Centre for Quantum Technologies

National University of Singapore

Equivalence of wave-particle duality to entropic uncertainty

arXiv:1403.4687

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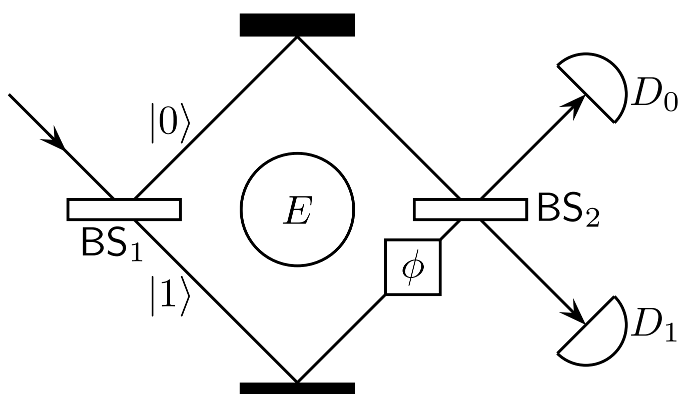


Warning

- No deep mathematics

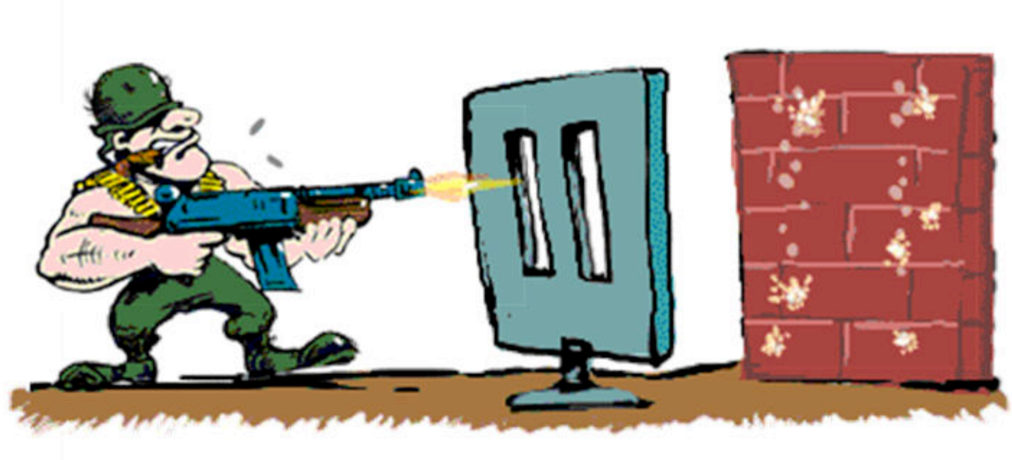
- Hopefully some deep physics

wave-particle duality follows from the uncertainty principle

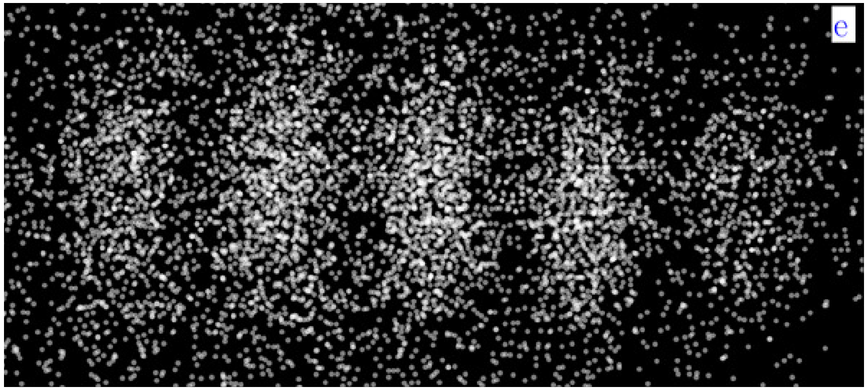
- Invitation to YOU: apply your heavy mathematical machinery to my topic

Wave-particle duality

Bullets show no interference pattern



… but electrons do



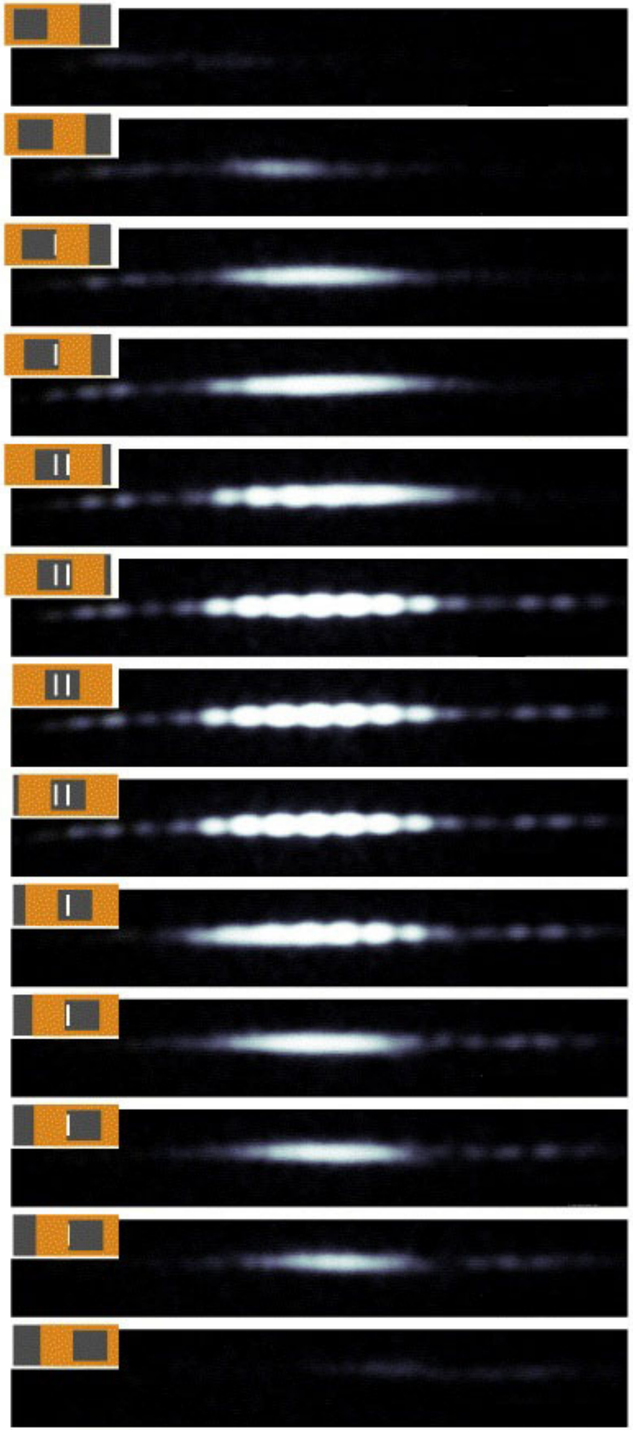
Data from: "Controlled double-slit electron diffraction" Bach et al. NJP (2013)

Bullets are just bunches of electrons mixed in with some protons and neutrons, so why the change in behavior?

Wave-particle duality

The transition (from no interference to interference) can even be seen with single electrons.

Data from: "Controlled double-slit electron diffraction" Bach et al. NJP (2013)

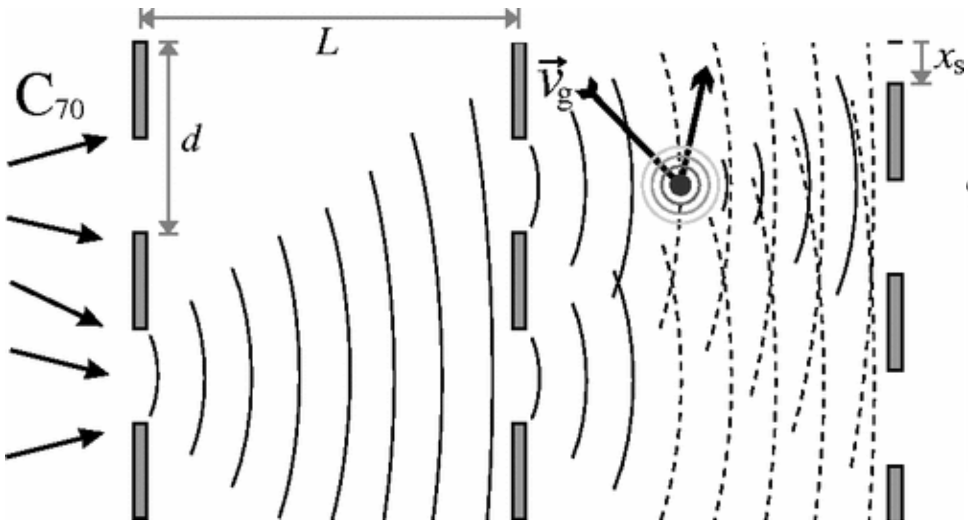


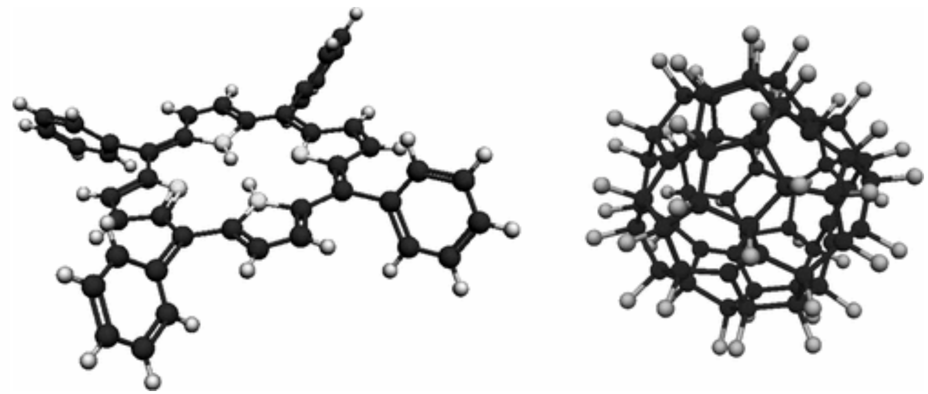
<u>The great mystery:</u>
Each kind of thing (bullet, electron, bacteria, …) has the ability to exhibit wave behavior, i.e., produce interference. Likewise, each can exhibit particle behavior, i.e., have a well-defined path. But the two behaviors compete – you either get one or the other.

Why? …. Nobody knows.

"You never get to understand quantum mechanics, you just get used to it."

Wave-particle duality: big molecules

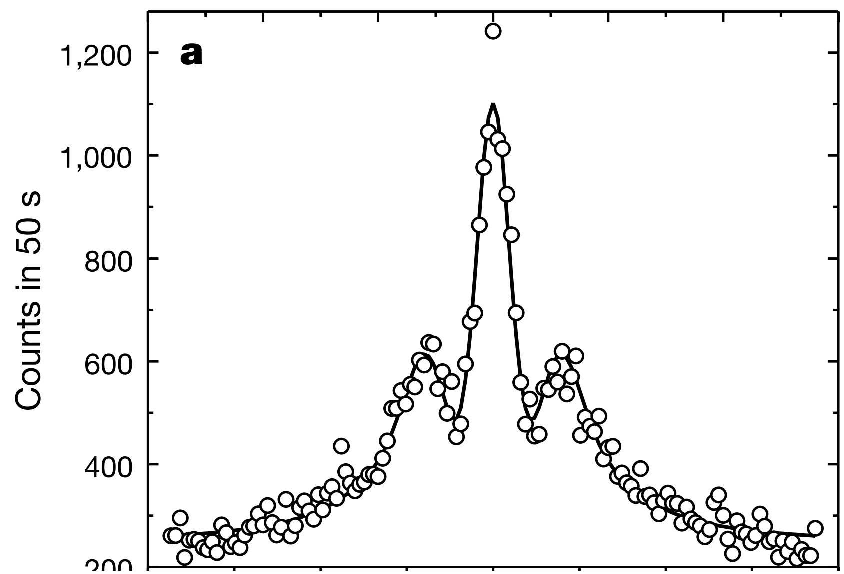




“Wave–particle duality of C60 molecules” Arndt et al. Nature (1999)

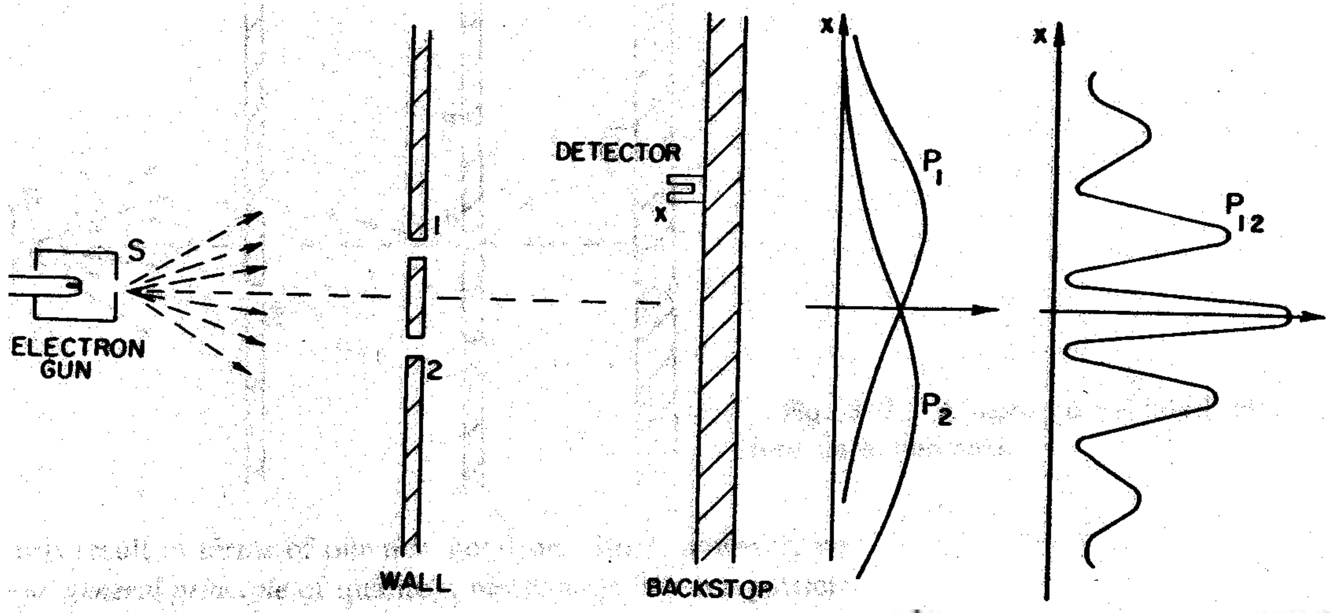
“Collisional Decoherence Observed in Matter Wave Interferometry” Hornberger et al. PRL (2003)

“Wave Nature of Biomolecules and Fluorofullerenes” Hackermüller et al. PRL (2003)



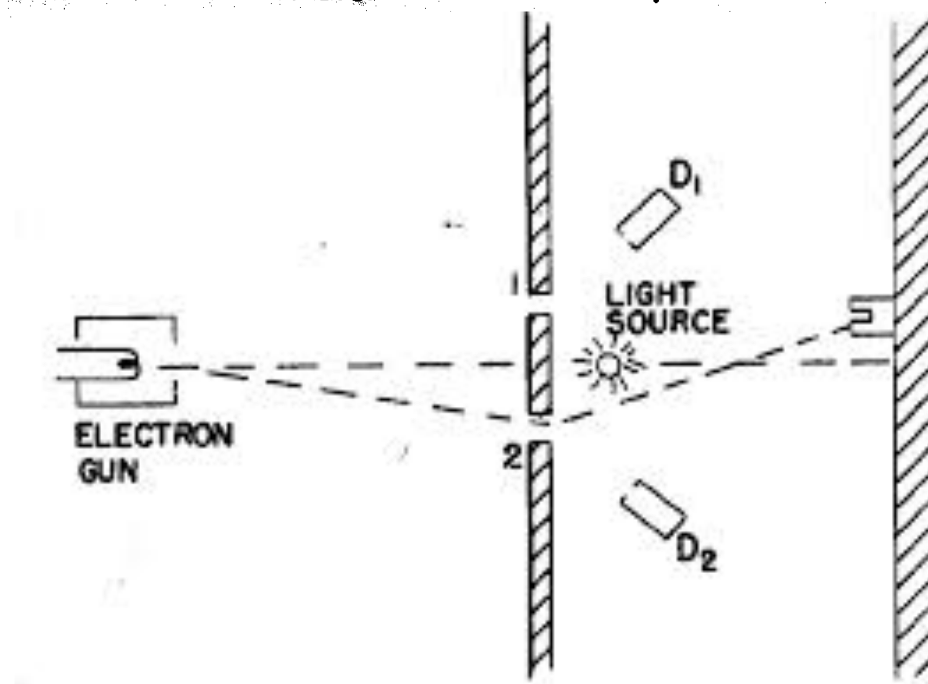
Wave-particle duality

While the behaviors are mysterious, we can get intuition for how they compete.



Feynman gives example of light source with variable wavelength….

… tradeoff between spatial resolution and momentum kick.

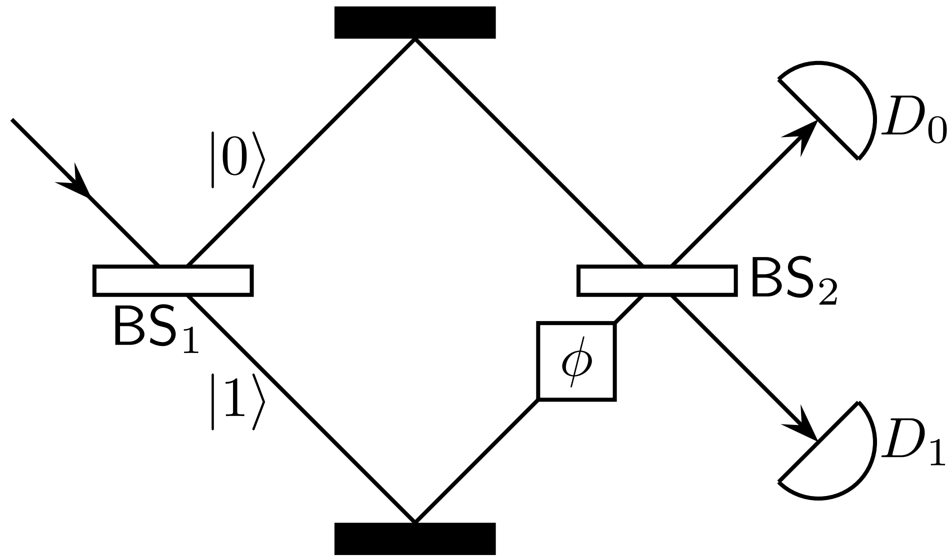


Wave-particle duality

getting quantitative

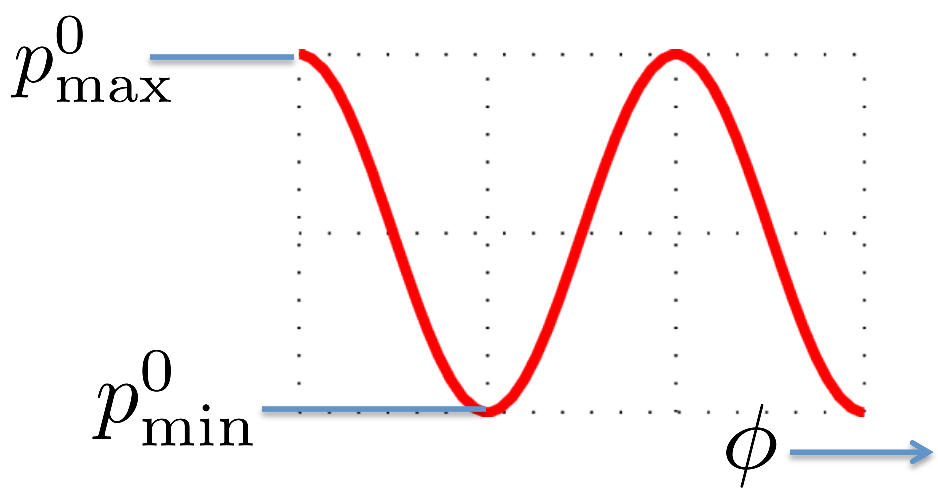
Simplification of double-slit:

Two-path interferometer for single photons (named after Mach and Zehnder).



<u>Fringe visibility</u>

$$\mathcal{V} = \frac{p^0_{\max} - p^0_{\min}}{p^0_{\max} + p^0_{\min}}$$



Wave-particle duality

getting quantitative

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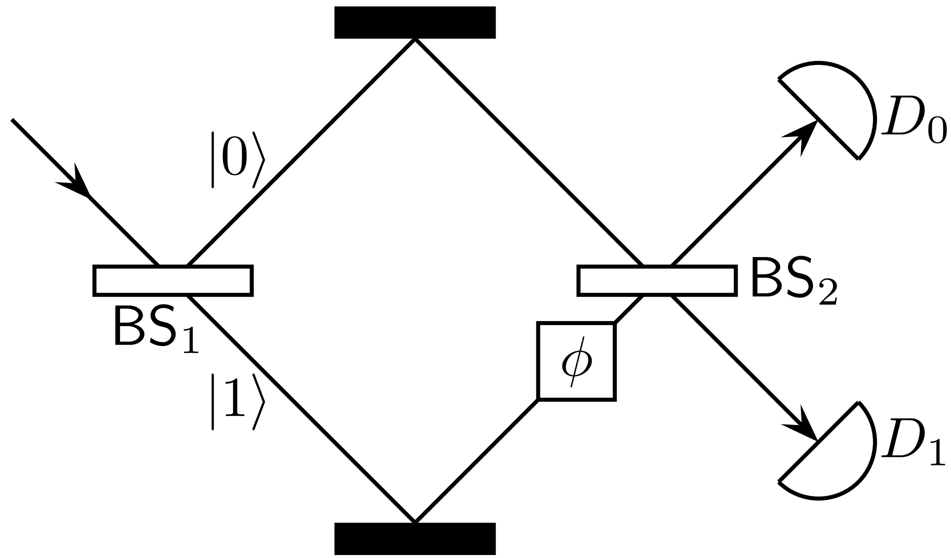


<u>Fringe visibility</u>

$$\mathcal{V} = \frac{p^0_{\max} - p^0_{\min}}{p^0_{\max} + p^0_{\min}}$$

<u>Path predictability</u> (e.g. asymmetric BS_1)

$$Z = \{|0\rangle, |1\rangle\}$$

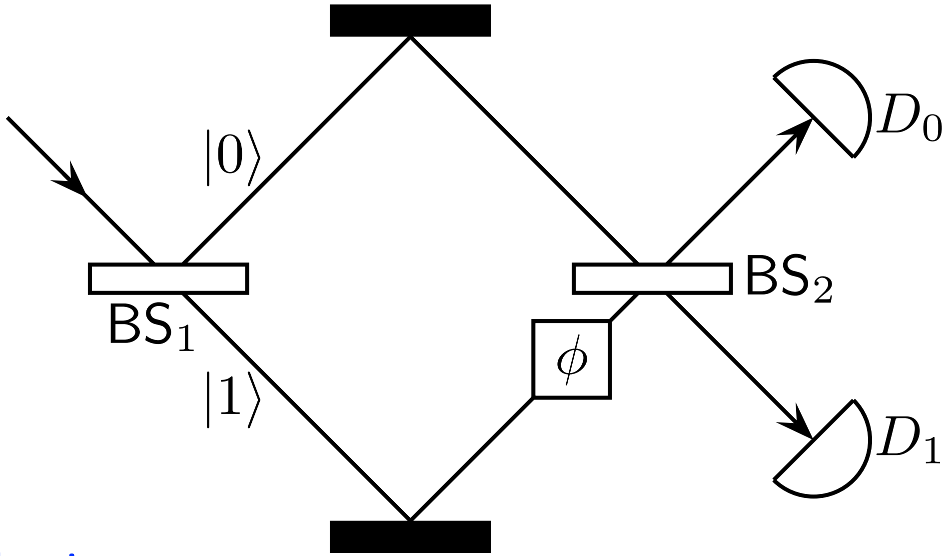
$$\mathcal{P} := 2p_{\text{guess}}(Z) - 1$$

probability of guessing Z correctly

Wave-particle duality

getting quantitative

Wooters, Zurek (1979)
Greenberger, Yasin (1988)
Englert (1996)



Wave-particle duality relation (WPDR):

$$\mathcal{P}^2 + \mathcal{V}^2 \leqslant 1$$

Full particle behavior → No wave behavior
Full wave behavior → No particle behavior

Fringe visibility

$$\mathcal{V} = \frac{p^0_{\max} - p^0_{\min}}{p^0_{\max} + p^0_{\min}}$$

Path predictability (e.g. asymmetric BS_1)

$$Z = \{|0\rangle, |1\rangle\}$$

$$\mathcal{P} := 2p_{\text{guess}}(Z) - 1$$

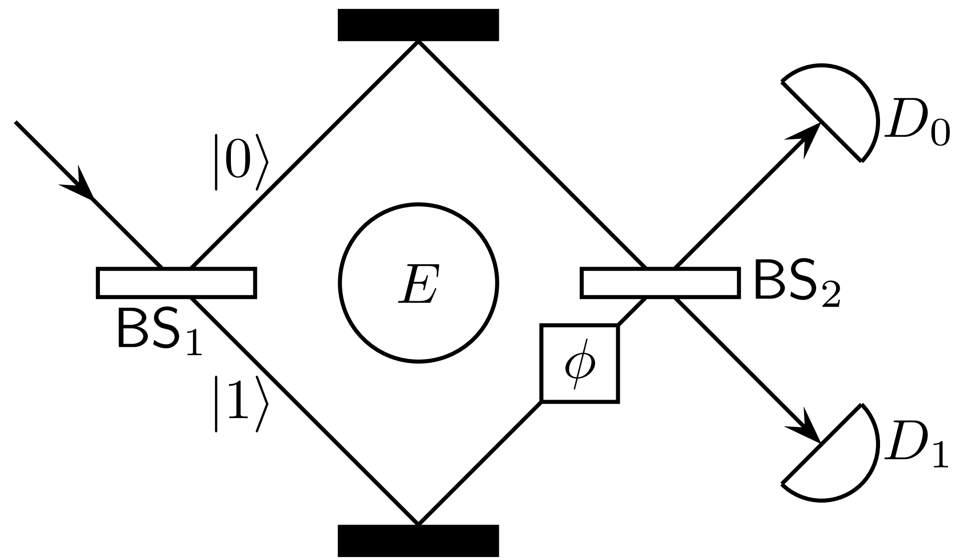
probability of guessing Z correctly

Wave-particle duality

getting quantitative

Jaeger, Shimony, Vaidman (1995)
Englert (1996)

Let *E* be a (partial) which-path detector. *E* could be gas of atoms whose internal state is sensitive to presence of photon.



Stronger WPDR:

$$\mathcal{D}^2 + \mathcal{V}^2 \leqslant 1$$

Path distinguishability

$$\mathcal{D} := 2p_{\text{guess}}(Z|E) - 1$$

probability of guessing Z correctly given E (i.e., given optimal measurement on E)

Fringe visibility

$$\mathcal{V} = \frac{p^0_{\max} - p^0_{\min}}{p^0_{\max} + p^0_{\min}}$$

WPDRs

Where do they come from?

$$\mathcal{P}^2 + \mathcal{V}^2 \leqslant 1$$
$$\mathcal{D}^2 + \mathcal{V}^2 \leqslant 1$$

Is wave-particle duality a fundamental principle of quantum mechanics, or is it a corollary of some other principle?

Englert: "... Does not make use of Heisenberg's uncertainty principle in any form"
"... There is only one observable involved"

Is it a consequence of position/momentum uncertainty principle?

$$\Delta q \Delta p \geqslant \hbar/2 \ \ ?$$

This was intensely debated in 1990's:

"Path detection and the uncertainty principle" Storey et al. Nature (1994).
"Complementarity and uncertainty" Englert, Scully, Walther. Nature (1995), and Reply by Storey et al.
"Uncertainty over complementarity?" Wiseman, Harrison. Nature (1995).

Looks to be inconclusive / still open to debate

Regardless, is it a consequence of the uncertainty principle for *qubits*?
(after all, a two-path interferometer is like a two-state system)

WPDRs

Where do they come from?

Consider: $$\mathcal{P}^2 + \mathcal{V}^2 \leqslant 1$$

Several authors showed that this WPDR is equivalent to Robertson's uncertainty relation for particular qubit observables

$$\Delta X \Delta Z \geqslant \frac{1}{2} |\langle \psi | [X, Z] | \psi \rangle|$$

Busch and Shilladay (2006)
Bjork et al. (1999)
Durr and Rempe (2000)
Bosyk et al. (2013)

Qubit observables:

$$\hat{P} = \sigma_z$$
$$\hat{V}_\phi = (\cos\phi)\ \sigma_x + (\sin\phi)\ \sigma_y$$

Variances:

$$(\Delta \hat{P})^2 = 1 - P^2$$
$$(\Delta \hat{V}_\phi)^2 = 1 - V^2 \cos^2(\theta - \phi)$$

Plugging into Robertson's relation gives:

$$(1 - P^2)[1 - V^2 \cos^2(\theta - \phi)]$$
$$\geqslant P^2 V^2 \cos^2(\theta - \phi) + V^2 \sin^2(\theta - \phi)$$

WPDRs

Where do they come from?

So we have

$$\mathcal{P}^2 + \mathcal{V}^2 \leqslant 1 \quad \Longleftrightarrow \quad \Delta X \Delta Z \geqslant \frac{1}{2} |\langle \psi | [X, Z] | \psi \rangle |$$

$$\mathcal{D}^2 + \mathcal{V}^2 \leqslant 1 \quad \Longleftrightarrow \quad \text{??????}$$

Note that distinguishability involves conditioning on system *E*. This is not so natural for standard deviation, but is quite natural for *entropies*. Could the *D*-*V* relation be related to the *entropic* uncertainty principle?

$$\mathcal{D} := 2p_{\text{guess}}(Z|E) - 1$$

WPDRs

Consider again: $$\mathcal{P}^2 + \mathcal{V}^2 \leqslant 1$$

Bosyk et al. [Phys. Scr. (2013)] considered entropic uncertainty relations (EURs), of the form:

$$H_q(P) + H_q(V) \geqslant \mathcal{B}_q$$

for Renyi entropies:

$$H_q(P) = \frac{1}{1-q} \ln\left[\left(\frac{1+P}{2}\right)^q + \left(\frac{1-P}{2}\right)^q\right]$$

$$H_q(V) = \frac{1}{1-q} \ln\left[\left(\frac{1+V}{2}\right)^q + \left(\frac{1-V}{2}\right)^q\right]$$

They argue that such EURs are <u>inequivalent</u> to the P-V relation!

But Maassen & Uffink (1988) proved an EUR that involves different *q*'s, for example,

$$H_\infty(P) + H_{1/2}(V) \geqslant 1$$

Our first result: This EUR is equivalent to the P-V relation!!!!

WPDRs

$$H_\infty(P) + H_{1/2}(V) \geqslant 1$$

INVITATION: Plug these formulas in to obtain *P-V* relation

$$H_\infty(P) = 1 - \log(1 + \mathcal{P})$$
$$H_{1/2}(V) = \log\left(1 + \sqrt{1 - \mathcal{V}^2}\right)$$

<u>So we have</u>

$$\mathcal{P}^2 + \mathcal{V}^2 \leqslant 1 \quad \Longleftrightarrow \quad H_\infty(P) + H_{1/2}(V) \geqslant 1$$

$$\mathcal{D}^2 + \mathcal{V}^2 \leqslant 1 \quad \Longleftrightarrow \quad ??????$$

APOLOGY: In what follows, I will switch notation:

$$H_\infty(P) \to H_{\min}(Z) \qquad H_{1/2}(V) \to H_{\max}(W)$$

Goals of our work

1.) Unify a vast literature on WPDRs. Many complicated versions of WPDRs have been formulated, for scenarios involving asymmetric beam splitters, quantum beam splitters, and photon polarization interactions. We show that all these WPDRs correspond to special cases of a single inequality.

2.) Show where WPDRs come from. Namely, show that they come from the entropic uncertainty relation (EUR) for the min- and max-entropies. Hence we unify the entropic uncertainty principle with the wave-particle duality principle.

3.) Provide a general, robust framework for discussing WPDRs and deriving novel WPDRs. Once WPDRs are reformulated as EURs, it becomes obvious how to apply them to novel interferometric models. It becomes clear that you can simply condition the entropy terms on various degrees of freedom and the relation still holds. We illustrate this by deriving a novel WPDR for a quantum beam splittter.

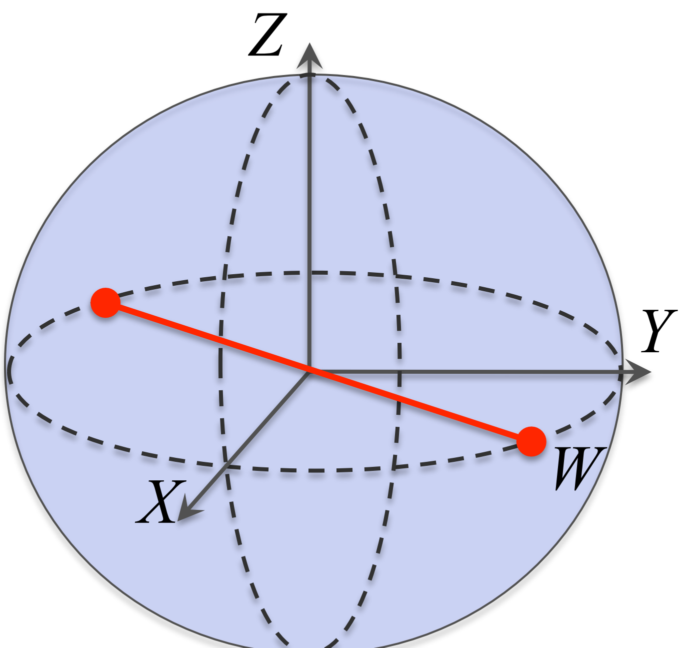
4.) Emphasize the distinction between preparation WPDRs and measurement WPDRs. That is, we emphasize that EURs can be applied in two conceptually different ways.

Main Result

For a two-path interferometer for single quantons, we identify particle and wave behaviors with the knowledge of specific (complementary) qubit observables, or lack of behavior with ignorance, as quantified by the min- and max-entropies commonly used in quantum information theory (e.g., QKD).

$$\text{lack of particle behavior: } H_{\min}(Z|J)$$

$$\text{lack of wave behavior: } \min_{W \in XY} H_{\max}(W|K)$$



Z : which-path observable

W : orthonormal basis observable in *XY* plane

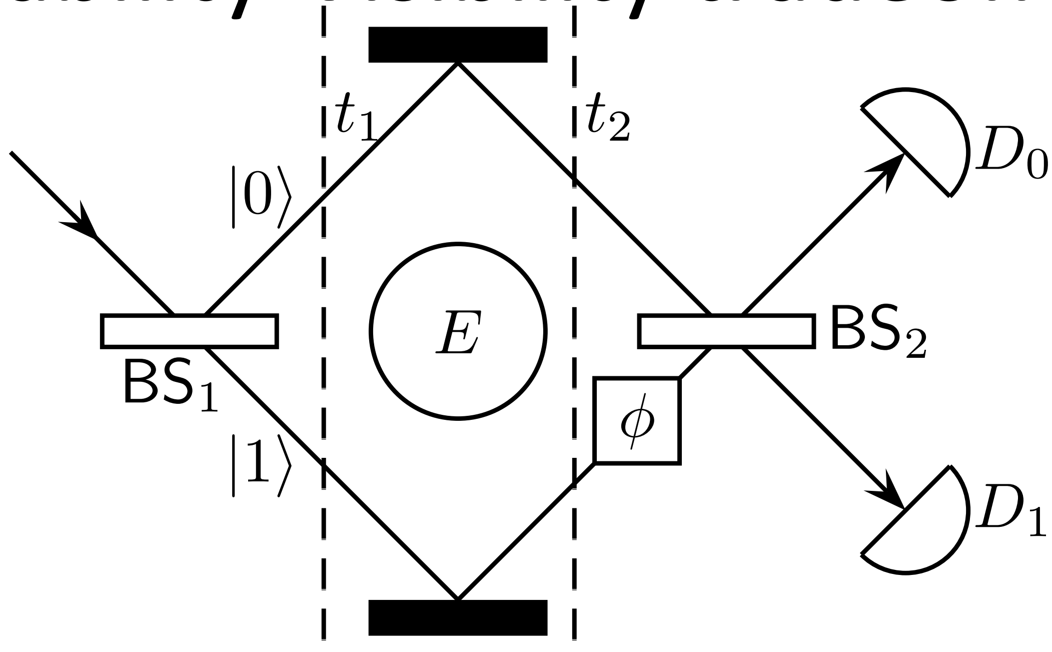
J, *K* : some other quantum systems that help to reveal the behavior

Our general WPDR:

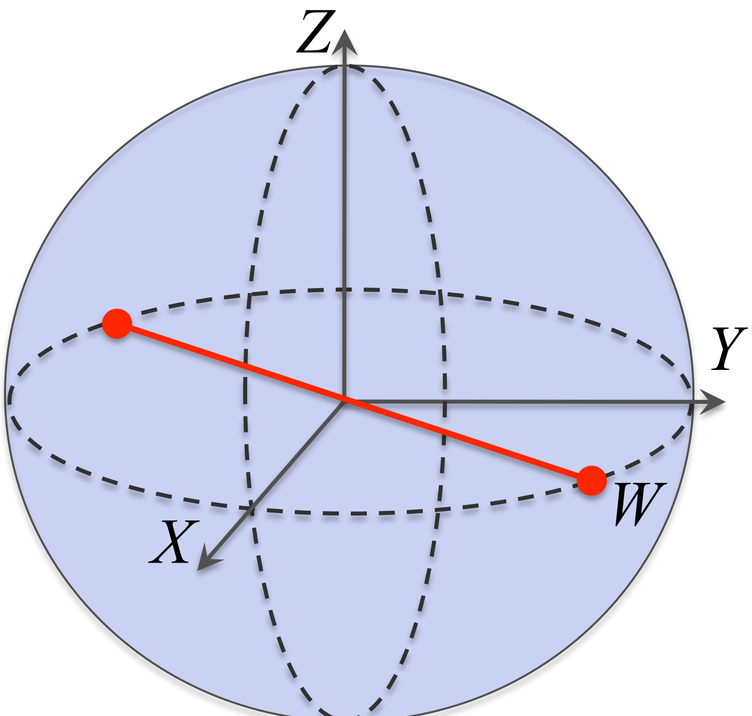
$$\boxed{H_{\min}(Z|J) + \min_{W \in XY} H_{\max}(W|K) \geqslant 1}$$

Interestingly, this has been used to prove security of quantum key distribution

Distinguishability-Visibility tradeoff



Recall scenario: photon interacts with *E* inside interferometer



Apply uncertainty relation at time t_2

$$\mathcal{P}^2 + \mathcal{V}^2 \leqslant 1$$

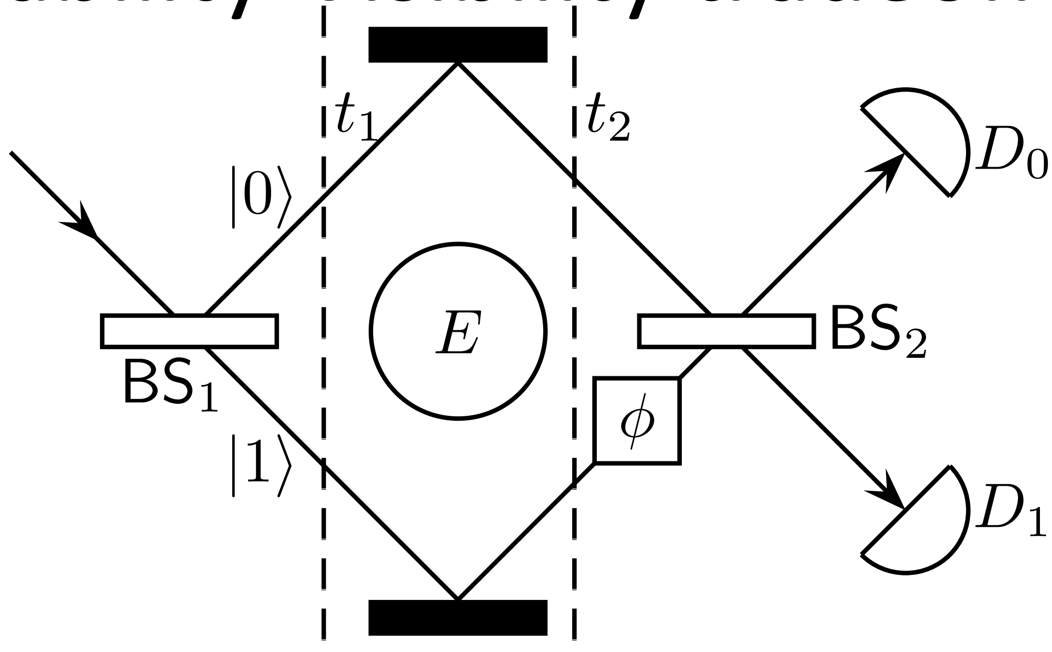
$$\boxed{H_{\min}(Z)_{t_2} + \min_{W \in XY} H_{\max}(W)_{t_2} \geqslant 1}$$

$$\mathcal{D}^2 + \mathcal{V}^2 \leqslant 1$$

$$\boxed{H_{\min}(Z|E)_{t_2} + \min_{W \in XY} H_{\max}(W)_{t_2} \geqslant 1}$$

Distinguishability-Visibility tradeoff

Recall scenario: photon interacts with *E* inside interferometer



$$\boxed{H_{\min}(Z|E)_{t_2} + \min_{W \in XY} H_{\max}(W)_{t_2} \geqslant 1}$$

$$H_{\min}(Z|E)_{t_2} = 1 - \log(1 + \mathcal{D})$$

$$\min_{W \in XY} H_{\max}(W)_{t_2} = \log(1 + \sqrt{1 - \mathcal{V}^2})$$

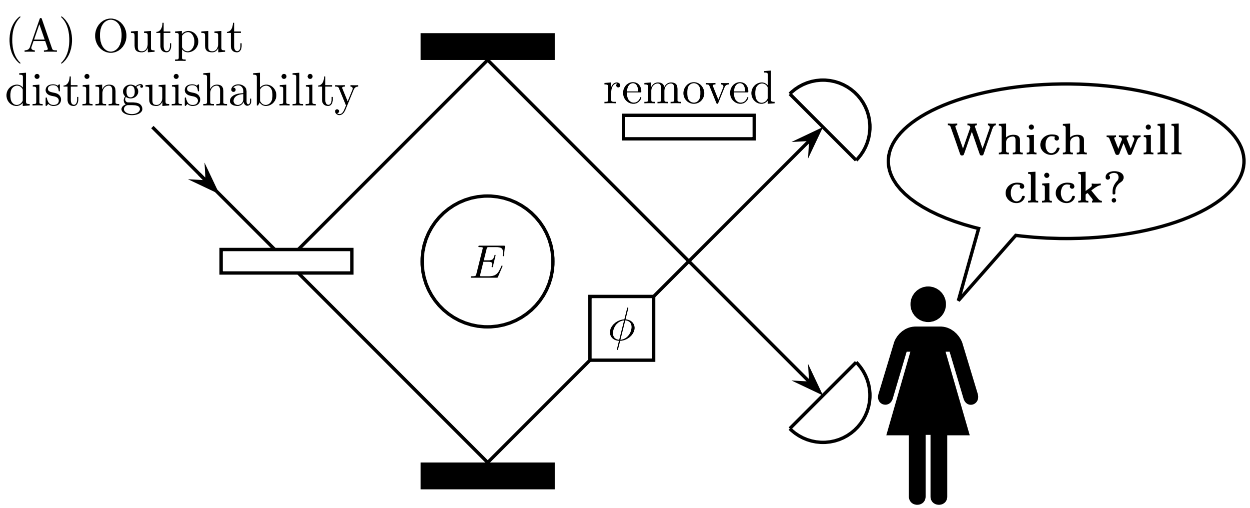
$$\Rightarrow \mathcal{D}^2 + \mathcal{V}^2 \leqslant 1$$

Preparation Uncertainty

<u>Remark</u>

We applied the preparation uncertainty relation at time t_2 to derive the WPDR. Preparation uncertainty restricts one's ability to predict *future* measurements. Englert noted in his 1996 PRL that, to measure *P* or *D*, one removes the second beam splitter (BS_2) and tries to predict which detector clicks. To be clear we call this output distinguishability.

$$\mathcal{D} := 2p_{\text{guess}}(Z|E)_{t_2} - 1$$



Measurement Uncertainty

But uncertainty relations can be applied in a conceptually different way. Instead of fixing the input state and considering complementary output measurements, one can fix the output measurement and consider complementary input ensembles:

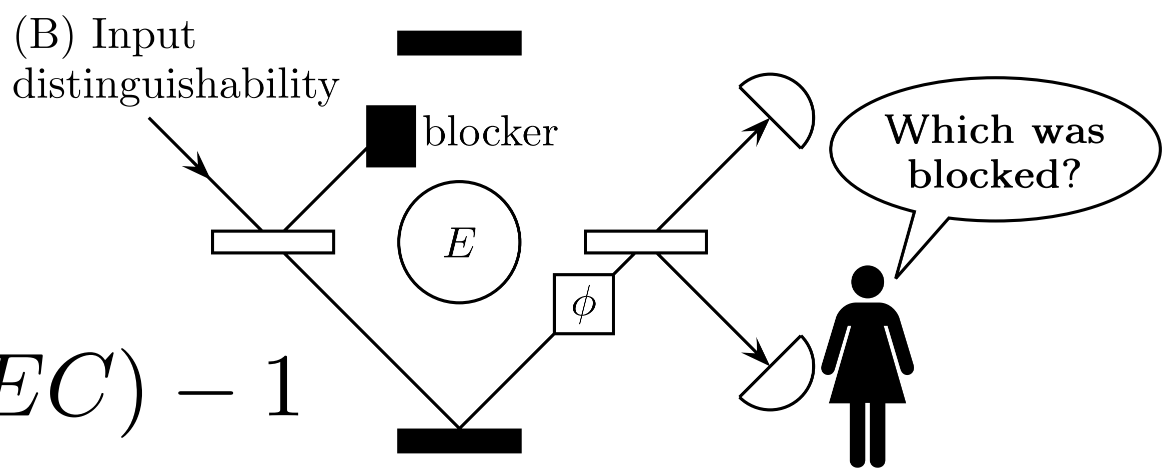
$$Z_i = \{|0\rangle, |1\rangle\}$$

$$W_i = \{|w_\pm\rangle\}$$

$$|w_\pm\rangle = (|0\rangle \pm e^{i\phi}|1\rangle)/\sqrt{2}$$

<u>Guessing game</u>

The Z_i states are generated by Bob flipping a coin and blocking either the top or bottom arm depending on flip outcome. Alice tries to guess Bob's coin flip, given E and given which detector clicks, denoted by C.



$$\mathcal{D}_i := 2p_{\text{guess}}(Z_i|EC) - 1$$

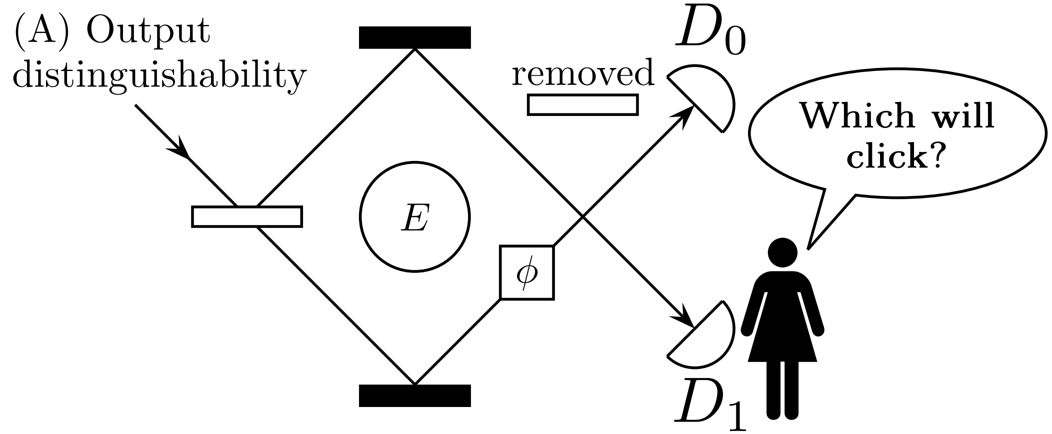
Preparation vs. Measurement Uncertainty

Output distinguishability

$$\mathcal{D} := 2p_{\text{guess}}(Z|E)_{t_2} - 1$$

Output visibility

$$\mathcal{V} = \frac{p^0_{\max} - p^0_{\min}}{p^0_{\max} + p^0_{\min}}$$

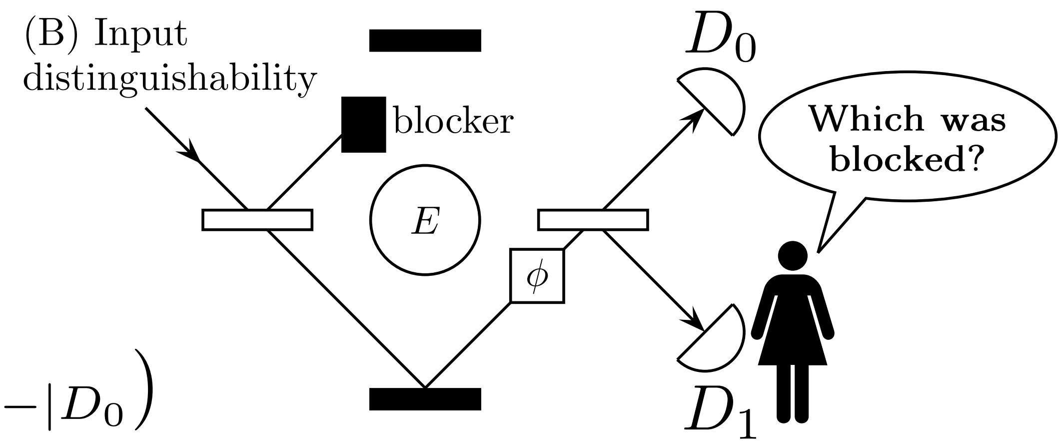


Input distinguishability

$$\mathcal{D}_i := 2p_{\text{guess}}(Z_i|EC) - 1$$

Input visibility

$$\mathcal{V}_i := \max_{W \in XY} \left(p_{w+|D_0} - p_{w-|D_0}\right)$$



Preparation vs. Measurement Uncertainty

Output distinguishability

$$\mathcal{D} := 2p_{\text{guess}}(Z|E)_{t_2} - 1$$

Output visibility

$$\mathcal{V} = \frac{p^0_{\max} - p^0_{\min}}{p^0_{\max} + p^0_{\min}}$$

“Preparation” WPDR

$$\boxed{\mathcal{D}^2 + \mathcal{V}^2 \leqslant 1}$$

Addresses question of how well Alice can prepare a state with low uncertainty in *Z* and *W*.

Input distinguishability

$$\mathcal{D}_i := 2p_{\text{guess}}(Z_i|EC) - 1$$

Input visibility

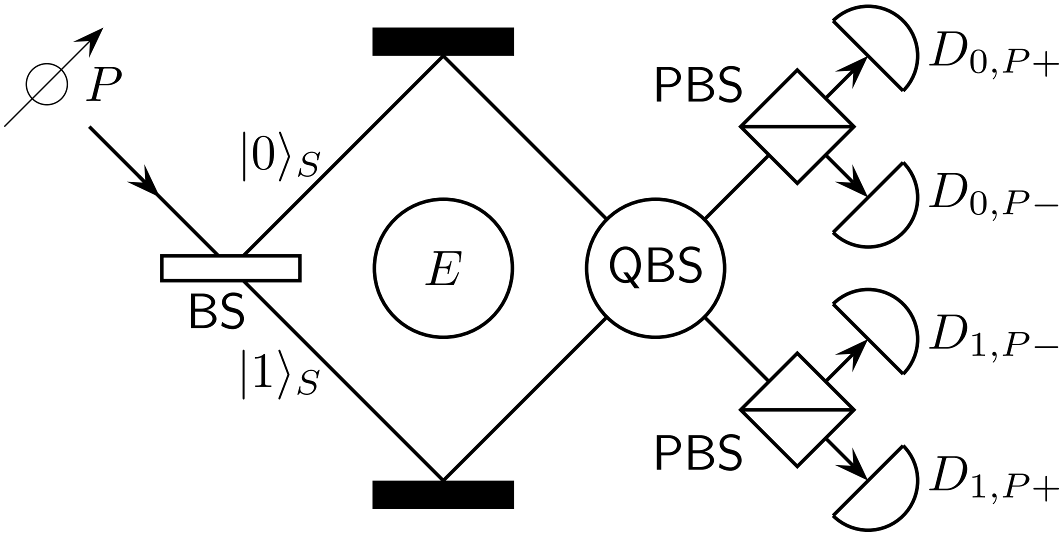
$$\mathcal{V}_i := \max_{W \in XY} \left(p_{w+|D_0} - p_{w-|D_0} \right)$$

“Measurement” WPDR

$$\boxed{\mathcal{D}_i^2 + \mathcal{V}_i^2 \leqslant 1}$$

Addresses question of how well Alice can jointly measure Bob’s *Z* and *W* observables

Example: quantum beam splitter

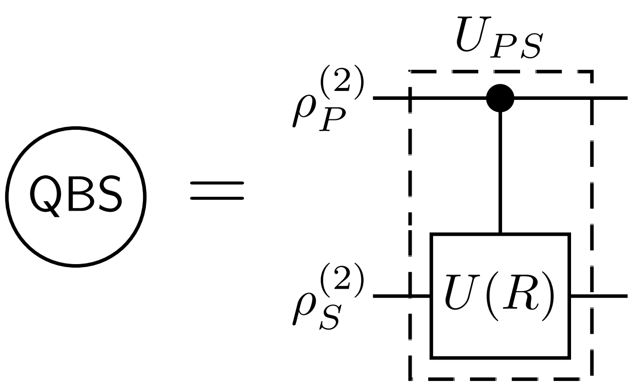


Feeding in a polarization superposition means that BS_2 is in a superposition of "absent" and "present".

$$\mathcal{D}_i^2 + \mathcal{V}_i^2 \leqslant 1$$

$$\mathcal{V}_i = \mathcal{V}$$

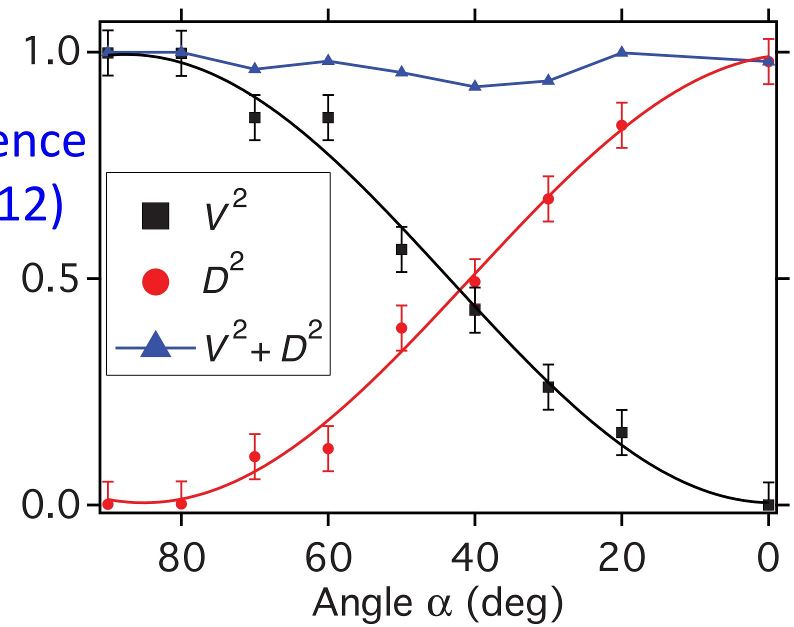
$$\boxed{\mathcal{D}_i^2 + \mathcal{V}^2 \leqslant 1}$$



$$\rho_P^{(2)} = |\psi_P^{(2)}\rangle\langle\psi_P^{(2)}|$$

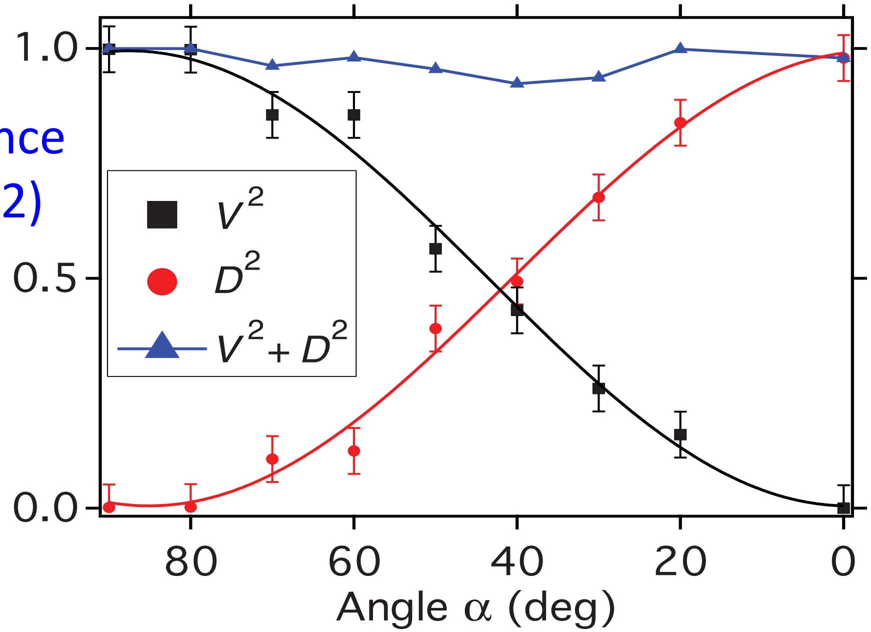
$$|\psi_P^{(2)}\rangle = \cos\alpha|H\rangle + \sin\alpha|V\rangle$$

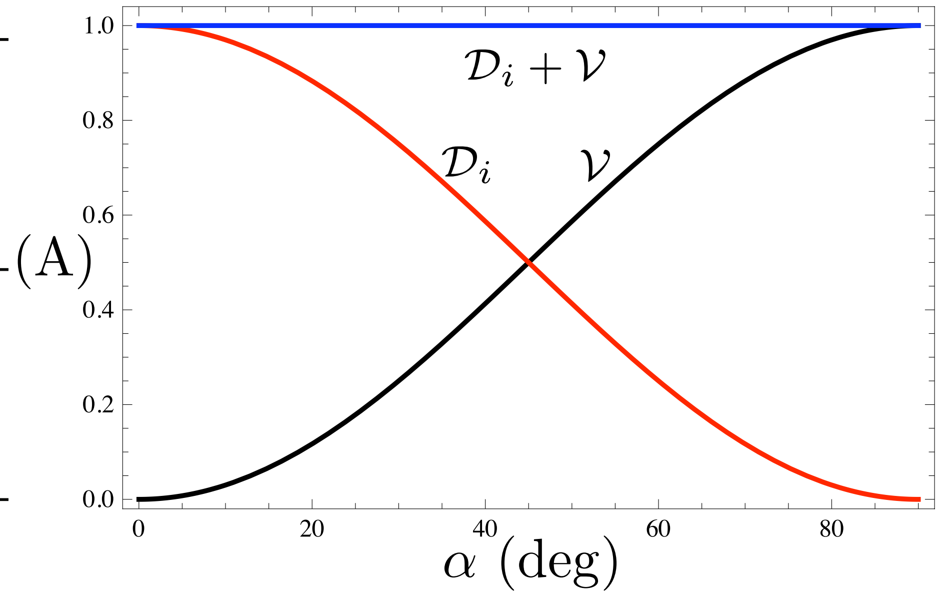
Science (2012)

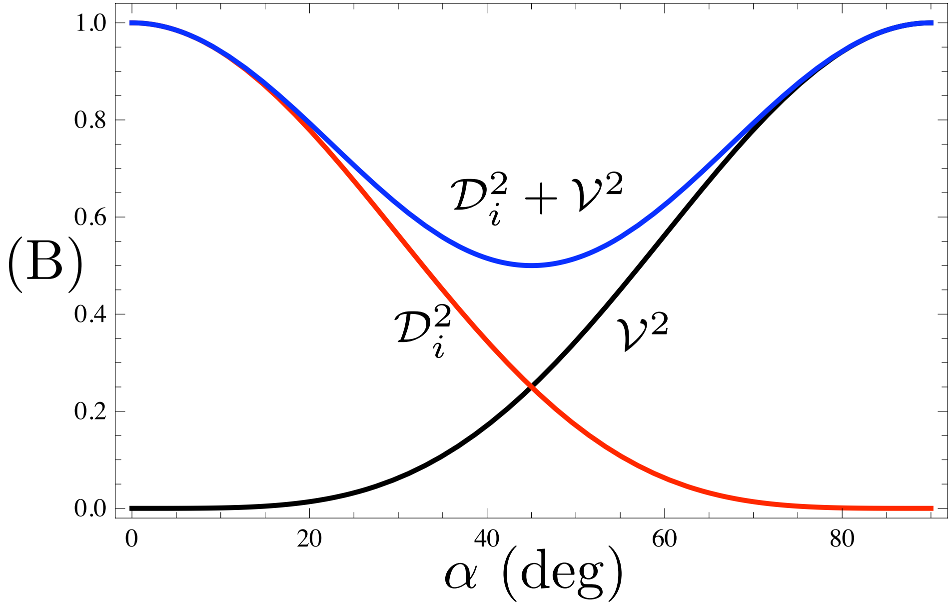


Example: quantum beam splitter

Science (2012)







$$\mathcal{D}_i^2 + \mathcal{V}^2 \leqslant 1$$

This relation is untight, and more importantly, does not capture beam splitter's coherence!!

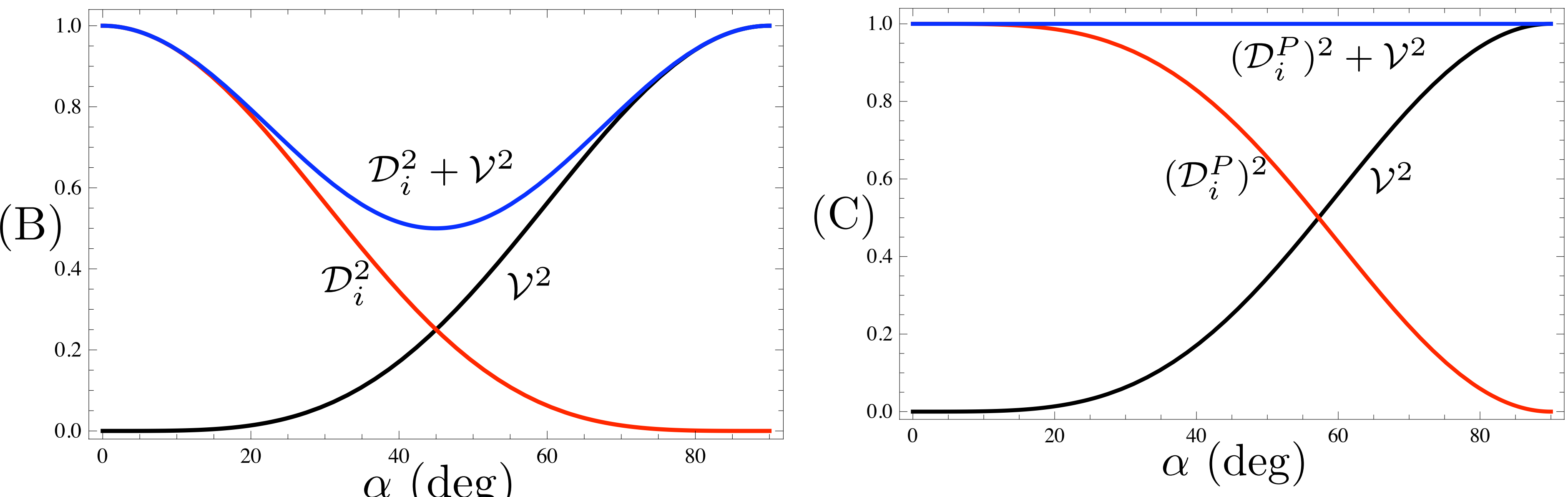
Example: quantum beam splitter

Our framework easily provides a tight relation that captures beam splitter's coherence.

Define polarization-enhanced distinguishability

$$\mathcal{D}_i^P := 2p_{\text{guess}}(Z|ECP) - 1$$

$$\mathcal{D}_i^2 + \mathcal{V}^2 \leqslant 1 \qquad (\mathcal{D}_i^P)^2 + \mathcal{V}^2 \leqslant 1$$



Final Remarks

- We have shown that WPDRs are EURs in disguise, namely, the uncertainty relation for the min- and max-entropies applied to qubit observables. Are WPDRs useful for quantum cryptography?

- All of our WPDRs hold if you replace both min and max with von Neumann.

- Our framework provides two classes of WPDRs associated with preparation uncertainty and measurement uncertainty.

- Our framework makes it obvious how to derive novel WPDRs. (We did this for the QBS.)

- Our framework can be applied fairly universally to single-quanton two-path interferometers. It would be interesting to extend this to mutli-photons or multi-paths.