The ZX-calculus is incomplete for quantum mechanics

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ZX-calculus

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- Diagramatic logical calculus for studying quantum information processing

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- Diagramatic logical calculus for studying quantum information processing
- Can be used as an alternative to traditional Hilbert space formalism
- Has been used to study:
  - Quantum algorithms
  - Quantum security protocols
  - Quantum error-correcting codes
  - and other problems involving quantum information
Atomic Diagrams (1)

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
= (1 0 0 0) = I
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
= \sigma
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
= \sigma
\]

\[
= \langle 00 | + \langle 11 |
\]

\[
= |00\rangle + |11\rangle
\]

\[
= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H
\]
The ZX-calculus is incomplete for quantum mechanics.

Atomic Diagrams (2)

\[
\begin{align*}
|0^m\rangle &\mapsto |0^n\rangle \\
|1^m\rangle &\mapsto e^{i\alpha} |1^n\rangle \\
\text{rest} &\mapsto 0
\end{align*}
\]

where \(\alpha \in [0, 2\pi)\)

\[
\begin{align*}
|+^m\rangle &\mapsto |+^n\rangle \\
|−^m\rangle &\mapsto e^{i\alpha} |−^n\rangle \\
\text{rest} &\mapsto 0
\end{align*}
\]
Compound Diagrams

\[
\begin{array}{c}
\psi_1 \\
\hline
\psi_2
\end{array} = D_1 \quad \text{and} \quad
\begin{array}{c}
\psi_1 \\
\hline
\psi_2
\end{array} = D_2
\]

then

\[
\begin{array}{c}
\psi_1 \\
\hline
\psi_2
\end{array} = D_1 \otimes D_2
\]

and

\[
\begin{array}{c}
\psi_1 \\
\hline
\psi_2
\end{array} = D_1 \circ D_2
\]
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"Only the topology matters"
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Axioms (5)

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Axioms (6)

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The ZX-calculus is incomplete for quantum mechanics.
Soundness, Completeness and Universality results

- The ZX-calculus is sound
Soundness, Completeness and Universality results

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  - $\vdash D_1 = D_2 \implies \mathcal{J}[D_1] = e^{i\phi}\mathcal{J}[D_2]$
Soundness, Completeness and Universality results

- The ZX-calculus is sound
  - $\forall U: \mathbb{Q}^n \rightarrow \mathbb{Q}^m, \exists D. \ J_D K = U$

- The ZX-calculus is universal

The ZX-calculus is incomplete for quantum mechanics
Soundness, Completeness and Universality results

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- Completeness of ZX (with the Euler decomposition of the Hadamard gate rule) was unknown.
  - \( \llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket \implies ZX \vdash D_1 = D_2 \)

- The ZX-calculus is complete for stabilizer quantum mechanics

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Soundness, Completeness and Universality results

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  \[ [D_1] = [D_2] \implies \text{ZX} \vdash D_1 = D_2 \]

- The ZX-calculus is complete for stabilizer quantum mechanics
  If \( D_1 \) and \( D_2 \) are ZX-SQM diagrams and \( [D_1] = [D_2] \) then \( \text{ZX} \vdash D_1 = D_2 \)
Recall that for single-qubit unitary maps:

\[
\begin{bmatrix}
D
\end{bmatrix} = e^{i\phi_1} \begin{bmatrix}
\alpha_1 \\
\beta_1 \\
\gamma_1
\end{bmatrix} = e^{i\phi_2} \begin{bmatrix}
\alpha_2 \\
\beta_2 \\
\gamma_2
\end{bmatrix}
\]

where \(\alpha_i, \beta_i, \gamma_i, \phi_i \in [0, 2\pi)\)
Alternative Models

Consider the following models:

\[
\begin{align*}
\begin{bmatrix}
\cdots \\
\alpha \\
\cdots \\
\end{bmatrix}_k & := 
\begin{bmatrix}
\cdots \\
\kappa \alpha \\
\cdots \\
\end{bmatrix} \\
\begin{bmatrix}
\cdots \\
\alpha \\
\cdots \\
\end{bmatrix}_k & := 
\begin{bmatrix}
\cdots \\
\kappa \alpha \\
\cdots \\
\end{bmatrix}, \quad \text{otherwise}
\end{align*}
\]

These models are sound when \( k = 4p + 1 \) for \( p \in \mathbb{Z} \).
The ZX-calculus is incomplete for quantum mechanics.

Counter-example diagrams:

\[ D_1 := \begin{align*} & \frac{\pi}{3} \\ & \frac{\pi}{3} \\ & \frac{2\pi}{3} \end{align*} \text{ and } D_2 := \begin{align*} & \pi \\ & \phi \\ & \beta \end{align*} \]
where

\[ \alpha := - \arccos \left( \frac{5}{2\sqrt{13}} \right) \approx 0.2561 \pi \]

\[ \beta := -2 \arcsin \left( \frac{\sqrt{3}}{4} \right) \approx -0.2851 \pi \]

\[ \phi := \arcsin \left( \frac{\sqrt{3}}{4} \right) - \alpha \approx 0.3987 \pi \]
Incomplete

We have:

\[
[D_1] = [D_2]
\]

but for any \( \lambda \in \mathbb{C} \):

\[
[D_1]_3 \neq \lambda [D_2]_3
\]

However, \([\cdot]_3\) is a sound model of ZX, so

\[
ZX \not\vdash D_1 = D_2
\]

and therefore the ZX-calculus is incomplete for quantum mechanics.
ZX is incomplete, what next?

- Nothing special about counter-example, others easily doable using the same approach
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- Color-swap rule might be needed for completeness
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\[
\begin{align*}
\alpha_2 &:= f_1(\alpha_1, \beta_1, \gamma_1) \\
\beta_2 &:= f_2(\alpha_1, \beta_1, \gamma_1) \\
\gamma_2 &:= f_3(\alpha_1, \beta_1, \gamma_1)
\end{align*}
\]

where

\[
\begin{align*}
f_1 &= ? \\
f_2 &= ? \\
f_3 &= ?
\end{align*}
\]
Even then, more challenges

\[ \alpha_1 \alpha_2 \alpha_1 \alpha_2 \beta_1 \beta_2 \beta_1 \beta_2 \gamma_1 \gamma_2 \gamma_1 \gamma_2 = b_1 b_2 b_1 b_2 c_1 c_2 c_1 c_2 \]

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Approximate completeness instead?

- Restrict generators, drop universality and work towards approximate completeness
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  - completeness holds
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- If diagram angles are of the form $\frac{k\pi}{2}$ (Stabilizer QM):
  - Completeness holds.
  - Calculus is not even approximately universal.
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- If diagram angles are of the form $\frac{k\pi}{2}$ (Stabilizer QM):
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- If diagram angles are of the form $\frac{k\pi}{4}$ (Clifford+$\mathbf{T}$):
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  - calculus is complete for line-graphs (next talk)
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  - completeness is unknown