情報論理・プログラムの数理
第1回演習レポート（4月12日出題）解答

1 (Exercise 1.16).

($\sim_f$ is an equivalence relation) By definition of equivalence relations, it suffices to show that $\sim_f$ satisfies reflexivity, symmetry and transitivity.

For each $x \in X$, we have $f(x) = f(x)$. Hence we have $x \sim_f x$ by definition of $\sim_f$. Therefore $\sim_f$ satisfies reflexivity.

Let $x, x' \in X$ and assume that $x \sim_f x'$. Then by definition of $\sim_f$, we have $f(x) = f(x')$. Therefore $f(x') = f(x)$ and we have $x' \sim_f x$ by definition. Hence $\sim_f$ satisfies symmetry.

Let $x, x', x'' \in X$ and assume that $x \sim_f x'$ and $x' \sim_f x''$. Then by definition of $\sim_f$, we have $f(x) = f(x')$ and $f(x') = f(x'')$. Therefore $f(x) = f(x'')$ and we have $x \sim_f x''$ by definition. Hence $\sim_f$ satisfies transitivity.

Therefore $\sim_f$ is an equivalence relation.

($\sim_f = \Delta_X$ iff f is injective) Assume that $\sim_f$ coincides with Δ_X. We further assume that $x, x' \in X$ satisfy $f(x) = f(x')$. By definition of $\sim_f$, we have $x \sim_f x'$. As $\sim_f = \Delta_X$, by definition of Δ_X, we have $x = x'$. Hence f is injective.

Conversely, assume that f is injective. Then we have:

$$\begin{aligned} x \sim_f x' &\Leftrightarrow f(x) = f(x') && \text{(by definition of } \sim_f\text{)} \\ &\Leftrightarrow x = x' && \text{(as } f \text{ is injective)} \\ &\Leftrightarrow x \Delta_X x' && \text{(by definition of } \Delta_X\text{)}\,. \end{aligned}$$

Hence $\sim_f$ coincides with Δ_X.

Therefore $\sim_f = \Delta_X$ if and only if f is injective.

2 (Exercise 1.19).

1. By definition of equivalence relations, it suffices to show that $\sim$ satisfies reflexivity, symmetry and transitivity.

For $x \in X$, by reflexivity of $\lesssim$, we have $x \lesssim x$. Hence we have $x(\lesssim \cap \gtrsim)x$ (i.e. $x \sim x$), and therefore $\sim$ satisfies reflexivity.

Let $x, x' \in X$ and assume that $x \sim x'$. Then by definition of $\sim$, we have $x \lesssim x'$ and $x \gtrsim x'$. Therefore we have $x' \sim x$ by definition of $\sim$, and this implies that $\sim$ satisfies symmetry.

Let $x, x', x'' \in X$ and assume that $x \sim x'$ and $x' \sim x''$. Then by definition of $\sim$, we have $x \lesssim x'$, $x \gtrsim x'$, $x' \lesssim x''$ and $x' \gtrsim x''$. By $x \lesssim x'$, $x' \lesssim x''$ and transitivity of $\lesssim$, we have $x \lesssim x''$. Similarly, by $x \gtrsim x'$, $x' \gtrsim x''$, we have $x \gtrsim x''$. Therefore we have $x \sim x''$ by definition, and this implies that $\sim_f$ satisfies transitivity.

Therefore $\sim$ is an equivalence relation.

2. Assume $x \sim x'$, $y \sim y'$ and $x \lesssim y$. By definition of $\sim$, $x \sim x'$ implies $x' \lesssim x$, and $y \sim y'$ implies $y \lesssim y'$. Therefore by transitivity of $\lesssim$, we have $x' \lesssim y'$.

3. By definition of partial orders, it suffices to show that $\lesssim$ satisfies reflexivity, antisymmetry and transitivity.

For each $x \in X$, by reflexivity of $\lesssim$, we have $x \lesssim x$. Hence we have $[x]_\sim \lesssim [x]_\sim$ by definition, and therefore $\lesssim$ on $X/\sim$ satisfies reflexivity.

Let $x, x' \in X$ and assume that $[x]_\sim \lesssim [x']_\sim$ and $[x]_\sim \gtrsim [x']_\sim$. Then by definition of $\lesssim$ on $X/\sim$, we have $x \lesssim x'$ and $x \gtrsim x'$. Then we have $x \sim x'$ by definition of $\sim$, and this implies $[x]_\sim = [x']_\sim$. Hence $\lesssim$ satisfies antisymmetry.

Let $x, x', x'' \in X$ and assume that $[x]_\sim \lesssim [x']_\sim$ and $[x']_\sim \lesssim [x'']_\sim$. Then by definition of $\lesssim$ on $X/\sim$, we have $x \lesssim x'$ and $x' \lesssim x''$. By transitivity of $\lesssim$ on X, we have $x \lesssim x''$. Therefore we have $[x]_\sim \lesssim [x'']$ by definition, and this implies that $\lesssim$ on $X/\sim$ satisfies transitivity.

Therefore $\lesssim$ is a partial order.