On monogamy of non-locality and macroscopic averages (examples and preliminary results)

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Overview

- Monogamy of violation of Bell inequalities from the no-signalling condition
  (Pawłowski & Brukner 2009: bipartite Bell inequalities)

- Average macro correlations arising from micro models
  (Ramanathan et al. 2011: QM models)

- General framework of Abramsky & Brandenburger (2011): generalise the results above
  provide a structural explanation related to Vorob’ev’s theorem (1962)

This talk: we mainly consider a simple illustrative example.
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- This talk: we mainly consider a simple illustrative example.
Monogamy of non-locality
Non-locality

\[ p(a_i, b_j = x, y) \]

Alice

Bob

\[ a_1, a_2 \]

\[ b_1, b_2 \]
Non-locality

\[ p(a_i, b_j = x, y) \]

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Non-locality

\[ p(a_i, b_j = x, y) \]

\[ B(A, B) \leq R \]

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Monogamy of non-locality

Alice

a₁, a₂

Bob

b₁, b₂

Charlie

c₁, c₂
Monogamy of non-locality

- Empirical model: no signalling probabilities

\[ p(a_i, b_j, c_k = x, y, z) \]

where \( x, y, z \) are possible outcomes.
Monogamy of non-locality

- Empirical model: no signalling probabilities

\[ p(a_i, b_j, c_k = x, y, z) \]

where \( x, y, z \) are possible outcomes.

- Consider the subsystem composed of \( A \) and \( B \) only, given by marginalisation (in QM, partial trace):

\[ p(a_i, b_j = x, y) = \sum_z p(a_i, b_j, c_k = x, y, z) \]

(this is independent of \( c_k \) due to no-signalling).

Similarly define \( p(a_i, c_k = x, z) \). (\( A \) and \( C \))
Monogamy of non-locality

Given a Bell inequality $B(-, -, R) \leq R,$
Monogamy of non-locality

Given a Bell inequality $B(-, -, ) \leq R$, 

![Diagram with Alice, Bob, and Charlie]
Monogamy of non-locality

Given a Bell inequality $B(-, -, \cdot) \leq R$,
Monogamy of non-locality

Given a Bell inequality $B(-, -) \leq R$,

- Alice:
  - $a_1, a_2$
- Bob:
  - $b_1, b_2$
- Charlie:
  - $c_1, c_2$
Monogamy of non-locality

Given a Bell inequality $\mathcal{B}(-, -, ) \leq R$, 

Monogamy relation: $\mathcal{B}(A, B) + \mathcal{B}(A, C) \leq 2R$
Macroscopic average behaviour
Macroscopic measurements

- (Micro) dichotomic measurement: a single particle is subject to an interaction $a$ and collides with one of two detectors: outcomes 0 and 1.
- The interaction is probabilistic: $p(a = x), x = 0, 1$. 

Consider beam (or region) of $N$ particles, differently prepared. Subject each particle to the interaction $a$: the beam gets divided into 2 smaller beams hitting each of the detectors. Outcome represented by the intensity of resulting beams: $I_a > 0, 1$ proportion of particles hitting the detector 1. We are concerned with the mean, or expected, value of such intensities.
Macroscopic measurements

- (Micro) dichotomic measurement: a single particle is subject to an interaction $a$ and collides with one of two detectors: outcomes 0 and 1.
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- Consider beam (or region) of $N$ particles, differently prepared.
- Subject each particle to the interaction $a$: the beam gets divided into 2 smaller beams hitting each of the detectors.
- Outcome represented by the intensity of resulting beams: $I_a \in [0, 1]$ proportion of particles hitting the detector 1.
- We are concerned with the mean, or expected, value of such intensities.
Macroscopic average behaviour

- This mean intensity can be interpreted as the average behaviour among the particles in the beam or region: if we would randomly select one of the $N$ particles and subject it to the microscopic measurement $a$, we would get outcome 1 with probability $I_a$:

$$I_a = \sum_{i=1}^{N} p_i(a = 1).$$

- The situation is analogous to statistical mechanics, where a macrostate arises as an averaging over an extremely large number of microstates, and hence several different microstates can correspond to the same macrostate.
Macroscopic average behaviour: multipartite

- Multipartite macroscopic measurements:
  - several ‘macroscopic’ sites consisting of a large number of microscopic sites/particles;
  - several (macro) measurement settings at each site.
- Average macroscopic Bell experiment: the (mean) values of the macroscopic intensities indicate the behaviour of a randomly chosen tuple of particles: one from each of the beams, or sites.

We shall show that, as long as there are enough particles (microscopic sites) in each macroscopic site, such average macroscopic behaviour is always local no matter which no-signalling model accounts for the underlying microscopic correlations.
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Macroscopic average behaviour: tripartite example

- Consider again the tripartite scenario:

\[
\begin{align*}
A & \quad b_1, b_2 \\
C & \quad c_1, c_2
\end{align*}
\]

We regard sites B and C as forming one 'macroscopic' site, M, and site A as forming another. In order to be 'lumped together', B and C must be symmetric/of the same type: the symmetry identifies the measurements \(b_1, c_1\) and \(b_2, c_2\), giving rise to 'macroscopic' measurements \(m_1\) and \(m_2\).

Given an empirical model \(p_{a_i, b_j, c^k, x, y, z}\), the 'macroscopic' average behaviour is a bipartite model (with two macro sites A and M) given by the following average of probabilities of the partial models:

\[
\begin{align*}
p_{a_i, m_j} & = \frac{1}{2} \left( p_{a_i, b_j} + p_{a_i, c_j} \right), \\
p_{a_i, m_j} & = \frac{1}{2} \left( p_{a_i, b_j} + p_{a_i, c_j} \right)
\end{align*}
\]
Macroscopic average behaviour: tripartite example

- Consider again the tripartite scenario.
- We regard sites $B$ and $C$ as forming one ‘macroscopic’ site, $M$, and site $A$ as forming another.
- In order to be ‘lumped together’, $B$ and $C$ must be symmetric/of the same type: the symmetry identifies the measurements $b_1 \sim c_1$ and $b_2 \sim c_2$, giving rise to ‘macroscopic’ measurements $m_1$ and $m_2$. 
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Given an empirical model $p(a_i, b_j, c_k = x, y, z)$, the ‘macroscopic’ average behaviour is a bipartite model (with two macro sites $A$ and $M$) given by the following average of probabilities of the partial models:

$$p_{a_i, m_j}(x, y) = \frac{p_{a_i, b_j}(x, y) + p_{a_i, c_j}(x, y)}{2}$$
Example: $W$ state

$Z$ and $X$ measurements on the $W$ state:

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(every entry should be divided by 24)
Example: $W$ state

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(every entry should be divided by 24)

This is **local**!
Another example model

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maximally non-local

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Again, this is **local**!
Monogamy and macroscopic averages
A simple observation
Consider any bipartite Bell inequality $B(–, –) \leq R$, given by a set of coefficients $\alpha(i, j, x, y)$ and a bound $R$. 
A simple observation
Consider any bipartite Bell inequality $B(-, -) \leq R$, given by a set of coefficients $\alpha(i, j, x, y)$ and a bound $R$.

\[ B(A, M) \leq R \]

\[ \iff \sum_{i,j,x,y} \alpha(i, j, x, y) p(a_i, m_j = x, y) \leq R \]

\[ \iff \sum_{i,j,x,y} \alpha(i, j, x, y) \frac{p(a_i, b_j = x, y) + p(a_i, c_j = x, y)}{2} \leq R \]

\[ \iff \sum_{i,j,x,y} \alpha(i, j, x, y) p(a_i, b_j = x, y) + \sum_{i,j,x,y} \alpha(i, j, x, y) p(a_i, c_j = x, y) \leq 2R \]

\[ \iff B(A, B) + B(A, C) \leq R \]
A simple observation
Consider any bipartite Bell inequality $B(\cdot, \cdot) \leq R$, given by a set of coefficients $\alpha(i, j, x, y)$ and a bound $R$.

$$B(A, M) \leq R$$

$\iff$

$$\sum_{i,j,x,y} \alpha(i, j, x, y) p(a_i, m_j = x, y) \leq R$$

$\iff$

$$\sum_{i,j,x,y} \alpha(i, j, x, y) \frac{p(a_i, b_j = x, y) + p(a_i, c_j = x, y)}{2} \leq R$$

$\iff$

$$\sum_{i,j,x,y} \alpha(i, j, x, y) p(a_i, b_j = x, y) + \sum_{i,j,x,y} \alpha(i, j, x, y) p(a_i, c_j = x, y) \leq 2R$$

$\iff$

$$B(A, B) + B(A, C) \leq R$$

The average model $p_{a_i, m_j}$ satisfies the inequality if and only if in the microscopic model Alice is monogamous with respect to violating it with Bob and Charlie.
A simple observation

- In the two examples above, the average models were local. Equivalently, the examples satisfied the monogamy relation for any Bell inequality.
- This is true for all no-signalling empirical models on the tripartite scenario under consideration, with two measurement settings per site.
- We give a structural explanation for this...
Vorob'ev’s theorem
Abramsky-Brandenburger framework

Measurement scenarios:

- a finite set of measurements $X$;
- a cover $\mathcal{U}$ of $X$ (or an abstract simplicial complex $\Sigma$ on $X$), indicating the **compatibility** of measurements.

Examples: Bell-type scenarios, KS configurations, and more.
Abramsky-Brandenburger framework

No-signalling empirical model:

- a family \( (p_C)_{C \in \mathcal{U}} \), where \( p_C \) is a probability distribution on the outcomes of measurements in context \( C \).
- compatibility condition: \( p_C \) and \( p_{C'} \) marginalise to the same distribution on the outcomes of measurements in \( C \cap C' \).

(on multipartite scenarios: no-signalling)
Abramsky-Brandenburger framework

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(on multipartite scenarios: no-signalling)

An empirical model admits a local/non-contextual hidden variable explanation (in the sense of Bell’s theorem) iff there exists a global distribution \( p_X \) (i.e. for all measurements at the same time) that marginalises to all the \( p_C \).

Obstructions to such extensions are witnessed by violations of general Bell inequalities.
Vorob'ev’s theorem

For which measurement compatibility structures $\mathcal{U}$ (or $\Sigma$) is it so that **any** no-signalling empirical model admits a global extension, i.e. is local/non-contextual?

Vorob'ev (1962) derived a necessary and sufficient combinatorial condition on $\Sigma$ (or $\mathcal{U}$) for this to be the case. Turns out to be equivalent to the notion of acyclicity of a database schema.
Vorob'ev's theorem

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Vorob'ev (1962) derived a necessary and sufficient combinatorial condition on \( \Sigma \) (or \( \mathcal{U} \)) for this to be the case.

- Turns out to be equivalent to the notion of acyclicity of a database schema.
Graham reduction step: delete a measurement that belongs to only one maximal context.
Acyclicity

- Graham reduction step: delete a measurement that belongs to only one maximal context.
- A cover is **acyclic** when it is Graham reducible to $\emptyset$. 
Acyclicity

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![Diagram showing the process of Graham reduction.](image)
Graham reduction step: delete a measurement that belongs to only one maximal context.

A cover is **acyclic** when it is Graham reducible to $\emptyset$.

**Theorem (Vorob'ev 1962, adapted)**

*All empirical models on $\Sigma$ are extendable iff $\Sigma$ is acyclic*
Sketch of proof of Vorob'ev's theorem

- If $\Sigma$ is acyclic,

\[
\begin{align*}
& b & d & e \\
\Rightarrow & b & d & e \\
\Rightarrow & b & d \\
\Rightarrow & b & d & e \\
\Rightarrow & b & d & e \\
\Rightarrow & \emptyset
\end{align*}
\]
Sketch of proof of Vorob'ev’s theorem

- If $\Sigma$ is acyclic,

Given distributions $P_{ab}$ over $\{a, b\}$ and $P_{bc}$ over $\{b, c\}$ compatible on $b$,

$$\sum_{x \in O} P(a, b = x, y) = \sum_{z \in O} P(b, c = y, z),$$

we can define an extension:

$$P(a, b, c = x, y, z) = \frac{P(a, b = x, y)P(b, c = y, z)}{P(b = y)}.$$
Sketch of proof of Vorob'ev’s theorem

- If $\Sigma$ is acyclic,
  
  ![Diagram showing acyclic case]

- If $\Sigma$ is not acyclic (Graham reduction fails),
  
  ![Diagram showing non-acyclic case]
Sketch of proof of Vorob'ev's theorem

- If $\Sigma$ is acyclic,

- If $\Sigma$ is not acyclic (Graham reduction fails).

There is a “cycle”!
A structural explanation
Measurement scenario: simplicial complex $\Delta_2 \ast \Delta_2 \ast \Delta_2$. 
- Measurement scenario: simplicial complex $\mathcal{D}_2 \ast \mathcal{D}_2 \ast \mathcal{D}_2$. 
Measurement scenario: simplicial complex $\Delta_2 \ast \Delta_2 \ast \Delta_2$. 
Structural Reason

- Measurement scenario: simplicial complex $D_2 \ast D_2 \ast D_2$.
- We identify $B$ and $C$: $b_1 \sim c_1$, $b_2 \sim c_2$.
- The macro scenario arises as a quotient.
Measurement scenario: simplicial complex $\mathcal{D}_2 \ast \mathcal{D}_2 \ast \mathcal{D}_2$.

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We identify $B$ and $C$: $b_1 \sim c_1, b_2 \sim c_2$.

The macro scenario arises as a quotient.
This quotient complex is **acyclic**.

Therefore, no matter which micro model \( p_{a_i,b_j,c_k} \) we start from, the averaged macro correlations \( p_{a_i,m_j} \) are local.

In particular, they satisfy any Bell inequality. Hence, the original tripartite model also satisfies a **monogamy relation** for any Bell inequality.
General multipartite scenarios
General multipartite scenarios

- $n$ macroscopic sites $A, B, C, \ldots$
- $k_i$ measurement settings at site $i$
- take $r_i$ copies of each site $i$, or $r_i$ micro sites constituting $i$.
  For a macro site $A$:
  - copies / micro sites: $A^{(1)}, \ldots, A^{(r_1)}$
  - measurement settings art $A^{(m)}$: $a_1^{(m)}, \ldots, a_{k_A}^{(m)}$

Scenario: $\Sigma_{n,k,r} := \mathcal{D}^{(*r_1)}_{k_1} \ast \cdots \ast \mathcal{D}^{(*r_n)}_{k_n}$. 
General multipartite scenarios

- $n$ macroscopic sites $A, B, C, \ldots$
- $k_i$ measurement settings at site $i$
- take $r_i$ copies of each site $i$, or $r_i$ micro sites constituting $i$. For a macro site $A$:
  - copies / micro sites: $A^{(1)}, \ldots, A^{(r_1)}$
  - measurement settings are $A^{(m)}$: $a_1^{(m)}, \ldots, a_{k_A}^{(m)}$

Scenario: $\Sigma_{n, \tilde{k}, \tilde{r}} := \mathcal{D}_{k_1}^{(*r_1)} \times \cdots \times \mathcal{D}_{k_n}^{(*r_n)}$.

- Symmetry by $S_{r_1} \times \cdots \times S_{r_n}$ identifies the copies at each macro site.

\[ a_j^{(1)} \sim \cdots \sim a_j^{(r_A)} \quad (\forall j \in \{1, \ldots, k_A\}) \],
\[ b_j^{(1)} \sim \cdots \sim a_j^{(r_A)} \quad (\forall j \in \{1, \ldots, k_A\}) \],
etc.
General multipartite scenarios

Proposition

The quotient of the measurement scenario $\Sigma_{n,k,r}$ by the symmetry above is acyclic iff one of the following holds:

- each site has at least as many microscopic sites or copies as it has measurement settings, i.e. $\forall i \in \{1,\ldots,n\} \cdot k_i \leq r_i$;
- one of the sites has a single copy and the condition above is satisfied by all the other sites, i.e. $\exists i_0 \cdot \left( r_{i_0} = 1 \wedge \forall i \in \{1,\ldots,i_0,\ldots,n\} \cdot k_i \leq r_i \right)$. 
General multipartite scenarios

Proposition

The quotient of the measurement scenario $\Sigma_{n, k, \tilde{r}}$ by the symmetry above is acyclic iff one of the following holds:

- each site has at least as many microscopic sites or copies as it has measurement settings, i.e. $\forall i \in \{1, \ldots, n\} \cdot k_i \leq r_i;

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  $$\exists i_0 \cdot \left( r_{i_0} = 1 \land \forall i \in \{1, \ldots, \tilde{i_0}, \ldots, n\} \cdot k_i \leq r_i \right).$$

We get monogamy relations

$$\sum_{m_B=1}^{r_B} \sum_{m_C=1}^{r_C} \ldots B(A, B^{(m_B)}, c^{(m_C)}, \ldots) \leq r_B r_C \ldots R$$
Conclusions
A symmetry ($G$-action) on $\Sigma$ identifies measurements.

A model satisfies a $G$-monogamy relation for a Bell inequality iff the macro average correlations (quotient model by $G$) satisfy the Bell inequality.

So, if the quotient scenario is acyclic, then any no-signalling empirical model is $G$-monogamous wrt to all Bell inequalities (since the average correlations, being defined in this quotient scenario, must be local/non-contextual).
In particular, we proved that this is the case for multipartite Bell-type scenarios provided the number of parties being identified as belonging to each ’macro’ site is larger than the number of measurement settings available to each of them.

Our approach highlights the reason why monogamy relations for general multipartite Bell inequalities follow from no-signalling alone, generalising the result of Pawłowski and Brukner (2009) from bipartite to multipartite. It also shows that what Ramanathan et al. proved holds not only for QM but for any no-signalling theory.

The approach is not restricted to multipartite Bell-type scenarios. More generally, we can apply the same ideas to derive monogamy relations for contextuality inequalities.
Questions...

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