情報論理・プログラムの数理
第7回演習レポート（6月1日出題）解答

論理学演習

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1 (Lemma 3.4.9).

1 ($\wedge$)

(if): Assume $A \in U', B \in U'$ and $A \wedge B \notin U'$. Because (U', V') is a maximally consitent pair, $A \wedge B \in V'$. However, this contradicts consistency of (U', V'), because $\vdash A, B \Rightarrow A \wedge B$ by

$$\frac{A \Rightarrow A \qquad B \Rightarrow B}{A, B \Rightarrow A \wedge B}\ .$$

(only if): Assume $A \wedge B \in U'$. We show $A \in U'$. If $A \notin U'$, by maximal consistency of (U', V') we have $A \in V'$, which contradicts $\vdash A \wedge B \Rightarrow A$.

$$\frac{A \Rightarrow A}{A \wedge B \Rightarrow A}$$

$B \in U'$ can be shown in the same way.

3 ($\supset$)

(if): Assume $A \notin U'$ or $B \in U'$, and $A \supset B \notin U'$.
(if-i) if $A \notin U'$: By maximal consistency of (U', V') we have $A, A \supset B \in V'$, which contradicts $\vdash \Rightarrow A, A \supset B$.

$$\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A, B}}{\Rightarrow A, A \supset B}$$

(if-ii) if $B \in U'$: By maximal consistency of (U', V') we have $A \supset B \in V'$, which contradicts $\vdash B \Rightarrow A \supset B$.

$$\dfrac{\dfrac{B \Rightarrow B}{A, B \Rightarrow B}}{B \Rightarrow A \supset B}$$

(only if): Assume $A \supset B \in U'$, $A \in U'$ and $B' \notin U'$. By maximal consistency of (U', V') we have $B \in V'$, which contradicts $\vdash A \supset B, A \Rightarrow B$.

$$\frac{A \Rightarrow A \qquad B \Rightarrow B}{A \supset B, A \Rightarrow B}$$

コメント

Consistency の意味を誤解しているのか、「$A \notin U'$ ならある論理式の列 $\Delta \subseteq V'$ が存在して $\vdash A \Rightarrow \Delta$ である」などといった仮定をおいて証明している答案が結構ありました (「$A \in U'$ なら、$\vdash A \Rightarrow \Delta$ となる $\Delta \subseteq V'$ は存在しない」の裏をとっている)($\Delta := A$ ならあっているのであながち間違いとも言い切れませんが…)。その他にも、「$\vdash A \wedge B \Rightarrow \Delta$ なら証明木の途中に $A \Rightarrow \Delta$ や $B \Rightarrow \Delta$ が出現する」と主張する答案が何枚か存在しましたが、例えば $\Delta := A \wedge B$ ならこれは正しくありません。

2 (Exercise 4.2).

$$(R(x,y))[f(x)/y] \equiv R(x, f(x))$$

$$(Q(x) \wedge \exists x.\, R(x,y))[g(y)/x] \equiv Q(g(y)) \wedge \exists x.\, R(x,y)$$

$$(Q(y) \wedge \exists x.\, R(x,y))[f(x)/y] \equiv Q(f(x)) \wedge \exists z.\, R(z, f(x))$$

(Renaming is needed.)

$$(\forall x.\, R(x,y))[f(x)/x] \equiv \forall x.\, R(x,y)$$

3.

Question

Consider the following formulas.

(a) $\forall x.\, (P(x) \supset Q(x)) \supset \forall x.\, P(x) \supset \forall x.\, Q(x)$

(b) $(\forall x.\, P(x) \supset \forall x.\, Q(x)) \supset \forall x.\, (P(x) \supset Q(x))$

(c) $\forall x.\, (P(x) \supset Q(x)) \supset \exists x.\, P(x) \supset \exists x.\, Q(x)$

(d) $(\exists x.\, P(x) \supset \exists x.\, Q(x)) \supset \forall x.\, (P(x) \supset Q(x))$

For each of those formulas, answer

- if it is valid (give a countermodel if it is not);
- if it is derivable (give a derivation tree if it is); and
- if it is satisfiable (give a model that satisfies the formula, if it is).

Answer

(a) Derivable, valid and satisfiable. A derivation tree is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(z) \Rightarrow P(z) \qquad Q(z) \Rightarrow Q(z)}{P(z) \supset Q(z), P(z) \Rightarrow Q(z)}}{P(z) \supset Q(z), \forall x.\, P(x) \Rightarrow Q(z)}}{\forall x.\, (P(x) \supset Q(x)), \forall x.\, P(x) \Rightarrow Q(z)}}{\forall x.\, (P(x) \supset Q(x)), \forall x.\, P(x) \Rightarrow \forall x.\, Q(x)}}{\forall x.\, (P(x) \supset Q(x)) \Rightarrow \forall x.\, P(x) \supset \forall x.\, Q(x)}}{\Rightarrow \forall x.\, (P(x) \supset Q(x)) \supset \forall x.\, P(x) \supset \forall x.\, Q(x)}$$

Since the formula is derivable, it is valid by soundness. It is satisfied by any model, so in particular a model

$$\mathbb{S} = (\{*\}, \emptyset, ([\![P]\!]_{\mathbb{S}}, [\![Q]\!]_{\mathbb{S}})) \tag{1}$$

satisfies it, where

$$[\![P]\!]_{\mathbb{S}} \colon * \mapsto \mathrm{tt}; \quad [\![Q]\!]_{\mathbb{S}} \colon * \mapsto \mathrm{tt} \ .$$

(b) Not derivable, not valid and satisfiable. A countermodel is, for example,

$$\mathbb{S} = (\{1, 2\}, \emptyset, ([\![P]\!]_{\mathbb{S}}, [\![Q]\!]_{\mathbb{S}})) \tag{2}$$

where

$$[\![P]\!]_{\mathbb{S}} \colon 1 \mapsto \mathrm{tt}, 2 \mapsto \mathrm{ff}; \quad [\![Q]\!]_{\mathbb{S}} \colon 1 \mapsto \mathrm{tt}, 2 \mapsto \mathrm{tt} \ .$$

The model given in (1) satisfies the formula.

(c) Derivable, valid and satisfiable. The derivation tree is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(z) \Rightarrow P(z) \qquad Q(z) \Rightarrow Q(z)}{P(z) \supset Q(z), P(z) \Rightarrow Q(z)}}{\forall x. (P(x) \supset Q(x)), P(z) \Rightarrow Q(z)}}{\forall x. (P(x) \supset Q(x)), P(z) \Rightarrow \exists x. Q(x)}}{\forall x. (P(x) \supset Q(x)), \exists x. P(x) \Rightarrow \exists x. Q(x)}}{\forall x. (P(x) \supset Q(x)) \Rightarrow \exists x. P(x) \supset \exists x. Q(x)}}{\Rightarrow \forall x. (P(x) \supset Q(x)) \supset \exists x. P(x) \supset \exists x. Q(x)}$$

Validity is easy by soundness. The trivial model (1) satisfies the formula.

(d) Not derivable, not valid and satisfiable. A countermodel is, for example, the model given by (2). The model given in (1) satisfies the formula.

コメント

モデルの定義で必要な P, Q の解釈 $[\![P]\!], [\![Q]\!] \colon U \to \{\mathrm{tt}, \mathrm{ff}\}$ を、$u \in U$ についてではなく、各変数 $x \in \mathbf{Var}$ について定めようとしている答案が多くありました。それは誤りなので、モデル (/構造) の定義 (教科書 4.3.1) を復習しましょう。