The ZX-calculus is incomplete for quantum mechanics

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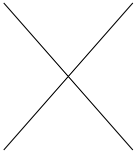
- Can be used as an alternative to traditional Hilbert space formalism
- Has been used to study:
  - Quantum algorithms
  - Quantum security protocols
  - Quantum error-correcting codes
  - and other problems involving quantum information

Atomic Diagrams (1)

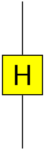
$$[\![\ \cdot\ ]\!] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$



$$[\![\ \cdot\ ]\!] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \sigma$$

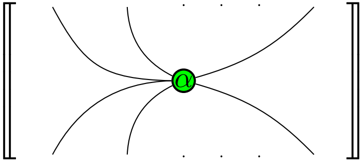
$$[\![\ \cdot\ ]\!] = \langle 00| + \langle 11|$$

$$[\![\ \cdot\ ]\!] = |00\rangle + |11\rangle$$

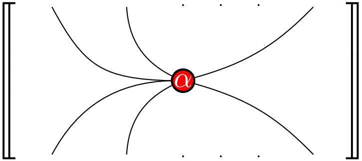


$$[\![\ \cdot\ ]\!] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H$$

Atomic Diagrams (2)



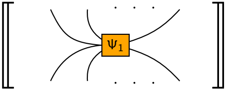
$$= \begin{cases} |0^m\rangle & \mapsto |0^n\rangle \\ |1^m\rangle & \mapsto e^{i\alpha}|1^n\rangle \\ \text{rest} & \mapsto 0 \end{cases}$$



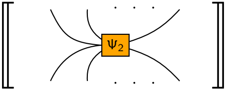
$$= \begin{cases} |+^m\rangle & \mapsto |+^n\rangle \\ |-^m\rangle & \mapsto e^{i\alpha}|-^n\rangle \\ \text{rest} & \mapsto 0 \end{cases}$$

where $\alpha \in [0, 2\pi)$

Compound Diagrams

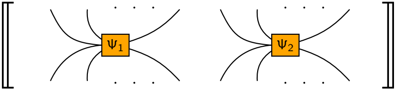


$= D_1$ and



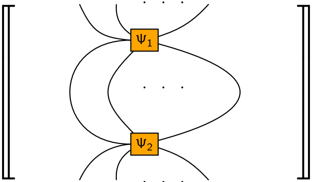
$= D_2$

then



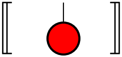
$= D_1 \otimes D_2$

and

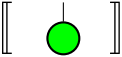


$= D_1 \circ D_2$

Examples


$= \sqrt{2}\,|0\rangle$

$= \sqrt{2}\,|1\rangle$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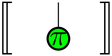
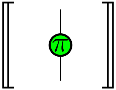
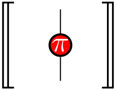


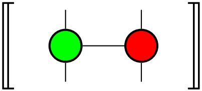
$= \sqrt{2}\,|-\rangle$



$= \mathrm{Z}$



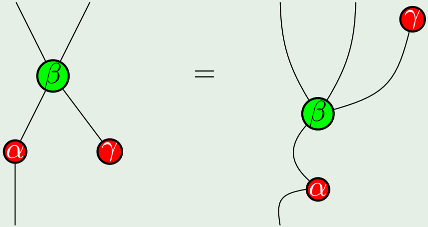
$= \mathrm{X}$


$= \mathrm{CNOT}$

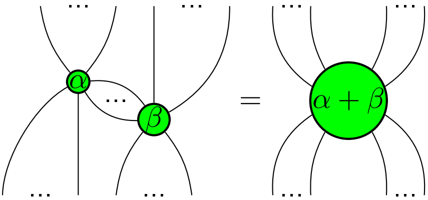
Axioms (1)

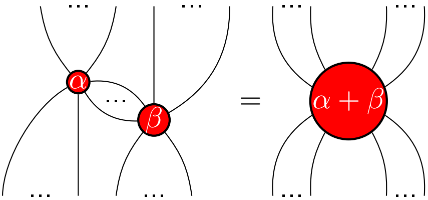
"Only the topology matters"

Example

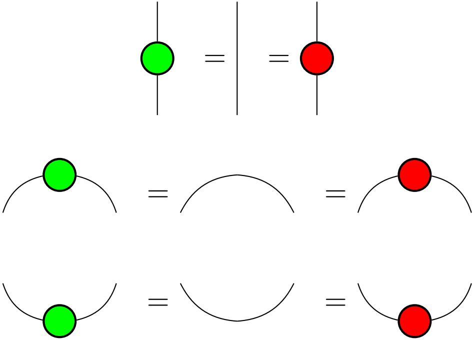


Axioms (2)

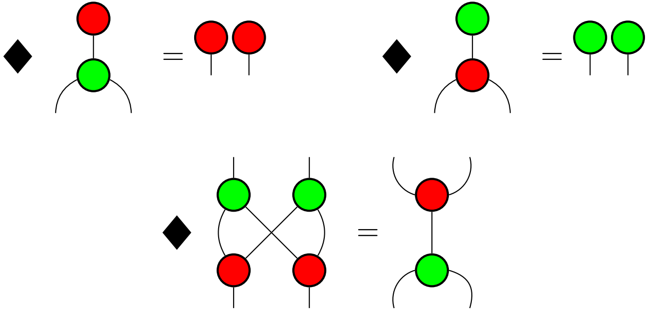




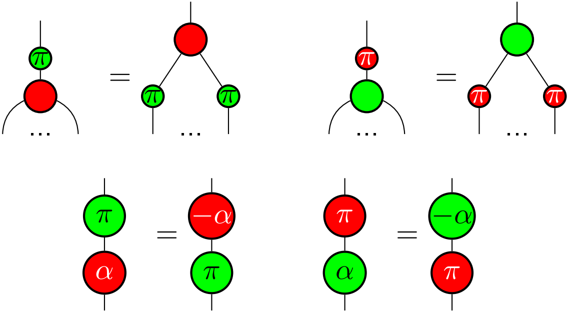
Axioms (3)



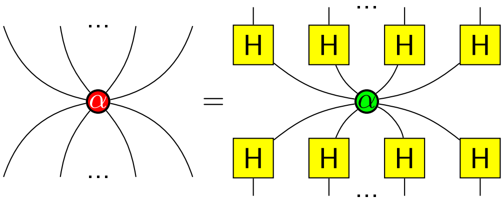
Axioms (4)

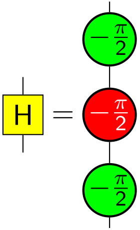


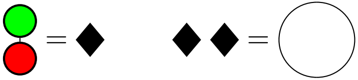
Axioms (5)



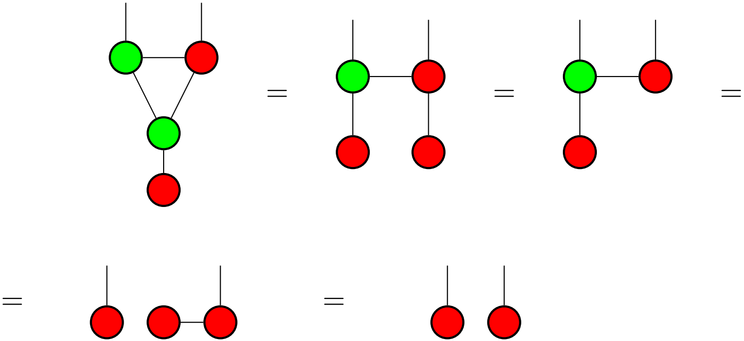
Axioms(6)







Example derivation



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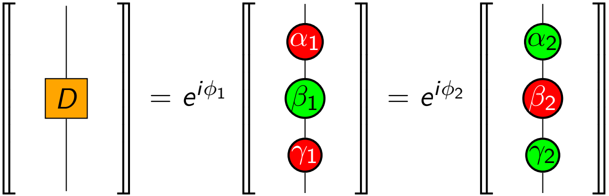
- The ZX-calculus is complete for stabilizer quantum mechanics
  - If D_1 and D_2 are ZX-SQM diagrams and $[\![D_1]\!] = [\![D_2]\!]$ then $ZX \vdash D_1 = D_2$

Euler Decomposition

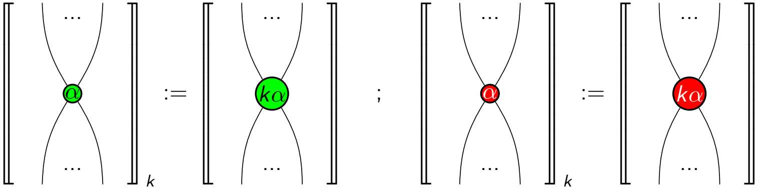
Recall that for single-qubit unitary maps:



where $\alpha_i, \beta_i, \gamma_i, \phi_i \in [0, 2\pi)$

Alternative Models

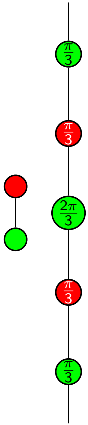
Consider the following models:

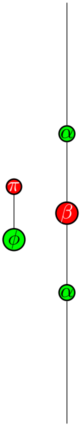


$$[\![\cdot]\!]_k := [\![\cdot]\!] \text{ , otherwise}$$

These models are sound when $k = 4p + 1$ for $p \in \mathbb{Z}$.

Counter-example diagrams

$D_1 :=$ 

and $D_2 :=$ 

Counter-example diagrams (cont.)

where

$$\alpha := -\arccos\left(\frac{5}{2\sqrt{13}}\right) \approx 0.2561\pi$$

$$\beta := -2\arcsin\left(\frac{\sqrt{3}}{4}\right) \approx -0.2851\pi$$

$$\phi := \arcsin\left(\frac{\sqrt{3}}{4}\right) - \alpha \approx 0.3987\pi$$

Incomplete

We have:

$$[\![D_1]\!] = [\![D_2]\!]$$

but for any $\lambda \in \mathbb{C}$:

$$[\![D_1]\!]_{-3} \neq \lambda [\![D_2]\!]_{-3}$$

However, $[\![\cdot]\!]_{-3}$ is a sound model of ZX, so

$$ZX \nvdash D1 = D2$$

and therefore the ZX-calculus is incomplete for quantum mechanics.

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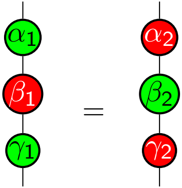
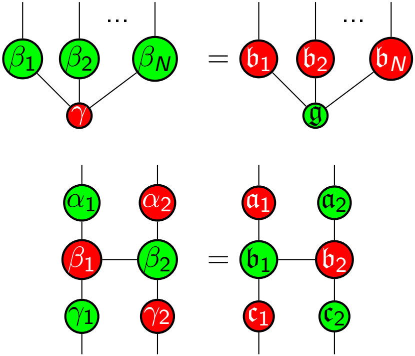


where

$$\alpha_2 := f_1(\alpha_1, \beta_1, \gamma_1) \quad ; \quad f_1 = ?$$
$$\beta_2 := f_2(\alpha_1, \beta_1, \gamma_1) \quad ; \quad f_2 = ?$$
$$\gamma_2 := f_3(\alpha_1, \beta_1, \gamma_1) \quad ; \quad f_3 = ?$$

Even then, more challenges



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