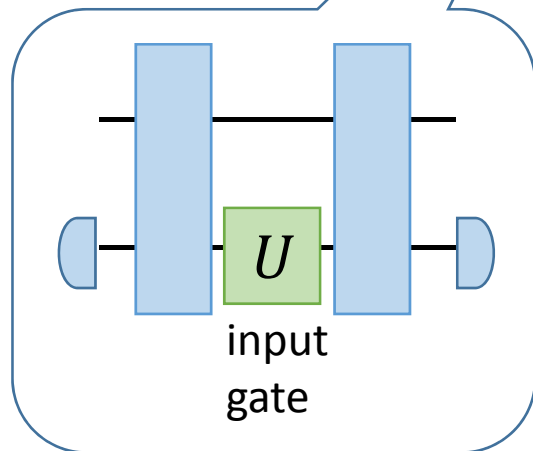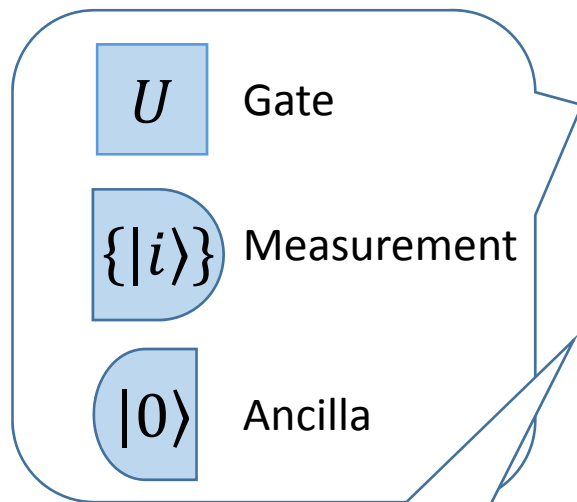# Circuit model implementation of controllization functional on unitary with and without fractional query

Akihito Soeda (University of Tokyo)

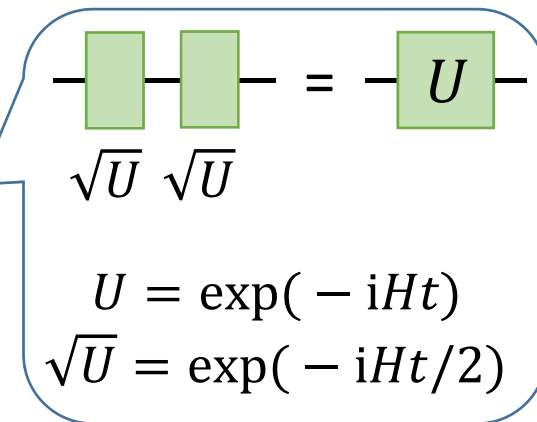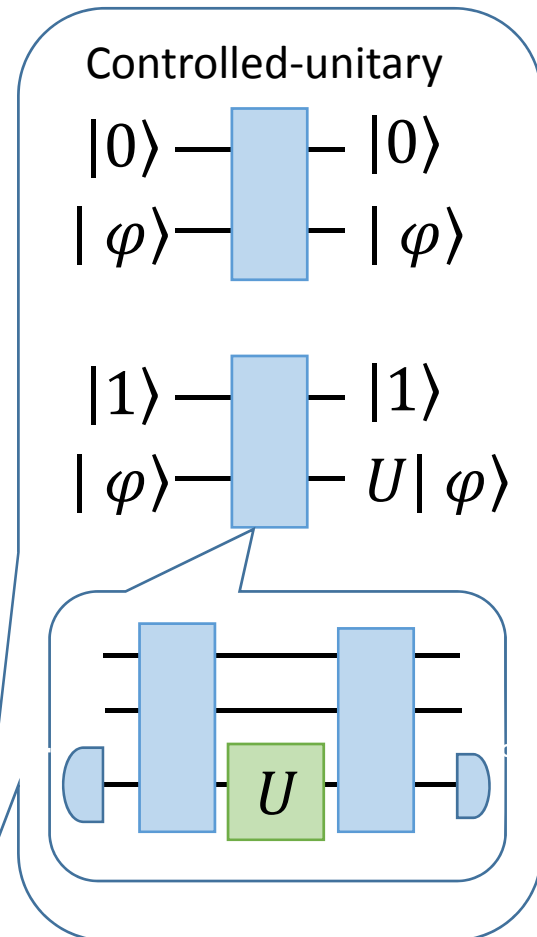Joint work with Shojun Nakayama and Mio Murao

QPL 2014

Introduction

Gate: U

Measurement: $\{|i\rangle\}$

Ancilla: $|0\rangle$

Circuit model

Functional

Controllization

Fractional query

Controlled-unitary

$|0\rangle$ — — $|0\rangle$
$|\varphi\rangle$ — — $|\varphi\rangle$

$|1\rangle$ — — $|1\rangle$
$|\varphi\rangle$ — — $U|\varphi\rangle$

input gate: U

$\sqrt{U}\ \sqrt{U} = U$

$U = \exp(-iHt)$
$\sqrt{U} = \exp(-iHt/2)$
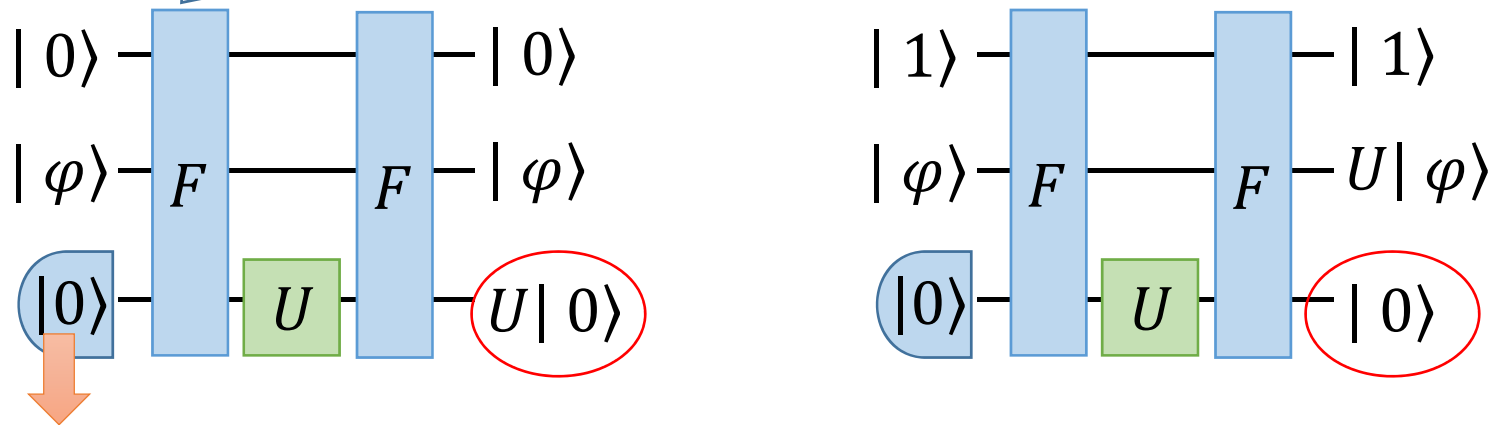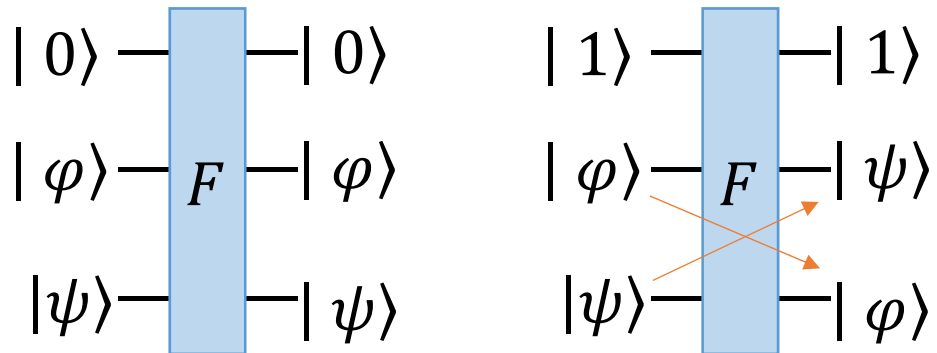
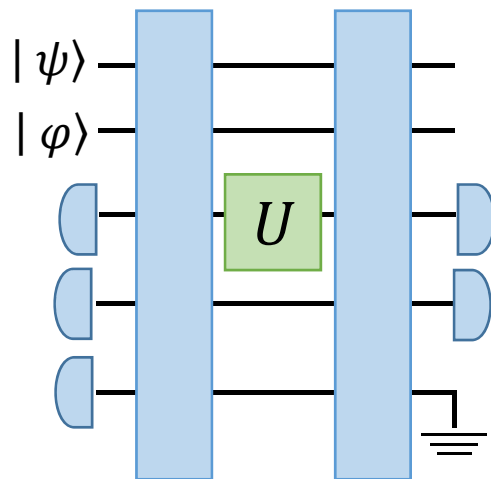# Failed attempt

Possible solution??:
Change to $|2\rangle$ such that $U|2\rangle = |2\rangle$ (Zhou et al. Nat. Comm. 2011)
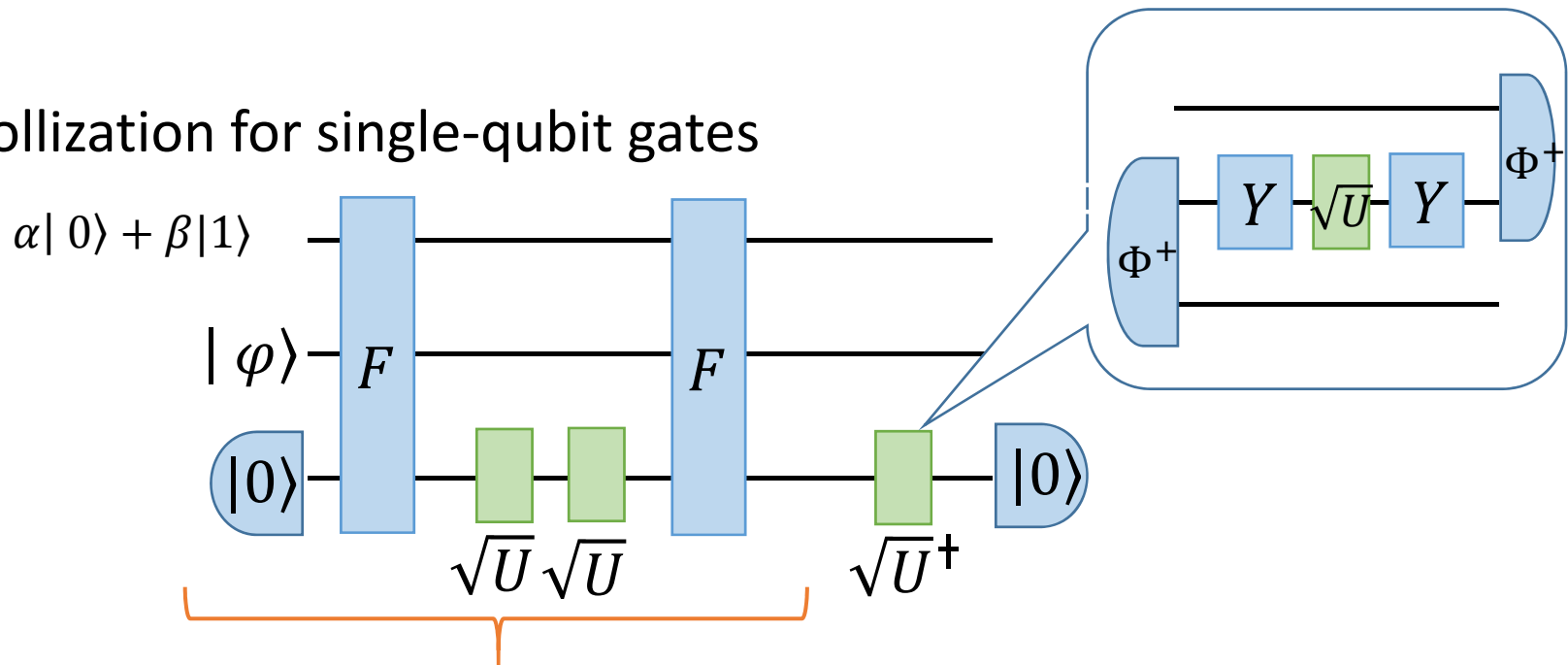
More failed attempts

$$|0\rangle\langle0| \otimes \mathbb{I} + |1\rangle\langle1| \otimes U$$

$$\exp(i\eta_U)$$

No-go for global-phase independent controllization by Araujo et al. (2013)

Controllization
with fractional query

Controllization for single-qubit gates

$$\alpha| 0\rangle|\varphi\rangle U| 0\rangle + \beta|1\rangle U|\varphi\rangle|0\rangle$$

$$\left(\alpha\langle 0|\sqrt{U}| 0\rangle\right)| 0\rangle|\varphi\rangle + \left(\beta\langle 0|\sqrt{U}^{\dagger}| 0\rangle\right)|1\rangle U|\varphi\rangle$$
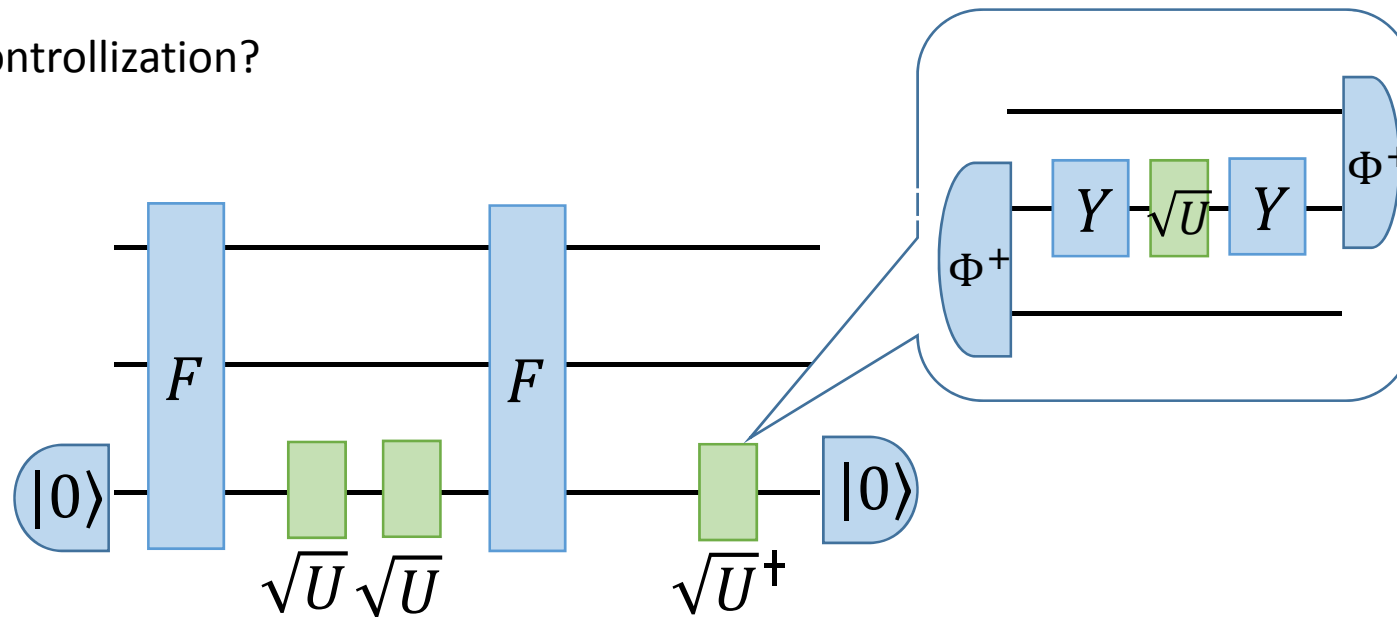
$$= \left|\langle 0|\sqrt{U}| 0\rangle\right|\{\alpha| 0\rangle|\varphi\rangle + \beta|1\rangle\exp(i\theta)U|\varphi\rangle\}$$

More success
with inverse fractional query
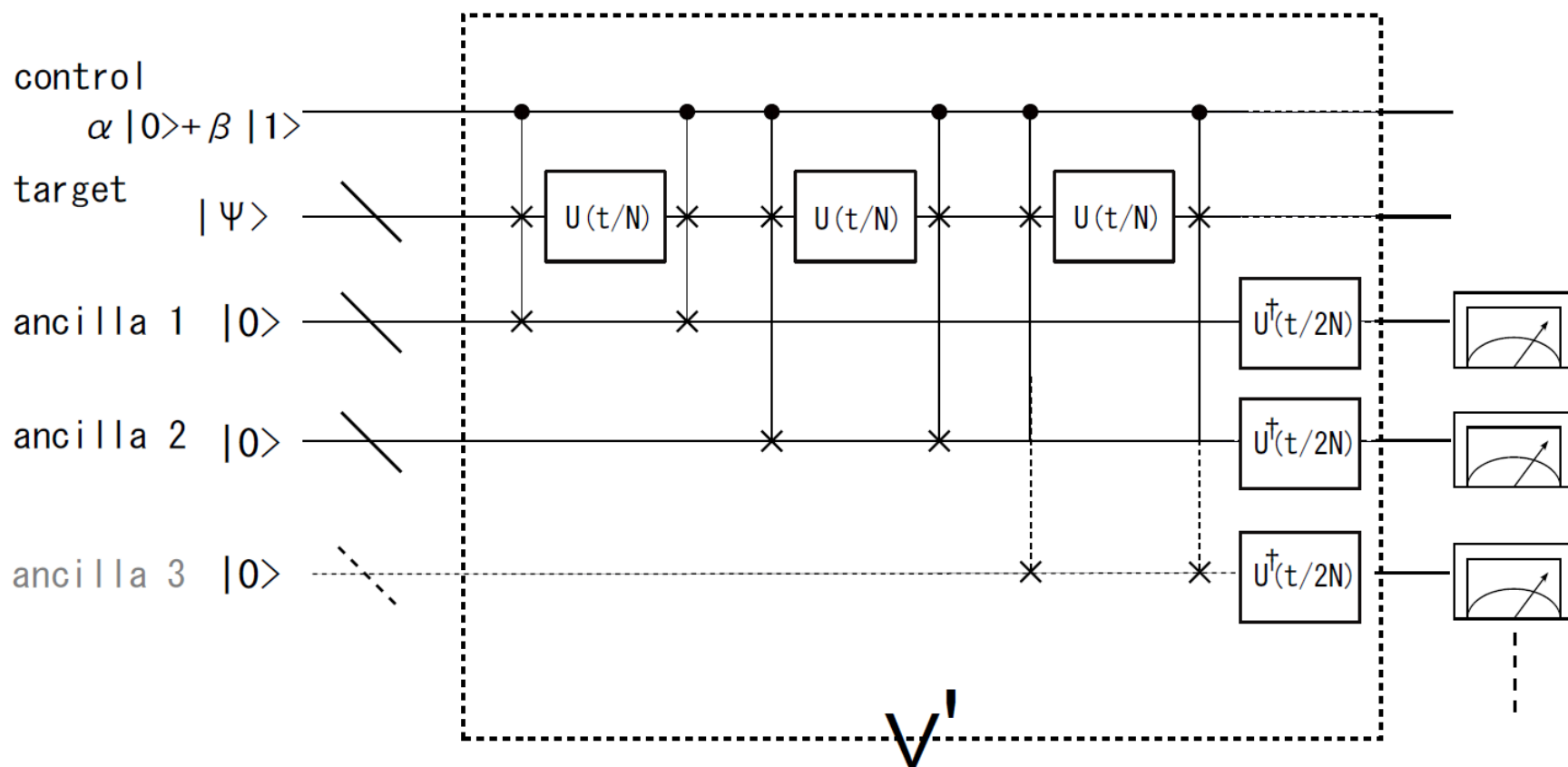
Back to controllization
without fractional query

Weak controllization?

- probabilistic (but nonzero for all input gates)
- requires multiple calls (but finite)

No-go without fractional query

$$= c_U \left(\begin{matrix} |0\rangle\langle 0| \otimes \mathbb{I} \\ +|1\rangle\langle 1| \otimes \exp(i\eta_U)\, U \end{matrix} \right)$$
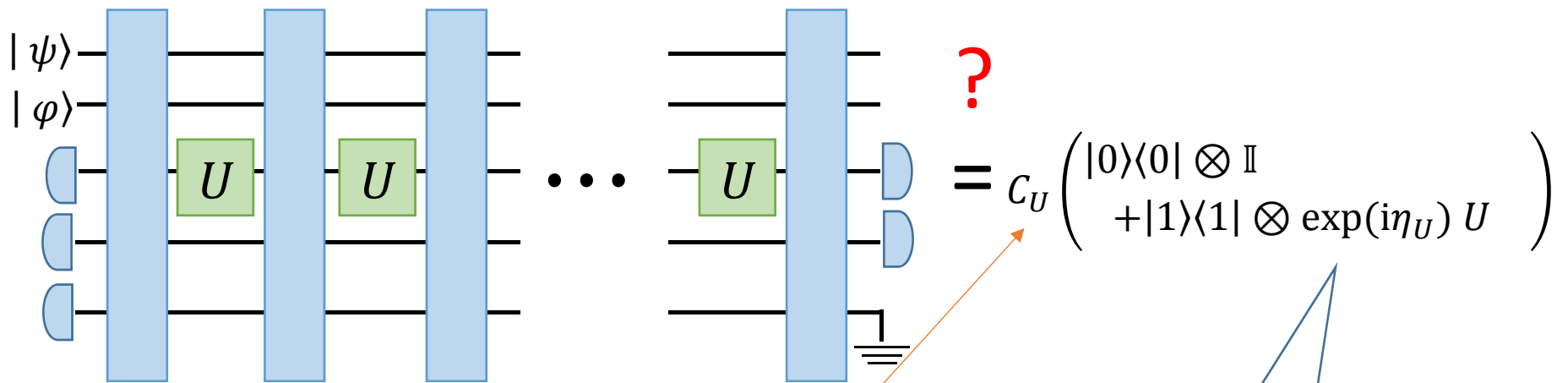
Success probability $= |c_U|^2$
(again, it must be finite for all U)

$$= \begin{bmatrix} c_U \mathbb{I} & \\ & c_U \exp(i\eta_U)\, U \end{bmatrix}$$

No-go when
$$U = \cos t\, I + \mathrm{i} \sin t\, (\sin \theta\, X + \cos \theta\, Z)$$

$$|\psi\rangle$$
$$|\varphi\rangle$$

U ⋯ U

N calls

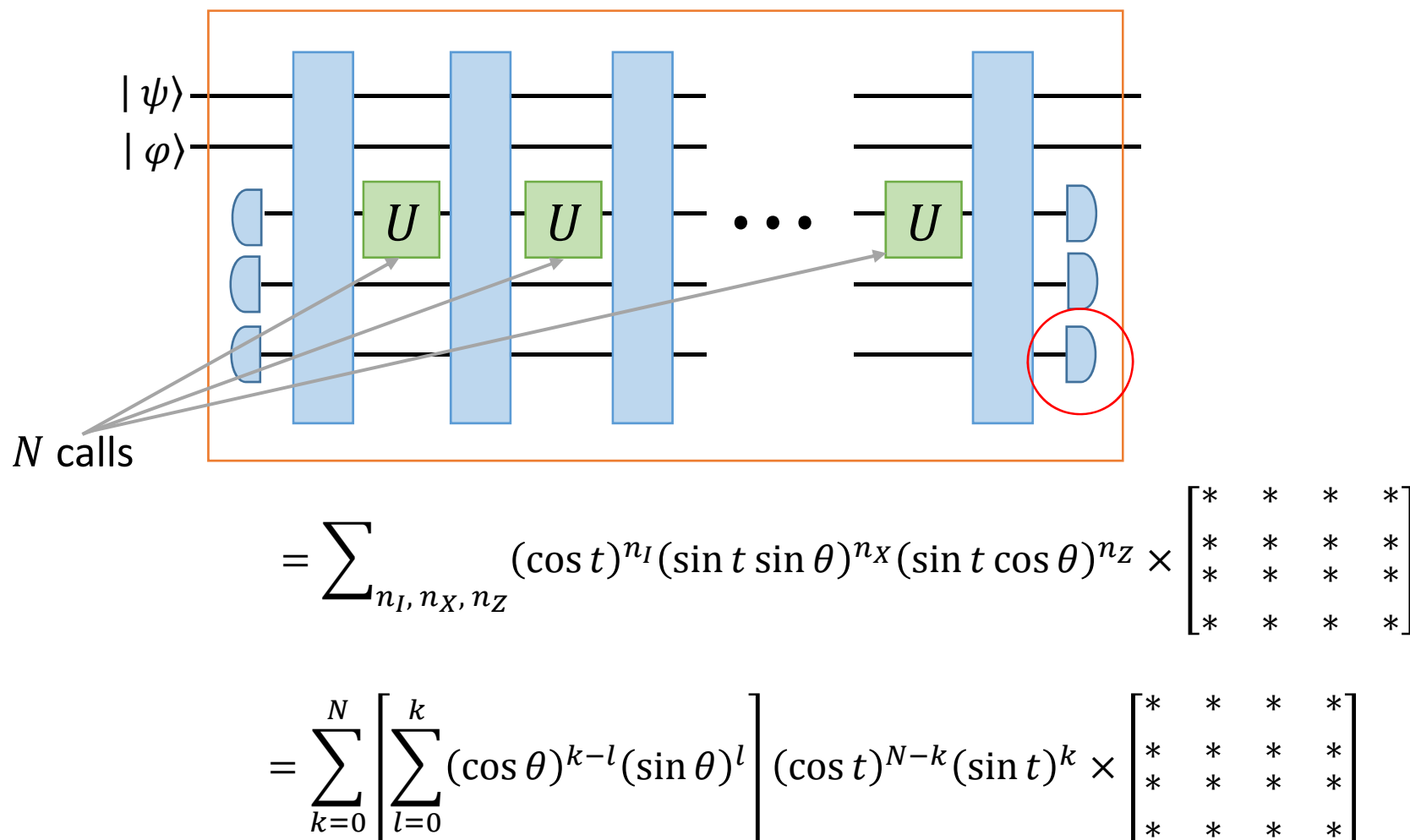$$= \sum_{n_I,\, n_X,\, n_Z} (\cos t)^{n_I} (\sin t \sin \theta)^{n_X} (\sin t \cos \theta)^{n_Z} \times \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

$$= \sum_{k=0}^{N} \left[\sum_{l=0}^{k} (\cos \theta)^{k-l} (\sin \theta)^{l} \right] (\cos t)^{N-k} (\sin t)^{k} \times \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

The contradiction

$$\sum_{k=0}^{N}\left[\sum_{l=0}^{k}(\cos\theta)^{k-l}(\sin\theta)^{l}\right](\cos t)^{N-k}(\sin t)^{k}\times\begin{bmatrix}*&*&*&*\\*&*&*&*\\*&*&*&*\\*&*&*&*\end{bmatrix}$$

$$\alpha_{kl}I+\beta_{kl}\mathrm{i}X+\delta_{kl}\mathrm{i}Y+\gamma_{kl}\mathrm{i}Z$$

?

$$=\begin{bmatrix}c_{U}\mathbb{I}\\&c_{U}\exp(\mathrm{i}\eta_{U})\,U\end{bmatrix}$$

$$c_{U}\exp(\mathrm{i}\eta_{U})\cos t\,I$$
$$+c_{U}\exp(\mathrm{i}\eta_{U})\sin t\sin\theta\,\mathrm{i}X$$
$$+c_{U}\exp(\mathrm{i}\eta_{U})\sin t\cos\theta\,\mathrm{i}Z$$

$$\beta_{N0}=0$$
$$c_{\mathrm{i}X}=\beta_{N0}$$

$$c_{\mathrm{i}X}=0$$

Conclusion

➢ Controllization of gates with and without fractional queries

- With fractional query, weak form of controllization (i.e., probabilistic and multiple calls) is possible.
  - If inverse fractional query is allowed, the success probability can be made arbitrarily high.

- Without fractional query, even the weak controllization is impossible.