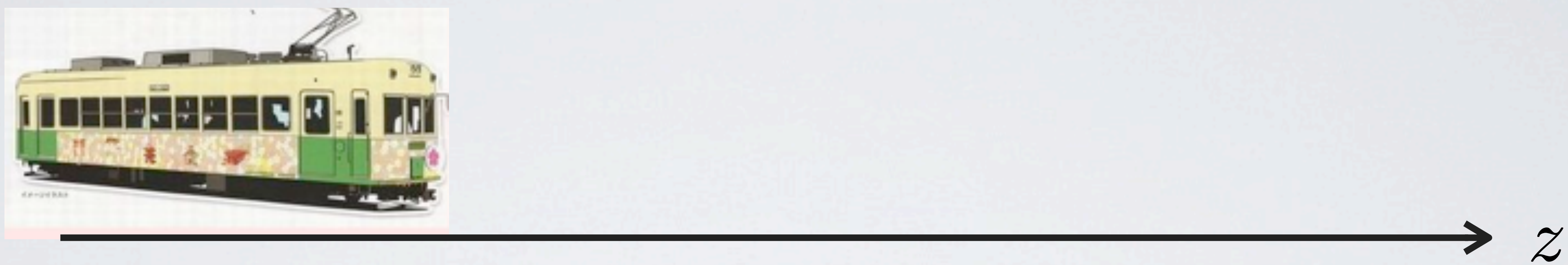
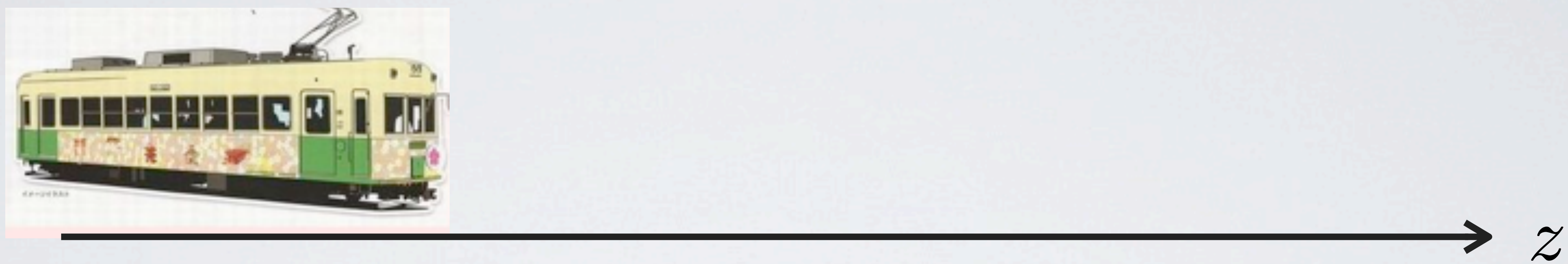
\end{array}$$

- **While** + reals + dt
- Not meant to be executed; for modeling

While$^{\mathrm{dt}}$: Example [Platzer '07]

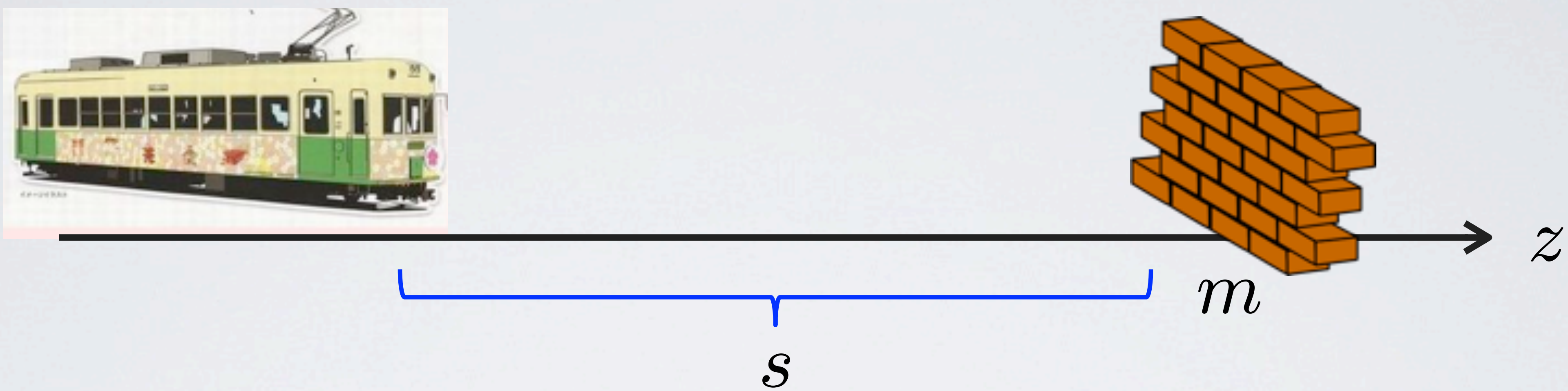


$\textbf{While}^{\mathrm{dt}}$: Example [Platzer '07]



$$
\begin{array}{l}
\texttt{while}\ t < \varepsilon\ \texttt{do}\ \{\\
\quad\quad t := t + \mathtt{dt}\,;\\
\quad\quad v := v + a \cdot \mathtt{dt}\,;\\
\quad\quad z := z + v \cdot \mathtt{dt}\\
\}
\end{array}
$$

$\mathbf{While}^{\mathrm{dt}}$: Example [Platzer '07]



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\qquad t := t + \mathrm{dt}\,; \\
\qquad v := v + a \cdot \mathrm{dt}\,; \\
\qquad z := z + v \cdot \mathrm{dt} \\
\}
\end{array}
$$

$$
\begin{array}{l}
\texttt{while}\ v > 0\ \texttt{do}\ \{ \\
\qquad t := 0\,; \\
\qquad \texttt{if}\ m - z < s\ \texttt{then}\ a := -b\ \texttt{else}\ a := 0\,; \\
\qquad \texttt{while}\ t < \varepsilon\ \texttt{do}\ \{ \\
\qquad\qquad t := t + \mathrm{dt}\,; \\
\qquad\qquad v := v + a \cdot \mathrm{dt}\,; \\
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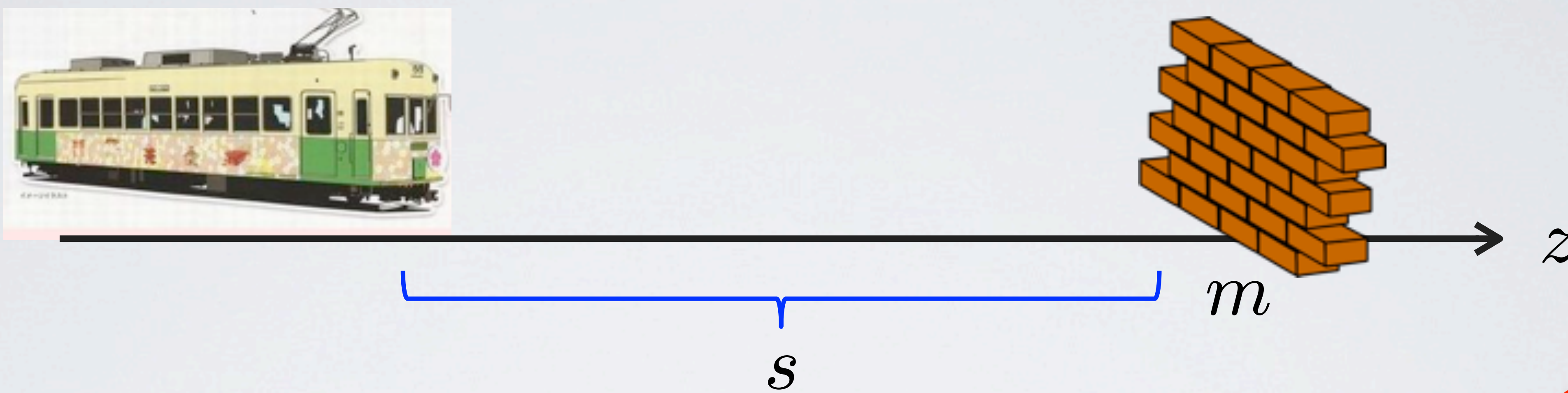
\qquad \}\}
\end{array}
$$

While$^{\mathrm{dt}}$: Example [Platzer '07]

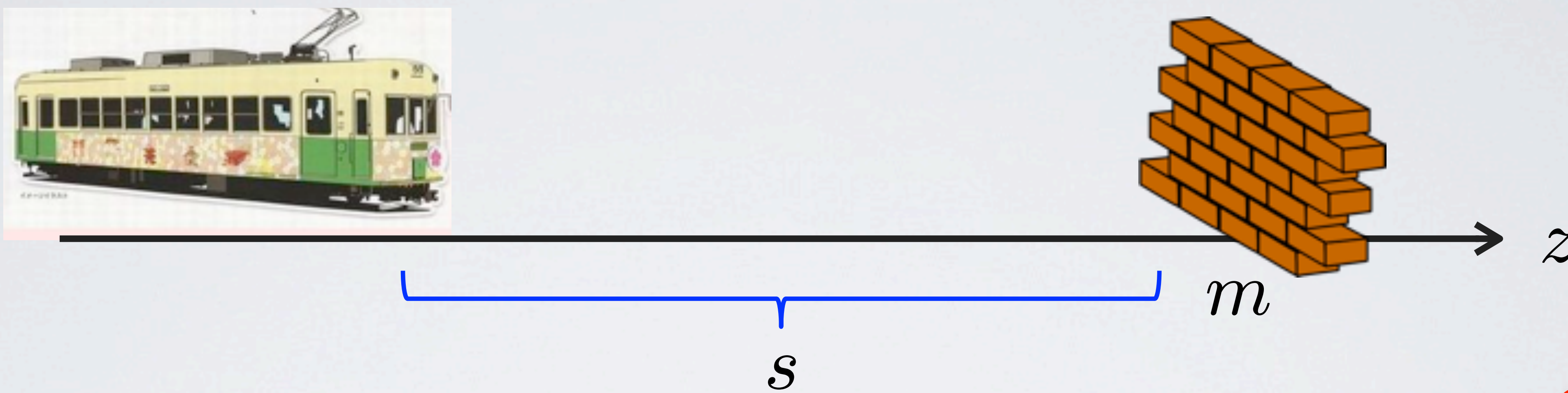



```
while t < ε do {
    t := t + dt ;
    v := v + a · dt ;
    z := z + v · dt
}
```

```
while v > 0 do {
    t := 0 ;
    if m − z < s then a := −b else a := 0 ;
    while t < ε do {
        t := t + dt ;
        v := v + a · dt ;
        z := z + v · dt
    }}
```

start braking

$\textbf{While}^{\mathrm{dt}}$: Example [Platzer '07]



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\begin{array}{l}
\texttt{while}\ t < \varepsilon\ \texttt{do}\ \{\\
\qquad t := t + \mathrm{dt}\,;\\
\qquad v := v + a \cdot \mathrm{dt}\,;\\
\qquad z := z + v \cdot \mathrm{dt}\\
\}
\end{array}
$$

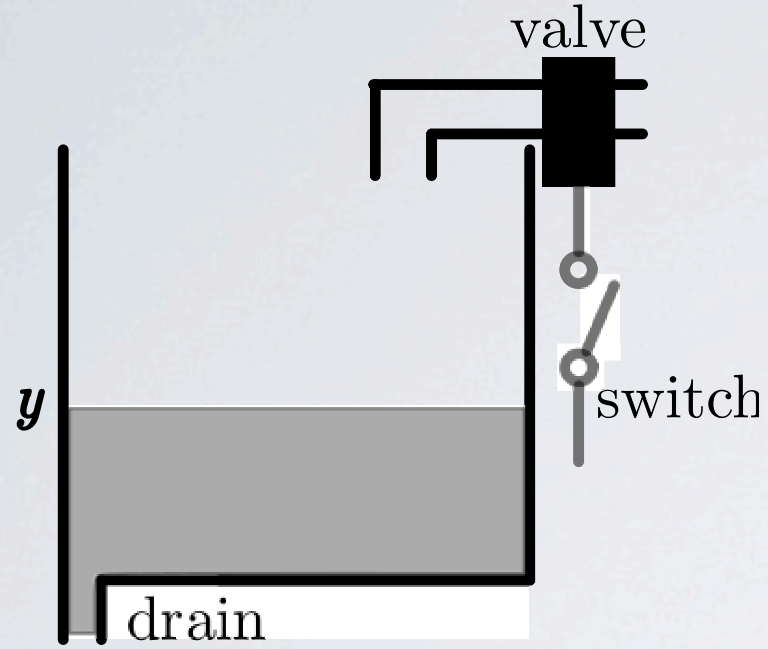
2nd derivative

(Unusual with hybrid automata)

$$
\begin{array}{l}
\texttt{while}\ v > 0\ \texttt{do}\ \{\\
\qquad t := 0\,;\\
\qquad \texttt{if}\ m - z < s\ \texttt{then}\ a := -b\ \texttt{else}\ a := 0\,;\\
\qquad \texttt{while}\ t < \varepsilon\ \texttt{do}\ \{\\
\qquad\qquad t := t + \mathrm{dt}\,;\\
\qquad\qquad v := v + a \cdot \mathrm{dt}\,;\\
\qquad\qquad z := z + v \cdot \mathrm{dt}\\
\qquad \}\}
\end{array}
$$

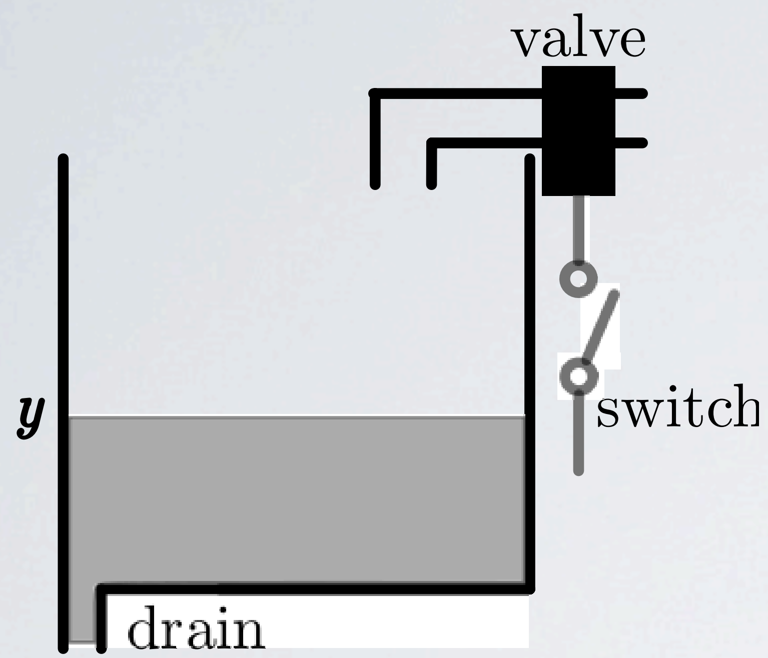
start braking

$\mathbf{While}^{\mathrm{dt}}$: Example [Alur et al. '95]



$$
\begin{array}{l}
x := 0;\ y := 1;\ s := 1;\ v := 1; \\
\texttt{while}\ t < t_{\mathsf{max}}\ \texttt{do}\quad \{ \\
\quad\quad x := x + \mathrm{dt};\quad t := t + \mathrm{dt}; \\
\quad\quad \texttt{if}\ v = 0\ \texttt{then}\ y := y - 2 \cdot \mathrm{dt}\ \texttt{else}\ y := y + \mathrm{dt}; \\
\quad\quad \texttt{case}\quad \{\ \begin{array}{ll}
s = 0\ \wedge\ v = 0\ \wedge\ y \leq 5 : & s := 1;\ x := 0; \\
s = 1\ \wedge\ v = 0\ \wedge\ x \geq 2 : & v := 1; \\
s = 1\ \wedge\ v = 1\ \wedge\ 10 \leq y : & s := 0;\ x := 0; \\
s = 0\ \wedge\ v = 1\ \wedge\ x \geq 2 : & v := 0; \\
\texttt{else} & \texttt{skip}\quad \}\}
\end{array}
\end{array}
$$

$\mathbf{While}^{\mathrm{dt}}$: Example [Alur et al. '95]



More involved jump structure

$$
\begin{array}{l}
x := 0;\; y := 1;\; s := 1;\; v := 1; \\
\texttt{while}\ t < t_{\mathsf{max}}\ \mathsf{do}\quad \{ \\
\qquad x := x + \mathrm{dt};\quad t := t + \mathrm{dt}; \\
\qquad \texttt{if}\ v = 0\ \texttt{then}\ y := y - 2 \cdot \mathrm{dt}\ \texttt{else}\ y := y + \mathrm{dt}; \\
\qquad \texttt{case}\quad \{\ \begin{array}{ll}
s = 0 \wedge v = 0 \wedge y \leq 5 : & s := 1;\; x := 0; \\
s = 1 \wedge v = 0 \wedge x \geq 2 : & v := 1; \\
s = 1 \wedge v = 1 \wedge 10 \leq y : & s := 0;\; x := 0; \\
s = 0 \wedge v = 1 \wedge x \geq 2 : & v := 0; \\
\texttt{else} & \texttt{skip}\quad \}\}
\end{array}
\end{array}
$$

$\mathbf{While}^{\mathrm{dt}}$: Semantics

$$
\begin{array}{l}
\boldsymbol{t} := \mathbf{0}\ ; \\
\texttt{while}\ (t \leq 1)\ \texttt{do}\ \{ \\
\quad t := t + \texttt{dt} \\
\}
\end{array}
$$

While$^{\mathrm{dt}}$: Semantics

$$t := 0\ ;$$
$$\texttt{while}\ (t \leq 1)\ \texttt{do}\ \{$$
$$\quad t := t + \mathtt{dt}$$
$$\}$$

$$t := 0\ ;$$
$$\texttt{while}\ (\texttt{true})\ \texttt{do}\ \{$$
$$\quad t := t + \mathtt{dt}$$
$$\}$$

While$^{\mathrm{dt}}$: Semantics

$$
\begin{array}{l}
\boldsymbol{t} := \boldsymbol{0}\ ; \\
\texttt{while}\ (t \leq 1)\ \texttt{do}\ \{ \\
\quad \boldsymbol{t} := \boldsymbol{t} + \texttt{dt} \\
\}
\end{array}
$$

$$\boldsymbol{t} = \boldsymbol{1} + \texttt{dt}$$

$$
\begin{array}{l}
\boldsymbol{t} := \boldsymbol{0}\ ; \\
\texttt{while}\ (\texttt{true})\ \texttt{do}\ \{ \\
\quad \boldsymbol{t} := \boldsymbol{t} + \texttt{dt} \\
\}
\end{array}
$$

$$\bot\ (\text{divergence})$$

$\mathbf{While}^{\mathrm{dt}}$: Semantics

$t := 0$;
while $(t \leq 1)$ do {
  $t := t + \mathrm{dt}$
}

??

$$t = 1 + \mathrm{dt}$$

$t := 0$;
while (true) do {
  $t := t + \mathrm{dt}$
}

??

$\perp$ (divergence)

While$^{\mathrm{dt}}$: Semantics

$$\begin{array}{l} t := 0\,; \\ \texttt{while}\ (t \leq 1)\ \texttt{do}\ \{ \\ \quad t := t + \mathtt{dt} \\ \} \end{array}$$

??

??

$$\begin{array}{l} t := 0\,; \\ \texttt{while}\ (\texttt{true})\ \texttt{do}\ \{ \\ \quad t := t + \mathtt{dt} \\ \} \end{array}$$

??

$$t = 1 + \mathtt{dt}$$

$\perp$ (divergence)

$\mathbf{While}^{\mathrm{dt}}$: Semantics

$$\begin{array}{l} t := 0\ ; \\ \texttt{while}\ (t \leq 1)\ \texttt{do}\ \{ \\ \quad t := t + \mathtt{dt} \\ \} \end{array}$$

??

??

$$\begin{array}{l} t := 0\ ; \\ \texttt{while}\ (\texttt{true})\ \texttt{do}\ \{ \\ \quad t := t + \mathtt{dt} \\ \} \end{array}$$

??

$$t = 1 + \mathtt{dt}$$

$$\perp \text{ (divergence)}$$

- Non-standard analysis (NSA) for "infinitesimal" dt
- While-loops ➜ sectionwise execution

II

“FIXING NOTATIONS” FOR NON-STANDARD ANALYSIS

Hyperreals

Defn.
The set of *hyperreal numbers* is

$$^*\mathbb{R} \;:=\; \mathbb{R}^{\mathbb{N}}/\sim_{\mathcal{F}}\,.$$

Hyperreals

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- Reals are hyperreals : $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$

Hyperreals

Defn.
The set of *hyperreal numbers* is

$$^*\mathbb{R} \;:=\; \mathbb{R}^{\mathbb{N}} / \sim_{\mathcal{F}} \,.$$

- Reals are hyperreals : $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$
- There are more than that:
  - An infinite ω : $\forall r \in \mathbb{R}.\ r < \omega$
  - An infinitesimal $\frac{1}{\omega}$: $\forall r \in \mathbb{R}.\ \left(0 < r \implies \frac{1}{\omega} < r\right)$

Hyperreals

Defn.
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Hyperreals

Defn.
The set of *hyperreal numbers* is

$$ {}^*\mathbb{R} := \mathbb{R}^{\mathbb{N}} / \sim_{\mathcal{F}} . \quad \ni \left[(a_0, a_1, a_2, \ldots) \right] $$

Hyperreals

Defn.
The set of *hyperreal numbers* is

$$ {}^*\mathbb{R} \;:=\; \mathbb{R}^{\mathbb{N}} / \sim_{\mathcal{F}} . \quad \ni \; \big[\,(a_0, a_1, a_2, \ldots)\,\big] $$

- An infinite

$$ \omega = \big[\,(1, 2, 3, \ldots)\,\big] $$

- An infinitesimal

$$ \omega^{-1} = \Big[\,(1, \frac{1}{2}, \frac{1}{3}, \ldots)\,\Big] $$

Hyperreals

Defn.
The set of *hyperreal numbers* is

$$^*\mathbb{R} := \mathbb{R}^{\mathbb{N}} / \sim_{\mathcal{F}} . \quad \ni \Big[\,(a_0, a_1, a_2, \ldots)\,\Big]$$

- An infinite $\quad \omega = \big[\,(1, 2, 3, \ldots)\,\big]$
- An infinitesimal $\quad \omega^{-1} = \Big[\,(1, \frac{1}{2}, \frac{1}{3}, \ldots)\,\Big]$
- A real (via $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$) $\quad r = \big[\,(r, r, r, \ldots)\,\big]$

(Prototype of) Hyperreals

Defn.
The set of *hyperreal numbers* is

$$^*\mathbb{R} \;:=\; \mathbb{R}^{\mathbb{N}} / \sim_{\mathcal{F}} . \quad \ni \quad \big[\,(a_0, a_1, a_2, \ldots)\,\big]$$

(Prototype of) Hyperreals

Defn.
The set of *hyperreal numbers* is

$$ {}^*\mathbb{R} \;:=\; \mathbb{R}^{\mathbb{N}} / \sim_{\mathcal{F}} . \quad \ni \; \big[\,(\boldsymbol{a_0}, \boldsymbol{a_1}, \boldsymbol{a_2}, \ldots)\,\big] $$

- "Infinite stream of reals"
  - <u>Operations</u>: pointwise

$$\begin{array}{ll} & \big[\,(a_0, a_1, \ldots)\,\big] \\ + & \big[\,(b_0, b_1, \ldots)\,\big] \\ = & \big[\,(a_0 + b_0, a_1 + b_1, \ldots)\,\big] \end{array}$$

(Prototype of) Hyperreals

Defn.
The set of *hyperreal numbers* is

$$^*\mathbb{R} \; := \; \mathbb{R}^{\mathbb{N}} / \sim_{\mathcal{F}} \, . \quad \ni \quad \big[\, (\boldsymbol{a_0}, \boldsymbol{a_1}, \boldsymbol{a_2}, \ldots) \,\big]$$

- "Infinite stream of reals"
  - Operations: pointwise

$$\begin{array}{ll} & [\,(\boldsymbol{a_0}, \boldsymbol{a_1}, \ldots)\,] \\ + & [\,(\boldsymbol{b_0}, \boldsymbol{b_1}, \ldots)\,] \\ = & [\,(\boldsymbol{a_0} + \boldsymbol{b_0}, \boldsymbol{a_1} + \boldsymbol{b_1}, \ldots)\,] \end{array}$$

  - Predicates: pointwise, "for almost every i"

$$\begin{array}{lll} [(\boldsymbol{a_i})_{i \in \mathbb{N}}] < [(\boldsymbol{b_i})_{i \in \mathbb{N}}] & & \\ \iff \quad \boldsymbol{a_i} < \boldsymbol{b_i} & & \text{for almost every } \boldsymbol{i} \\ \iff \quad \{\boldsymbol{i} \in \mathbb{N} \mid \boldsymbol{a_i} \not< \boldsymbol{b_i}\} & & \text{is finite} \end{array}$$

(Prototype of) Hyperreals

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The set of *hyperreal numbers* is

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- "Infinite stream of reals"
  - Operations: pointwise
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$$
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 & \big[\,(a_0, a_1, \ldots)\,\big] \\
+ & \big[\,(b_0, b_1, \ldots)\,\big] \\
= & \big[\,(a_0 + b_0, a_1 + b_1, \ldots)\,\big]
\end{array}
$$

$$
\begin{array}{lll}
\big[(a_i)_{i\in\mathbb{N}}\big] < \big[(b_i)_{i\in\mathbb{N}}\big] & & \\
\iff \quad a_i < b_i & & \text{for almost every } i \\
\iff \quad \{ i \in \mathbb{N} \mid a_i \not< b_i \} & & \text{is finite}
\end{array}
$$

"For sufficiently large i"
"Except for finitely many i"

An Infinitesimal

$$\forall r \in \mathbb{R}.\ (\ 0 < r \implies \omega^{-1} < r\ )$$

$$\left(1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots \frac{1}{N}, \frac{1}{N+1}, \frac{1}{N+2}, \ldots\right)$$

$$\wedge$$

$$?$$

$$\left(\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots\ \frac{1}{N},\ \frac{1}{N},\ \frac{1}{N}, \ldots\right)$$

An Infinitesimal

$$\boxed{\forall r \in \mathbb{R}.\ \left(\ 0 < r \implies \omega^{-1} < r\ \right)}$$

ω^{-1}

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{N}, \frac{1}{N+1}, \frac{1}{N+2}, \ldots\right)$$

$$\wedge$$

$$? \quad \left(\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots \quad \frac{1}{N}, \quad \frac{1}{N}, \quad\quad \frac{1}{N}, \ldots \right)$$

An Infinitesimal

$$\forall r \in \mathbb{R}.\ \left(0 < r \implies \omega^{-1} < r\right)$$

ω⁻¹

$$\wedge \quad \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{N}, \frac{1}{N+1}, \frac{1}{N+2}, \ldots\right)$$

$$? \quad \left(\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots \quad \frac{1}{N}, \quad \frac{1}{N}, \quad \frac{1}{N}, \ldots\right)$$

$1/N$

An Infinitesimal

$$\boxed{\forall r \in \mathbb{R}.\ (\ 0 < r \implies \omega^{-1} < r\ )}$$

ω^{-1}

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{N}, \frac{1}{N+1}, \frac{1}{N+2}, \ldots\right)$$

$$\wedge \quad \cancel{\wedge}$$

$$? \quad \left(\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots\right)$$

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$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \frac{1}{N}, \frac{1}{N+1}, \frac{1}{N+2}, \dots\right)$$

$\wedge$ $\quad \cancel{\wedge} \quad \cancel{\wedge}$

?

$$\left(\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \dots \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \dots\right)$$

$1/N$

An Infinitesimal

$$\boxed{\forall r \in \mathbb{R}.\ \left(0 < r \implies \omega^{-1} < r \right)}$$

ω^{-1}

$$\wedge \quad \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{N}, \frac{1}{N+1}, \frac{1}{N+2}, \ldots\right)$$

$$\cancel{\wedge} \quad \cancel{\wedge} \quad \cancel{\wedge}$$

$$? \quad \left(\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots\right)$$

$1/N$

An Infinitesimal

$$\forall r \in \mathbb{R}.\ (\ 0 < r \implies \omega^{-1} < r\ )$$

ω^-1

$$\wedge \quad \begin{array}{l} \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{N}, \frac{1}{N+1}, \frac{1}{N+2}, \ldots\right) \\ \quad \cancel{\wedge} \quad \cancel{\wedge} \quad \cancel{\wedge} \quad \cancel{\wedge} \\ ?\ \left(\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots \frac{1}{N}, \frac{1}{N}, \quad \frac{1}{N}, \ldots\right) \end{array}$$

1/N

An Infinitesimal

$$\forall r \in \mathbb{R}.\ \left(\, 0 < r \implies \omega^{-1} < r \,\right)$$

ω^{-1}

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{N}, \frac{1}{N+1}, \frac{1}{N+2}, \ldots\right)$$

$\wedge$

$\cancel{\wedge}$ $\cancel{\wedge}$ $\cancel{\wedge}$ $\cancel{\wedge}$ ...

?

$$\left(\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots\right)$$

$1/N$

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$$\boxed{\forall r \in \mathbb{R}.\ \left(0 < r \implies \omega^{-1} < r \right)}$$

ω^{-1}

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \frac{1}{N}, \frac{1}{N+1}, \frac{1}{N+2}, \dots\right)$$

$$\wedge \quad \cancel{\wedge}\ \ \cancel{\wedge}\ \ \cancel{\wedge}\ \ \cancel{\wedge}\ \ \dots\ \ \cancel{\wedge}$$

$$? \quad \left(\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \dots \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \dots\right)$$

$1/N$

An Infinitesimal

$$\boxed{\forall r \in \mathbb{R}.\ \left(0 < r \implies \omega^{-1} < r\right)}$$

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{N}, \frac{1}{N+1}, \frac{1}{N+2}, \ldots\right)$$

ω^{-1}

$$\wedge \quad \cancel{\wedge} \quad \cancel{\wedge} \quad \cancel{\wedge} \quad \cancel{\wedge} \quad \ldots \quad \cancel{\wedge} \quad \wedge$$

?

$$\left(\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots\right)$$

$1/N$

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ω^{-1}

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{N}, \frac{1}{N+1}, \frac{1}{N+2}, \ldots\right)$$

$$\wedge \qquad \cancel{\wedge} \quad \cancel{\wedge} \quad \cancel{\wedge} \quad \cancel{\wedge} \quad \ldots \quad \cancel{\wedge} \quad \wedge \quad \wedge$$

?

$$\left(\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots\right)$$

$1/N$

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ω^{-1}

$$\overset{?}{\wedge} \quad \cancel{\wedge}\ \cancel{\wedge}\ \cancel{\wedge}\ \cancel{\wedge}\ \ldots\ \cancel{\wedge}\ \wedge\ \wedge\ \ldots$$

$$\left(\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots\right)$$

$1/N$

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$$\boxed{\forall r \in \mathbb{R}.\ \left(0 < r \implies \omega^{-1} < r \right)}$$

ω⁻¹

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{N}, \frac{1}{N+1}, \frac{1}{N+2}, \ldots\right)$$

$$\wedge \quad \cancel{\wedge} \quad \cancel{\wedge} \quad \cancel{\wedge} \quad \cancel{\wedge} \quad \ldots \quad \cancel{\wedge} \quad \wedge \quad \wedge \quad \ldots$$

?

✓

$$\left(\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots \frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots\right)$$

1/N

Infinites

$$(1, 1, 1, 1, \ldots) \overset{?}{>} (1, 2, 3, 4, \ldots)$$

$$(1, 2, 3, 4, \ldots) \overset{?}{>} (0, 1, 2, 3, \ldots)$$

Infinites

I

(1, 1, 1, 1, ...)

^

?

(1, 2, 3, 4, ...)

(1, 2, 3, 4, ...)

^

?

(0, 1, 2, 3, ...)

Infinites

I

(1, 1, 1, 1, ...)

^

?

(1, 2, 3, 4, ...)

ω

(1, 2, 3, 4, ...)

^

?

(0, 1, 2, 3, ...)

Infinites

1

(1, 1, 1, 1, …)

$\wedge$

?

(1, 2, 3, 4, …)

ω

ω

(1, 2, 3, 4, …)

$\wedge$

?

(0, 1, 2, 3, …)

Infinites

1

(1, 1, 1, 1, ...)

$\wedge$
?

(1, 2, 3, 4, ...)

ω

ω

(1, 2, 3, 4, ...)

$\wedge$
?

(0, 1, 2, 3, ...)

ω − 1

Infinites

1

(1, 1, 1, 1, …)

∧ $\cancel{\wedge}$

(1, 2, 3, 4, …)

?

ω

ω

(1, 2, 3, 4, …)

∧

(0, 1, 2, 3, …)

?

ω − 1

Infinites

1

(1, 1, 1, 1, …)

$\wedge$?

$\cancel{\wedge}$ $\wedge$

(1, 2, 3, 4, …)

ω

ω

(1, 2, 3, 4, …)

$\wedge$?

(0, 1, 2, 3, …)

ω − 1

Infinites

1

(1, 1, 1, 1, ...)

^ ? $\cancel{\wedge}$ ^ ^

(1, 2, 3, 4, ...)

ω

ω

(1, 2, 3, 4, ...)

^ ?

(0, 1, 2, 3, ...)

ω − 1

Infinites

1

$(1, 1, 1, 1, \ldots)$

$\wedge$?

$\cancel{\wedge}$ $\wedge$ $\wedge$ $\wedge$

$(1, 2, 3, 4, \ldots)$

ω

ω

$(1, 2, 3, 4, \ldots)$

$\wedge$?

$(0, 1, 2, 3, \ldots)$

$\omega - 1$

Infinites

$\wedge$?

$(1, 1, 1, 1, \ldots)$ 1

$\cancel{\wedge}\ \wedge\ \wedge\ \wedge\ \ldots$

$(1, 2, 3, 4, \ldots)$ ω

$\wedge$?

$(1, 2, 3, 4, \ldots)$ ω

$(0, 1, 2, 3, \ldots)$ ω − 1

Infinites

1

$(1, 1, 1, 1, \ldots)$

$\wedge$ $\cancel{\wedge}$ $\wedge$ $\wedge$ $\wedge$ $\ldots$

$(1, 2, 3, 4, \ldots)$

ω

?

✓

ω

$(1, 2, 3, 4, \ldots)$

$\wedge$

$(0, 1, 2, 3, \ldots)$

ω − 1

?

Infinites

1

$(1, 1, 1, 1, \ldots)$

$\wedge$ $\cancel{\wedge}$ $\wedge$ $\wedge$ $\wedge$ $\ldots$

? ✓

$(1, 2, 3, 4, \ldots)$

ω

ω

$(1, 2, 3, 4, \ldots)$

$\wedge$ $\cancel{\wedge}$

?

$(0, 1, 2, 3, \ldots)$

$\omega - 1$

Infinites

1

$(1, 1, 1, 1, \ldots)$

$\wedge \quad \cancel{\wedge} \;\; \wedge \;\; \wedge \;\; \wedge \;\; \ldots$

?

✓

$(1, 2, 3, 4, \ldots)$

ω

ω

$(1, 2, 3, 4, \ldots)$

$\wedge \quad \cancel{\wedge} \;\; \cancel{\wedge}$

?

$(0, 1, 2, 3, \ldots)$

ω − 1

Infinites

1

$(1, 1, 1, 1, \ldots)$

$\wedge \quad \cancel{\wedge} \quad \wedge \quad \wedge \quad \wedge \quad \ldots$

?

✓

$(1, 2, 3, 4, \ldots)$

ω

ω

$(1, 2, 3, 4, \ldots)$

$\wedge \quad \cancel{\wedge} \quad \cancel{\wedge} \quad \cancel{\wedge}$

?

$(0, 1, 2, 3, \ldots)$

$\omega - 1$

Infinites

1

(1, 1, 1, 1, …)

$\cancel{\wedge}$ $\wedge$ $\wedge$ $\wedge$ …

(1, 2, 3, 4, …)

ω

$\bigwedge$

?

✓

ω

(1, 2, 3, 4, …)

$\cancel{\wedge}$ $\cancel{\wedge}$ $\cancel{\wedge}$ $\cancel{\wedge}$

(0, 1, 2, 3, …)

ω − 1

$\bigwedge$

?

Infinites

1

$$\wedge \qquad \begin{array}{c} (1,\ 1,\ 1,\ 1,\ \ldots) \\ \cancel{\wedge}\ \ \wedge\ \ \wedge\ \ \wedge\ \ \ldots \\ (1,\ 2,\ 3,\ 4,\ \ldots) \end{array}$$

?

✓

ω

ω

$$\wedge \qquad \begin{array}{c} (1,\ 2,\ 3,\ 4,\ \ldots) \\ \cancel{\wedge}\ \ \cancel{\wedge}\ \ \cancel{\wedge}\ \ \cancel{\wedge}\ \ \ldots \\ (0,\ 1,\ 2,\ 3,\ \ldots) \end{array}$$

?

ω − 1

Infinites

$\wedge$

?

✓

1

$$(1, 1, 1, 1, \ldots)$$

$$\cancel{\wedge}\quad \wedge\quad \wedge\quad \wedge\quad \ldots$$

$$(1, 2, 3, 4, \ldots)$$

ω

$\cancel{\wedge}$

?

ω

$$(1, 2, 3, 4, \ldots)$$

$$\cancel{\wedge}\quad \cancel{\wedge}\quad \cancel{\wedge}\quad \cancel{\wedge}\quad \ldots$$

$$(0, 1, 2, 3, \ldots)$$

$\omega - 1$

Trouble... Resolved

$$\mathbf{0} \qquad\qquad \left[\left(\,\mathbf{1}, -\mathbf{1},\ \mathbf{1}, -\mathbf{1}, \ldots\right)\right]$$

Trouble... Resolved

$$\mathbf{0} \quad \begin{matrix} > \\ = \\ < \\ ?? \end{matrix} \quad \left[(\, \mathbf{1}, -\mathbf{1},\ \mathbf{1}, -\mathbf{1}, \ldots \,) \right]$$

Trouble... Resolved

$$\mathbf{0} \quad \begin{matrix} > \\ = \\ < \\ {\color{red}??} \end{matrix} \quad \left[\left(\mathbf{1}, -\mathbf{1}, \mathbf{1}, -\mathbf{1}, \ldots \right)\right]$$

- Meaning of "almost every i" extended
  - ... so that

For each $\boldsymbol{S} \subset \mathbb{N}$, exactly one of

$$\boldsymbol{S} \quad \text{and} \quad \mathbb{N} \setminus \boldsymbol{S}$$

is "almost all $\boldsymbol{i}$."

Trouble... Resolved

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??

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Filters & Ultrafilters

Defn.
An *ultrafilter* $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is such that:

1. For each $X \subseteq \mathbb{N}$, exactly one of

$$X \quad \text{and} \quad \mathbb{N} \setminus X$$

   is in $\mathcal{F}$.

2. $X, Y \in \mathcal{F} \implies X \cap Y \in \mathcal{F}$
3. $X \in \mathcal{F}, X \subseteq Y \implies Y \in \mathcal{F}$
4. $\emptyset \notin \mathcal{F}$

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Given $X \subseteq \mathbb{N}$ is
"yes, almost all!" or "no!"

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Given $\boldsymbol{X} \subseteq \mathbb{N}$ is
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An *ultrafilter* $\boldsymbol{\mathcal{F}} \subseteq \boldsymbol{\mathcal{P}}(\mathbb{N})$ is such that:

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$$\boldsymbol{X} \quad \text{and} \quad \mathbb{N} \setminus \boldsymbol{X}$$
is in $\boldsymbol{\mathcal{F}}$.
2. $\boldsymbol{X}, \boldsymbol{Y} \in \boldsymbol{\mathcal{F}} \implies \boldsymbol{X} \cap \boldsymbol{Y} \in \boldsymbol{\mathcal{F}}$
3. $\boldsymbol{X} \in \boldsymbol{\mathcal{F}}, \boldsymbol{X} \subseteq \boldsymbol{Y} \implies \boldsymbol{Y} \in \boldsymbol{\mathcal{F}}$
4. $\emptyset \notin \boldsymbol{\mathcal{F}}$

Defn.
A *filter* $\boldsymbol{\mathcal{F}} \subseteq \boldsymbol{\mathcal{P}}(\mathbb{N})$ is that which satisfies Cond. 2.–4.

Filters & Ultrafilters

Given $X \subseteq \mathbb{N}$ is
"yes, almost all!" or "no!"

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A *filter* $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is that which satisfies Cond. 2.–4.

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Filters & Ultrafilters

Given $X \subseteq \mathbb{N}$ is
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Fix one such

Hyperreals

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The set of *hyperreal numbers* is

$$^*\mathbb{R} \;:=\; \mathbb{R}^{\mathbb{N}} / \sim_{\mathcal{F}} \, .$$

Hyperreals

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- Predicates: pointwise, "for almost every i"

$$\begin{aligned} & [(a_i)_{i\in\mathbb{N}}] < [(b_i)_{i\in\mathbb{N}}] \\ \Longleftrightarrow \quad & a_i < b_i \qquad \text{for "almost every } i\text{"} \\ \Longleftrightarrow \quad & \{i \in \mathbb{N} \mid a_i < b_i\} \in \mathcal{F} \end{aligned}$$

(For the other predicates, too)

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$$ \big[(a_i)_{i \in \mathbb{N}} \big] < \big[(b_i)_{i \in \mathbb{N}} \big] $$
$$ \iff \quad a_i < b_i \qquad \text{for "almost every } i\text{"} $$
$$ \iff \quad \{ i \in \mathbb{N} \mid a_i < b_i \} \in \mathcal{F} $$

(For the other predicates, too)

- Consequences: ω is infinite; ω^{-1} is infinitesimal;
$^*\mathbb{R}$ is an ordered field; [(0,1,0,1,...)] is either 0 or 1; ...

The Sectionwise Paradigm

Thm. (Łos)
For any first-order formula $\varphi(x)$ and a hyperreal $\mathbf{a} = \left[(a_i)_{i\in\mathbb{N}}\right]$,

$$^*\mathbb{R} \models \varphi(\mathbf{a}) \iff \{i \in \mathbb{N} \mid \mathbb{R} \models \varphi(a_i)\} \in \mathcal{F}.$$

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Thm. (Łos) (Constants for reals, not hyperreals)
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0th section

1st section

2nd section

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0th section

1st section

2nd section

a satisfies φ

iff

almost every section does

The Transfer Principle

Thm. (Łos)
For any first-order formula $\varphi(x)$ and a hyperreal $\mathbf{a} = \big[(a_i)_{i\in\mathbb{N}}\big]$,

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For any first-order sentence φ,

$$\mathbb{R} \models \varphi \iff {}^*\mathbb{R} \models \varphi \;.$$

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$\forall x.\, \psi$

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For any first-order sentence φ,

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$\forall x \in \mathbb{R}.\, \psi$

*-transform

$\forall x \in {}^*\mathbb{R}.\, \psi$

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Thm. (Constants for reals, not hyperr

For any first-order sentence φ,

$$\mathbb{R} \models \varphi \quad \Longleftrightarrow \quad ^*\mathbb{R} \models \varphi \; .$$

$$\forall x, y.\, (x < y \lor x = y \lor x > y)$$

$$\forall x.\, (x \neq 0 \implies \exists y.\, (xy = 1))$$

$$\forall x \in \mathbb{R}.\, \psi$$

*-transform

$$\forall x \in {}^*\mathbb{R}.\, \psi$$

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  with Naoki Kobayashi
- Program verification, static analysis, type systems
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[Fixed pt. obs., Braga, PT, 2007]

III

$\mathbf{While}^{\mathrm{dt}}$

"SECTIONWISE" SEMANTICS

$\mathbf{While}^{\mathrm{dt}}$: Semantics

$$
\begin{array}{l}
t := 0\ ; \\
\texttt{while}\ (t \leq 1)\ \texttt{do}\ \{ \\
\quad t := t + \mathtt{dt} \\
\}
\end{array}
$$

$$
\begin{array}{l}
t := 0\ ; \\
\texttt{while}\ (\texttt{true})\ \texttt{do}\ \{ \\
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$\mathbf{While}^{\mathrm{dt}}$: Semantics

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t := 0\ ; \\
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$$

$$t = 1 + \mathtt{dt}$$

$$
\begin{array}{l}
t := 0\ ; \\
\texttt{while}\ (\texttt{true})\ \texttt{do}\ \{ \\
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\}
\end{array}
$$

$$\perp\ \text{(divergence)}$$

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$$

$[\![\mathtt{dt}]\!]$

$:= \omega^{-1}$

$= [\,(1, \frac{1}{2}, \frac{1}{3}, \ldots)\,]$

$\perp$ (divergence)

$\mathbf{While}^{\mathtt{dt}}$: Semantics

$$\begin{array}{l} t := 0\ ; \\ \mathtt{while}\ (t \leq 1)\ \mathtt{do}\ \{ \\ \quad t := t + \mathtt{dt} \\ \} \end{array}$$

??

$$\begin{array}{l} t := 0\ ; \\ \mathtt{while}\ (\mathtt{true})\ \mathtt{do}\ \{ \\ \quad t := t + \underline{\mathtt{dt}} \\ \} \end{array}$$

$[\![\mathtt{dt}]\!]$

$:= \omega^{-1}$

$= \left[\, (1, \frac{1}{2}, \frac{1}{3}, \ldots) \,\right]$

$t = 1 + \mathtt{dt}$

$\perp$ (divergence)

Sectionwise Semantics

- Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (t < 1)
  t := t + dt;
```

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```

Sectionwise Semantics

- Execute sectionwise and bundle up the outcomes!

```
t := (0,0,0,...);
while (t < (1,1,1,...))
```

$$\texttt{t := t +} \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right) \texttt{;}$$

Sectionwise Semantics

- Execute sectionwise and bundle up the outcomes!

0th section

```
t := 0;
while (t < 1)
   t := t + 1 ;
```

1st section

```
t := 0;
while (t < 1)
   t := t + 1/2 ;
```

2nd section

```
t := 0;
while (t < 1)
   t := t + 1/3 ;
```

...

Sectionwise Semantics

- Execute sectionwise and bundle up the outcomes!

0th section	1st section	2nd section	
`t := 0;` `while (t < 1)` `t := t + 1 ;`	`t := 0;` `while (t < 1)` $\mathtt{t := t +} \frac{1}{2}$ `;`	`t := 0;` `while (t < 1)` $\mathtt{t := t +} \frac{1}{3}$ `;`	...
`t = 1`	`t = 1`	`t = 1`	...

Sectionwise Semantics

- Execute sectionwise and bundle up the outcomes!

```
t := (0,0,0,...);
while (t < (1,1,1,...))
```

$$\texttt{t := t +}\left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right) \texttt{;}$$

```
t = (1,1,1,...)
```

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t := 0;
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2nd section

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Sectionwise Semantics

- Execute sectionwise and bundle up the outcomes!

0th section	1st section	2nd section	
`t := 0;` `while (t <= 1)` $t := t + 1\ ;$	`t := 0;` `while (t <= 1)` $t := t + \frac{1}{2}\ ;$	`t := 0;` `while (t <= 1)` $t := t + \frac{1}{3}\ ;$	...
$t = 1 + 1$	$t = 1 + \frac{1}{2}$	$t = 1 + \frac{1}{3}$	...

Sectionwise Semantics

- Execute sectionwise and bundle up the outcomes!

```
t := (0,0,0,...);
while (t <= (1,1,1,...))

  t := t + (1, 1/2, 1/3, ...) ;
```

$$t = (1,1,1,\ldots) + \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$$

Sectionwise Semantics

- Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (t <= 1)
  t := t + dt;
```

```
t = 1 + dt
```

Sectionwise Semantics

- Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (true)
  t := t + dt;
```

Sectionwise Semantics

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...

Sectionwise Semantics

- Execute sectionwise and bundle up the outcomes!

0th section	1st section	2nd section	

`t := 0;`
`while (true)`
`t := t +` 1 `;`
	`t := 0;`
	`while (true)`
	`t := t +` $\frac{1}{2}$ `;`
		`t := 0;`
		`while (true)`
		`t := t +` $\frac{1}{3}$ `;`
			...

$\perp$	$\perp$	$\perp$	...

Sectionwise Semantics

- Execute sectionwise and bundle up the outcomes!

```
t := (0,0,0,...);
while (true)
```

$$t := t + \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right);$$

$$t = (\perp, \perp, \perp, \ldots)$$

Sectionwise Semantics

- Execute sectionwise and bundle up the outcomes!

```
t := 0;
while (true)
  t := t + dt;
```

$\perp$

Sectionwise Semantics

Defn. (Section)
Let $\boldsymbol{e}$: a $\textsc{While}^{\mathtt{dt}}$-expr., and $\boldsymbol{i} \in \mathbb{N}$.
$\boldsymbol{e}|_{\boldsymbol{i}}$, the $\boldsymbol{i}$-*th section* of $\boldsymbol{e}$, is obtained by replacing

$$\mathtt{dt} \mapsto \frac{\mathbf{1}}{\boldsymbol{i}+\mathbf{1}} \quad \text{and} \quad \infty \mapsto \boldsymbol{i}+\mathbf{1}\ .$$

Sectionwise Semantics

Defn. (Section)
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A **While**-expression
➜ standard semantics

Sectionwise Semantics

Defn. (Section)
Let e: a WHILE$^{\mathsf{dt}}$-expr., and $i \in \mathbb{N}$.
$e|_i$, the i-*th section* of e, is obtained by replacing

$$\mathsf{dt} \mapsto \frac{1}{i+1} \quad \text{and} \quad \infty \mapsto i+1 \ .$$

A **While**-expression
➜ standard semantics

$[\![\texttt{while } b' \texttt{ do } c']\!]\sigma = \sigma' \quad \overset{\text{def}}{\Longleftrightarrow}$

- $\sigma = \sigma' = \bot$;
- there exists a finite sequence $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_n = \sigma'$ such that: $[\![b']\!]\sigma_n = \mathrm{ff}$; and for each $j \in [0, n)$. $\big([\![b']\!]\sigma_j = \mathrm{tt} \ \& \ [\![c']\!]\sigma_j = \sigma_{j+1} \big)$; or
- such a finite sequence does not exist and $\sigma' = \bot$.

$\mathbf{While}^{\mathrm{dt}}$: Denotational Semantics

$$[\![x]\!]\sigma := \sigma(x) \qquad [\![\mathtt{c}_r]\!]\sigma := r \quad \text{for each } r \in \mathbb{R}$$

$$[\![a_1 \ \mathtt{aop}\ a_2]\!]\sigma := [\![a_1]\!]\sigma \ \mathtt{aop}\ [\![a_2]\!]\sigma$$

$$[\![\mathtt{dt}]\!]\sigma := \omega^{-1} = \big[\,(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots)\,\big] \qquad [\![\infty]\!]\sigma := \omega = \big[\,(1, 2, 3, \dots)\,\big]$$

$$[\![\mathtt{true}]\!]\sigma := \mathtt{tt} \qquad [\![\mathtt{false}]\!]\sigma := \mathtt{ff}$$

$$[\![b_1 \wedge b_2]\!]\sigma := [\![b_1]\!]\sigma \wedge [\![b_2]\!]\sigma \qquad [\![\neg b]\!]\sigma := \neg([\![b]\!]\sigma)$$

$$[\![a_1 < a_2]\!]\sigma := [\![a_1]\!]\sigma < [\![a_2]\!]\sigma$$

$$[\![\mathtt{skip}]\!]\sigma := \sigma \qquad [\![x := a]\!]\sigma := \sigma\big[\,x \mapsto [\![a]\!]\sigma\,\big] \qquad [\![c_1; c_2]\!]\sigma := [\![c_2]\!]\big(\,[\![c_1]\!]\sigma\,\big)$$

$$[\![\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2]\!]\sigma := \begin{cases} [\![c_1]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathtt{tt} \\ [\![c_2]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathtt{ff} \end{cases}$$

$$[\![\mathtt{while}\ b\ \mathtt{do}\ c]\!]\sigma := \Big(\,[\![\,(\mathtt{while}\ b\ \mathtt{do}\ c)|_i\,]\!](\sigma|_i)\,\Big)_{i\in\mathbb{N}}$$

$\mathbf{While}^{\mathrm{dt}}$: Denotational Semantics

Hyperstate (stores hyperreals)

$$
\begin{aligned}
&[\![x]\!]\sigma := \sigma(x) \qquad && [\![\mathtt{c}_r]\!]\sigma := r \quad \text{for each } r \in \mathbb{R}\\
&[\![a_1 \ \mathtt{aop}\ a_2]\!]\sigma := [\![a_1]\!]\sigma \ \mathtt{aop}\ [\![a_2]\!]\sigma &&\\
&[\![\mathtt{dt}]\!]\sigma := \omega^{-1} = \left[\,(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots)\,\right] && [\![\infty]\!]\sigma := \omega = \left[\,(1, 2, 3, \dots)\,\right]
\end{aligned}
$$

$$
\begin{aligned}
&[\![\mathtt{true}]\!]\sigma := \mathrm{tt} \qquad && [\![\mathtt{false}]\!]\sigma := \mathrm{ff}\\
&[\![b_1 \wedge b_2]\!]\sigma := [\![b_1]\!]\sigma \wedge [\![b_2]\!]\sigma && [\![\neg b]\!]\sigma := \neg([\![b]\!]\sigma)\\
&[\![a_1 < a_2]\!]\sigma := [\![a_1]\!]\sigma < [\![a_2]\!]\sigma &&
\end{aligned}
$$

$$
[\![\mathtt{skip}]\!]\sigma := \sigma \qquad [\![x := a]\!]\sigma := \sigma\left[\,x \mapsto [\![a]\!]\sigma\,\right] \qquad [\![c_1; c_2]\!]\sigma := [\![c_2]\!]\left(\,[\![c_1]\!]\sigma\,\right)
$$

$$
[\![\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2]\!]\sigma := \begin{cases} [\![c_1]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{tt}\\ [\![c_2]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{ff}\end{cases}
$$

$$
[\![\mathtt{while}\ b\ \mathtt{do}\ c]\!]\sigma := \Big(\,[\![\,(\mathtt{while}\ b\ \mathtt{do}\ c)|_i\,]\!](\sigma|_i)\,\Big)_{i \in \mathbb{N}}
$$

While$^{\mathrm{dt}}$: Denotational Semantics

Hyperstate (stores hyperreals)

$$
\begin{array}{lll}
[\![x]\!]\sigma & := & \sigma(x) \\
[\![a_1 \ \mathtt{aop}\ a_2]\!]\sigma & := & [\![a_1]\!]\sigma \ \mathtt{aop}\ [\![a_2]\!]\sigma \\
[\![\mathtt{dt}]\!]\sigma & := & \omega^{-1} = \big[\,(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots)\,\big]
\end{array}
\qquad
\begin{array}{lll}
[\![\mathtt{c}_r]\!]\sigma & := & r \quad \text{for each } r \in \mathbb{R} \\
& & \\
[\![\infty]\!]\sigma & := & \omega = \big[\,(1, 2, 3, \dots)\,\big]
\end{array}
$$

$$
\begin{array}{lll}
[\![\mathtt{true}]\!]\sigma & := & \mathrm{tt} \\
[\![b_1 \wedge b_2]\!]\sigma & := & [\![b_1]\!]\sigma \wedge [\![b_2]\!]\sigma \\
[\![a_1 < a_2]\!]\sigma & := & [\![a_1]\!]\sigma < [\![a_2]\!]\sigma
\end{array}
\qquad
\begin{array}{lll}
[\![\mathtt{false}]\!]\sigma & := & \mathrm{ff} \\
[\![\neg b]\!]\sigma & := & \neg([\![b]\!]\sigma)
\end{array}
$$

$$
[\![\mathtt{skip}]\!]\sigma := \sigma \qquad [\![x := a]\!]\sigma := \sigma\big[\,x \mapsto [\![a]\!]\sigma\,\big] \qquad [\![c_1; c_2]\!]\sigma := [\![c_2]\!]\big([\![c_1]\!]\sigma\big)
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$$
[\![\mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2]\!]\sigma := \begin{cases} [\![c_1]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{tt} \\ [\![c_2]\!]\sigma & \text{if } [\![b]\!]\sigma = \mathrm{ff} \end{cases}
$$

$$
[\![\mathtt{while}\ b\ \mathtt{do}\ c]\!]\sigma := \Big(\,\big[\!\big[\,(\mathtt{while}\ b\ \mathtt{do}\ c)|_i\,\big]\!\big](\sigma|_i)\,\Big)_{i \in \mathbb{N}}
$$

Section of a program

$\mathbf{While}^{\mathrm{dt}}$: Denotational Semantics

Hyperstate (stores hyperreals)

$$\llbracket x \rrbracket \sigma := \sigma(x) \qquad \llbracket \mathtt{c}_r \rrbracket \sigma := r \quad \text{for each } r \in \mathbb{R}$$

$$\llbracket a_1 \ \mathtt{aop}\ a_2 \rrbracket \sigma := \llbracket a_1 \rrbracket \sigma \ \mathtt{aop}\ \llbracket a_2 \rrbracket \sigma$$

$$\llbracket \mathtt{dt} \rrbracket \sigma := \omega^{-1} = \left[\, (1, \tfrac{1}{2}, \tfrac{1}{3}, \dots) \,\right] \qquad \llbracket \infty \rrbracket \sigma := \omega = \left[\, (1, 2, 3, \dots) \,\right]$$

$$\llbracket \mathtt{true} \rrbracket \sigma := \mathrm{tt} \qquad \llbracket \mathtt{false} \rrbracket \sigma := \mathrm{ff}$$

$$\llbracket b_1 \wedge b_2 \rrbracket \sigma := \llbracket b_1 \rrbracket \sigma \wedge \llbracket b_2 \rrbracket \sigma \qquad \llbracket \neg b \rrbracket \sigma := \neg(\llbracket b \rrbracket \sigma)$$

$$\llbracket a_1 < a_2 \rrbracket \sigma := \llbracket a_1 \rrbracket \sigma < \llbracket a_2 \rrbracket \sigma$$

$$\llbracket \mathtt{skip} \rrbracket \sigma := \sigma \qquad \llbracket x := a \rrbracket \sigma := \sigma\left[\, x \mapsto \llbracket a \rrbracket \sigma \,\right] \qquad \llbracket c_1 ; c_2 \rrbracket \sigma := \llbracket c_2 \rrbracket \left(\llbracket c_1 \rrbracket \sigma \right)$$

$$\llbracket \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2 \rrbracket \sigma := \begin{cases} \llbracket c_1 \rrbracket \sigma & \text{if } \llbracket b \rrbracket \sigma = \mathrm{tt} \\ \llbracket c_2 \rrbracket \sigma & \text{if } \llbracket b \rrbracket \sigma = \mathrm{ff} \end{cases}$$

$$\llbracket \mathtt{while}\ b\ \mathtt{do}\ c \rrbracket \sigma := \Big(\llbracket (\mathtt{while}\ b\ \mathtt{do}\ c)|_i \rrbracket (\sigma|_i) \Big)_{i \in \mathbb{N}}$$

Section of a program

Applied to a section of a memory state

$\mathbf{While}^{\mathrm{dt}}$: Denotational Semantics

Hyperstate (stores hyperreals)

$$[\![x]\!]\sigma := \sigma(x) \qquad [\![\mathtt{c}_r]\!]\sigma := r \quad \text{for each } r \in \mathbb{R}$$

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$$[\![\mathtt{skip}]\!]\sigma := \sigma \qquad [\![x := a]\!]\sigma := \sigma\left[\,x \mapsto [\![a]\!]\sigma\,\right] \qquad [\![c_1; c_2]\!]\sigma := [\![c_2]\!]\left([\![c_1]\!]\sigma\right)$$

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$$[\![\mathtt{while}\ b\ \mathtt{do}\ c]\!]\sigma := \Big(\,[\![\,\underline{(\mathtt{while}\ b\ \mathtt{do}\ c)|_i}\,]\!](\underline{\sigma|_i})\,\Big)_{\underline{i\in\mathbb{N}}}$$

Bundled up

Section of a program

Applied to a section
of a memory state

"Sectionwise" Lemmas

Sectionwise Execution Lemma. For any expr. $\boldsymbol{e}$ and $\boldsymbol{i} \in \mathbb{N}$,

$$[\![e]\!]\boldsymbol{\sigma} = \big[\, \big([\![e|_{\boldsymbol{i}}]\!](\boldsymbol{\sigma}|_{\boldsymbol{i}}) \big)_{\boldsymbol{i}\in\mathbb{N}} \,\big] \ .$$

Sectionwise Satisfaction Lemma. For any hyperstate $\boldsymbol{\sigma}$ and an $\textsc{Assn}^{\mathrm{dt}}$ formula φ:

$$\boldsymbol{\sigma} \models \varphi \quad \Longleftrightarrow \quad \boldsymbol{\sigma}|_i \models \varphi|_i \quad \text{for almost every } \boldsymbol{i}.$$

Q. Is a **While**^{dt} program executable?

Q. Is a **While**^{dt} program executable?

- A. Not exactly.
  - A modeling language
    - Advantage: close to a common programming style

Q. Does the choice of `dt` matter?

Q. Does the choice of dt matter?

- A. Yes, for "pathological" programs

```
t := 0;
while (t ≠ 1)
  t := t + dt;
```

Terminates with dt = (1, 1/2, 1/3, …)
Doesn't with dt = (1/π, 1/2π, 1/3π, …)

IV

$\mathbf{Assn}^{dt}$ AND $\mathbf{Hoare}^{dt}$

Assertion Language $\mathbf{Assn}^{\mathrm{dt}}$

$$\boldsymbol{A} \quad ::= \quad \mathtt{true} \mid \mathtt{false} \mid \boldsymbol{A_1} \wedge \boldsymbol{A_2} \mid \neg \boldsymbol{A} \mid \boldsymbol{a_1} < \boldsymbol{a_2} \mid \forall \boldsymbol{x} \in {}^*\mathbb{N}.\, \boldsymbol{A} \mid \forall \boldsymbol{x} \in {}^*\mathbb{R}.\, \boldsymbol{A}$$

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$$\boldsymbol{A} \quad ::= \quad \texttt{true} \mid \texttt{false} \mid \boldsymbol{A_1} \wedge \boldsymbol{A_2} \mid \neg \boldsymbol{A} \mid \boldsymbol{a_1} < \boldsymbol{a_2} \mid \underline{\forall \boldsymbol{x} \in {}^*\mathbb{N}.}\ \boldsymbol{A} \mid \underline{\forall \boldsymbol{x} \in {}^*\mathbb{R}.}\ \boldsymbol{A}$$

- Only hyperquantifiers are allowed

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- Only hyperquantifiers are allowed
- Standard quantifiers ➜ failure of
  - sectionwise semantics
  - soundness of Hoare logic ...

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- Only hyperquantifiers are allowed
- Standard quantifiers ➜ failure of
  - sectionwise semantics
  - soundness of Hoare logic ...
- Drawback: can't say "infinitely close": $\forall r \in \mathbb{R}.\ |x - y| < r$

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$$\boldsymbol{\sigma} \models \boldsymbol{\varphi} \quad \Longleftrightarrow$$
$$\boldsymbol{\sigma}|_{i} \models \boldsymbol{\varphi}|_{i} \quad \text{for almost every } \boldsymbol{i}.$$

"Sectionwise" Lemmas

Sectionwise Execution
For any expr. e and $i \in \mathbb{N}$,

$$[\![e]\!]\boldsymbol{\sigma} = \big[\,\big(\,[\![e|_i]\!](\boldsymbol{\sigma}|_i)\,\big)$$

- $\mathtt{dt} \mapsto \frac{1}{i+1}$
  (same for ∞)
- $\forall x \in {}^*\mathbb{R} \quad \mapsto \quad \forall x \in \mathbb{R}$
  (same for $\mathbb{N}$, $\exists$)

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Łos' Theorem

Wednesday, November 9, 2011

Program Logic **Hoare**^{dt}

- Hoare triple

$$\{A\}\, c\, \{B\}$$

- Hoare logic: rule-based derivation of valid Hoare triples
  - ➜ Formal/automated verif. of programs (hybrid systems!)

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If A holds prior to execution, (precondition)

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$$\frac{}{\{A\}\ \texttt{skip}\ \{A\}}\ (\textsc{Skip}) \qquad \frac{}{\{\,A[a/x]\,\}\ x := a\ \{A\}}\ (\textsc{Assign})$$

$$\frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1; c_2\ \{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2\ \{B\}}\ (\textsc{If})$$

$$\frac{\{A \wedge b\}\ c\ \{A\}}{\{A\}\ \texttt{while}\ b\ \texttt{do}\ c\ \{A \wedge \neg b\}}\ (\textsc{While}) \qquad \frac{\models A \Rightarrow A' \quad \{A'\}\ c\ \{B'\} \quad \models B' \Rightarrow B}{\{A\}\ c\ \{B\}}\ (\textsc{Conseq})$$

- Precisely the same rules as with **Hoare**

Program Logic **Hoare**^{dt}

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$$\frac{\{A\}\ c_1\ \{C\} \quad \{C\}\ c_2\ \{B\}}{\{A\}\ c_1; c_2\ \{B\}}\ (\textsc{Seq}) \qquad \frac{\{A \wedge b\}\ c_1\ \{B\} \quad \{A \wedge \neg b\}\ c_2\ \{B\}}{\{A\}\ \mathtt{if}\ b\ \mathtt{then}\ c_1\ \mathtt{else}\ c_2\ \{B\}}\ (\textsc{If})$$

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- Precisely the same rules as with **Hoare**

Thm.
HOARE^{dt} rules are *sound* and *relatively complete*.

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Verification example

```
t := 0; x := 0;
v := 0; a := 1;
while (t < 4) {
  v' := v + a * dt;
  x' := x + v * dt;
  v := v'; x := x';
  if (t < 2) then a := 1
  else a:= -1;
  t := t + dt;
}
```

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  else a:= -1;
  t := t + dt;
}
```

Loop invariant:

$\exists n \in {}^*\mathbb{N}.\ t = n * dt\ \&$

$t < 2 + dt \rightarrow$

$v = n * dt\ \&\ a = 1\ \&$

$x = (n-1)n * dt^2 / 2$

$t >= 2 + dt \rightarrow$

$v = (2n_0 + 4 - n) * dt\ \&$

$a = -1\ \&$

$x = x_0 + (3n_0 + 7 - n)(n - n_0 - 2) * dt^2 / 2$

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Verification example

$\vdots$ Using the loop invariant

$\{\mathbf{true}\}$

```
t := 0; x := 0; v := 0; a := 1;
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  x' := x + v * dt;
  v := v'; x := x';
  t := t + dt;
  a := (t < 2) ? 1 : -1;
}
```
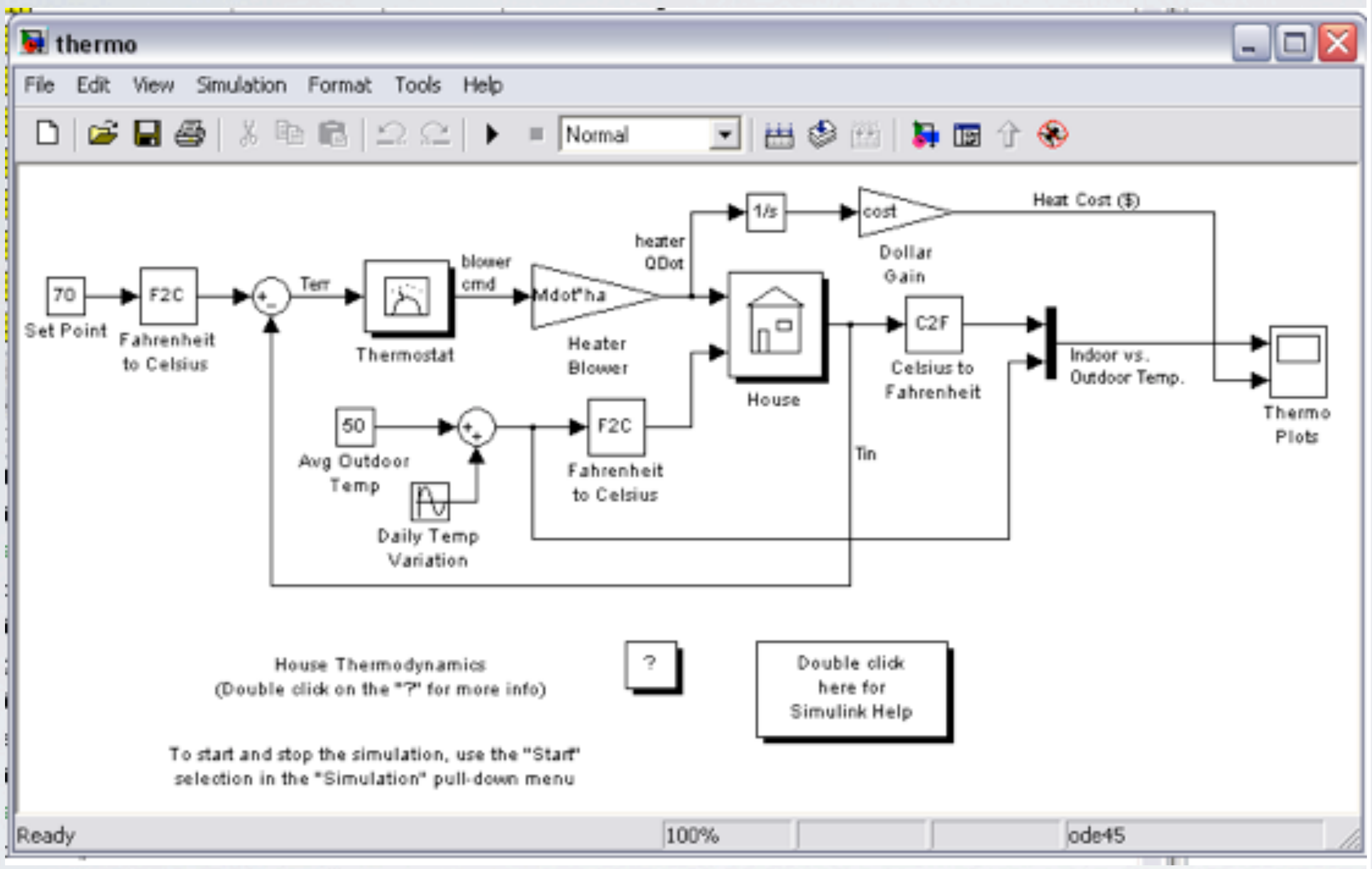
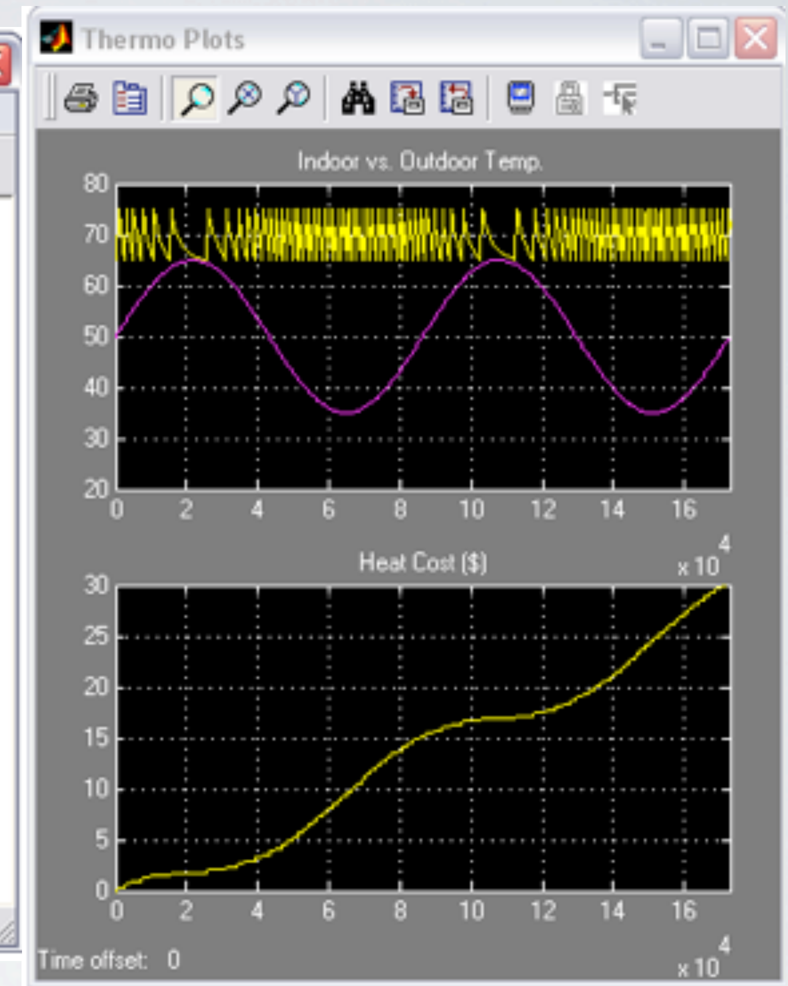

$\{x < 4.01\}$

V

RELATED & FUTURE WORK

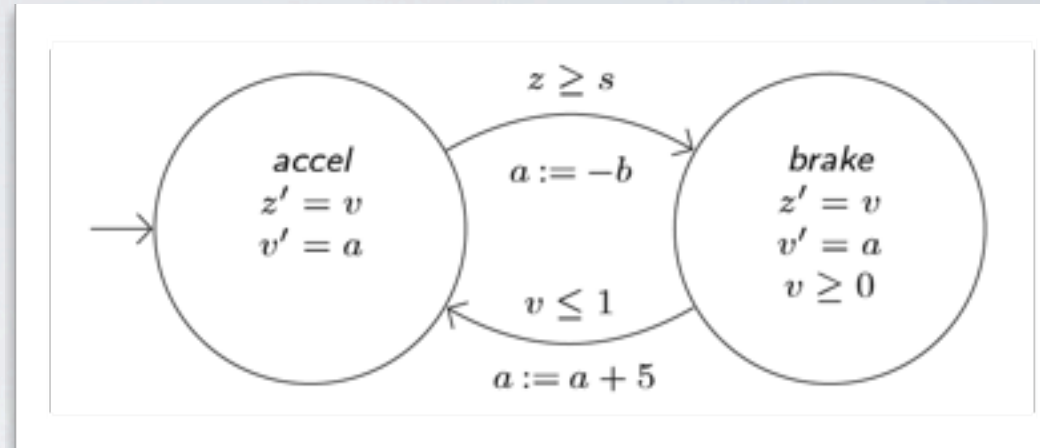
Related Work



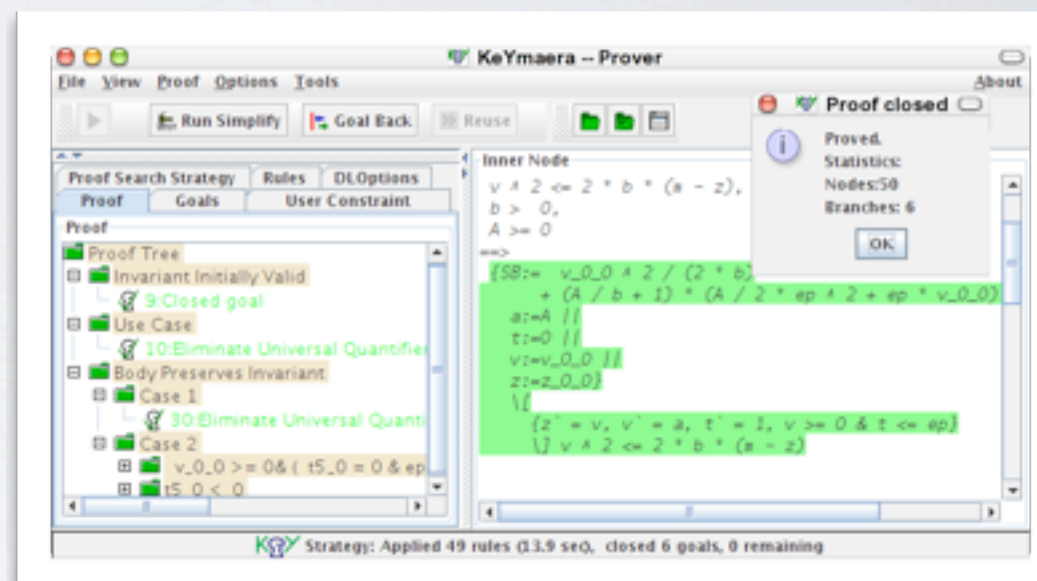


- Simulink [Mathwork Inc.]
  - Industrial standard for hybrid system design
  - Control-theory oriented
  - Test/simulation, rather than verification

Related Work



- Hybrid automaton [Alur & others, '90s-]
  - Model-checking: "push-button" verif. algorithms
  - Flow-dynamics restricted



- Differential dynamic logic [Platzer & others, '07-]
  - Dynamic logic (≈ Hoare logic) + differential equations
  - Automatic prover KeYmaera

Related Work

- Use of NSA for hybrid systems
  [Benveniste et al., Bliudze & Krob, Nakamura & Fusaoka, Rust]

  - Ours: clean integration with existing verification framework; actually proves something!

Future Work: Practical

- Automatic prover (prototype under development)
- (Any program verification techniques)^{dt}
- Simulink as (a stream processing language)^{dt}

Future Work: Theoretical

- Internalize the whole framework in a topos
  - Ultrapower construction of toposes?
- Characterize computing power of $\mathbf{While}^{\mathrm{dt}}$
  - Preliminary results: [Miyabe-Suenaga]

Summary

While

Programming lang.

```
while (t<a) do {
  t:=t+1;
  if ...
}
```

Assn

First-order assertion lang.

∃z(x=2*z ∧ y=3*z)

Hoare

Hoare-style program logic

$$\frac{\{A \wedge b\}\; c\; \{A\}}{\{A\}\; \texttt{while}\; b\; \texttt{do}\; c\; \{A \wedge \neg b\}}$$

Summary

While^{dt}

Programming lang.

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Rigorous semantics by non-standard analysis

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Rigorous semantics by non-standard analysis

- **Hoare**$^{\text{dt}}$: sound and relatively complete
- Program verification/static analysis of hybrid systems
- Actual verification with NSA

Thank you for your attention!
Ichiro Hasuo (Dept. CS, U Tokyo)
http://www-mmm.is.s.u-tokyo.ac.jp/~ichiro/

Summary

While$^{\mathrm{dt}}$

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```
while (t<a) do {
  t:=t+1;
  if ...
}
```

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Rigorous semantics by non-standard analysis

- **Hoare**$^{\mathrm{dt}}$: sound and relatively complete
- Program verification/static analysis of hybrid systems
- Actual verification with NSA