Entropic Formulation of Heisenberg's Measurement-Disturbance Relation

arXiv:1311.7637

QPL 2014, Kyoto, 6. June 2014

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JSPS 日本学術振興会 Japan Society for the Promotion of Science

KAKENHI grant No. 24-02793

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Different Uncertainty Trade-Offs

Uncertainty Relation: Limitation due to complementarity of observables

Always: X ($\mathbb{X}$) and Z ($\mathbb{Z}$) complementary observables ($[X,Z] \neq$

$X \leftrightarrow \{\mathbb{X}_x\}_{x\in X}$ and $Z \leftrightarrow \{\mathbb{Z}_z\}_{z\in Z}$ POVM's

Heisenberg '27

- **Preparation Uncertainty**

Kennard '27

Robertson '27

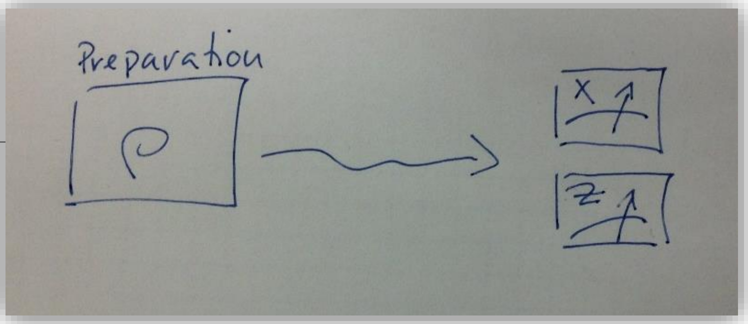
There exists no preparation in which X and Z are both predetermined.

- **Measurement Uncertainty**

Joint Measurability: *There exists no observable which jointly measures X and Z.*

Measurement-Disturbance: *An attempt to measure X generally disturbs Z.*

Preparation Uncertainty



Robertson: $\Delta\mathbb{X}\Delta\mathbb{Z} \geq \frac{1}{2}| < \psi, [\mathbb{X}, \mathbb{Z}]\psi > |$

- Variances depends on eigenvalues of X and Z
- Constant vanishes for certain states

Massen & Uffink '88: $H(X)_\rho + H(Z)_\rho \geq -\log \max_{x,z} \|\mathbb{X}_x^{1/2}\mathbb{Z}_z^{1/2}\|$

- how does quantum information affect uncertainty

Berta et al. 2010: $H(X|B)_\rho + H(Z|B)_\rho \geq -\log \max_{x,z} \|\mathbb{X}_x^{1/2}\mathbb{Z}_z^{1/2}\| + H(A|B)$

Application in Quantum Information Science
Entropies have statistical meaning!

Entropies to Measure Uncertainty

Measurement Uncertainty (incomplete!)

- Constructions of joint position and momentum operators (e.g., von Neumann, Holevo, Werner, Busch,…): Positive Formulation!

- Noise operator approach to determine inaccuracy (e.g., Arthur & Kelly, Appleby, Ozawa,…)

$$\epsilon(Z, A, \psi) := \langle U(\psi \otimes \Psi) | (Z - A)^2 U(\psi \otimes \Psi) \rangle^{1/2}$$

  - Inspired by classical measurement theory (not always operational!)

Previous formulation suffer often from drawbacks:

- operationally not entirely clear
- Depend also on the eigenvalues, not only on POVM’s
- maturity as in preparation uncertainty relations not yet achieved

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Goal

A **measurement-disturbance relation** that

- has a clear and operationally motivated setup which is state-dependent (-> applications)
- uses faithful and operational error measures (they vanish if no deviation
- uses entropic measures which have statistical interpretation in information theory (-> applications)

Outlook

1) Introduction to Measurement Uncertainty

2) Our New Measurement-Disturbance Relation

3) Extension to Quantum Memories

4) Applications (Qubits and Position-Momentum)

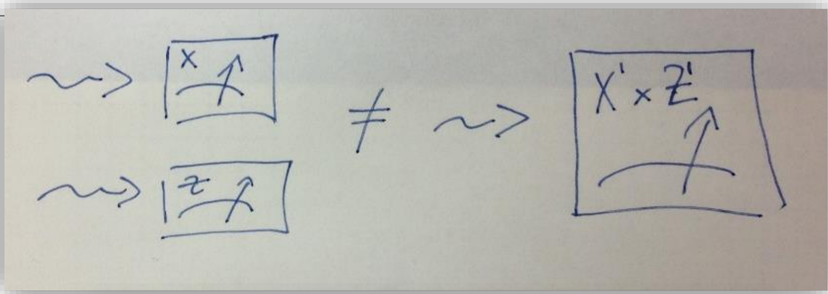
Measurement Uncertainty

1) Joint Measurability

There exists **no** joint observable

of X and Z with marginals being X and Z:

- Measurement Errors of X’ w.r.t. X and Z’ w.r.t Z are in trade-off



POVM $E_{x,z}$: $X_x = \sum_z E_{x,z}$

$Z_z = \sum_x E_{x,z}$

Measurement Uncertainty

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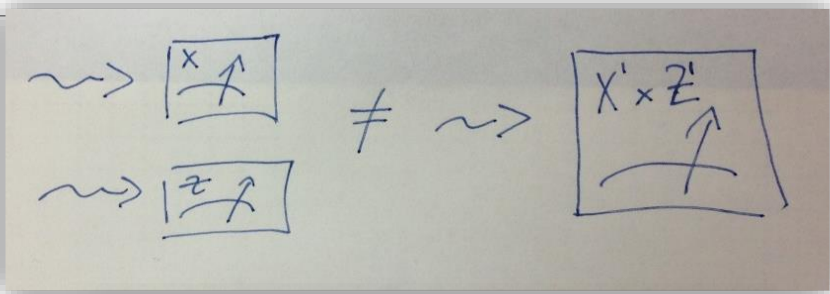
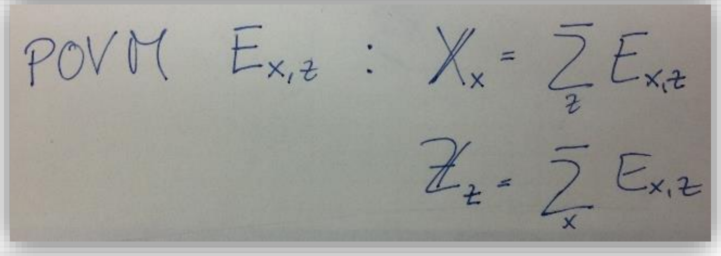


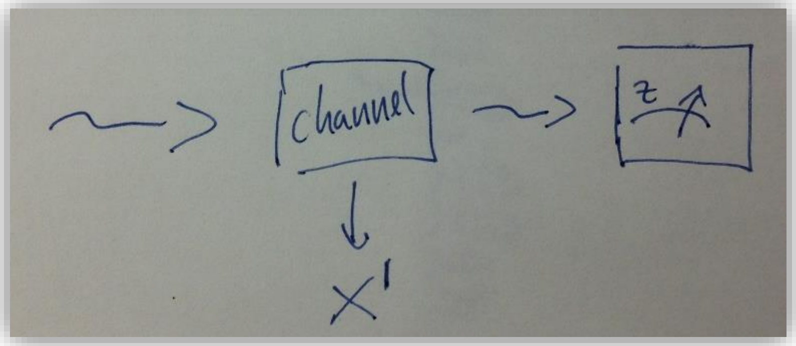


2) Measurement-Disturbance

An attempt to measure X will

disturb a subsequent Z measurement:

- The error about X is in trade-off with the disturbance in Z



Measurement Uncertainty

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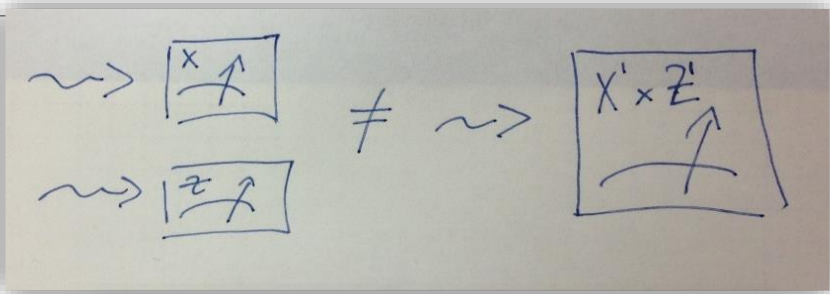

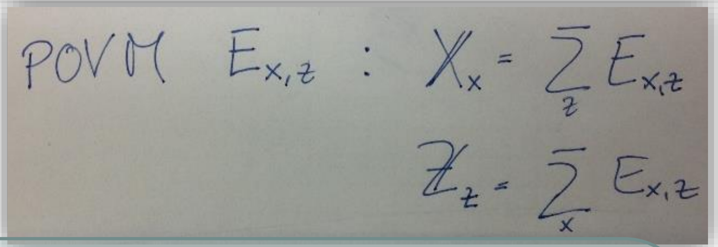

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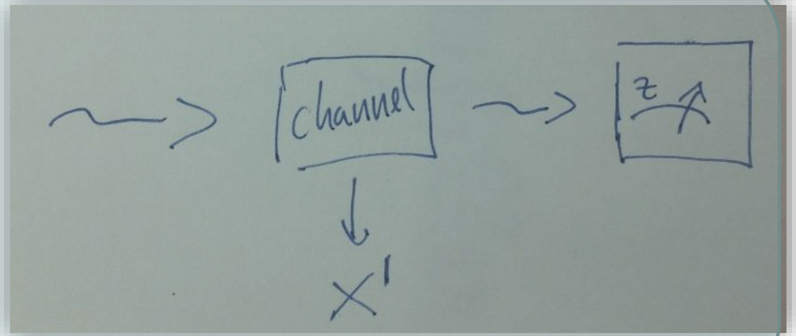
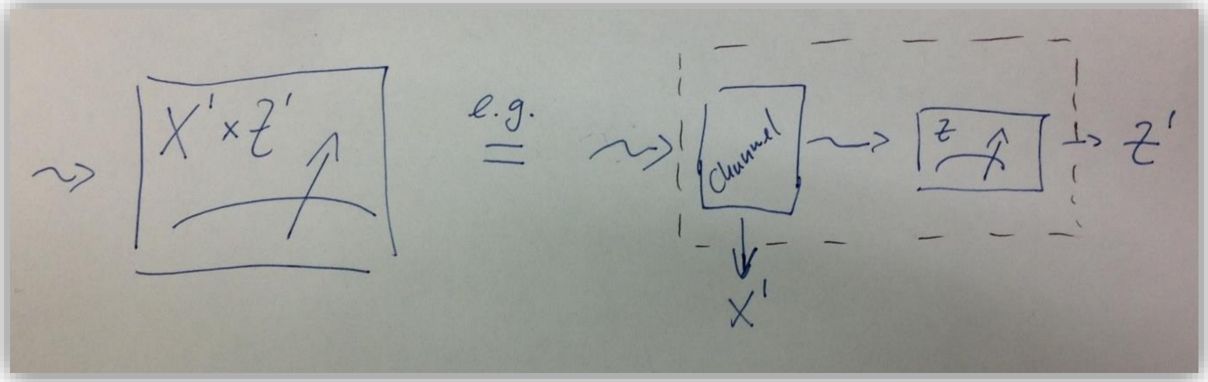


Connection between joint measurability and measurement-disturbance

A subsequent measurement is a joint measurement:



→ A trade-off for joint measurability implies one for error-disturbance

But only if we are interested in the potential of the channel to perform an accurate X measurement!

Recent Advances in State-independent Joint Measurement Relations

1) Bush, Lathi, Werner, PRL 111, 160405 (2013)**:**

- Position-Momentum Observables for worst-case calibration errors (testing Q’ with infinitely localized position states)
- Generalization to qubits

2) Buscemi, Hall, Ozawa, Wilde, PRL 112, 050401 (2014)

- Entropic error measures motivated via an information theoretic task

3**) Renes & Scholz,** arXiv:1402.6711 (2014)

- CP-Norm between channels

Interpretation as measurement-disturbance relations:

Any channel that extracts information about X-eigenstates must disturb the Z-eigenstates!

State-dependent Error Measures for Joint-Measurability Problem

Ozawa's Relation, Physics Letters A 320, 367 (2004)

For any fixed state ρ

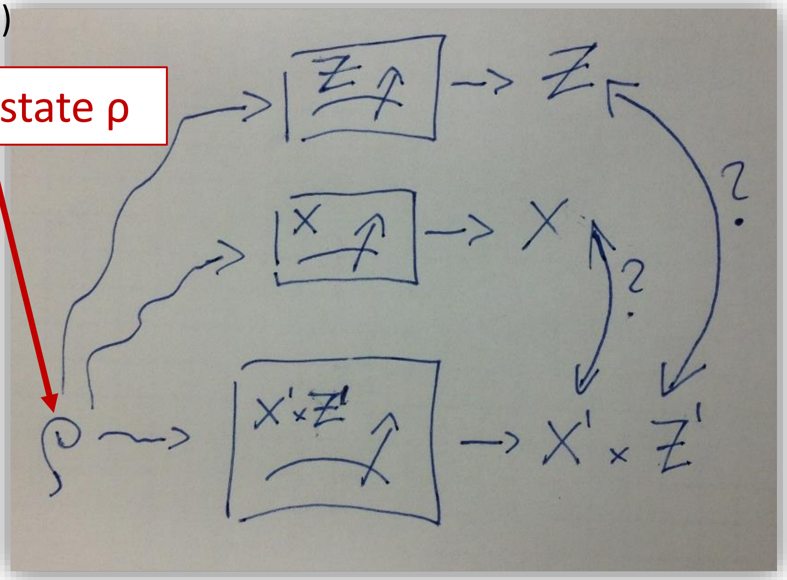
$$E_\psi(X)D_\psi(Z) + E_\psi(X)\Delta Z + D_\psi(Z)\Delta X \geqslant C_{X,Z}(|\psi\rangle). \quad (3)$$

$$C_{X,Z}(|\psi\rangle) := \tfrac{1}{2}|\langle\psi|[X,Z]|\psi\rangle|.$$

Error

Disturbance

Initial Uncertainties

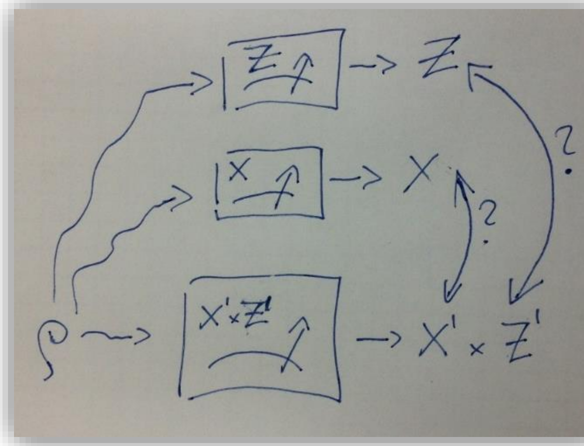


Insight: Initial uncertainty is crucial!

State-dependent Error Measures for Joint-Measurability Problem

Operational Error and Disturbance:

Error/Disturbance has to be detectable on the level of the outcome distributions (i.e. the probabilities)



Faithful Error and Disturbance:

- Error: $E(X,X';\rho) = 0$ iff probabibility distribution of X is same as for X'
- Disturbance: $D(Z,Z';\rho) = 0$ iff probabibility distribution of Z is same as for Z'

Problem for state-dependent trade-off relation (Korzekwa, Jennings and Rudolph, Phys. Rev. A 89, 052108 (2014)):

If error and disturbance are faithful no trade-off can hold!

State-dependent Measurement-Disturbance

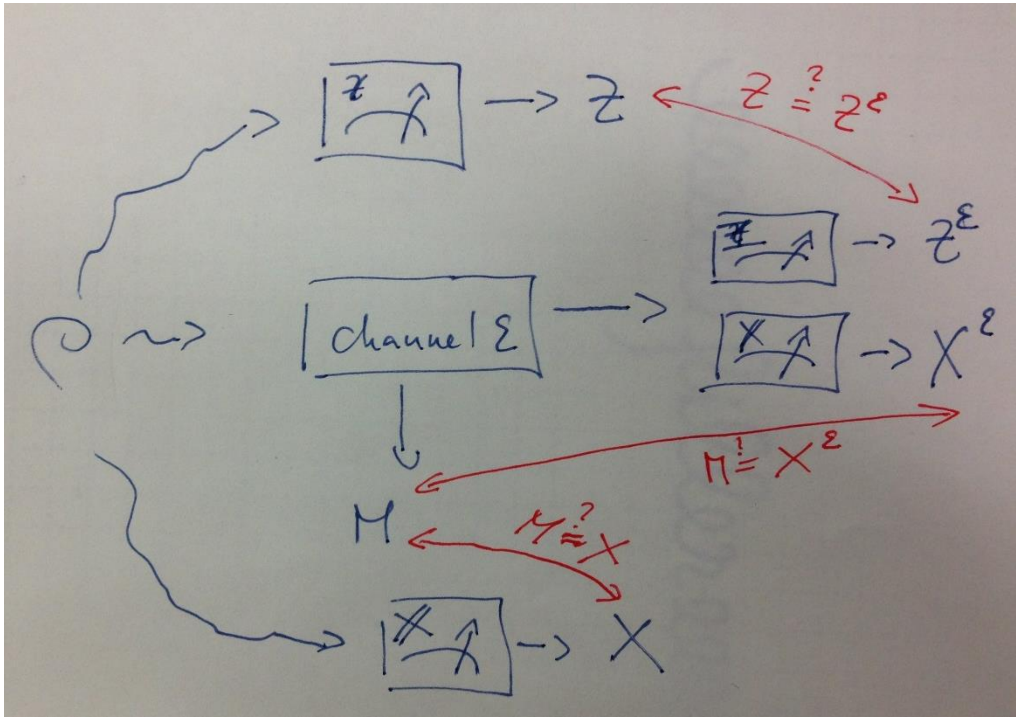
Operational Disturbance:

statistical distance between Z and $Z^{\mathcal{E}}$

2 Possible Measurement Errors:

1) can one infer X from M? (retrodictive error*)

2) can one infer $X^{\mathcal{E}}$ from M? (predictive error*)



*Appleby, International Journal of Theoretical Physics, 37, 5, 1491 (1998)

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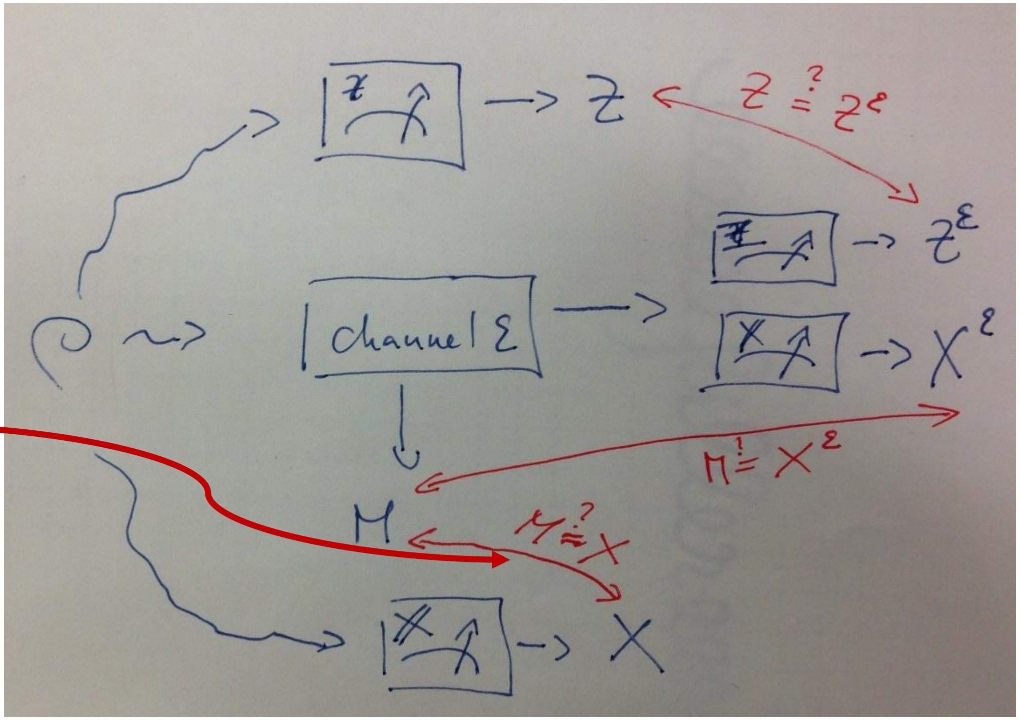

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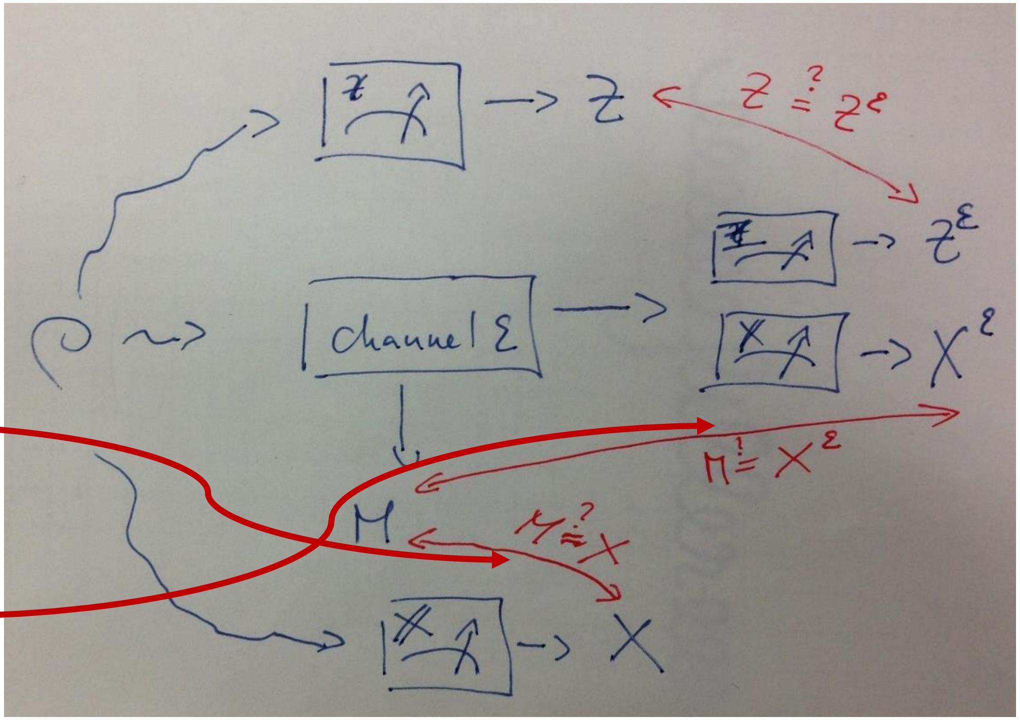

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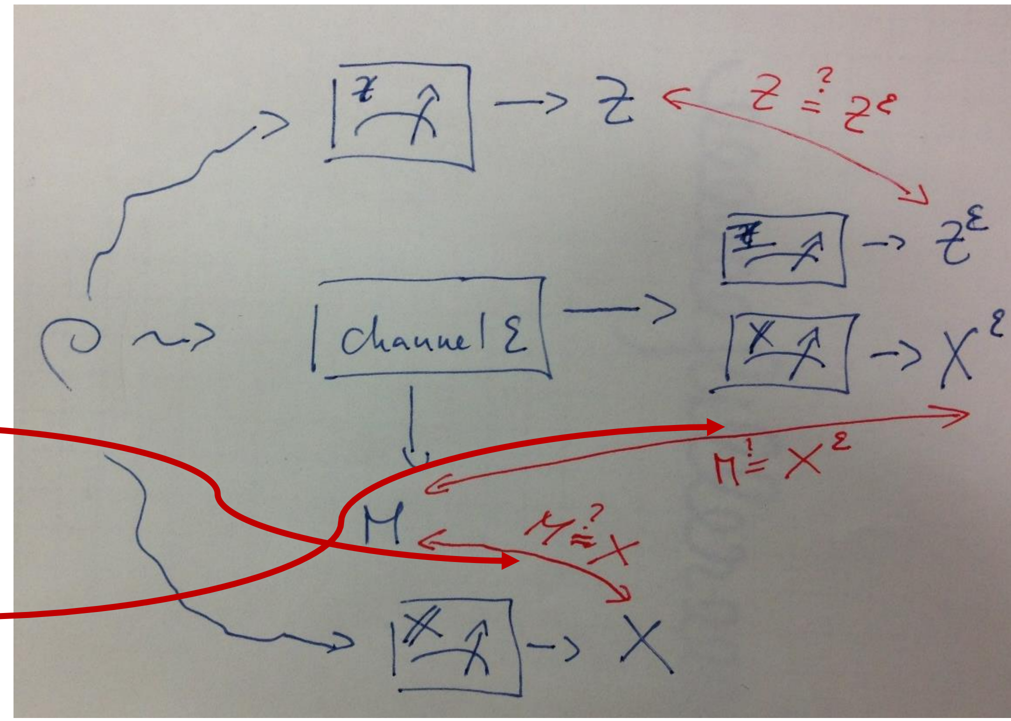

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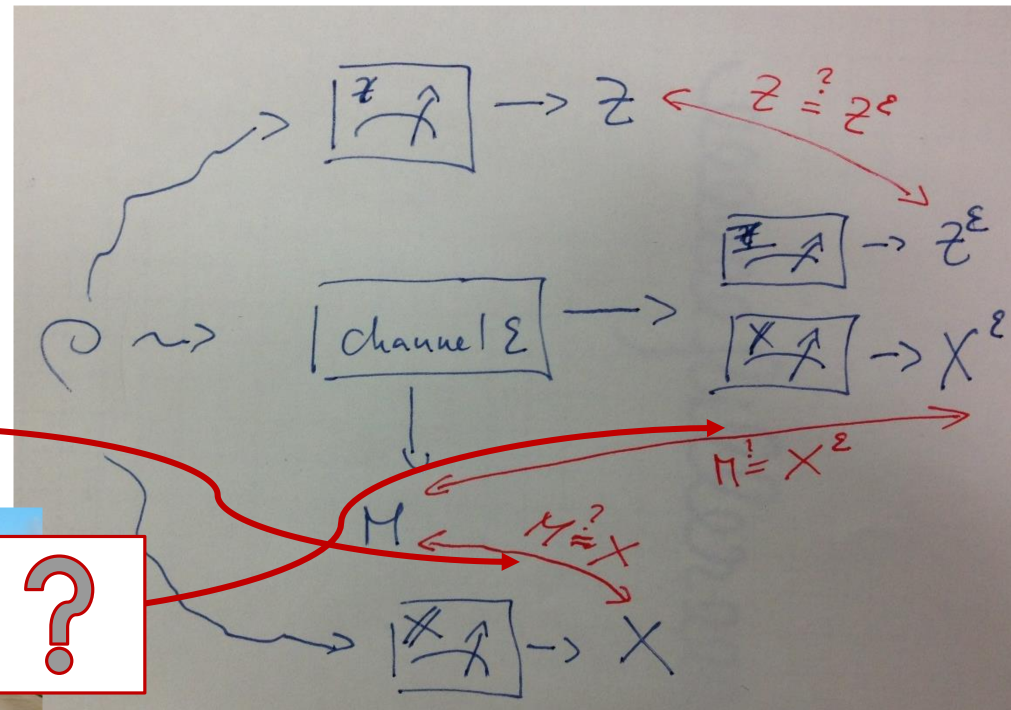

2 Possible M

1) can one
(retrodic

2) can one
(predict

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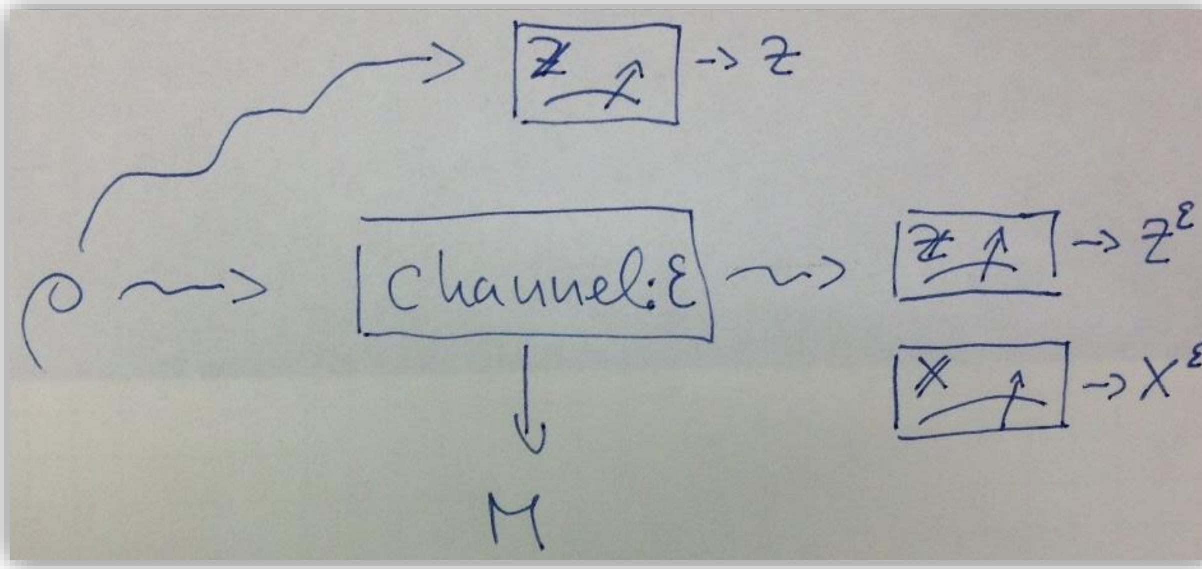
Outlook

1) Introduction to Measurement Uncertainty
2) **Our New Measurement-Disturbance Relation**
3) Extension to Quantum Memories
4) Applications (Qubits and Positoin-Momentum)

2) Setup for our Measurement-Disturbance Relation

Setup



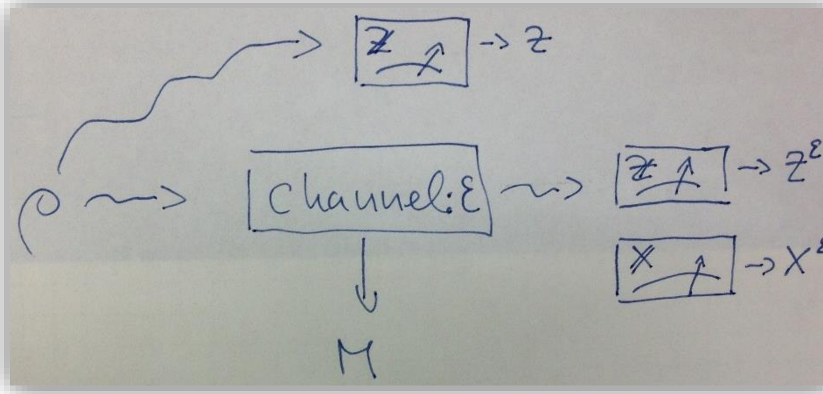
Operational Quantities:

- $P_Z(z) = \mathrm{tr}(\rho \mathbb{Z}_z)$
- $P_Z^\mathcal{E}(z) = \mathrm{tr}\,(\mathcal{E}(\rho)\mathbb{Z}_z)$
- $Q_{XM}^\mathcal{E} = \mathrm{tr}\,(\mathcal{E}(\rho)\mathbb{M}_m \otimes \mathbb{X}_x)$ (joint probability of M and $X^\mathcal{E}$)

Faithful Disturbance Measure

Distance between $P_Z^{\mathcal{E}}(z)$ and $P_Z(z)$ quantified with **relative entropy**

$$\mathsf{D}(\rho_S, \mathbb{Z}, \mathcal{E}) := D(P_Z || P_Z^{\mathcal{E}}).$$



- $D(P_Z||P_Z^{\mathcal{E}}) = \sum P_Z(z) \log(P_Z(z)/P_Z^{\mathcal{E}}(z))$
- Operational statistical meaning in hypothesis testing
- faithful

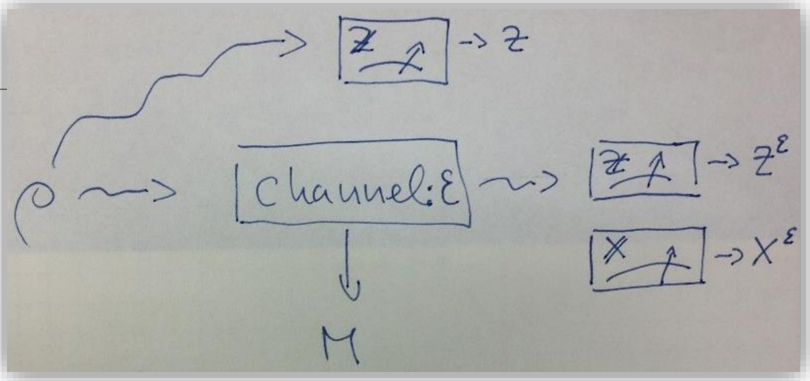
Side Remark: our result holds for arbitrary Renyi relative entropy

E.g., $D_{\frac{1}{2}}(P_Z||P_Z^{\mathcal{E}}) = \log F(P_Z, P_Z^{\mathcal{E}})$ (F=Fidelity)

Faithful Predictive Error

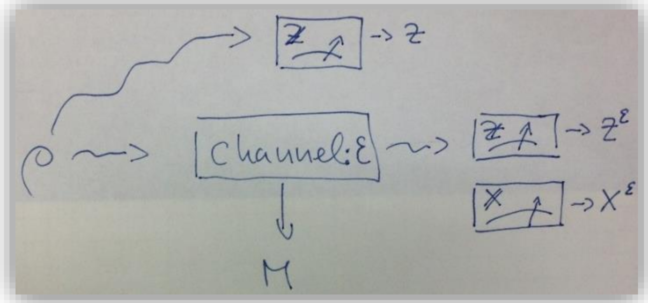
Entropy of X given M:

$$\mathsf{E}(\rho_S, \mathbb{X}, \mathcal{E}) := H_{\max}(X|M)_{Q^{\mathcal{E}}}$$



- $Q^{\mathcal{E}}_{XM}$ = joint probability distr. of M and $X^{\mathcal{E}}$
- $H_{\max}(X|M)_{Q^{\mathcal{E}}}$: **conditional max-entropy** of $X^{\mathcal{E}}$ given M (1/2-Renyi entropy)
- $H_{\max}(X|M)_{Q^{\mathcal{E}}} = \log \sum_m Q^{\mathcal{E}}_M(m) \exp(H_{1/2}\ (Q^{\mathcal{E}}_{X|M})$
- Faithful
- One-shot entropy related to the amount of data which one has to be supplied in order to reconstruct X from M

Measurement-Disturbance Trade-off



Result: For any channel $\mathcal{E}$:

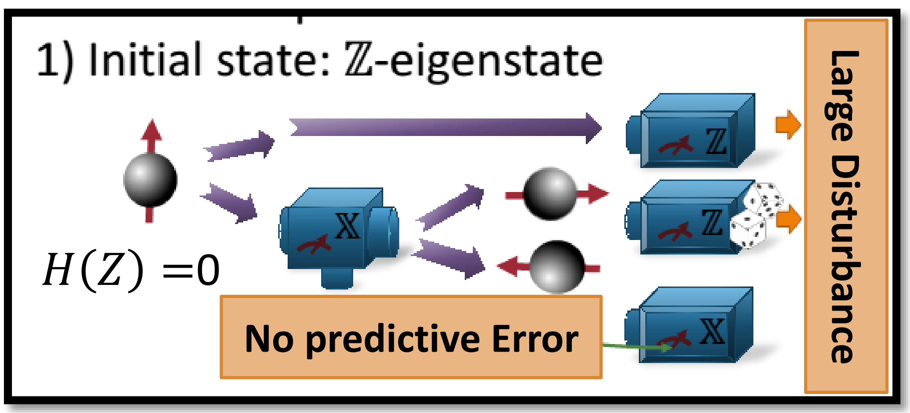
$$\boxed{\mathsf{D}(\rho_S, \mathbb{Z}, \mathcal{E}) + \mathsf{E}(\rho_S, \mathbb{X}, \mathcal{E}) \geqslant \log 1/c - H(Z)_P}$$

- $c = \max_{x,z} \|\sqrt{\mathbb{X}_x}\sqrt{\mathbb{Z}_z}\|_\infty^2$ (state independent)
- $H(Z)_P = -\sum P_Z(z) \log P_Z(z)$, the von Neumann entropy of the initial Z distribution

Lower Bound Must Depend on Initial Uncertainty

$$\mathsf{D}(\rho_S, \mathbb{Z}, \mathcal{E}) + \mathsf{E}(\rho_S, \mathbb{X}, \mathcal{E}) \geqslant \log 1/c - H(Z)_P$$

Example: Q-bit system with Pauli X and Z observables and $\mathcal{E}$ a perfect X instrument



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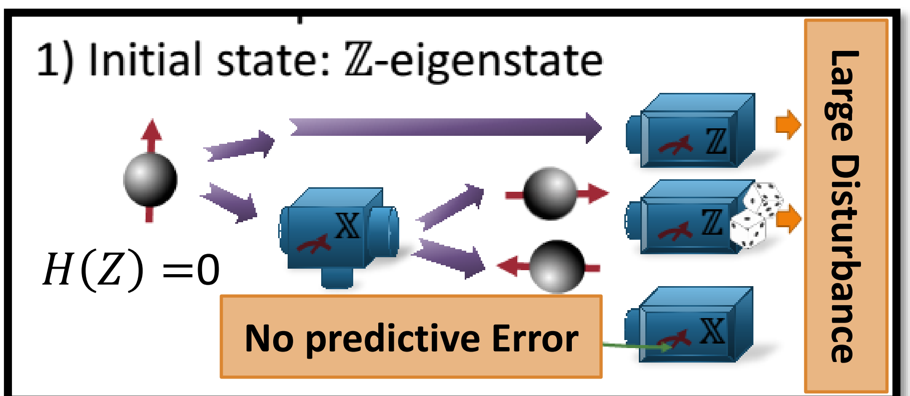
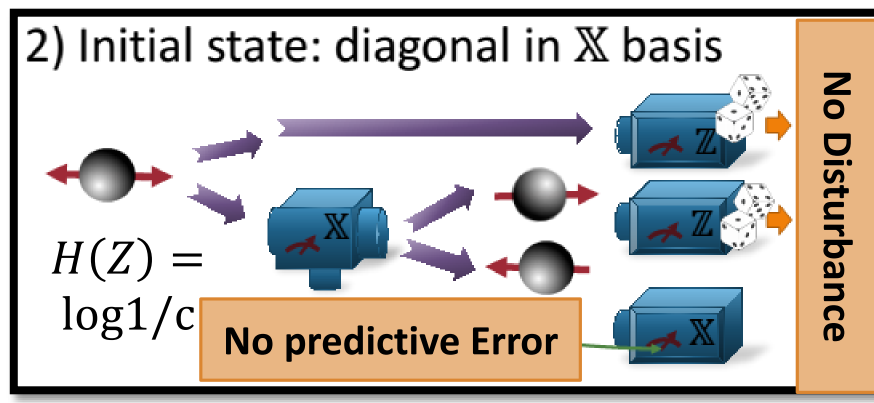




Proof of the Relation (without QM memory)

$$\mathsf{D}(\rho_S,\mathbb{Z},\mathcal{E})+\mathsf{E}(\rho_S,\mathbb{X},\mathcal{E})\geqslant \log 1/c - H(Z)_P$$

$$\mathsf{D}(\rho_S,\mathbb{Z},\mathcal{E}) := D(P_Z\|P_Z^{\mathcal{E}}).$$

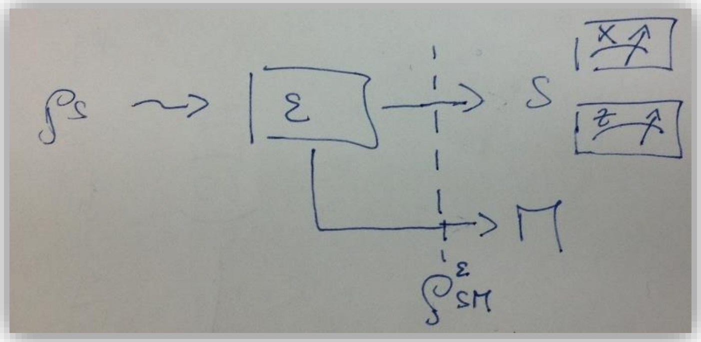
$$\mathsf{E}(\rho_S,\mathbb{X},\mathcal{E}) := H_{\max}(X|M)_{Q^{\mathcal{E}}}$$

Ingredient 1:

Preparation Uncertainty applied to $\rho_{SM}^{\mathcal{E}}$*:

$$H_{\min}(Z)_{P^{\mathcal{E}}} + H_{\max}(X|M)_{Q^{\mathcal{E}}} \geqslant \log 1/c$$

$$H_{\min}(Z)_{P^{\mathcal{E}}} = -\log \max_z P_Z^{\mathcal{E}}(z)$$



Ingredient 2:

Bound of min entropy of $P^{\mathcal{E}}$ by the distance betw. P and $P^{\mathcal{E}}$:

$$D(P_Z\|P_Z^{\mathcal{E}}) + H(Z)_P = -\sum_z P_Z(z)\log P_Z^{\mathcal{E}}(z)$$

$$\geqslant -\sum_z P_Z(z)\log \max_z P_Z^{\mathcal{E}}(z)$$

*M. Tomamichel and R. Renner, PRL 106,110506 (2011)

Outlook

1) Introduction to Measurement Uncertainty

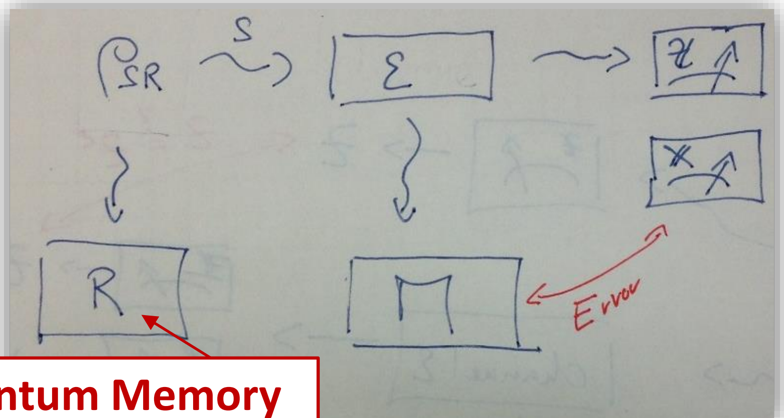
2) Our New Measurement-Disturbance Relation

3) **Extension to Quantum Memories**

4) Applications (Qubits and Positoin-Momentum)

3) Extension to Quantum Memory

System initially correlated to a quantum system R:



Quantum Memory

→ Interaction also disturbs the correlation to R!

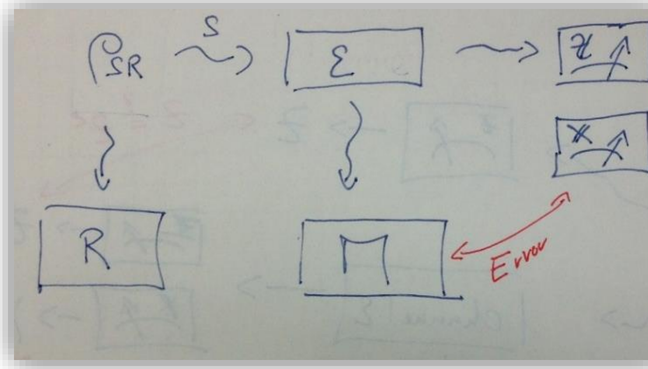
Then:

$$\mathsf{D}(\rho_{SR},\mathbb{Z},\mathcal{E}) + \mathsf{E}(\rho_S,\mathbb{X},\mathcal{E}) \geqslant \log 1/c - H(Z|R)_\rho$$

- $\mathsf{D}(\rho_{SR},\mathbb{Z},\mathcal{E}) = D(\rho_{ZR}||\rho^{\mathcal{E}}_{ZR})$ = distance between ρ_{ZR} and $\rho^{\mathcal{E}}_{ZR}$
- H(Z|R) = H(ZR) – H(R) , the conditional von Neumann entropy

Extension to Quantum Memory (example)

$$\rho_{SR} = \frac{1}{\sqrt{2}}\left(|00> \ +|11>\right)$$



Then…

- ρ_S is maximally mixed
- no trade-off if R is not taken into account (H(Z) = log1/c)
- If you can check correlation between Z and R -> disturbance (H(Z|R) = 0)
- Non-trivial relation

$$\mathsf{D}(\rho_{SR}, \mathbb{Z}, \mathcal{E}) + \mathsf{E}(\rho_S, \mathbb{X}, \mathcal{E}) \geqslant \log 1/c$$

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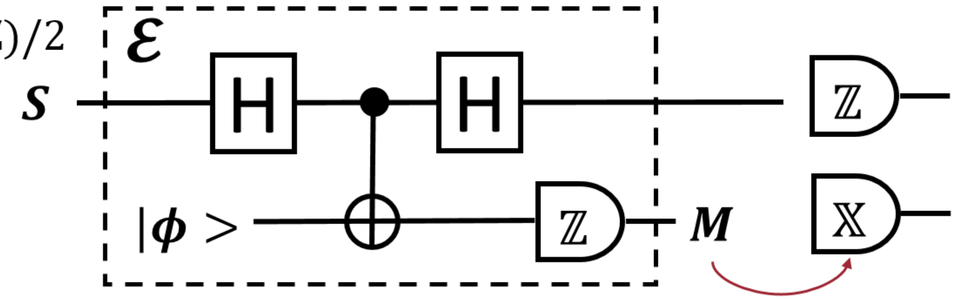
4) **Applications (Qubits and Position-Momentum)**

Qubit Example and Tightness

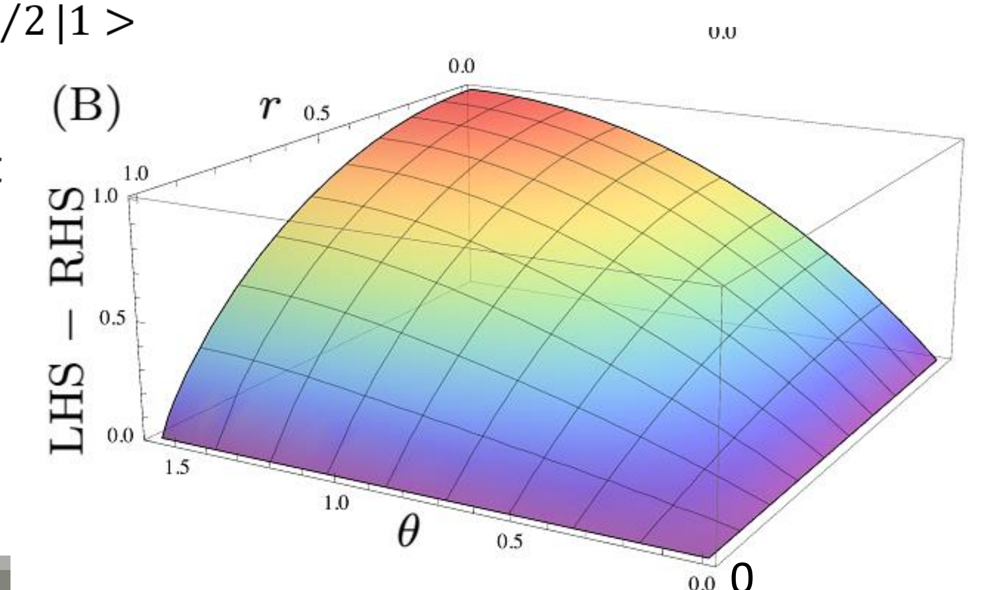
Weak Pauli X measurement (Phys. Rev. Lett. 109, 100404, 2012):

$\rho_S = (\mathbb{I} + r\mathbb{Z})/2$



$|\phi> = cos\,\theta/2|0> + sin\,\theta/2\,|1>$

- θ = 0: perfect X Measurement
- θ = π/2: identity channel
- Perfect tight for pure input state and all measurement strength



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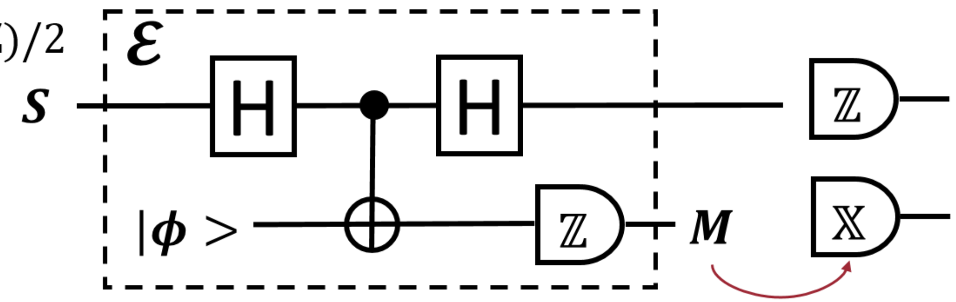

$|\phi> = cos\,\theta/2|0> + sin\,\theta/2\,|1>$

Including Quantum Memory (if ρ_S is not pure):

If ρ_{SR} is purification of ρ_S :

Perfectly tight for all r and θ

(Even classical memory is enough)

Extension to Position-Momentum Measurements

- Original setup considered by Heisenberg
- Observables $\mathbb{P}$ and $\mathbb{Q}$ with $[\mathbb{Q}, \mathbb{P}] = -i$

Same measurement disturbance holds if entropies are changed to differential entropies:

$$\boxed{\mathsf{D}(\rho_{SR}, \mathbb{P}, \mathcal{E}) + \mathsf{e}(\rho_S, \mathbb{Q}, \mathcal{E}) \geqslant \log 2\pi\hbar - h(P|R)_\rho}$$

- $e(\rho_S, \mathbb{Q}, \mathcal{E}) \coloneqq h_{\max}(Q|M)_{\rho^\varepsilon}$ (differential quantum conditional max-entropy*)
- $h(P|R)$ = the differential quantum conditional von Neumann entropy*

* Berta, Christandle,FF, Scholz,Tomamichel, arXiv:1308.4527

Application to Coarse-Grained Position Measurement

What is the momentum disturbance if a coarse grained position measurement with a binning δ_q has been performed?

Applying our relation we obtain:

$$\boldsymbol{\delta_q \; d_P \geq \hbar/2}$$

- $d_P = \left(\frac{2^{h(P)_\rho}}{4\pi}\right) 2^{D(\rho_S,\mathbb{P},\mathcal{E})}$

→ d_P depends on the initial P distribution

→ if initial momentum is approximately sharp, then disturbance is larger

→ interplay between measurement and preparation uncertainty

Conclusion and Outlook

- Presented a trade-off between disturbance and predictability of two complementary observables
- Operational disturbance and error measures with interpretation in information theory
- Tight for recent experiments
- Applies to position and momentum operators
- Application to quantum information theory (e.g., cryptography)?
- State-independent predictive error?

Thank you for your attention!

arXiv:1311.7637