情報論理・プログラムの数理
第9回演習レポート（6月15日出題）解答

1 (Exercise 6.4).

Proof. As $P, Q \subseteq \mathbb{N}^n$ are PR predicates, their characteristic functions $\chi_P, \chi_Q : \mathbb{N}^n \to 2 \hookrightarrow \mathbb{N}$ are all PR (see Def. 6.1.13).

We show that $\chi_{\neg P}, \chi_{P \vee Q}, \chi_{P \wedge Q} : \mathbb{N}^n \to \mathbb{N}$ are all PR.

For $\neg P$, we have:

$$\chi_{\neg P}(\vec{x}) = \begin{cases} 0 & (\vec{x} \in \mathbb{N}^n \setminus P) \\ 1 & (\text{otherwise}) \end{cases} = \begin{cases} 0 & (\chi_P(\vec{x}) = 1) \\ 1 & (\text{otherwise}) \end{cases} = 1 \dot{-} \chi \neg P(\vec{x})\,.$$

As the constant $1 = \mathsf{succ}(\mathsf{zero}) : \mathbb{N}^0 \to \mathbb{N}$ and the normalized subtraction $\dot{-} : \mathbb{N}^2 \to \mathbb{N}$ (Example 6.1.9) are both PR, $\chi_{\neg P}$ is also PR.

For $P \vee Q$, we have:

$$\chi_{P \vee Q}(\vec{x}) \;=\; \begin{cases} 0 & (\vec{x} \in P \cup Q) \\ 1 & (\text{otherwise}) \end{cases} \;=\; \begin{cases} 0 & (\chi_P(\vec{x}) = 0 \text{ or } \chi_P(\vec{x}) = 0) \\ 1 & (\text{otherwise}) \end{cases} \;=\; \mathsf{mult}(\chi_P(\vec{x}), \chi_Q(\vec{x}))\,.$$

As the multiplication $\mathsf{mult} : \mathbb{N}^2 \to \mathbb{N}$ is PR, $\chi_{P \vee Q}(\vec{x})$ is also PR.

For $P \wedge Q$, we have:

$$P \wedge Q = \mathbb{N}^n \setminus \big((\mathbb{N}^n \setminus P) \cup (\mathbb{N}^n \setminus Q)\big) = \neg(\neg P \vee \neg Q)\,.$$

As we have already shown that $\neg$ and $\vee$ preserve PR predicates. Hence $P \wedge Q$ is also a PR predicate. □

2 (Exercise 6.6).

For $\vec{x} \in \mathbb{N}^2$, we have:

$$\max(\vec{x}) = \begin{cases} \mathsf{proj}^2_0(\vec{x}) & (\text{if } \mathsf{proj}^2_0(\vec{x}) \geq \mathsf{proj}^2_1(\vec{x}) \text{ is true}) \\ \mathsf{proj}^2_1(\vec{x}) & (\text{if } \mathsf{proj}^2_0(\vec{x}) < \mathsf{proj}^2_1(\vec{x}) \text{ is true})\,. \end{cases}$$

By Example 6.1.14, $(\_{}_1 \leq \_{}_2) \subseteq \mathbb{N}^2$ is a PR predicate. Hence by Lem. 6.1.17, $(\_{}_1 \geq \_{}_2) = (\_{}_2 \leq \_{}_1) \subseteq \mathbb{N}^2$ is a PR predicate. By Lem. 6.1.15, $(\_{}_1 < \_{}_2) = \big((\_{}_1 \leq \_{}_2) \wedge (\neg(\_{}_1 = \_{}_2))\big) \subseteq \mathbb{N}^2$ is also a PR predicate. By Lemma 6.1.18, the case distinction is a PR function. Hence $\max : \mathbb{N}^2 \to \mathbb{N}$ is PR.

Similarly, for $\vec{x} \in \mathbb{N}^2$, we have:

$$\min(\vec{x}) = \begin{cases} \mathsf{proj}^2_0(\vec{x}) & (\text{if } \mathsf{proj}^2_0(\vec{x}) \leq \mathsf{proj}^2_1(\vec{x}) \text{ is true}) \\ \mathsf{proj}^2_1(\vec{x}) & (\text{if } \mathsf{proj}^2_0(\vec{x}) > \mathsf{proj}^2_1(\vec{x}) \text{ is true})\,. \end{cases}$$

Therefore $\min : \mathbb{N}^2 \to \mathbb{N}$ is also PR.