Tensors, !-graphs, and non-commutative quantum structures

Aleks Kissinger David Quick

QPL
June 2014

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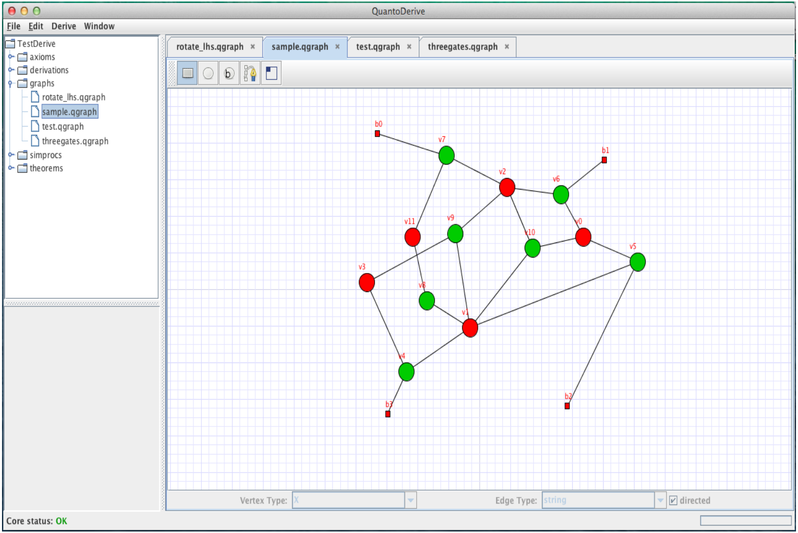
Quanto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threegates.qgraph
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v7
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v9
v3
v5
b2
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Edge Type:
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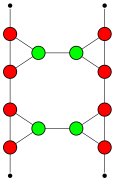
Core status: OK

Defining Nodes

Given the graph: 

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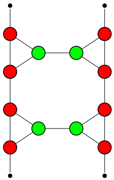
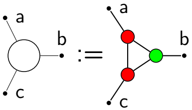
Given the graph: 

Could we define: 

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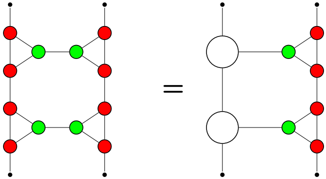

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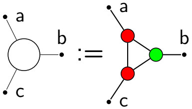

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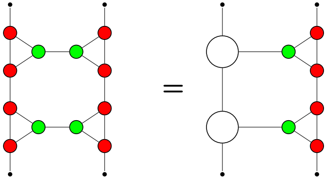

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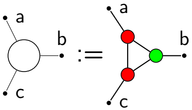

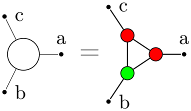

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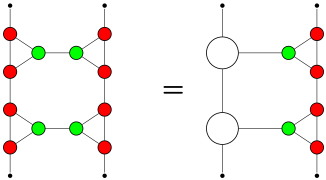

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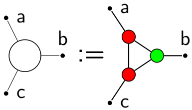

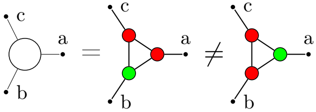

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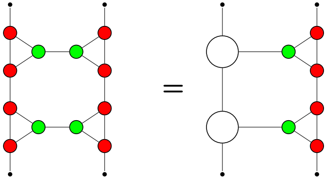

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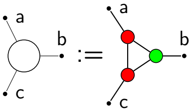

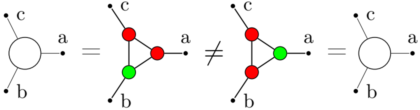

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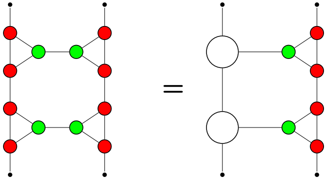

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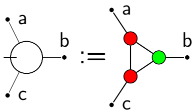

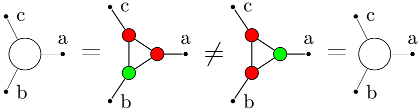

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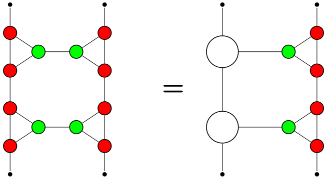

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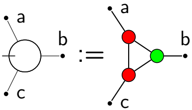

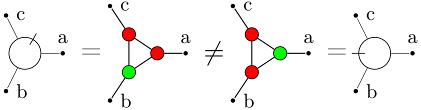

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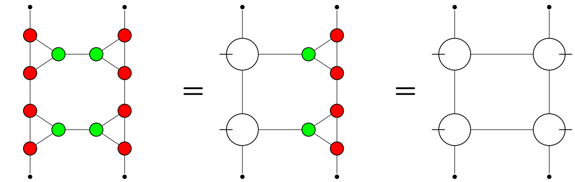

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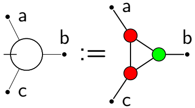
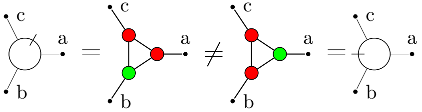
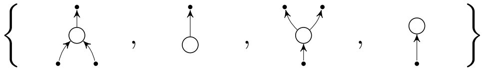
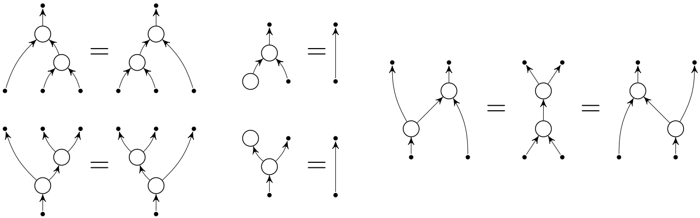




Recursively Defining Nodes

Given $\Sigma =$  satisfying:



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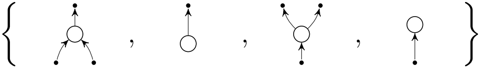
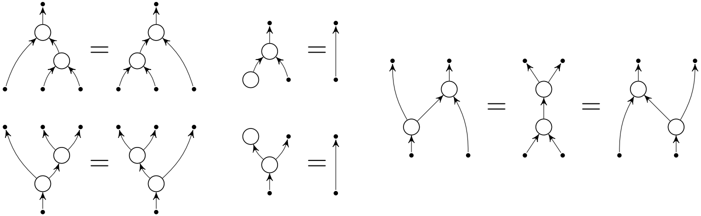
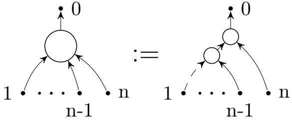
Given $\Sigma =$  satisfying:



We may want to define nodes of the form:



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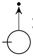
0 inputs: := 

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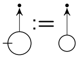
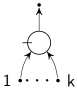
0 inputs: 

Given k inputs: 

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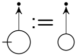

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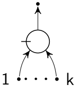

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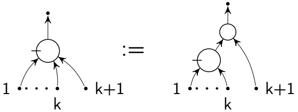

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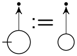
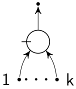
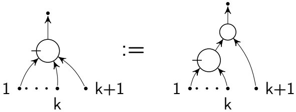
0 inputs: 

Given k inputs: 

define k+1 inputs: 

Recursive definitions suggest proof by induction. To do this we need to be more formal with variable arity nodes.

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Use !-boxes for multiple copies of a section from a graph.

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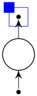
Then we have operations allowing deletion of !-boxes and creation of a new instance of the contents:



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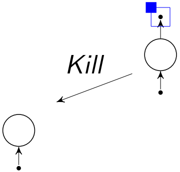

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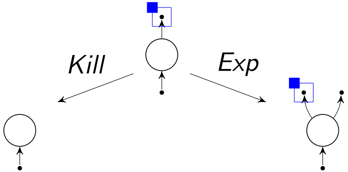

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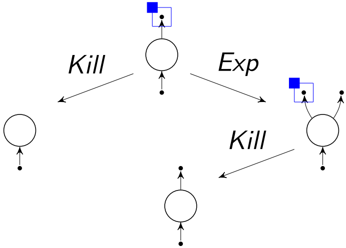

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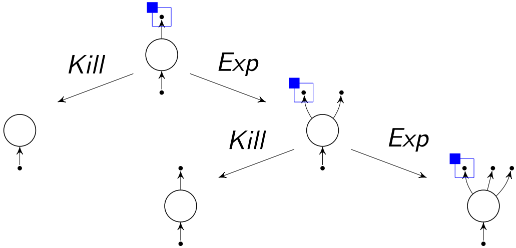

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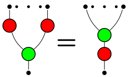

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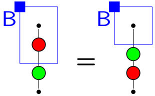

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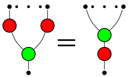

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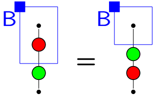
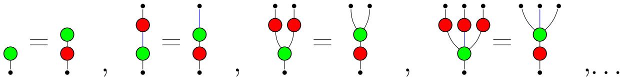
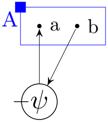


Which represents concrete equations like:



!-Box Expansion

This !-box (labelled A) has many different possible expansions:



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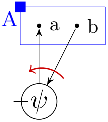

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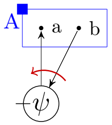

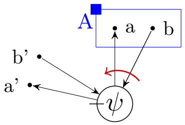

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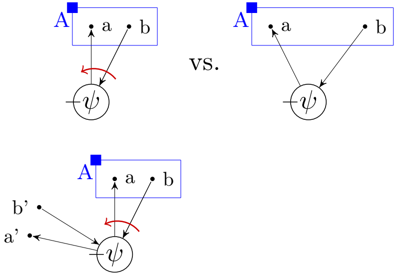

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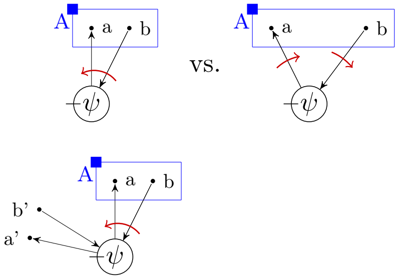

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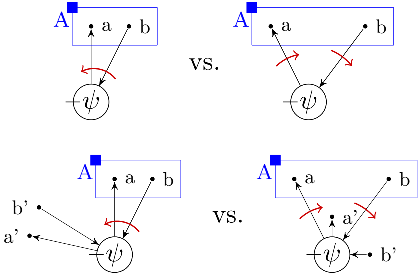

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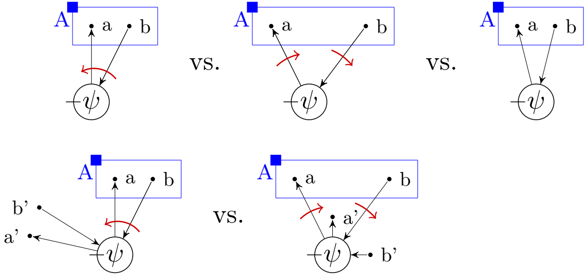

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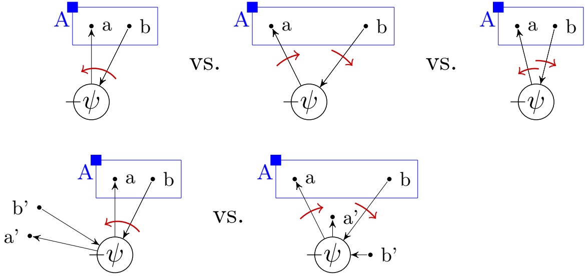

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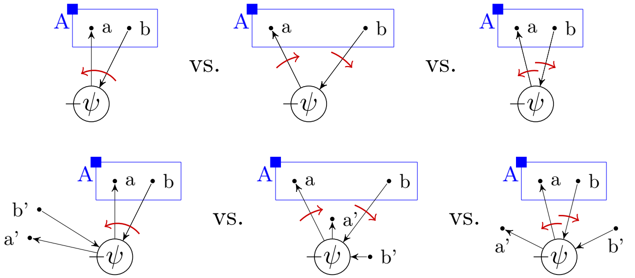

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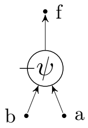

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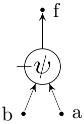

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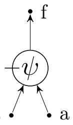

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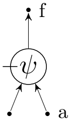
$\psi_{\hat{f}\breve{a}\check{b}} =$ 

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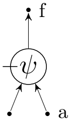
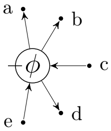
$\psi_{\hat{f}\check{a}\check{b}} =$ 

$\phi_{\hat{a}\hat{b}\check{c}\hat{d}\check{e}} =$ 

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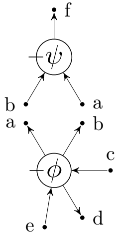

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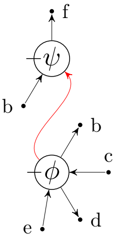

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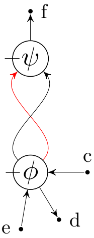
$$\psi_{\hat{f}\check{a}\check{b}}\phi_{\hat{a}\hat{b}\check{c}\hat{d}\check{e}} \quad = $$



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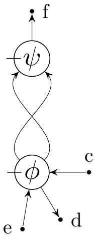

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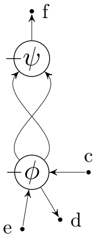
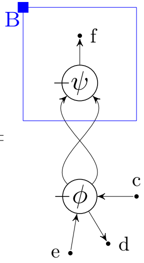


$$= \quad \psi_{\hat{f}\check{x}\check{y}}\phi_{\hat{x}\hat{y}\check{c}\hat{d}\check{e}}$$

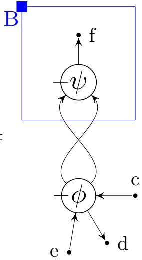
Building a Tensor

$$\psi_{\hat{f}\check{a}\check{b}}\phi_{\hat{a}\hat{b}\check{c}\hat{d}\check{e}} \quad = $$



Building a Tensor

$$\left[\psi_{\hat{f}\check{a}\check{b}}\right]^{B} \phi_{\hat{a}\hat{b}\check{c}\hat{d}\check{e}} =$$



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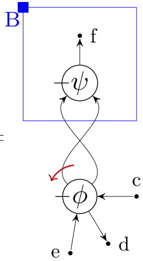

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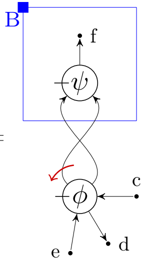

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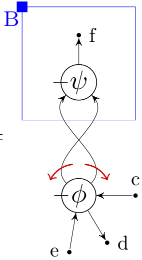

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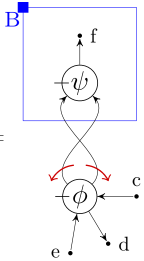

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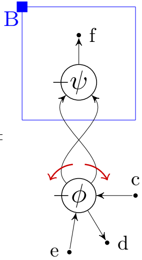
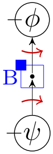
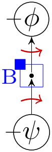


An Example



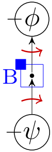
An Example

$$\psi_{[\hat{a}\rangle^B}\phi_{\langle\check{a}]^B} \quad = $$



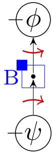
An Example

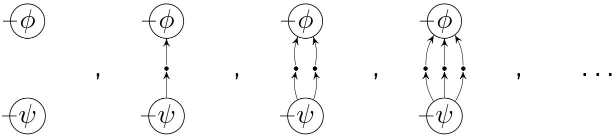
$$\psi_{[\hat{a}\rangle^B}\phi_{\langle\check{a}]^B}\left[1\right]^B \quad = $$



An Example

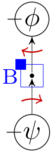
$$\psi_{[\hat{a}\rangle^B}\,\phi_{\langle\check{a}]^B}\left[1\right]^B \quad =$$





Twisted Example

$$\psi_{[\hat{a}\rangle^B}\phi_{[\check{a}\rangle^B}\left[1\right]^B \quad = $$



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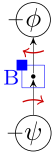
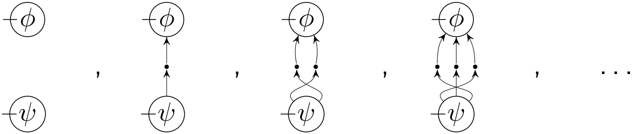
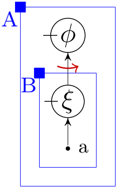




Nesting Example (Inner)

$$\left[\left[\xi_{\hat{b}\check{a}}\right]^B \phi_{\langle \check{b}]^B}\right]^A =$$



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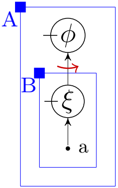

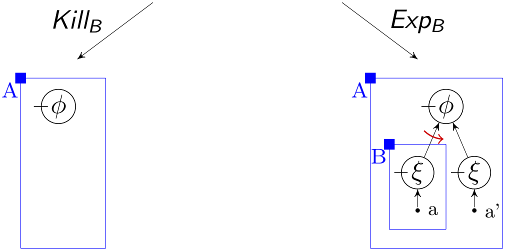

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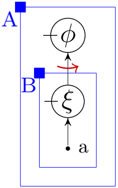
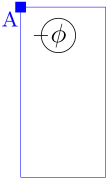
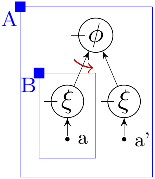


$Kill_B$

Exp_B



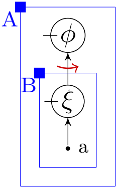
$$[\phi]^A$$



$$\left[\left[\xi_{\hat{b}\check{a}}\right]^B \xi_{\hat{b}'\check{a}'} \phi_{\check{b}'\langle\check{b}]^B}\right]^A$$

Nesting Example (Outer)

$$\left[\left[\xi_{\hat{b}\check{a}}\right]^{B}\phi_{\langle\check{b}]^{B}}\right]^{A} =$$



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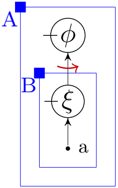

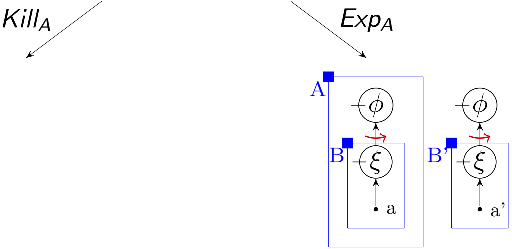

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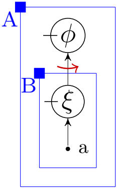
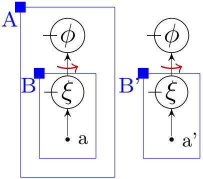


$Kill_A$ $\qquad$ Exp_A

1



$$\left[\left[\xi_{\hat{b}\check{a}}\right]^{B} \phi_{\langle \check{b}]^{B}}\right]^{A} \left[\xi_{\hat{b}'\check{a}'}\right]^{B'} \phi_{\langle \check{b}']^{B'}}$$

Table of Contents

Edgeterms

Definition

Fix disjoint, infinite sets $\mathcal{E}$ and $\mathcal{B}$ of edge names and !-box names, respectively. The set of *edgeterms* $\mathcal{T}_e$ is defined recursively as follows:

- $\epsilon \in \mathcal{T}_e$ (i.e empty)
- $\check{a}, \hat{a} \in \mathcal{T}_e$ $a \in \mathcal{E}$
- $[e\rangle^A, \langle e]^A \in \mathcal{T}_e$ $e \in \mathcal{T}_e,\ A \in \mathcal{B}$
- $ef \in \mathcal{T}_e$ $e, f \in \mathcal{T}_e$

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Two edgeterms are equivalent if one can be transformed into the other by:

$$\epsilon e \equiv e \equiv e\epsilon \qquad e(fg) \equiv (ef)g \qquad [\epsilon\rangle^A \equiv \epsilon \equiv \langle\epsilon]^A$$

!-Tensors

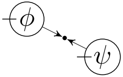
Definition

The set of all !-tensor expressions $\mathcal{T}_\Sigma$ for a signature Σ is defined recursively as:

- $1 \in \mathcal{T}_\Sigma$ (empty tensor)
- $1_{\hat{a}\check{b}} \in \mathcal{T}_\Sigma$ $a, b \in \mathcal{E}$
- $\phi_e \in \mathcal{T}_\Sigma$ $e \in \mathcal{T}_e, \phi \in \Sigma$
- $\left[G\right]^A \in \mathcal{T}_\Sigma$ $G \in \mathcal{T}_\Sigma,\ A \in \mathcal{B}$
- $GH \in \mathcal{T}_\Sigma$ $G, H \in \mathcal{T}_\Sigma$

Satisfying conditions F1-2, C1-3

Conditions: F1-F2

F1: No directed edge name can appear more than once as these can not be plugged together: $\phi_{\hat{a}}\psi_{\hat{a}} \quad =$ 

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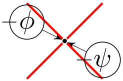
F1: No directed edge name can appear more than once as these can not be plugged together: $\phi_{\hat{a}}\psi_{\hat{a}} \quad =$ 

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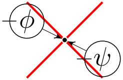
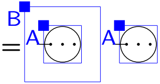


F2: No box can appear more than once to prevent overlap (which we do not allow in this formalism): $[[\ldots]^{A}]^{B}[\ldots]^{A} \quad =$



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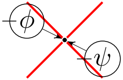

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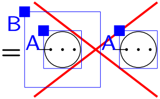
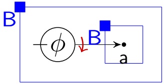


Conditions: C1-C3

C1: An edge entering a !-box can't be on a node already in that

!-Box: $\left[\phi_{[\hat{a}\rangle^{B}}\right]^{B} \quad =$ 

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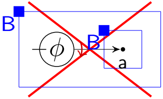
!-Box: $\left[\phi_{[\hat{a}\rangle^B}\right]^B \quad =$ 

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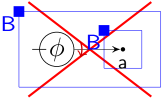
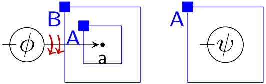


C2: Nested !-Boxes around an edge must be nested in the same way

in the rest of the tensor: $\phi_{[[\hat{a}\rangle^A\rangle^B}\left[\psi\right]^A \quad =$



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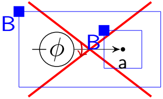

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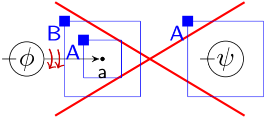

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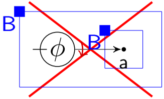

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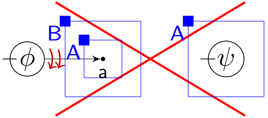

C3: Bound edges must be in compatible !-boxes:

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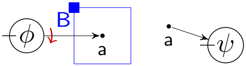

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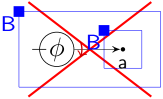
!-Box: $\left[\phi_{[\hat{a}\rangle^B}\right]^B \quad =$ 

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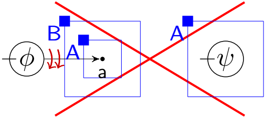
in the rest of the tensor: $\phi_{[[\hat{a}\rangle^A\rangle^B}\left[\psi\right]^A \quad =$ 

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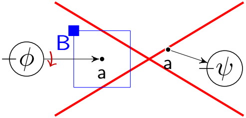
$\phi_{[\hat{a}\rangle^B}\psi_{\check{a}} \quad =$ 

Operations

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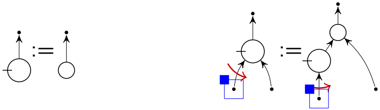
$Exp_B := [\left[G\right]^B \mapsto \left[G\right]^B \mathbf{fr}(G), [e\rangle^B \mapsto [e\rangle^B \mathbf{fr}(e), \langle e]^B \mapsto \mathbf{fr}(e)\langle e]^B]$

Table of Contents

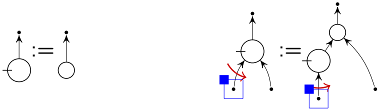
Spider Node

Definition

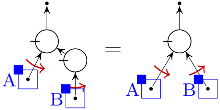


Spider Node

Definition



Theorem



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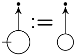
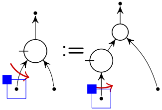
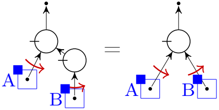
Definition





Theorem



We would like to do induction on !-box B splitting into a base case (after $Kill_B$) and an inductive step (proving Exp_B from original).

Induction

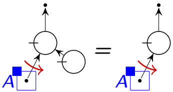
$$\frac{Kill_B(G = H) \qquad\qquad (G = H) \implies Exp_B(G = H)}{G = H} \text{ (Induction)}$$

Induction

$$\frac{Kill_B(G = H) \qquad Fix_B(G = H) \implies Exp_B(G = H)}{G = H} \text{ (Induction)}$$

Spider Theorem ($Kill_B$)

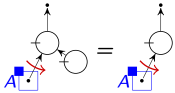
Theorem



(base)

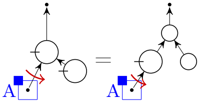
Spider Theorem ($Kill_B$)

Theorem



(base)

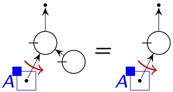
Proof.



□

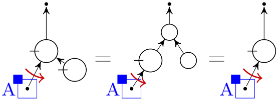
Spider Theorem ($Kill_B$)

Theorem



(base)

Proof.



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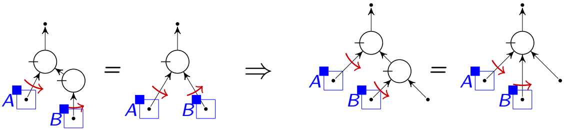
Spider Theorem ($Fix_B \implies Exp_B$)

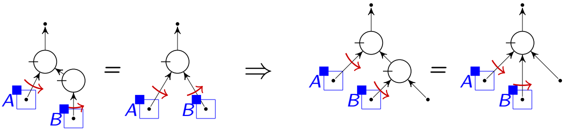
Theorem



(step)

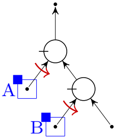
Spider Theorem ($Fix_B \implies Exp_B$)

Theorem



(step)

Proof.



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Spider Theorem ($Fix_B \implies Exp_B$)

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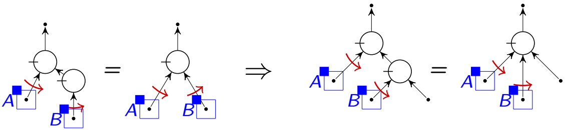
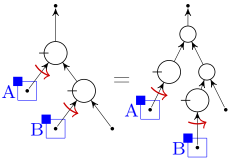


(step)

Proof.



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Spider Theorem ($Fix_B \implies Exp_B$)

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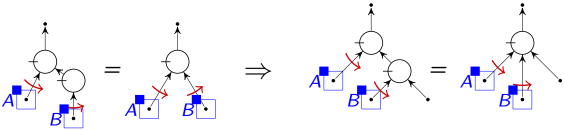

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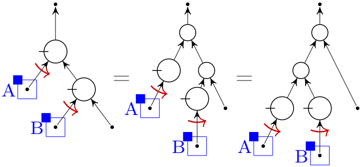

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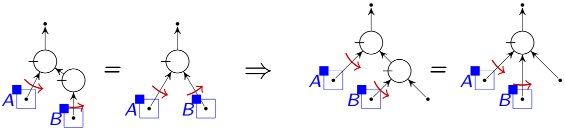
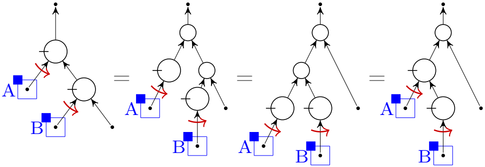


(step)

Proof.



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Spider Theorem ($Fix_B \implies Exp_B$)

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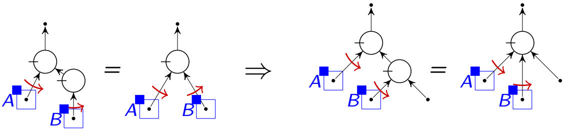

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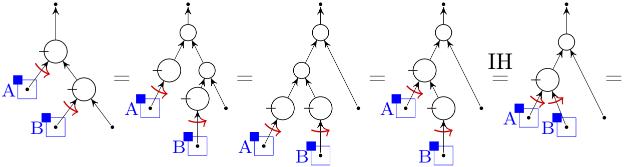

□

Spider Theorem ($Fix_B \implies Exp_B$)

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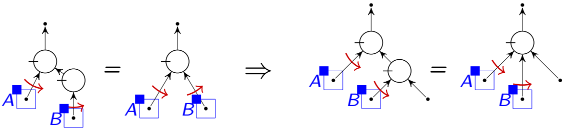
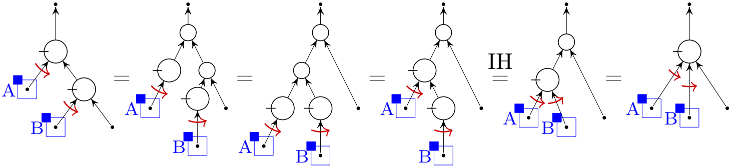


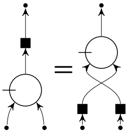
(step)

Proof.



□

Anti-Homomorphism



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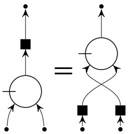

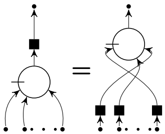

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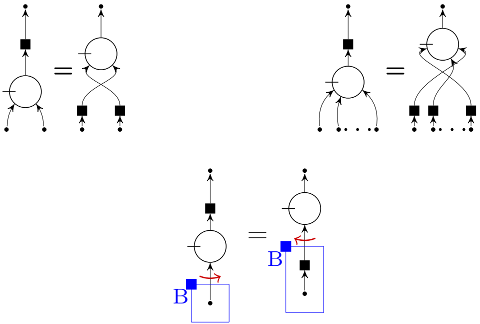

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Summary

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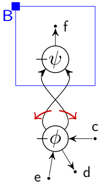
- Diagrammatic language:



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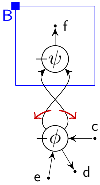

- Tensor notation: $\left[\psi_{\hat{f}\check{a}\check{b}}\right]^{B}\phi_{\langle\hat{a}]^{B}[\hat{b}\rangle^{B}\check{c}\hat{d}\check{e}}$

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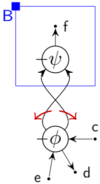

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$$\left[\psi_{\hat{f}\check{a}\check{b}}\right]^{B}\phi_{\langle\hat{a}]^{B}[\hat{b}\rangle^{B}\check{c}\hat{d}\check{e}}$$

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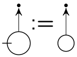

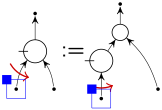

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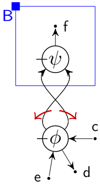

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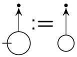

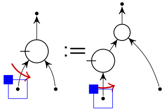

- !-Box Induction:

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