Dichromatic and Trichromatic Calculus for Qutrit Systems

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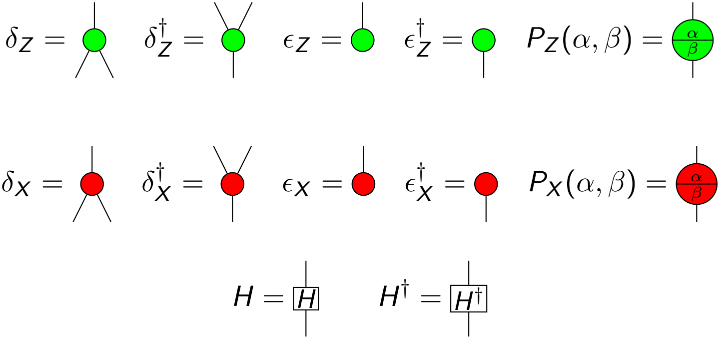
Contents

Some Backgrounds

- [1] B. Coecke, R. Duncan. Interacting quantum observables: Categorical algebra and diagrammatics.
  - Provided a general framework of dagger symmetric monoidal categories for axiomatising complementarity of quantum observables.
  - Introduced the intuitive and universal ZX Calculus for qubits.
- [2] A. Lang and B. Coecke. Trichromatic open digraphs for understanding qubits.
  - Introduced a trichromatic graphical calculus.
  - 'Dichromatic ZX Calculus + Euler angle decomposition of the Hadamard gate = Trichromatic calculus'.

Qutrit RG Generators

We define a category **RG** where the objects are n-fold monoidal products of an object $*$, denoted $*^n (n \geq 0)$. In **RG**, a morphism from $*^m$ to $*^n$ is a finite undirected open graph from m wires to n wires, built from



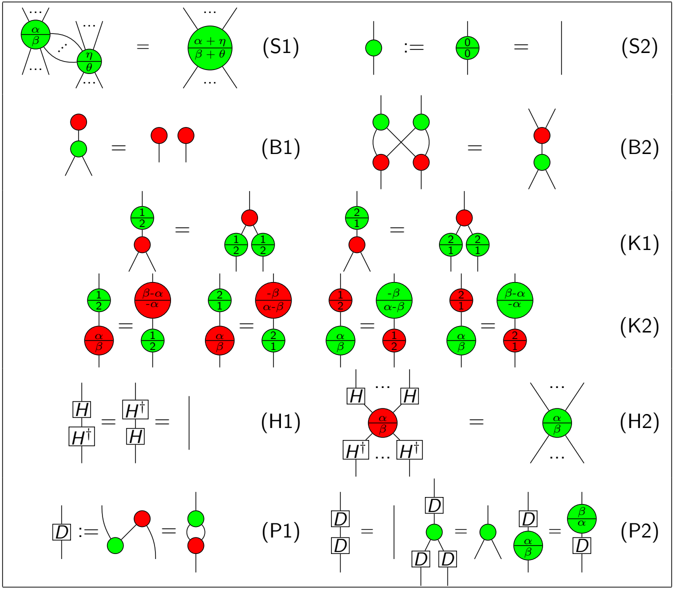
where $\alpha, \beta \in [0, 2\pi)$. For convenience, we denote the frequently used angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ by 1 and 2 respectively.

Qutrit RG Rules

RG morphisms are also subject to the following equations:

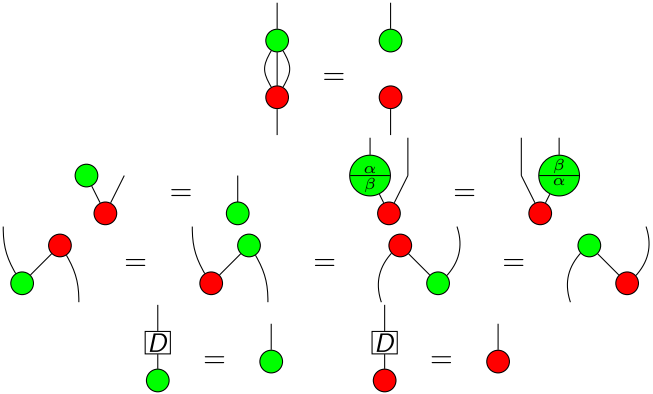
1. Equations in the following figure.
2. All equations hold under flip of graphs, negation of angles, and exchange of H and $H^\dagger$.
3. All equations hold under flip of colours (except for rules $K2$ and $H2$).

Qutrit RG Rules



Some Derived Rules

These equations are very useful when wanting to demonstrates some more complex equalities in describing quantum protocols[4] and algorithms[6].



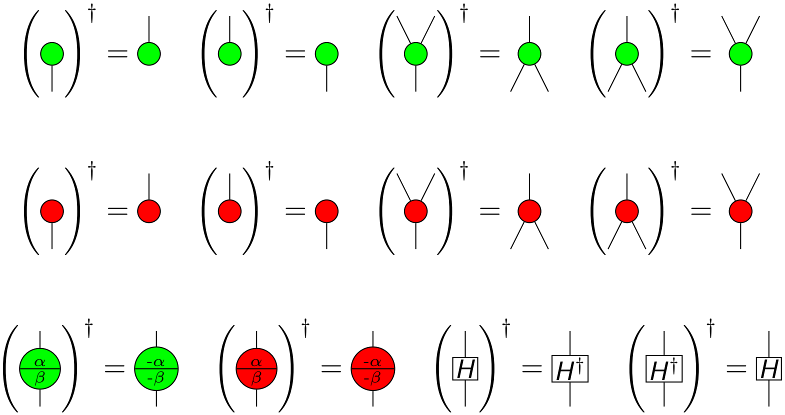
(1)

(2)

(3)

(4)

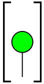
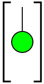
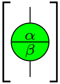
Dagger Functor



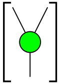
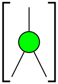
RG is a dagger symmetric monoidal category.

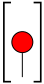
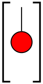
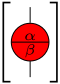
RG Interpretation

We give an interpretation $[\cdot]_{RG} : \mathbf{RG} \to \mathbf{FdHilb_Q}$

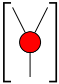
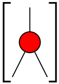
 $= |+\rangle$  $= \langle +|$ 

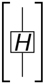
$= |0\rangle\langle 0| + e^{i\alpha}|1\rangle\langle 1| + e^{i\beta}|2\rangle\langle 2|$

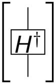
 $= |0\rangle\langle 00| + |1\rangle\langle 11| + |2\rangle\langle 22|$  $= |00\rangle\langle 0| + |11\rangle\langle 1| + |22\rangle\langle 2|$

 $= |0\rangle$  $= \langle 0|$ 

$= |+\rangle\langle ++| + e^{i\alpha}|\omega\rangle\langle\omega| + e^{i\beta}|\bar{\omega}\rangle\langle\bar{\omega}|$

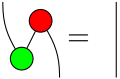
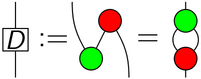
 $= |+\rangle\langle ++| + |\omega\rangle\langle\omega\omega| + |\bar{\omega}\rangle\langle\bar{\omega}\bar{\omega}|$  $= |++\rangle\langle +| + |\omega\omega\rangle\langle\omega| + |\bar{\omega}\bar{\omega}\rangle\langle\bar{\omega}|$

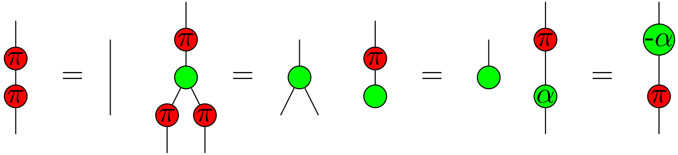


$= |+\rangle\langle 0| + |\omega\rangle\langle 1| + |\bar{\omega}\rangle\langle 2|$ 

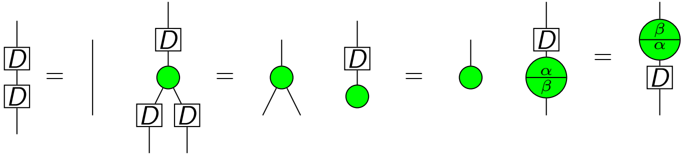
$= |0\rangle\langle +| + |1\rangle\langle\omega| + |2\rangle\langle\bar{\omega}|$

Differences Between Qutrit and Qubit Rules I

- In qubit case,  . For qutrit duliser, 
- The qubit dualizer (identical permutation) is an even permutation, while the qutrit dualizer is an odd permutation.
- There is only one odd permutation  in qubit case satisfying

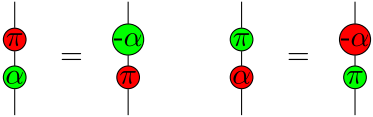


The qutrit dualizer satisfies

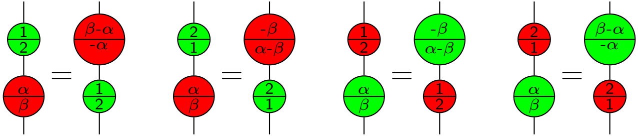


Differences Between Qutrit and Qubit Rules II

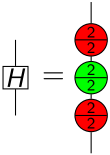
- In qubit case the $K2$ rule still holds when flipping the colours,



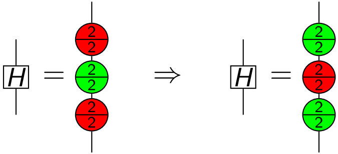
Whereas it doesn't hold in qutrit case



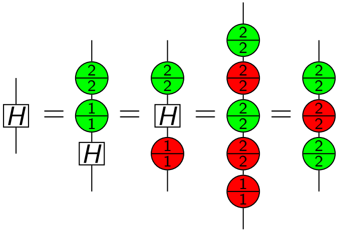
Decomposition of the Hadamard Gate

- Euler decomposition of the Hadamard gate: 

- The Euler decomposition is not unique:



- Proof:



Euler Decomposition Not Derivable

- In the qubit case, Duncan and Perdrix [5] proved that the Euler decomposition is not derivable from ZX calculus.
- The Euler decomposition is not derivable from **RG**.
- Proof: We define an alternative interpretation functor $[\cdot]_0 : \mathbf{RG} \to \mathbf{FdHilb_Q}$ exactly as $[\cdot]_{RG}$ with the following change:

$$[P_X(\alpha,\beta)]_0 = [P_X(0,0)]_{RG} \qquad [P_Z(\alpha,\beta)]_0 = [P_Z(0,0)]_{RG}$$

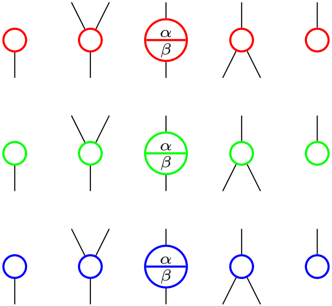
This functor preserves all the rules, so its image is indeed a valid model of the theory. However we have the following inequality

$$[H]_0 \neq [P_X(\frac{4\pi}{3},\frac{4\pi}{3})]_0 \circ [P_Z(\frac{4\pi}{3},\frac{4\pi}{3})]_0 \circ [P_X(\frac{4\pi}{3},\frac{4\pi}{3})]_0$$

hence the Euler decomposition is not derivable from **RG**.

Qutrit RGB Generators

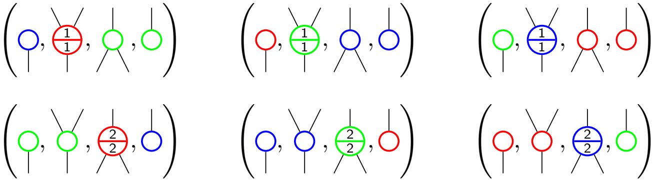
Similarly, we define a category **RGB** where morphism from $*^m$ to $*^n$ is a finite undirected open graph from m wires to n wires, built from

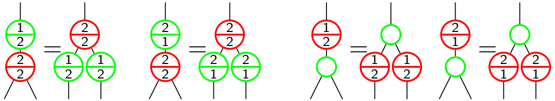


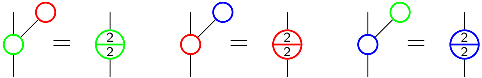
Qutrit RGB Rules

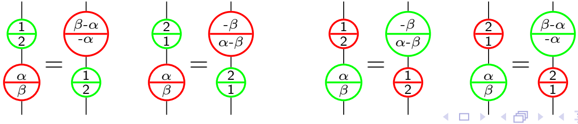
RGB morphisms are subject to the following equations

1. Each colour respects the equations (S1) and (S2).
2. All equations hold under flip of graphs and negation of angles.
3. The following quadruples form bialgebras:



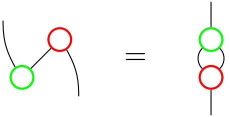
4. 

5. 

6. 

Qutrit RGB Rules

7.



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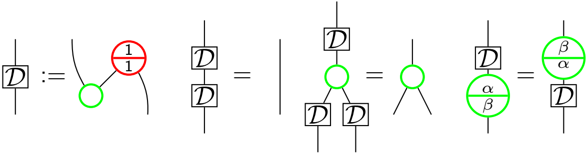

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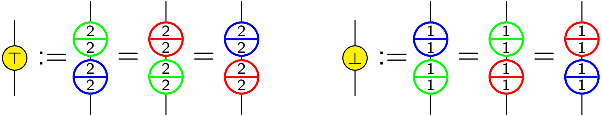
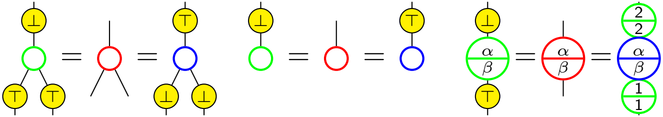
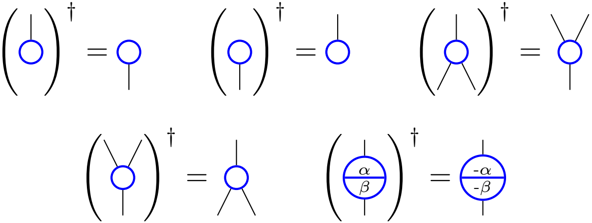


10.



Dagger Functor

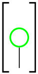
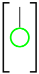
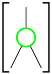
- Dagger functor (only showing the blue one):

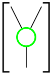
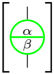


- **RGB** is a dagger symmetric monoidal category.

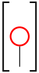
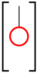
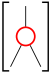
RGB Interpretation

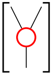
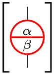
We give an interpretation $[\cdot]_{RGB} : \mathbf{RGB} \to \mathbf{FdHilb_Q}$

 $= |+\rangle$  $= \langle +|$  $= |00\rangle\langle 0| + |11\rangle\langle 1| + |22\rangle\langle 2|$

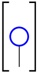
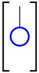
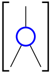
 $= |0\rangle\langle 00| + |1\rangle\langle 11| + |2\rangle\langle 22|$ 

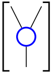
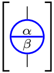
$= |0\rangle\langle 0| + e^{i\alpha}|1\rangle\langle 1| + e^{i\beta}|2\rangle\langle 2|$

 $= |u\rangle$  $= \langle u|$  $= |++\rangle\langle +| + \omega|\omega\omega\rangle\langle\omega| + \omega|\bar{\omega}\bar{\omega}\rangle\langle\bar{\omega}|$

 $= |+\rangle\langle ++| + \bar{\omega}|\omega\rangle\langle\omega\omega| + \bar{\omega}|\bar{\omega}\rangle\langle\bar{\omega}\bar{\omega}|$ 

$= |+\rangle\langle +| + e^{i\alpha}|\omega\rangle\langle\omega| + e^{i\beta}|\bar{\omega}\rangle\langle\bar{\omega}|$

 $= |0\rangle$  $= \langle 0|$  $= |uu\rangle\langle u| + |tt\rangle\langle t| + |vv\rangle\langle v|$

 $= |u\rangle\langle uu| + |t\rangle\langle tt| + |v\rangle\langle vv|$ 

$= |u\rangle\langle u| + e^{i\alpha}|t\rangle\langle t| + e^{i\beta}|v\rangle\langle v|$

RGB Interpretation

where $\omega = e^{\frac{2}{3}\pi i}, \bar{\omega} = e^{\frac{4}{3}\pi i}$, and

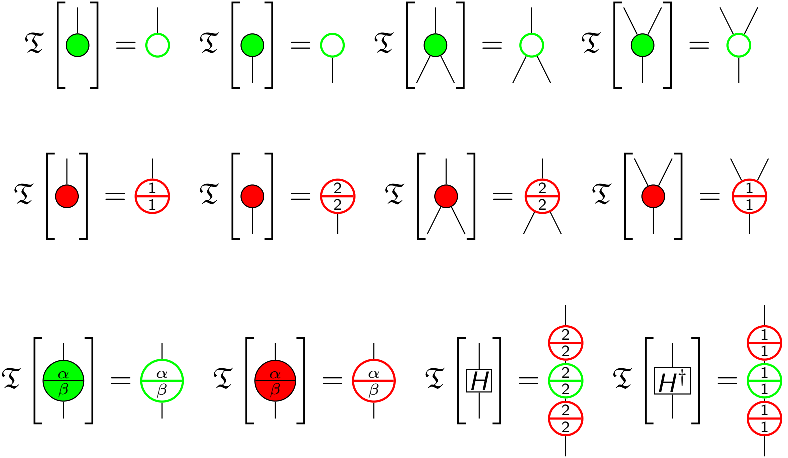
$$\begin{cases} |+\rangle & = & |0\rangle + |1\rangle + |2\rangle \\ |\omega\rangle & = & |0\rangle + \omega\,|1\rangle + \bar{\omega}\,|2\rangle \\ |\bar{\omega}\rangle & = & |0\rangle + \bar{\omega}\,|1\rangle + \omega\,|2\rangle \end{cases}$$

and

$$\begin{cases} |u\rangle & = & |0\rangle + \bar{\omega}\,|1\rangle + \bar{\omega}\,|2\rangle \\ |t\rangle & = & |0\rangle + |1\rangle + \omega\,|2\rangle \\ |v\rangle & = & |0\rangle + \omega\,|1\rangle + |2\rangle \end{cases}$$

RG to RGB Translation

We have a functor $\mathfrak{T}$: **RG** $\to$ **RGB**



Single Qutrit Quantum Algorithm

- Recently, Gedik[6] introduces a simple algorithm using only a single qutrit to determine the parity of permutations.
- Like Deutsch's algorithm, a speed-up relative to corresponding classical algorithms is obtained.
- The algorithm can be depicted by the dichromatic calculus:

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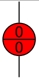
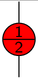
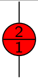
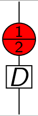
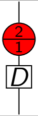
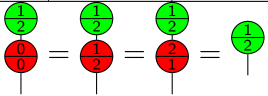
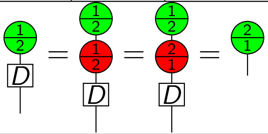
U_f						
$U_f \|w\rangle$						
Parity	Even			Odd		

Qudit ZX Calculus is Universal

The proof [3] that qudit ZX calculus is universal for quantum mechanics is based on the facts that the d-dimensional phase gates Z_d, X_d are sufficient to simulate all single qudit unitary transforms, where

$$Z_d(b_0, b_1 ..., b_{d-1}) : b_0 \left|0\right\rangle + b_1 \left|1\right\rangle + ... + b_{d-1} \left|d-1\right\rangle \mapsto \left|d-1\right\rangle$$

(the d complex coefficients, $b_0, b_1 ..., b_{d-1}$ are normalized to unity),

$$X_d(\phi) : \begin{cases} \left|d-1\right\rangle & \mapsto & e^{i\phi}\left|d-1\right\rangle \\ \left|p\right\rangle & \mapsto & \left|p\right\rangle \text{ for } p \neq d-1 \end{cases}$$

Qudit ZX Calculus is Universal

It is proved [3] that some Z_d phase gates can be realized by X phase gate $\Lambda_X(\alpha_1, \alpha_2, ..., \alpha_{d-1})$ in the qudit ZX calculus. However, not every Z_d phase gate can be represented by $\Lambda_X(\alpha_1, \alpha_2, ..., \alpha_{d-1})$. In fact, to realize any $Z_d(b_0, b_1..., b_{d-1})$ in this way, we need to find $\alpha_1, \alpha_2, ..., \alpha_{d-1}$ such that

$$c_j b_0 + c_{j-1} b_1 + ... c_0 b_j + c_{d-1} b_{j+1} + ... c_{j+1} b_{d-1} = d \tag{1}$$

$$c_k b_0 + c_{k-1} b_1 + ... c_0 b_k + c_{d-1} b_{k+1} + ... c_{k+1} b_{d-1} = 0, \quad \forall k \neq j \tag{2}$$

where $c_k = 1 + \sum_{l=1}^{d-1} \eta^{r_k(l)} e^{i\alpha_l}$, r_k permutes the entries 1 (there is one r_k for each k), $j = d$.

Qudit ZX Calculus is Universal

Since $\sum_{k=0}^{d-1} c_k = d$, summing up all the equations in (1) and (2), we have $\sum_{k=0}^{d-1} b_k = 1$. Of course, not every unit complex vector $(b_0, b_1 ..., b_{d-1})$ satisfies $\sum_{k=0}^{d-1} b_k = 1$ or $\sum_{k=0}^{d-1} b_k = e^{i\alpha}$ up to a global phase.

For example, $(b_0, b_1 ..., b_{d-1}) = (0, 1/\sqrt{2}, 1/\sqrt{2}, 0, ..., 0), d > 2$, is such a counterexample.

The above argument means that we need to find a proof of universality of qudit ZX calculus in another way. We solve this problem by the theory of Lie algebra.

Qudit ZX Calculus is Universal

Let

$$H = \left\{ \begin{pmatrix} e^{i\alpha_0} & & \\ & \ddots & \\ & & e^{i\alpha_{d-1}} \end{pmatrix} \middle| \alpha_0, ..., \alpha_{d-1} \in \mathbb{R} \right\}$$

$V = \frac{1}{\sqrt{d}} \sum_{j,k=0}^{d-1} \omega^{jk} \left|j\right\rangle \left\langle k\right| , \omega = e^{i\frac{2\pi}{d}}, H' = VHV^{-1}$. We give an outline of the proof here, the details will be shown in a forthcoming arXiv paper.

First step: Both H and H' are closed connected subgroups of the compact Lie group of unitaries $G = U(d)$.

Qudit ZX Calculus is Universal

Second step: H and H' generate a dense subgroup of G.

By [7], it amounts to showing that $\mathfrak{h}$ and $\mathfrak{h}'$ generate $\mathfrak{g}$ as a Lie algebra, where $\mathfrak{h} = \text{Lie } H, \mathfrak{h}' = \text{Lie } H', \mathfrak{g} = \text{Lie } G$.

Qudit ZX Calculus is Universal

The basis vectors of Lie algebra $\mathfrak{g}$ consist of

$$\sigma_x^{(jk)}, (0 \leq j < k \leq d-1), \sigma_y^{(jk)}, (0 \leq j < k \leq d-1)$$

$$\sigma_z^{(jk)}, (j = 0, 1 \leq k \leq d-1), iI_d$$

where

$$\sigma_x^{(jk)} = i \left|j\right\rangle \left\langle k\right| + i \left|k\right\rangle \left\langle j\right|, \sigma_y^{(jk)} = \left|j\right\rangle \left\langle k\right| - \left|k\right\rangle \left\langle j\right|$$

$$\sigma_z^{(jk)} = i \left|j\right\rangle \left\langle j\right| - i \left|k\right\rangle \left\langle k\right|$$

The basis vectors of $\mathfrak{h}$ consist of

$$\sigma_z^{(0k)}, (1 \leq k \leq d-1), iI_d$$

thus the basis vectors of $\mathfrak{h}'$ consist of

$$V\sigma_z^{(0k)}V^{-1}, iI_d$$

By direct calculation, we can prove that $\mathfrak{h}$ and $\mathfrak{h}'$ generate $\mathfrak{g}$.

Qudit ZX Calculus is Universal

Third step: H and H' generate G, i.e., H and V generate G. Thus up to a global phase $e^{i\alpha}$, Λ_X and Λ_Z generate $U(d)$. Here we use the following lemma from [7].

Lemma: Let G be a compact Lie group. If $H_1, ..., H_k$ are closed connected subgroups and they generate a dense group of G, then in fact they generate G.

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Thanks!