Weighted Rewriting

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(ongoing joint work with **Martin Avanzini**)

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Background

- We now know probabilistic systems, e.g.:

$$\mathrm{f}(\mathrm{s}(x)) \hookrightarrow \left\{\frac{1}{2} : \mathrm{f}(x),\ \frac{1}{2} : \mathrm{f}\Big(\mathrm{s}\big(\mathrm{s}(x)\big)\Big)\right\}$$

- but no good correspondence to rewriting
  - can be seen as an ARS, but over **sub-multi-distributions**

    e.g.)

$$\{1 : \mathrm{f}(1)\} \hookrightarrow^{\mathrm{M}} \left\{\frac{1}{2} : \mathrm{f}(0),\ \frac{1}{2} : \mathrm{f}(2)\right\}$$
$$\hookrightarrow^{\mathrm{M}} \left\{\frac{1}{2} : \mathrm{f}(0),\ \frac{1}{4} : \mathrm{f}(1), \frac{1}{4} : \mathrm{f}(3)\right\}$$
$$\hookrightarrow^{\mathrm{M}} \left\{\frac{1}{2} : \mathrm{f}(0),\ \frac{1}{8} : \mathrm{f}(0), \frac{1}{8} : \mathrm{f}(2), \frac{1}{8} : \mathrm{f}(2), \frac{1}{8} : \mathrm{f}(4)\right\}$$

  - for **termination**:

    only ranking functions (interpretations, supermartingale)
  - for **confluence**: hard to formulate

Outline

- Weighted Abstract Reduction Systems
- Instances
- Termination-like properties
- Bound analysis

Weighted Abstract Reduction System

- **wARS:** $\rightsquigarrow \subseteq \mathbb{R}_{\geq 0} \times A \times A$
  - $\rightsquigarrow^{[w]} := \{\langle a, b \rangle \mid \langle w, a, b \rangle \in \rightsquigarrow\}$
- **weighted order:** a wARS $\succ$ which is
  - **reflexive**: $a \succ^{[0]} a$
  - **transitive**: $a \succ^{[w]} b \succ^{[v]} c \implies a \succ^{[w+v]} c$
- $\widehat{\rightsquigarrow}$: the least weighted order extending $\rightsquigarrow$
  - $\rightsquigarrow^{w} := \widehat{\rightsquigarrow}^{[w]}$
  - $\rightsquigarrow^{+} := \bigcup_{w>0} \rightsquigarrow^{w}$
  - $\rightsquigarrow^{*} := \bigcup_{w\geq 0} \rightsquigarrow^{w}$
  - $\mathrm{NF}_{\rightsquigarrow} := \{a \mid \nexists b.\ a \rightsquigarrow^{+} b\}$
  - **confluence**: $a \leftsquigarrow^{*} \circ \rightsquigarrow^{*} b \implies a \rightsquigarrow^{*} \circ \leftsquigarrow^{*} b$

ARS

- ARS: $\mapsto \subseteq A \times A$
- **uniformly weighted** ARS
  $$\check{\mapsto} := \{1\} \times \mapsto = \{\langle 1, a, b\rangle \mid a \mapsto b\}$$
- **Remarks**:
  - $\check{\mapsto}^n = \mapsto^n$
  - $\check{\mapsto}^+ = \mapsto^+$
  - $\check{\mapsto}^* = \mapsto^*$
  - $\mathrm{NF}_{\check{\mapsto}} = \mathrm{NF}_{\mapsto}$

Relative ARS

- for two ARSs $\mapsto, \dashrightarrow \subseteq A \times A$,
- **relative ARS** $(\mapsto / \dashrightarrow) := (\dashrightarrow^* \circ \mapsto \circ \dashrightarrow^*)$
- let wARS: $\overline{\mapsto / \dashrightarrow} := (\{1\} \times \mapsto) \cup (\{0\} \times \dashrightarrow)$
- **Remarks**:
  - $\overline{\mapsto / \dashrightarrow}^{\,n} = (\mapsto / \dashrightarrow)^n$ for $n > 0$
  - $\overline{\mapsto / \dashrightarrow}^{\,0} = \dashrightarrow^*$
  - $\mathrm{NF}_{\overline{\mapsto / \dashrightarrow}} = \mathrm{NF}_{\mapsto / \dashrightarrow}$

Weighted Term Rewriting

- **wTRS** $\mathcal{R}$: wARS over $T(F,V)$

$$0 + x \; \mathcal{R}^{[1]} \; x$$
$$x + s(y) \; \mathcal{R}^{[2]} \; s(x+y)$$
$$x + y \; \mathcal{R}^{[0]} \; y + x$$

- $\mathcal{R}$ is **closed under contexts** and **substitutions** if
  - $s \; \mathcal{R}^{[w]} \; t \Longrightarrow f(\dots, s, \dots) \; \mathcal{R}^{[w]} \; f(\dots, t, \dots)$ for $f \in F$
  - $s \; \mathcal{R}^{[w]} \; t \;\; \Longrightarrow \;\; s\sigma \; \mathcal{R}^{[w]} \; t\sigma$
- $\underset{\mathcal{R}}{\rightarrow}$: least weighted order closed under contexts & substs (**weighted rewrite order**) extending $\mathcal{R}$

$$x + s(0) \underset{\mathcal{R}}{\rightarrow}^{2} s(x+0) \underset{\mathcal{R}}{\rightarrow}^{0} s(0+x) \underset{\mathcal{R}}{\rightarrow}^{1} s(x)$$

Distribution reduction system

- **dARS** $\rightarrowtail$: wARS over distributions

  e.g.) $\left\{\frac{1}{2}\,\mathrm{HC}l, \frac{1}{2}\,\mathrm{NaOH}\right\} \rightarrowtail^{[56.5]} \left\{\frac{1}{2}\,\mathrm{NaC}l, \frac{1}{2}\,\mathrm{H_2O}\right\}$

- $\rightarrowtail$ is **closed under convex sum** (CUSC): iff

  for $\sum_i p_i = 1$,

  $\forall i.\, \mu_i \;\rightarrowtail^{[w_i]}\; \nu_i \Longrightarrow \left(\sum_i p_i \cdot \mu_i\right) \rightarrowtail^{[\sum_i p_i w_i]} \left(\sum_i p_i \cdot \nu_i\right)$

- $\widehat{\rightarrowtail}$: least CUCS weighted order, extending $\rightarrowtail$

  $\{0.2\ \mathrm{HC}l, 0.8\ \mathrm{NaOH}\} \;\rightarrowtail^{11.3}$

  $\{0.1\ \mathrm{NaC}l, 0.1\ \mathrm{H_2O}, 0.1\ \mathrm{HC}l, 0.7\ \mathrm{NaOH}\} \;\rightarrowtail^{11.3}$

  $\{0.2\ \mathrm{NaC}l, 0.2\ \mathrm{H_2O}, 0.6\ \mathrm{NaOH}\} \quad \in \mathrm{NF}_{\rightarrowtail}$

  … WN but not SN

Probabilistic ARS

- **pARS**: $\hookrightarrow \subseteq A \times \mathrm{Dist}(A)$

  e.g.) $\mathrm{f}(\mathrm{s}(x)) \hookrightarrow \left\{\frac{1}{2}: \mathrm{f}(x),\ \frac{1}{2}: \mathrm{f}\left(\mathrm{s}(\mathrm{s}(x))\right)\right\}$
- see as wARS $\dot{\hookrightarrow}$ over $\mathrm{MDist}(A)$:

  $\{1: a\} \dot{\hookrightarrow}^{[1]} \mu \iff a \hookrightarrow \mu$

  In MDist, $\{0.5: a, 0.5: a\} \neq \{1: a\}$
- wARS $\rightsquigarrow$ over $\mathrm{MDist}(A)$ is **closed under convex mset sum** (CUCMS) if $\forall i.\, \mu_i \rightsquigarrow^{[w_i]} \nu_i \implies$

  $(\biguplus_i p_i \cdot \mu_i) \rightsquigarrow^{[\sum_i p_i w_i]} (\biguplus_i p_i \cdot \nu_i)$ for $\sum_i p_i = 1$
- $\widehat{\dot{\hookrightarrow}}$: least CUCMS weighted order, extending $\dot{\hookrightarrow}$

  $\{1: \mathrm{f}(1)\} \dot{\hookrightarrow}^{1} \left\{\frac{1}{2}: \mathrm{f}(0),\ \frac{1}{2}: \mathrm{f}(2)\right\}$

  $\dot{\hookrightarrow}^{1/2} \left\{\frac{1}{2}: \mathrm{f}(0),\ \frac{1}{4}: \mathrm{f}(1), \frac{1}{4}: \mathrm{f}(3)\right\}$

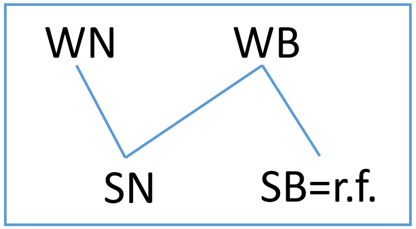
  $\dot{\hookrightarrow}^{1/2} \left\{\frac{1}{2}: \mathrm{f}(0),\ \frac{1}{8}: \mathrm{f}(0), \frac{1}{8}: \mathrm{f}(2), \frac{1}{8}: \mathrm{f}(2), \frac{1}{8}: \mathrm{f}(4)\right\} \hookrightarrow^{3/8} \ldots$

Outline

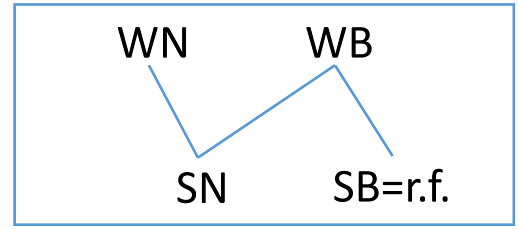
- Weighted Abstract Reduction Systems
- Instances
- **Termination-like properties**
- Bound analysis

Termination-like properties for wARS

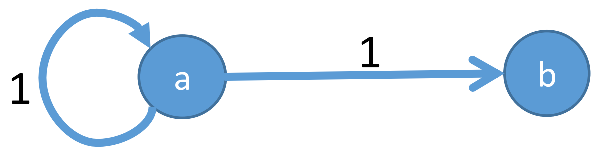
- wARS $\rightsquigarrow$ is
  - **normalizing(??)** on $S \subseteq A$ if $\qquad \mathrm{WN}_{\rightsquigarrow}(S)$
    $\forall a \in S.\ \exists b \in \mathrm{NF}_{\rightsquigarrow}.\ a \rightsquigarrow^{*} b$
  - **terminating** on $S \subseteq A$ if $\qquad \mathrm{SN}_{\rightsquigarrow}(S)$
    There is no infinite seq. $S \ni a_0 \rightsquigarrow^{[w_0]} a_1 \rightsquigarrow^{[w_1]} \dots$
  - **weakly bounded** on $S \subseteq A$ if $\qquad \mathrm{WB}_{\rightsquigarrow}(S)$
    $S \ni a_0 \rightsquigarrow^{[w_0]} a_1 \rightsquigarrow^{[w_1]} \dots \Longrightarrow \exists v \in \mathbb{R}_{\geq 0}.\ \sum_{i \in \mathbb{N}} w_i \leq v$
  - **strongly bounded** on $S \subseteq A$ if $\qquad \mathrm{SB}_{\rightsquigarrow}(S)$
    $\forall a_0 \in S.\ \exists v \in \mathbb{R}_{\geq 0}.\ a_0 \rightsquigarrow^{[w_0]} a_1 \rightsquigarrow^{[w_1]} \dots \Longrightarrow \sum_{i \in \mathbb{N}} w_i \leq v$
- **Remark**:
  - ranking functions (interpretation method) on $\mathbb{R}_{\geq 0}$ are sound & complete for SB



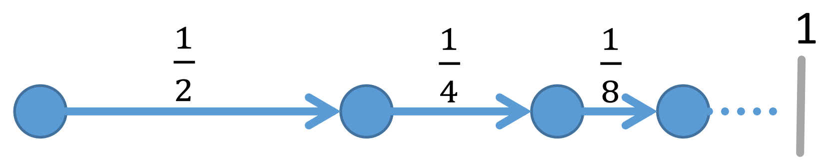
Counterexamples



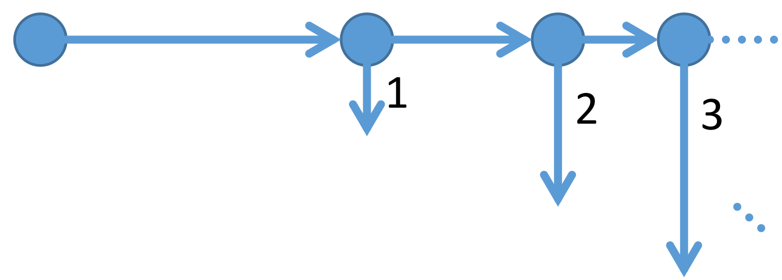
- WN $\not\Rightarrow$ SN



- SB $\not\Rightarrow$ WN

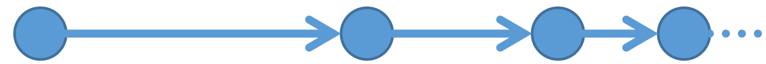


- WB $\not\Rightarrow$ SB



generalizes a counterexample in [Avanzini+, FLOPS 2018]
r.f. are incomplete for "positive almost sure termination"

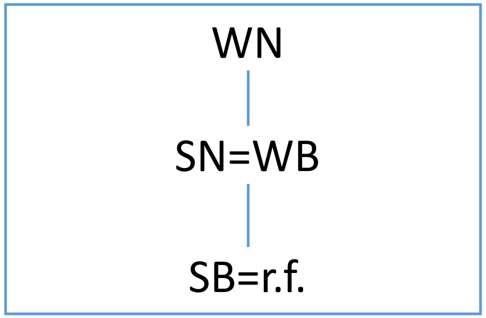
Non-Zeno



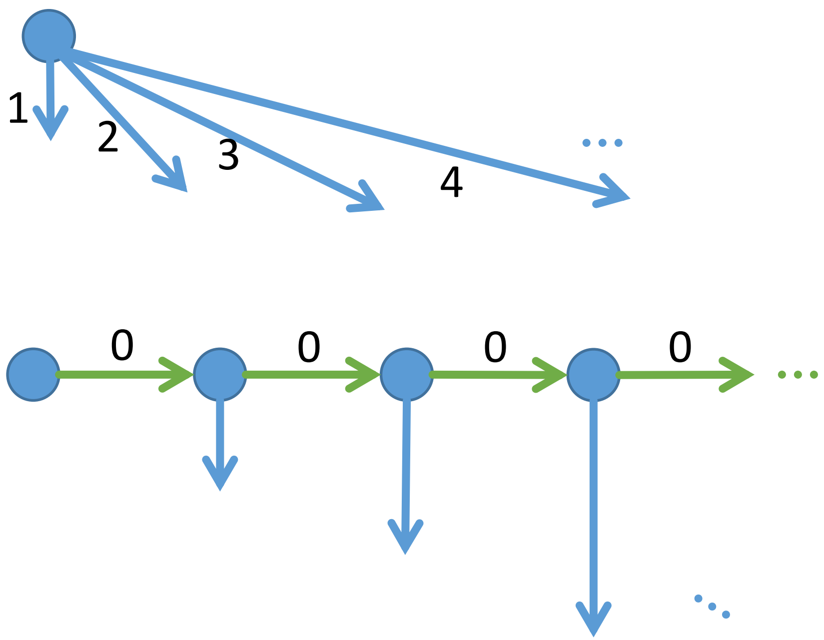
- sequence $a_0 \rightsquigarrow^{[w_0]} a_1 \rightsquigarrow^{[w_1]} \cdots$ is **Zeno** if
  - $\sum_{i\in\mathbb{N}} w_i < \infty$
  - but $\sum_{i=0\dots n} w_i < \sum_{i\in\mathbb{N}} w_i$ for any n
    - i.e., $\nexists n.\, w_n = w_{n+1} = \cdots = 0$
- wARS $\rightsquigarrow$ is **non-Zeno** if it admits no Zeno sequence
- **Proposition**: If $\rightsquigarrow$ is non-Zeno, then

$$\mathrm{WB}_{\rightsquigarrow} \iff \mathrm{SN}_{\rightsquigarrow}$$

- **Remark**:
  - ranking functions are **sound** for SN
  - ARSs, relative ARSs are non-Zeno
  - but pARS/dARS are not

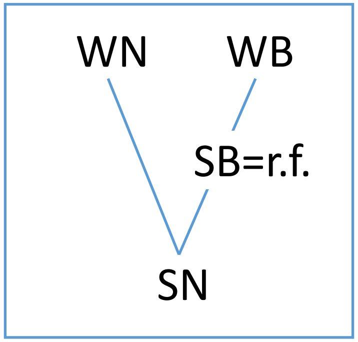


Non-Zeno ∧ WB ⇏ SB

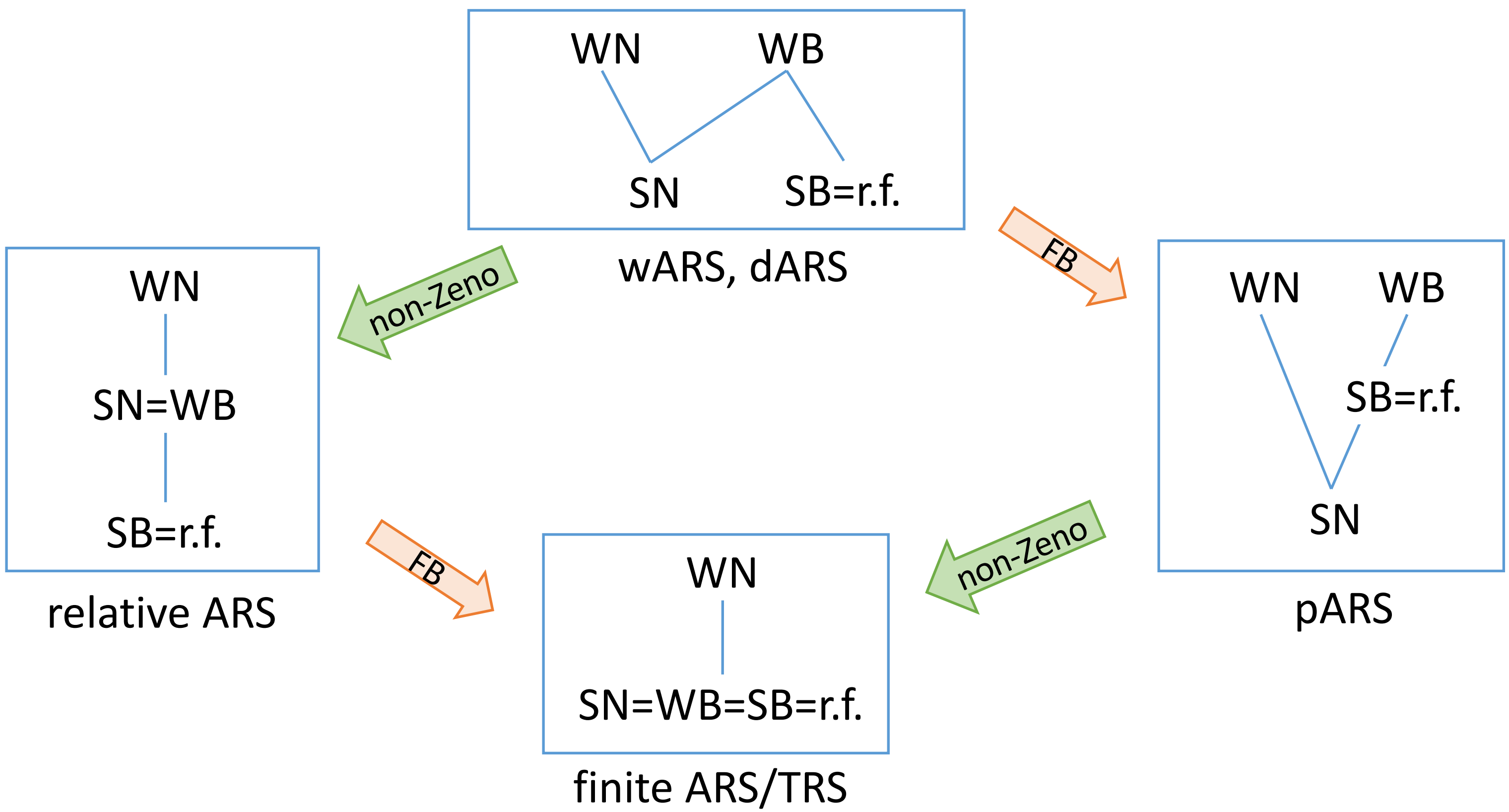


Finite branching

- wARS $\rightsquigarrow$ is **FB** if for every $a \in A$,
  - the set $\{b \mid \exists w.\, a \rightsquigarrow^{[w]} b\}$ is finite
- **Proposition**: if $\rightsquigarrow$ is FB then
$$\mathrm{SN}_{\rightsquigarrow} \Longrightarrow \mathrm{SB}_{\rightsquigarrow}$$
- **Remark**:
  - ranking functions are **complete** (maybe unsound) for SN
  - finite ARSs (TRSs), pARSs are FB
  - relative ARSs, dARSs are not



Summary of termination properties



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Potential

- The **potential** of $a \in A$ w.r.t. wARS $\rightsquigarrow$:
  - $\mathrm{pot}_{\rightsquigarrow}(a) \coloneqq \sup\{w \mid \exists b.\, a \rightsquigarrow^{w} b\} \leq \infty$
- **Remark**:
  - for ARS $\mapsto$, $\mathrm{pot}_{\overset{\smile}{\mapsto}}(a) = \mathrm{dh}_{\mapsto}(a)$, ***derivation height***
  - For pARS $\hookrightarrow$, $\mathrm{pot}_{\dot{\hookrightarrow}}(a) = \mathbb{E}\big(\mathrm{dh}_{\hookrightarrow}(a)\big)$
- **Proposition**:

$$\mathrm{SB}_{\rightsquigarrow}(S) \Leftrightarrow \mathrm{pot}_{\rightsquigarrow}(S) \subseteq [0, \infty)$$

Embedding

- Let $\rightsquigarrow$ be wARS on A and $\succ$ wARS on A'
- $\eta : A \to A'$ is an **embedding of** $\rightsquigarrow$ **to** $\succ$ if
$$a \rightsquigarrow^{[w]} b \Longrightarrow \eta(a) \succ^{[w]} \eta(b)$$
- η is a **pre-embedding** if $\qquad \rightsquigarrow \trianglelefteq^{\eta} \succ$
$$a \rightsquigarrow^{[w]} b \Longrightarrow \eta(a) \succ^{\geq w} \eta(b)$$
- **Lemma**:
$$\rightsquigarrow \trianglelefteq^{\eta} \succ \Longrightarrow \mathrm{pot}_{\rightsquigarrow}(a) \leq \mathrm{pot}_{\succ}\big(\eta(a)\big)$$
- **Corollary**:

  Suppose $\rightsquigarrow \trianglelefteq^{\eta} \succ$. Then $\mathrm{SB}_{\rightsquigarrow}(S)$ if $\mathrm{SB}_{\succ}\big(\eta(S)\big)$

Ranking function

- wARS $\succ_{\mathbb{R}}$ on $[0, \infty]$:

  $$a \succ_{\mathbb{R}}^{w} b :\Leftrightarrow a = w + b$$

- η is a **ranking function** for $\rightsquigarrow$:

  $$\rightsquigarrow \trianglelefteq^{\eta} \succ_{\mathbb{R}}$$

- **Lemma**:

  $$\mathrm{pot}_{\succ_{\mathbb{R}}}(a) = a$$

- **Corollary**:

  if η is a r.f. for $\rightsquigarrow$ then $\mathrm{pot}_{\rightsquigarrow}(a) \leq \eta(a)$

- **Theorem**:

  $\mathrm{SB}_{\rightsquigarrow}(S) \Leftrightarrow$ there is r.f. η with $\eta(S) \subseteq [0, \infty)$

for (weighted) TRS...

- recall **wTRS** $\mathcal{R}$: wARS over $T(F,V)$
- $\underset{\mathcal{R}}{\rightarrow}$: least (**weighted**) **rewrite order** extending $\mathcal{R}$
- **Lemma**:

  If $\mathcal{R} \trianglelefteq^{\eta} \succ$ for rewrite order $\succ$, then

  $\mathrm{pot}_{\underset{\mathcal{R}}{\rightarrow}}(a) \leq \mathrm{pot}_{\succ}\big(\eta(a)\big)$
- **Corollary**:

  $\mathrm{SB}_{\underset{\mathcal{R}}{\rightarrow}}(S)$ iff

  $\mathcal{R} \trianglelefteq^{\eta} \succ$ for rewrite order $\succ$ s.t. $\mathrm{SB}_{\succ}\big(\eta(S)\big)$

F-algebra

- ***F*-algebra** $\mathcal{A}$ over A:
  - fixes interpretation $f_{\mathcal{A}} : A^n \to A$ for n-ary $f \in F$
- **evaluation** of term under assignment $\alpha : V \to A$:
  - $[\![x]\!]^{\alpha}_{\mathcal{A}} = \alpha(x)$
  - $[\![f(s_1, \dots)]\!]^{\alpha}_{\mathcal{A}} = f_{\mathcal{A}}([\![s_1]\!]^{\alpha}_{\mathcal{A}}, \dots)$
- $\mathcal{A}$ is **monotone** w.r.t. weighted order $\succ$ if
  - $\forall i. s_i \succ^{w_i} t_i \Longrightarrow f_{\mathcal{A}}(s_1, \dots) \succ^{\sum_i w_i} f_{\mathcal{A}}(t_1, \dots)$
- define $\succ_{\mathcal{A}}$ by $s \succ^{w}_{\mathcal{A}} t :\Longleftrightarrow \forall \alpha. [\![s]\!]^{\alpha}_{\mathcal{A}} \succ^{\geq w} [\![t]\!]^{\alpha}_{\mathcal{A}}$
- **Theorem**: $\mathrm{SB}_{\underset{\mathcal{R}}{\to}}(S)$ iff

  $\mathcal{R} \subseteq \succ_{\mathcal{A}}$ for monotone F-algebra with $\mathrm{SB}_{\succ}(S)$

for dARS…

- Recall **dARS** $\rightarrowtail$: wARS over $\mathrm{Dist}(A)$
- $\widehat{\rightarrowtail}$: least CUCS weighted order extending $\rightarrowtail$
- <u>**Lemma**</u>:

  If $\rightarrowtail \trianglelefteq^{\eta} \succ$ for CUCS weighted order $\succ$, then
  $\mathrm{pot}_{\rightarrowtail}(a) \leq \mathrm{pot}_{\succ}\big(\eta(a)\big)$, so
  so, $\mathrm{SB}_{\rightarrowtail}(S) \Leftarrow \mathrm{SB}_{\succ}\big(\eta(S)\big)$

Barycentric algebra

- **barycentric algebra** fixes $\mathbb{E} : \mathrm{Dist}(A) \to A$ s.t.
  - $\mathbb{E}(\{1\colon a\}) = a$
  - $\mathbb{E}(\sum_i p_i \cdot \mu_i) = \sum_i p_i \, \mathbb{E}(\mu_i)$
  - $\mathbb{E}$ is **monotone** w.r.t. wARS $\succ$ over A iff
    $\forall i.\, a_i \succ^{[w_i]} b_i \implies \mathbb{E}(\{p_i\colon a_i\}_i) \succ^{[\sum_i p_i w_i]} \mathbb{E}(\{p_i\colon b_i\}_i)$
- define $\succ_{\mathbb{E}}$ by $\mu \succ_{\mathbb{E}}^{w} \nu :\Longleftrightarrow \mathbb{E}(\mu) \succ^{w} \mathbb{E}(\nu)$
- **Lemma**: $\mathrm{pot}_{\succ_{\mathbb{E}}}(\mu) = \mathrm{pot}_{\succ}\big(\mathbb{E}(\mu)\big)$
- **Theorem**: If $\rightarrowtail \trianglelefteq^{\eta} \succ_{\mathbb{E}}$ for dARS $\rightarrowtail$, $\mathbb{E}$ mono $\succ$, then
  - $\mathrm{pot}_{\rightarrowtail}(\mu) \le \mathrm{pot}_{\succ}\Big(\mathbb{E}\big(\eta(\mu)\big)\Big)$
  - $\mathrm{SB}_{\rightarrowtail}(S) \Longleftarrow \mathrm{SB}_{\succ}\Big(\mathbb{E}\big(\eta(S)\big)\Big)$

remark

- from $\succ_{\mathbb{R}}$ (i.e. $a \succ_{\mathbb{R}}^{w} b :\Leftrightarrow a = w + b$)
- we get $\succ_{\mathbb{RE}}$ ($\mu \succ_{\mathbb{RE}}^{w} \nu :\Leftrightarrow \mathbb{E}(\mu) = w + \mathbb{E}(\nu)$)

For pARS...

- Recall **pARS**: $\hookrightarrow \subseteq A \times \mathrm{Dist}(A)$
- ...same story as dARS.
- η s.t. $\hookrightarrow \trianglelefteq^{\eta} \succ_{\mathbb{RE}}$ is called a
  - probabilistic ranking function [Bournez&Garnier'05]
  - Lyapnov ranking function [Ferrer-Fioriti&Hermanns'11]
  - ranking super-martingale [Chakarov&Sankaranarayanan'13]

Summary

- Introduced weighted ARSs
  - reduction steps have non-uniform weight
  - generalizes ARSs, relative ARSs, probabilistic ARSs
  - termination, boundedness, cost analysis
  - (omitted) incremental cost analysis
- Future work
  - Implement in NaTT?