Completeness results for the $\mathcal{ZX}$-calculus

Miriam Backens

Department of Computer Science, University of Oxford

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Outline

Motivation

Approximate completeness for single qubits

Completeness for stabilizer diagrams

Conclusions & Outlook
Motivation

- $\text{zx}$-calculus is incomplete, even for single qubits; not obvious how to complete it
- instead: look for fragments of the general calculus which are complete
- approximate universality: small sets of operators suffice to approximate arbitrary unitaries to any accuracy
- $approximate$ $completeness$: completeness for such a set
Outline

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Completeness for stabilizer diagrams

Conclusions & Outlook
The single-qubit Clifford+T group

An approximately universal group, generated by:

- single-qubit Clifford group $C_1 = \langle S, H \rangle$, where
  
  $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$
  
  $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

- T gate

  $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$
The \( \text{ZX-calculus for the single-qubit Clifford+T group} \)

Generated by \( \bullet \frac{\pi}{2}, \quad \text{H}, \text{and} \quad \bullet \frac{\pi}{4} \) —or \( \bullet \frac{\pi}{4} \) and \( \bullet \frac{\pi}{2} \)

- *single-qubit*: restrict diagrams to line graphs
- *Clifford+T group*: restrict phases to multiples of \( \frac{\pi}{4} \)

(Ignore global phases.)
The $zx$-calculus for the single-qubit Clifford+T group

Generated by $\pi/2$, $H$, and $\pi/4$ — or $\pi/4$ and $\pi/2$

- **single-qubit**: restrict diagrams to line graphs
- **Clifford+T group**: restrict phases to multiples of $\pi/4$

(Ignore global phases.)

Rules:

\[
\begin{align*}
\bullet \alpha & = \bullet \alpha + \beta \\
\bullet \beta & = \bullet \alpha + \beta \quad \text{and} \quad \bullet \pi & = \bullet -\alpha \\
\bullet \pi/2 & = \bullet \pi/2 \quad \text{for integer } n
\end{align*}
\]
Any single-qubit Clifford diagram can be written uniquely as one of the following, with $\alpha, \beta, \gamma \in \{0, \pi/2, \pi, -\pi/2\}$:

\[ \begin{align*}
\alpha & \quad \beta \\
\gamma & \quad \pm\pi/2 \\
\pi/2 & 
\end{align*} \]

and

\[ \begin{align*}
\beta & \quad \alpha \\
\gamma & \quad \pm\pi/2 \\
\pi/2 & 
\end{align*} \]
Any single-qubit Clifford diagram can be written uniquely as one of the following, with $\alpha, \beta, \gamma \in \{0, \pi/2, \pi, -\pi/2\}$:

\[
\begin{align*}
\alpha & \quad \beta \\
\alpha & \quad \beta & \quad \pm \pi/2
\end{align*}
\]

or, equivalently, one of

\[
\begin{align*}
a \pi & \quad \beta \\
\pm \pi/2 & \quad \gamma
\end{align*}
\]

where $a, b \in \{0, 1\}$ and $\beta, \gamma$ as above.
The normal form for single-qubit Clifford+T diagrams

Following [Matsumoto & Amano 2008], any single-qubit Clifford+T diagram is either pure Clifford or it can be written as

\[
\begin{align*}
W & \in \left\{ \begin{array}{c}
\frac{\pi}{2} \\
\frac{\pi}{2}
\end{array} \right\} \\
V_k & \in \left\{ \begin{array}{c}
\frac{\pi}{4} \\
\frac{3\pi}{4} \\
\frac{\pi}{2} \\
\frac{\pi}{2}
\end{array} \right\} \quad \text{for } 1 \leq k \leq n \\
U & \in \left\{ \begin{array}{c}
\frac{\pi}{4} + \alpha \\
\beta \\
\pm \frac{\pi}{2} \\
\frac{\pi}{2}
\end{array} \right\}
\end{align*}
\]

with \( n \) a non-negative integer and \( \alpha, \beta, \gamma \in \{0, \frac{\pi}{2}, \pi, -\frac{\pi}{2}\} \).
Diagrams are rewritten into normal form by pushing phases towards the bottom of the diagram as much as possible.

- can push $\alpha$ and $\pi$ past $\pi/4$
- can push $\alpha$ and $\pi$ past $\pi/2$
- can push $\pi$ and $\pi$ past $V \in \{\pi/4, \pi/2, 3\pi/4, \pi/2\}$
Diagrams are rewritten into normal form by pushing phases towards the bottom of the diagram as much as possible.

- Can push $\alpha$ and $\pi$ past $\pi/4$.
- Can push $\alpha$ and $\pi$ past $\pi/2$.
- Can push $\pi$ and $\pi$ past $V \in \{\pi/4, \pi/2, 3\pi/4\}$.

Write Clifford operators as $\{a\pi, b\pi/2, \pm \pi/2, \gamma\}$, then

$$C = \left\{ \begin{array}{l}
\pi/4 & \pi/4 \\
(\pi/2, 3\pi/4, \pi/2, \pi/2) & (\pi/2, 3\pi/4, \pi/2, \pi/2)
\end{array} \right\}$$
Pushing phases down, part 1

Diagrams are rewritten into normal form by pushing phases towards the bottom of the diagram as much as possible.

- can push \( \alpha \) and \( \pi \) past \( \pi/4 \)
- can push \( \alpha \) and \( \pi \) past \( \pi/2 \)
- can push \( \pi \) and \( \pi \) past \( V \in \{ \pi/4, \pi/2, 3\pi/4 \} \)

Write Clifford operators as \( \{ a\pi, b\pi/2, \pm\pi/2, \pm\pi/4 \} \), then

\[
C = \begin{cases}
    a\pi + \beta + \pi/4 & \text{or} \\
    b\pi/2 + (-1)^c \pi/2 \\
\end{cases}
\]
Pushing phases down, part 1

Diagrams are rewritten into normal form by pushing phases towards the bottom of the diagram as much as possible.

- can push $\alpha$ and $\pi$ past $\pi/4$.
- can push $\alpha$ and $\pi$ past $\pi/2$.
- can push $\pi$ and $\pi$ past $V \in \{\pi/4, \pi/2, 3\pi/4\}$.

Write Clifford operators as

\[ \{a\pi, \beta, b\pi/2, \pm \pi/2, \gamma, (\pm 1)^c \pi/2\} \]

then

\[
C_{\pi/4} = \begin{cases} 
\{a\pi, \beta, b\pi/2\} & \text{or} \\
\{\gamma, (\pm 1)^c \pi/2\} & \end{cases}
\]
Pushing phases down, part 1

Diagrams are rewritten into normal form by pushing phases towards the bottom of the diagram as much as possible.

- can push $\alpha$ and $\pi$ past $\frac{\pi}{4}$
- can push $\alpha$ and $\pi$ past $\frac{\pi}{2}$
- can push $\pi$ and $\pi$ past $V \in \{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{2} \}$

Write Clifford operators as $\{ a \pi \beta b \pi/2 \pm \pi/2 \gamma \}$, then

$$C_{\pi/4} = \left\{ \begin{array}{c} (-1)^{a\pi/4} \text{ or } \frac{\pi}{4} \\ \frac{\pi}{4} \end{array} \right\} \begin{array}{c} b\pi/2 \\ \frac{\pi}{2} \\ \frac{3\pi}{2} \\ \frac{\pi}{2} \end{array} \begin{array}{c} \beta \\ -(\frac{\pi}{2})^c \end{array} \right\}$$
Pushing phases down, part 1

Diagrams are rewritten into normal form by pushing phases towards the bottom of the diagram as much as possible.

- can push $\alpha$ and $\pi$ past $\pi/4$
- can push $\alpha$ and $\pi$ past $\pi/2$
- can push $\pi$ and $\pi$ past $V \in \{\pi/4, \pi/2, 3\pi/4\}$

Write Clifford operators as \[
\left\{ \begin{array}{c}
\pi/4 \\
-a\pi/2 \\
a\pi \\
\beta \\
b\pi/2 \\
\pm\pi/2 \\
\gamma \\
\end{array} \right. \right\}, \text{ then}
\]

\[
C_{\pi/4} = \left\{ \begin{array}{c}
\pi/4 \\
-a\pi/2 \\
a\pi \\
\beta \\
b\pi/2 \\
(\pm1)^c \pi/2 \\
\gamma \\
\pi/4 \\
\end{array} \right. \right\} \quad \text{or} \quad \left\{ \begin{array}{c}
\pi/4 \\
-a\pi/2 \\
a\pi \\
\beta \\
b\pi/2 \\
(\pm1)^c \pi/2 \\
\gamma \\
\pi/4 \\
\end{array} \right. \right\}
\]
Diagrams are rewritten into normal form by pushing phases towards the bottom of the diagram as much as possible.

- can push $\alpha$ and $\pi$ past $\pi/4$
- can push $\alpha$ and $\pi$ past $\pi/2$
- can push $\pi$ and $\pi$ past $\forall \in \{\pi/2, 3\pi/4\}$

Write Clifford operators as $
\begin{cases}
\{ \alpha \pi \beta \}, \text{ then} \quad \begin{cases}
\begin{array}{ll}
\pi/4 & \text{or } \pi/4 \\
\alpha \pi & \pm \pi/2 \\
\beta & \gamma \\
\beta + a\pi/2 & \pi/4 \\
\end{array}
\end{cases}
\end{cases}$
Pushing phases down, part 1

Diagrams are rewritten into normal form by pushing phases towards the bottom of the diagram as much as possible.

- can push $\alpha$ and $\pi$ past $\pi/4$
- can push $\alpha$ and $\pi$ past $\pi/2$
- can push $\pi$ and $\pi$ past $V \in \{\pi/4, \pi/2, 3\pi/4\}$

Write Clifford operators as $\{a\pi, a\pi/2, b\pi/2, b\pi, c \pi/2, (\pm \pi/2, \pm \pi/2, \pm \pi/2, \pm \pi/2, \pm \pi/2, \pm \pi/2 ) \},$ then

$$C = \begin{cases} \pi/4, a\pi, \beta + a\pi/2 \lor b\pi/2, (\pm \pi/2, \pm \pi/2, \pm \pi/2, \pm \pi/2, \pm \pi/2, \pm \pi/2 ) \end{cases}$$
Pushing phases down, part 1

Diagrams are rewritten into normal form by pushing phases towards the bottom of the diagram as much as possible.

- can push $\alpha$ and $\pi$ past $\pi/4$
- can push $\alpha$ and $\pi$ past $\pi/2$
- can push $\pi$ and $\pi$ past $V \in \{\pi/4, \pi/2, 3\pi/4\}$

Write Clifford operators as $\{a\pi\beta b\pi/2 \pm \pi/2 \gamma\}$, then

\[
C = \begin{cases}
\pi/4 \\
a\pi \\
\beta + a\pi/2
\end{cases}
\quad \text{or} \quad
\begin{cases}
b\pi/2 \\
\pm \pi/2 \\
\gamma
\end{cases}
\]

or

\[
C = \begin{cases}
\pi/4 \\
a\pi \\
\beta + a\pi/2
\end{cases}
\quad \text{or} \quad
\begin{cases}
\pi/4 \\
b\pi/2 \\
\pi/2
\end{cases}
\]

or

\[
C = \begin{cases}
\pi/4 \\
a\pi \\
\beta + a\pi/2
\end{cases}
\quad \text{or} \quad
\begin{cases}
\pi/4 \\
c\pi \\
\pi/4
\end{cases}
\]
Pushing phases down, part 1

Diagrams are rewritten into normal form by pushing phases towards the bottom of the diagram as much as possible.

- can push $\alpha$ and $\pi$ past $\pi/4$
- can push $\alpha$ and $\pi$ past $\pi/2$
- can push $\pi$ and $\pi$ past $V \in \{ \pi/4, \pi/2, 3\pi/4 \}$

Write Clifford operators as $\{ \pm, a\pi, b\pi/2, \beta, \pm \pi/2, \gamma, c\pi, \pi/4, \pi/2, \} \quad \text{, then}$

$C = \{ \pi/4, a\pi, \beta + a\pi/2 \}$ or $\{ \pi/4, b\pi/2, \gamma + c\pi/2 \}$
Pushing phases down, part 1

Diagrams are rewritten into normal form by pushing phases towards the bottom of the diagram as much as possible.

- Can push $\alpha$ and $\pi$ past $\pi/4$.
- Can push $\alpha$ and $\pi$ past $\pi/2$.
- Can push $\pi$ and $\pi$ past $V \in \{\pi/4, \pi/2, 3\pi/4\}$.

Write Clifford operators as $\{a\pi, b\pi/2, \pm \pi/2, \pm \gamma\}$, then

\[
C = \left\{ \begin{array}{c}
\pi/4 \\
\pi/4 \\
a\pi \\
\beta + a\pi/2 \\
\gamma + c\pi/2
\end{array} \right. \\
\text{or}
\left\{ \begin{array}{c}
\pi/4 \\
\pi/4 \\
b\pi/2 \\
\pm \pi/2 \\
\gamma
\end{array} \right. = W
\]

or

\[
\left\{ \begin{array}{c}
\pi/4 \\
\pi/4 \\
b\pi/2 \\
\pm \pi/2 \\
\gamma
\end{array} \right. = W
\]

or

\[
\left\{ \begin{array}{c}
\pi/4 \\
\pi/4 \\
b\pi/2 \\
\pm \pi/2 \\
\gamma
\end{array} \right. = W
\]
Pushing phases down, part 2

Have $C_{\pi/4} = W_{\pi/4}^{d\pi\delta}$, and
d
$W_{\pi/2}^{d\pi\delta} = \left\{ \begin{array}{l} d\pi \\ e\pi \\ \delta \\ \pi/2 \end{array} \right\}$ or

$W_{\pi/2}^{\pi/2} = \left\{ \begin{array}{l} d\pi \\ \delta \\ e\pi \\ \pi/2 \\ \pi/2 \end{array} \right\}$.
Pushing phases down, part 2

Have \( C = \pi/4 \), and

\[
\begin{align*}
\delta \quad d\pi \quad \pi/2 \\
\end{align*}
\]

\[
\begin{align*}
W = \frac{\pi}{4} \\
d\pi \\
\delta \\
\end{align*}
\]

therefore

\[
\begin{align*}
V_C &= V'_d e^\pi W \\
U_C &= U'_W \\
V_b &= V'_e f^\pi \\
\end{align*}
\]

or

\[
\begin{align*}
\frac{d\pi}{\pi/2} \\
\frac{(-1)^e}{\pi/2} \\
\end{align*}
\]
Pushing phases down, part 2

Have

\[ C = \frac{\pi}{4}, \quad \delta = \pi, \quad \frac{\pi}{2} \]

and

\[
\begin{aligned}
\delta &= \pi \\
\frac{\pi}{2} &= \left\{ \begin{array}{l}
ed \pi \\
\pi/2 \\
d \pi \\
\end{array} \right. \\
\end{aligned}
\]

\[
\begin{aligned}
\delta &= \pi \\
\frac{\pi}{2} &= \left\{ \begin{array}{l}
ed \pi \\
\pi/2 \\
\end{array} \right. \\
\end{aligned}
\]

or

\[
\begin{aligned}
d \pi &= \pi/2 \\
(-1)^{\pi/2} \pi &= \pi/2 \\
\end{aligned}
\]
Have

\[ C = \pi/4 \]

, and

\[ W = \{ \begin{array}{c} d\pi \\ d\pi \\ e\pi \\ d\pi \end{array} \] or

\[ \{ \begin{array}{c} d\pi \\ d\pi \\ e\pi \\ d\pi \end{array} \]
Have $C = \frac{\pi}{4}$, and

\[ d\pi \delta \pi/2 = \begin{cases} 
\frac{\pi}{2} e\pi (d + e)\pi \\
\frac{\pi}{2} 
\end{cases} \]

or

\[ d\pi (-1)^{e\pi} /2 \pi/2 \]

Therefore

\[ V_C = V'_d e\pi W, \quad U_C = U'_W, \quad V_b = V'_f e\pi f \pi. \]
Pushing phases down, part 2

Have $\frac{\pi}{4} C = \frac{\pi}{4} W$, and

\[
\begin{align*}
\pi / 4 & = d \pi + \delta \\
(\pi / 2) & = e \pi + (d + e) \pi \\

\end{align*}
\]

Therefore $V C = V' d \pi e \pi W$, $U C = U' W$, and $V b = V' e \pi f \pi$. 

or

\[
\begin{align*}
(-1)^{d + e} \pi / 2 & = d \pi \\
\pi / 2 & = \pi / 2
\end{align*}
\]
Have 
\[ C = \pi/4 \]

\[ W = \frac{\pi}{4} d \delta \]

\[ \pi/2 \]

\[ (d + e)\pi \]

\[ (-1)^{d+e} \pi/2 \]

\[ \pi/2 \]

\[ d \pi \]

, and

\[ V_C = V_{C'} \]

\[ U_C = U_{C'} \]

\[ V_b = V_{b'} e^{\pi/2} f \]
Pushing phases down, part 2

Have $C = \frac{\pi}{4}$, and

$$\begin{align*}
d\pi & \quad \delta \\
\frac{\pi}{2} & \quad \end{align*} = \begin{cases} 
\frac{\pi}{2} \quad e\pi \\
(d + e)\pi
\end{cases}$$

or

$$\begin{align*}
\frac{\pi}{2} & \quad (d + e)\pi \\
\frac{\pi}{2} & \quad d\pi
\end{align*}$$
Pushing phases down, part 2

Have

\[ C = \pi/4 \]

\[ \delta \]

\[ \pi/2 \]

\[ d \pi \]

\[ e \pi \]

\[ (d + e)\pi \]

\[ \pi/2 \]

\[ (-1)^d + e \pi/2 \]

\[ (d + e)\pi \]

\[ e\pi \]

\[ \omega \]

\[ \pi/4 \]

\[ d \pi \]

\[ \delta \]

\[ \pi/2 \]

\[ \pi/2 \]

\[ (d + e)\pi \]

\[ e\pi \]

therefore

\[ V_C = V'_d \pi e \pi W \]

\[ U_C = U'_W \]

\[ V_b = V'_e f \pi \]
Pushing phases down, part 2

Have

\[ C = \pi / 4 \]

\[ \pi / 2 \]

\[ d \pi \]

\[ \delta \]

\[ \pi / 2 \]

\[ (d + e) \pi \]

\[ d \pi \]

\[ \pi / 2 \]

\[ \pi / 2 \]

\[ d \pi \]

\[ \pi / 2 \]

\[ (d + e) \pi \]

\[ \pi / 2 \]

\[ \pi / 2 \]

\[ (d + e) \pi \]

\[ d \pi \]

\[ \pi / 2 \]

\[ \pi / 2 \]

\[ (d + e) \pi \]

\[ d \pi \]

\[ \pi / 2 \]

\[ \pi / 2 \]

\[ (d + e) \pi \]

\[ d \pi \]

\[ \pi / 2 \]

\[ \pi / 2 \]

\[ (d + e) \pi \]

\[ d \pi \]

\[ \pi / 2 \]

\[ \pi / 2 \]

\[ (d + e) \pi \]

\[ d \pi \]

\[ \pi / 2 \]

\[ \pi / 2 \]

\[ (d + e) \pi \]

\[ d \pi \]

\[ \pi / 2 \]

\[ \pi / 2 \]

\[ (d + e) \pi \]

\[ d \pi \]
Pushing phases down, part 2

Have

\[ C = \frac{\pi}{4} \]

\[ W = \frac{\pi}{4} d \pi \delta \]

\[ d \pi \delta \pi/2 = \begin{cases} 
\pi/2 e \pi (d + e) \pi d \pi 
\end{cases} \]

or

\[ = \begin{cases} 
\pi/2 \pi/2 (d + e) \pi d \pi \pi/2 
\end{cases} \]

\[ = \begin{cases} 
\pi/2 g \pi h \pi 
\end{cases} \]

therefore

\[ V_C = V'_d e \pi W \]

\[ U_C = U'_e W \]

\[ V_b = V'_d f \pi/2 \]

\[ \pi/2 g \pi h \pi \]
Have $\pi/4$, and $\delta_d \pi = \begin{cases} \pi/2 & \text{or} \\ (d + e)\pi & \end{cases}$, and therefore

$$\begin{align*}
\frac{C}{V} &= \frac{W}{V'}, \\
\frac{C}{U} &= \frac{W}{U'}, 
\end{align*}$$
Rewriting diagrams into normal form, starting with a Clifford diagram

The single-qubit Clifford+T group is generated by $\frac{\pi}{4}$ and $\frac{\pi}{2}$ so it suffices to check what happens when a pure Clifford operator or a normal form diagram is composed with one of these.

Now for any Clifford unitary $C$,

$\begin{array}{cc}
\text{is pure Clifford, and} & \\
C & C
\end{array}$

$= U \in \left\{ \frac{\pi}{4+\alpha}, \frac{\pi}{4+\gamma}, \frac{\pi}{2}, \pm \frac{\pi}{2}, \frac{3\pi}{2} \right\}$.
Rewriting diagrams into normal form, starting with a normal form diagram

For any $W \in \left\{ \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2} \right\}$ and $V_{n} \in \left\{ \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4} \right\}$,

$W = \frac{\pi}{2} = \frac{\pi}{2} = \frac{\pi}{2}$

$V_{n} = \left\{ \frac{3\pi}{4}, \frac{\pi}{2}, \frac{\pi}{2} \right\}$

Case $n = 0$ is similar.
Rewriting diagrams into normal form, starting with a normal form diagram

For any $W \in \{ \pi/2, \pi/2, \pi/2 \}$ and $V_n \in \{ \pi/4, \pi/2, 3\pi/4 \}$,

\[
\begin{align*}
W & \quad \pi/2 \\
V_n & \quad \pi/2 \\
W' & = C \\
V_n' & = V_n \\
& \quad a\pi \\
& \quad b\pi \\
\end{align*}
\]

and

\[
\begin{align*}
W & \quad \pi/4 \\
V_n & \quad \pi/4 \\
W' & = \left\{ \begin{array}{l}
C' \\
V_{n+1}/V_n
\end{array} \right. \\
\end{align*}
\]

Case $n = 0$ is similar.
No normal form diagram represents the identity operator

Can write any single-qubit density operator as \( xX + yY + zZ \), where \( x, y, z \in \mathbb{R} \) and \( X, Y, Z \) are the Pauli matrices. Clifford unitaries act on the vectors \((x, y, z)\) by permuting the elements and adding minus signs; \( T \) sends

\[
(x, y, z) \mapsto \frac{1}{\sqrt{2}} \left( x - y, x + y, z\sqrt{2} \right).
\]
No normal form diagram represents the identity operator

Can write any single-qubit density operator as $xX + yY + zZ$, where $x, y, z \in \mathbb{R}$ and $X, Y, Z$ are the Pauli matrices. Clifford unitaries act on the vectors $(x, y, z)$ by permuting the elements and adding minus signs; $T$ sends

$$(x, y, z) \mapsto \frac{1}{\sqrt{2}} \left( x - y, x + y, z\sqrt{2} \right).$$

States resulting from application of a normal form operator to $|0\rangle$ will have vectors of the form

$$\frac{1}{\sqrt{2}^m} \left( x_1 + x_2\sqrt{2}, y_1 + y_2\sqrt{2}, z_1 + z_2\sqrt{2} \right)$$

where $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{Z}$. By parity arguments, can show none of them represent $|0\rangle$, thus no normal form operator is equal to the identity.
The inverse of a normal form diagram has the same number of $\pi/4$ phase shifts as the original.

We have $U^\dagger = C \pi/4$ and $V^\dagger = V \pi/2$, so
The inverse of a normal form diagram has the same number of $\pi/4$ phase shifts as the original.

We have $U^\dagger = C\frac{\pi}{4}$ and $V^\dagger = V\frac{\pi}{2}$, so

\[
\begin{align*}
U^\dagger & \quad V_1^\dagger & \quad \cdots & \quad V_n^\dagger & \quad W^\dagger \\
C & \quad & \cdots & \quad & \cdots \\
\frac{\pi}{4} & \quad & \frac{\pi}{2} & \quad & \frac{\pi}{2} \\
\frac{\pi}{2} & \quad & \frac{\pi}{2} & \quad & \frac{\pi}{2} \\
\end{align*}
\]
The inverse of a normal form diagram has the same number of $\pi/4$ phase shifts as the original.

We have $U^\dagger = \begin{array}{c} C \\
\pi/4 \end{array}$ and $V^\dagger = \begin{array}{c} V \\
\pi/2 \end{array}$, so
The inverse of a normal form diagram has the same number of $\pi/4$ phase shifts as the original.

We have $U^\dagger = C_{\pi/4}$ and $V^\dagger = V_{\pi/2}$, so

\[ W^\dagger V^\dagger n V^\dagger 1 U^\dagger \ldots = V^1_n V^n W^\dagger \]
The inverse of a normal form diagram has the same number of $\pi/4$ phase shifts as the original.

We have $U^\dagger = C_{\pi/4}$ and $V^\dagger = V_{\pi/2}$, so

\[
\begin{align*}
U^\dagger & = C_{\pi/4} \\
V_1 & = V_{\pi/2} \\
V_n & = V_{\pi/2} \\
W^\dagger & = V_{\pi/2}
\end{align*}
\]
The inverse of a normal form diagram has the same number of $\pi/4$ phase shifts as the original.

We have $U^\dagger = C_{\pi/4}$ and $V^\dagger = V_{\pi/2}$, so

$$
\begin{align*}
C_{\pi/4} & = U^\dagger \\
V^\dagger & = V_{\pi/2}
\end{align*}
$$
Normal forms are unique

Given two normal form diagrams that are not identical, composing one diagram with the inverse of the other will yield a non-trivial diagram.

- Assume diagrams are such that the topmost nodes differ (in colour or phase, or both).
- Inverting a diagram does not change the number of $\pi/4$ phases.
- Distinguish cases to see that the resulting diagram never collapses to the trivial one.
Outline

Motivation

Approximate completeness for single qubits

Completeness for stabilizer diagrams

Conclusions & Outlook
Stabilizer quantum mechanics

Stabilizer operations:

- preparation of qubits in state $|0\rangle$
- Clifford unitaries, generated by

\[ S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \Lambda_X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \]

- measurements in computational basis
Stabilizer quantum mechanics

Stabilizer operations:

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- Clifford unitaries, generated by

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- measurements in computational basis

ZX-calculus:

where $\alpha, \beta$ are multiples of $\pi/2$
Graph states in the $\text{zx}$-calculus

**Definition**
Let $G$ be a finite simple undirected graph. The $\text{zx}$-calculus diagram for the corresponding graph state consists of:
- for each node in $G$, a green node with one output, and
- for each edge in $G$, an edge with a Hadamard node on it.

E.g.

\[
\begin{array}{c}
\text{2} \\
\text{3} \\
\text{4} \\
\text{1}
\end{array}
\]
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Local complementations

The local Clifford group consists of all tensor products of the single-qubit Clifford operators $\langle S, H \rangle$.

**Theorem (Van den Nest et. al, 2004)**

*Any stabilizer state is local Clifford-equivalent to some graph state.*
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*Two graph states are local Clifford-equivalent if and only if they are related by a sequence of local complementations.*

A local complementation about a vertex $v$ inverts the subgraph generated by the neighbourhood of $v$: e.g.

```
2
/  \
|   |
/  \
3
```

```
1   4
```
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\[ \begin{array}{ccc}
1 & 2 & 3 \\
2 & 3 & 4 \\
\end{array} \rightarrow \begin{array}{ccc}
1 & 2 & 3 \\
2 & 3 & 4 \\
\end{array} \]
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\[
\begin{align*}
2 & \quad 3 \\
1 & \quad 4 \\
3 & \quad \rightarrow \\
2 & \quad 3 \\
1 & \quad 4 \\
2 & \quad \rightarrow
\end{align*}
\]
Stabilizer completeness proof (overview)

[Duncan & Perdrix, 2009] show that local complementations can be derived from the rules of the $\mathcal{ZX}$-calculus. Use this to show that the results from [Van den Nest et al., 2004] hold in the $\mathcal{ZX}$-calculus:

- Any stabilizer $\mathcal{ZX}$-calculus diagram can be rewritten into a graph state with local Clifford operators, e.g.

  ![Diagram with $H$, $\pi/2$, $-\pi/2$ operators]

- There exists a terminating algorithm that, given two stabilizer diagrams, will rewrite them to be identical if possible.

Conclusions & Outlook

- **zx-calculus is not complete in general but fragments of it are complete, e.g.**
  - line graphs where all phases are multiples of $\pi/4$ (single-qubit Clifford+T group)
  - diagrams where all phases are multiples of $\pi/2$ (stabilizer quantum mechanics)
- can these results be combined to multi-qubit Clifford+T operators?
- can we introduce some notion of approximate equality in the zx-calculus?
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Thank you!