

Complete non-orders and fixed points

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Introduction

- Interactive Theorem Proving is appreciated for reliability
- But it's also engineering tool for mathematics (esp. Isabelle/jEdit)
 - refactoring proofs and claims
 - sledgehammer
 - quickcheck/nitpick(/nunchaku)
- We develop an Isabelle library of **order theory** (as a case study)
 - ⇒ we could generalize many known results, like:
 - completeness conditions: duality and relationships
 - Knaster-Tarski fixed-point theorem
 - Kleene's fixed-point theorem

Order

A binary relation (\sqsubseteq)

- **reflexive** $\Leftrightarrow x \sqsubseteq x$
- **transitive** $\Leftrightarrow x \sqsubseteq y$ and $y \sqsubseteq z$ implies $x \sqsubseteq z$
- **antisymmetric** $\Leftrightarrow x \sqsubseteq y$ and $y \sqsubseteq x$ implies $x = y$
- **partial order** \Leftrightarrow reflexive + transitive + antisymmetric

Order

A binary relation (\sqsubseteq)

locale less_eq_syntax = **fixes** less_eq :: 'a \Rightarrow 'a \Rightarrow bool (**infix** " \sqsubseteq " 50)

• **reflexive** $\Leftrightarrow x \sqsubseteq x$

locale reflexive = ... **assumes** " $x \sqsubseteq x$ "

• **transitive** $\Leftrightarrow x \sqsubseteq y$ and $y \sqsubseteq z$ implies $x \sqsubseteq z$

locale transitive = ... **assumes** " $x \sqsubseteq y \Rightarrow y \sqsubseteq z \Rightarrow x \sqsubseteq z$ "

• **antisymmetric** $\Leftrightarrow x \sqsubseteq y$ and $y \sqsubseteq x$ implies $x = y$

locale antisymmetric = ... **assumes** " $x \sqsubseteq y \Rightarrow y \sqsubseteq x \Rightarrow x = y$ "

• **partial order** \Leftrightarrow reflexive + transitive + antisymmetric

locale partial_order = reflexive + transitive + antisymmetric

Quasi-order

A binary relation (\sqsubseteq)

locale less_eq_syntax = **fixes** less_eq :: 'a \Rightarrow 'a \Rightarrow bool (**infix** " \sqsubseteq " 50)

• **reflexive** $\Leftrightarrow x \sqsubseteq x$

locale reflexive = ... **assumes** " $x \sqsubseteq x$ "

• **transitive** $\Leftrightarrow x \sqsubseteq y$ and $y \sqsubseteq z$ implies $x \sqsubseteq z$

locale transitive = ... **assumes** " $x \sqsubseteq y \Rightarrow y \sqsubseteq z \Rightarrow x \sqsubseteq z$ "

~~• **antisymmetric** $\Leftrightarrow x \sqsubseteq y$ and $y \sqsubseteq x$ implies $x = y$~~

~~**locale** antisymmetric = ... **assumes** " $x \sqsubseteq y \Rightarrow y \sqsubseteq x \Rightarrow x = y$ "~~

• **quasi-order** \Leftrightarrow reflexive + transitive

locale quasi_order = reflexive + transitive

Pseudo-order [Skala 1971]

A binary relation (\sqsubseteq)

locale less_eq_syntax = **fixes** less_eq :: 'a \Rightarrow 'a \Rightarrow bool (**infix** " \sqsubseteq " 50)

• **reflexive** $\Leftrightarrow x \sqsubseteq x$

locale reflexive = ... **assumes** " $x \sqsubseteq x$ "

~~• **transitive** $\Leftrightarrow x \sqsubseteq y$ and $y \sqsubseteq z$ implies $x \sqsubseteq z$~~

~~**locale** transitive = ... **assumes** " $x \sqsubseteq y \Rightarrow y \sqsubseteq z \Rightarrow x \sqsubseteq z$ "~~

• **antisymmetric** $\Leftrightarrow x \sqsubseteq y$ and $y \sqsubseteq x$ implies $x = y$

locale antisymmetric = ... **assumes** " $x \sqsubseteq y \Rightarrow y \sqsubseteq x \Rightarrow x = y$ "

• **pseudo order** \Leftrightarrow reflexive + antisymmetric

locale pseudo_order = reflexive + antisymmetric

Non-order

A binary relation (\sqsubseteq)

locale less_eq_syntax = **fixes** less_eq :: 'a \Rightarrow 'a \Rightarrow bool (**infix** " \sqsubseteq " 50)

~~• **reflexive** $\iff x \sqsubseteq x$~~

~~**locale** reflexive = ... **assumes** " $x \sqsubseteq x$ "~~

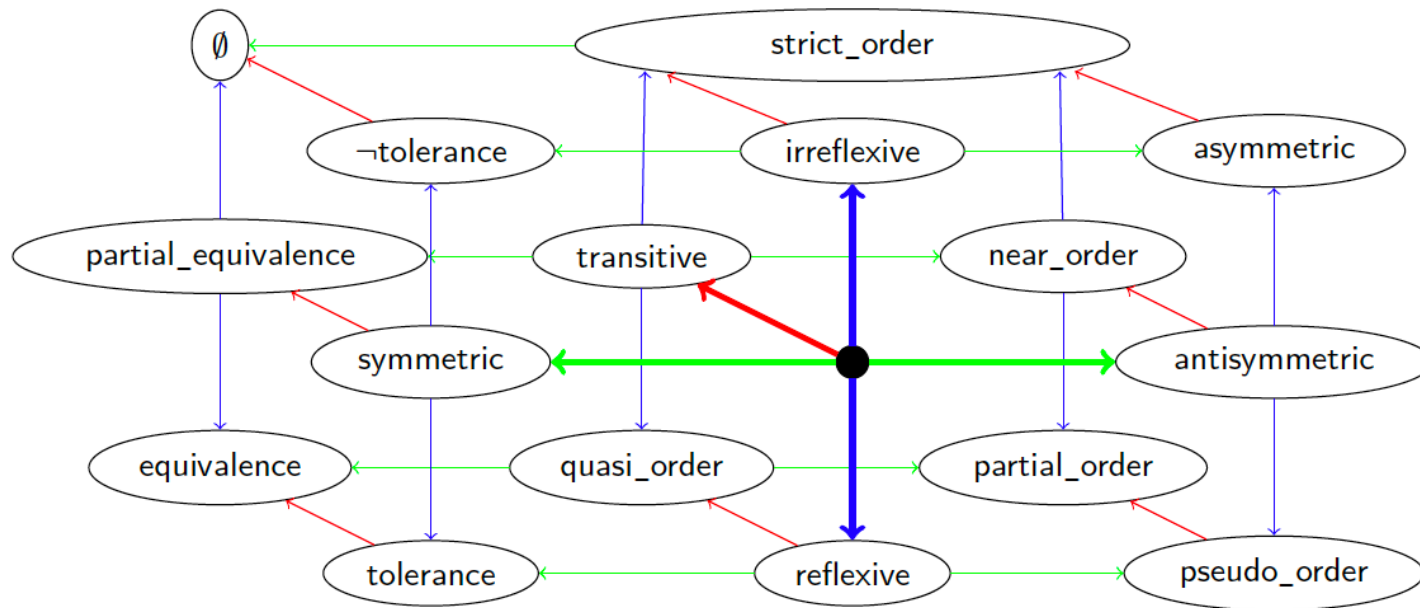
~~• **transitive** $\iff x \sqsubseteq y$ and $y \sqsubseteq z$ implies $x \sqsubseteq z$~~

~~**locale** transitive = ... **assumes** " $x \sqsubseteq y \implies y \sqsubseteq z \implies x \sqsubseteq z$ "~~

~~• **antisymmetric** $\iff x \sqsubseteq y$ and $y \sqsubseteq x$ implies $x = y$~~

~~**locale** antisymmetric = ... **assumes** " $x \sqsubseteq y \implies y \sqsubseteq x \implies x = y$ "~~

Locale combinations



Complete non-orders

- upper/lower **bounds**:

definition "bound $(\sqsubseteq) X b \equiv \forall x \in X. x \sqsubseteq b$ "

- **greatest/least** elements:

definition "extreme $(\sqsubseteq) X e \equiv e \in X \wedge (\forall x \in X. x \sqsubseteq e)$ "

- **suprema/infima** (l.u.b./g.l.b.):

abbreviation "extreme_bound $(\sqsubseteq) X s \equiv \text{extreme } (\exists) \{b. \text{bound } (\sqsubseteq) X b\} s$ "

- **complete** \iff any set X of elements has a supremum

locale complete = **assumes** " $\exists s. \text{extreme_bound } (\sqsubseteq) X s$ "

Proposition: The dual of complete **non-order** is complete

sublocale complete \subseteq dual: complete " (\supseteq) "

Knaster–Tarski fixed points

Knaster–Tarski: Part 1

- **Theorem** (Knaster–Tarski, part 1)

Any monotone map f on a complete order \sqsubseteq has a fixed point
(monotone: $x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$)
(fixed point: $f(x) = x$)

- **Theorem** [Stauti & Maaden 2013]

Any monotone map f on a complete **pseudo**-order \sqsubseteq has a fixed point
(relaxed transitivity)

Theorem [this work]

Any monotone map f on a complete **non**-order \sqsubseteq has a **quasi**-fixed point
(relaxed transitivity, reflexivity, antisymmetry)

(quasi-fixed point: $f(x) \sim x$ i.e., $f(x) \sqsubseteq x$ and $x \sqsubseteq f(x)$)

Proof sketch (by Stauti & Maaden)

definition AA **where** " $AA \equiv \{A. f ` A \subseteq A \wedge (\forall B \subseteq A. \sqcup B \in A)\}$ "

lemma " $\exists c \in \bigcap AA. f c = c$ "

proof

define c **where** " $c \equiv \sqcup (\bigcap AA)$ "

show " $c \in \bigcap AA$ "...

show " $f c = c$ "

proof (rule antisym)

show " $f c \sqsubseteq c$ "...

show " $c \sqsubseteq f c$ "...

qed

qed

Proof sketch (**minus reflexivity**)

definition AA **where** " $AA \equiv$
 $\{A. f ` A \subseteq A \wedge (\forall B \subseteq A. \sqcup B \in A)\}$ "

lemma " $\exists c \in \bigcap AA. f c = c$ "

proof

define c **where** " $c \equiv \sqcup (\bigcap AA)$ "

show " $c \in \bigcap AA$ "...

show " $f c = c$ "

proof (rule antisym)

show " $f c \sqsubseteq c$ "...

show " $c \sqsubseteq f c$ "...

qed

qed

works!

Proof sketch (**minus antisymmetry**)

definition AA **where** " $AA \equiv \{A. f ` A \subseteq A \wedge (\forall B \subseteq A. \forall s. \text{extreme_bound } (\sqsubseteq) B s \longrightarrow s \in A)\}$ "

supremum is not unique

lemma " $\exists c \in \bigcap AA. f c \sim c$ "

proof-

obtain c **where** " $\text{extreme_bound } (\sqsubseteq) (\bigcap AA) c$ "...

show " $c \in \bigcap AA$ "...

show " $f c \sim c$ "

proof (~~rule antisym~~)

show " $f c \sqsubseteq c$ "...

show " $c \sqsubseteq f c$ "...

qed

qed

$f c \sqsubseteq c$ and $c \sqsubseteq f c$ doesn't mean $f c = c$

Knaster–Tarski, Part 1: Existence

- **Main result 1**

theorem (in complete)

assumes "monotone $(\sqsubseteq) (\sqsubseteq) f$ " **shows** " $\exists x. f\ x \sim x$ "

Knaster–Tarski, Part 2: Completeness

- **Theorem** (Knaster–Tarski, Part 2)

For any monotone map on a complete order,
the set of fixed points is complete

- **Theorem** [Stauti & Maaden 2013]

Any monotone map on a complete **pseudo order** has a **least** fixed point

- **Conjecture?**

Any monotone map on a complete **non-order** has a **least** **quasi**-fixed point?

Least quasi-fixed points?

- **Counterexample** [Nitpick]

non**theorem** (**in** complete)

assumes "monotone (\sqsubseteq) (\sqsubseteq) f" **shows** " $\exists p$. extreme (\exists) {s. f s ~ s} p"

nitpick

f = (λx . _) (a₁ := a₃ , a₂ := a₃ , a₃ := a₃ , a₄ := a₁)

(\sqsubseteq) = (λx . _)

(a₁ := (λx . _) (a₁ := False, a₂ := True, a₃ := True, a₄ := True),

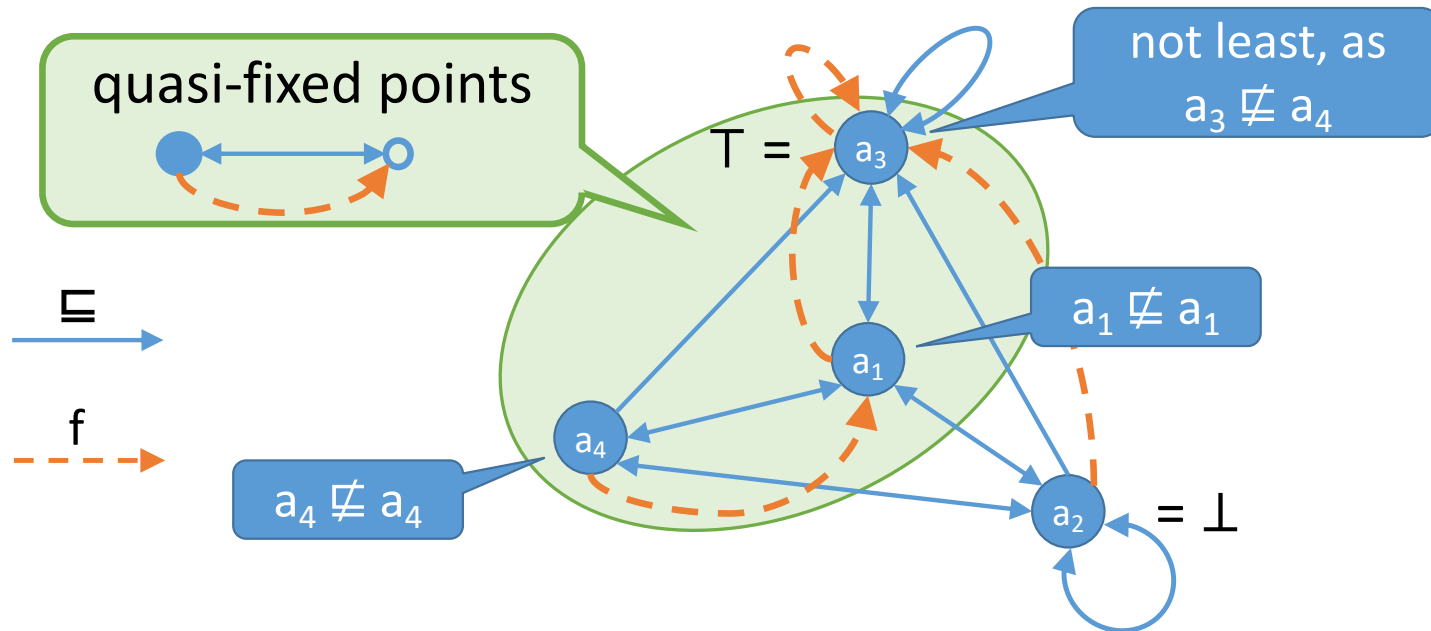
a₂ := (λx . _) (a₁ := True, a₂ := True, a₃ := True, a₄ := True),

a₃ := (λx . _) (a₁ := True, a₂ := False, a₃ := True, a₄ := False),

a₄ := (λx . _) (a₁ := True, a₂ := True, a₃ := True, a₄ := False))

least quasi-fixed points?

- **Counterexample** [Nitpick]



Argument by Stauti & Maaden

definition AA where " $AA \equiv \{A. f ` A \subseteq A \wedge (\forall B \subseteq A. \sqcup B \in A)\}$ "

lemma " $\exists c \in \bigcap AA. f c = c$ " ...

from previous proof

definition A where " $A \equiv \{a. \text{bound} (\exists) \{p. f p = p\} a\}$ "

lemma " $A \in AA$ "

proof

show " $f ` A \subseteq A$ " ..

by dropping antisymmetry, proof breaks here!

show " $\forall B \subseteq A. \sqcup B \in A$ " ...

qed

$FP = \{p. f p = p\}$

i.e., least fixed point

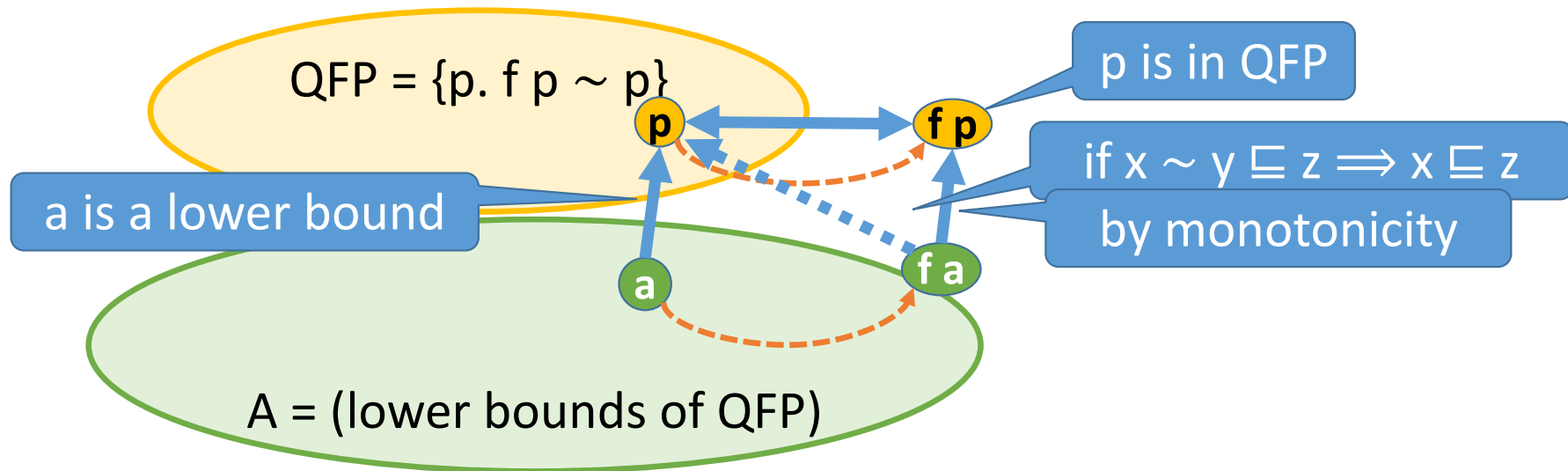
c

$A = (\text{lower bounds of FP})$

$\in AA !$

So $c \in A (\cap FP)$

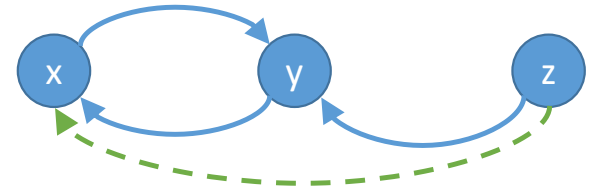
Proof of " $f \setminus A \subseteq A$ "



Attractivity

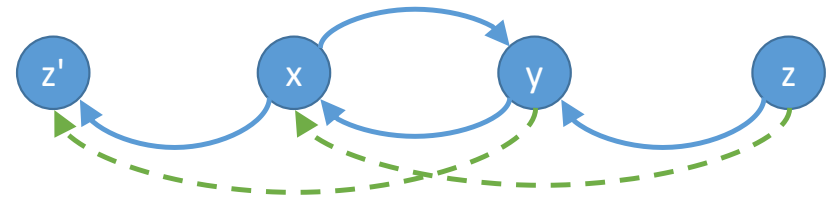
locale semiattractive =

assumes " $x \sqsubseteq y \Rightarrow y \sqsubseteq x \Rightarrow y \sqsubseteq z \Rightarrow x \sqsubseteq z$ "

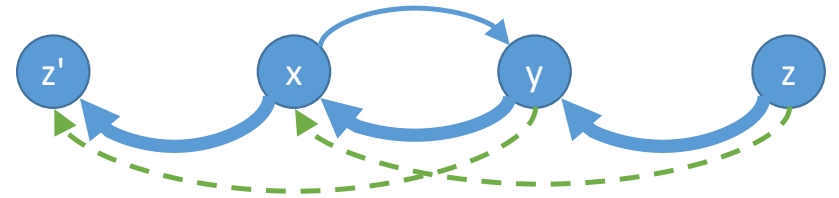


Attractivity

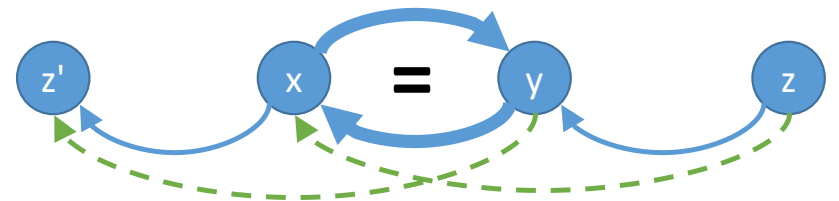
locale attractive =
semiattractive + dual: semiattractive " (\exists) "



sublocale transitive \subseteq attractive



sublocale antisymmetric \subseteq attractive

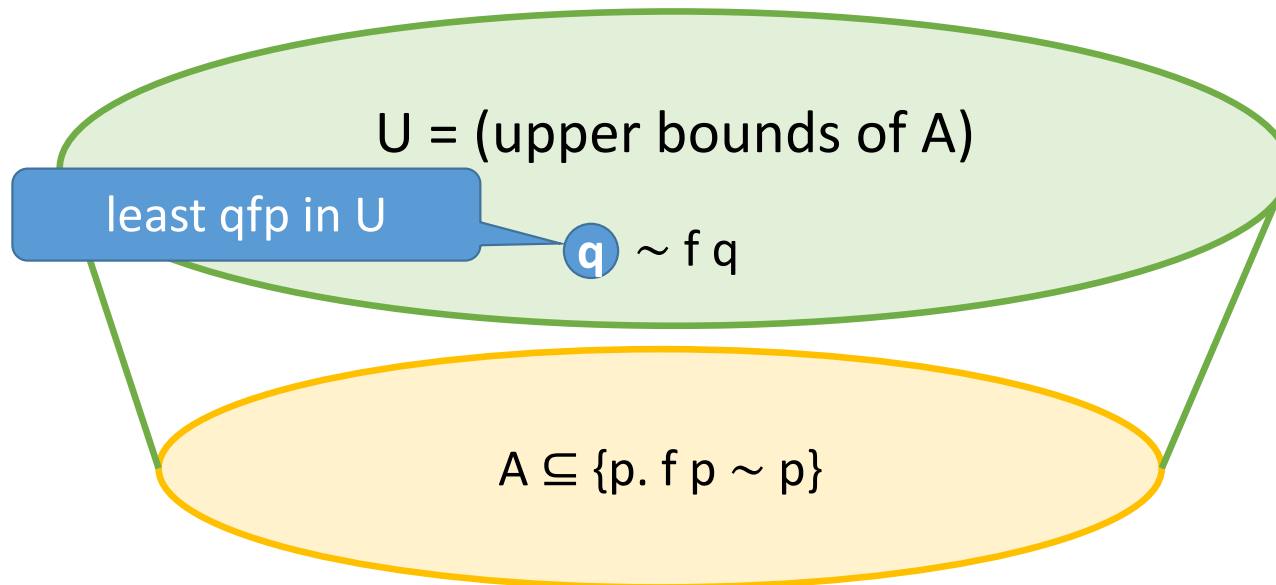


Knaster-Tarski, part 2: Completeness

- **Main result 2:**

theorem (in complete_attractive)

assumes "monotone (\sqsubseteq) (\sqsubseteq) f" **shows** "complete_in (\sqsubseteq) {p. f p ~ p}"



Knaster-Tarski, part 2: Completeness

- **Main result 2:**

- theorem** (in complete_attractive)

- assumes** "monotone (\sqsubseteq) (\sqsubseteq) f" **shows** "complete_in (\sqsubseteq) {p. f p \sim p}"

- In pseudo order, $x \sim y \iff x = y$. So

- corollary** (in complete_pseudo_order)

- assumes** "monotone (\sqsubseteq) (\sqsubseteq) f" **shows** "complete_in (\sqsubseteq) {p. f p = p}"

Completes Stauti & Maaden's work!
... but is reflexivity necessary?

Completeness only with antisymmetry

- **conjecture** (in `complete_antisymmetric`)
 assumes "monotone (\sqsubseteq) (\sqsubseteq) f " **shows** "`complete_in (\sqsubseteq) {p. f p = p}`"

$U = (\text{upper bounds of } A)$

least quasi-fixed point in U ,
but...

$$q \sim f q$$

$$r = f r$$

there might be a
smaller non-quasi fixed point!

$$A \subseteq \{p. f p = p\}$$

Completeness only with antisymmetry

- a key lemma

lemma qfp_interpolant:

assumes "complete (\sqsubseteq)"

and "monotone (\sqsubseteq) (\sqsubseteq) f"

and " $\forall a \in A. \forall b \in B. a \sqsubseteq b$ "

and " $\forall a \in A. f a = a$ "

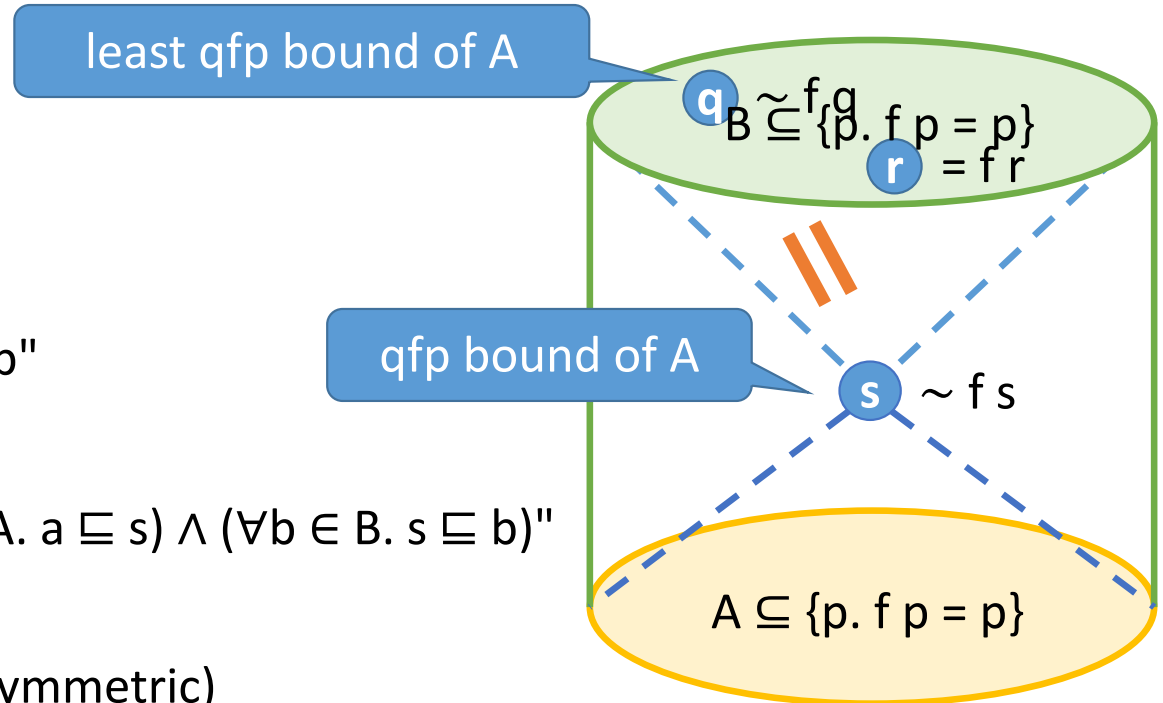
and " $\forall b \in B. f b = b$ "

shows " $\exists s. f s \sim s \wedge (\forall a \in A. a \sqsubseteq s) \wedge (\forall b \in B. s \sqsubseteq b)$ "

- **Main result 3**

theorem (in complete_antisymmetric)

assumes "monotone (\sqsubseteq) (\sqsubseteq) f" **shows** "complete_in (\sqsubseteq) {p. f p = p}"



Kleene fixed points

Kleene fixed points, part 1: Construction

- **Theorem** (Kleene, part 1)

Let f be a Scott-continuous map on a directed-complete order.
Then $\sqcup_n f^n(\perp)$ exists and is a fixed point.

- **Theorem** [Mashburn 1983]

Let f be an ω -continuous map on a ω -complete order.
Then $\sqcup_n f^n(\perp)$ exists and is a fixed point.

Theorem [this work]

Let f be an ω -continuous map on a ω -complete **non-order**.

Let \perp be a least element.

Then $\{f^n(\perp) \mid n \in \mathbb{N}\}$ has suprema, and they are all **quasi**-fixed point.

ω -completeness

- **ω -chain:** a sequence c_0, c_1, \dots s.t. $i \leq j$ implies $c_i \sqsubseteq c_j$

definition " ω _chain $C \equiv \exists c :: \text{nat} \Rightarrow 'a. \text{monotone } (\leq) (\sqsubseteq) c \wedge C = \text{range } c$ "

- **ω -complete:** any ω -chain has a supremum

locale ω _complete =

assumes " ω _chain $C \Rightarrow \exists s. \text{extreme_bound } (\sqsubseteq) C s$ "

- **ω -continuity:** f preserves all suprema of ω -chains

- **definition** " ω _continuous $f \equiv$

$\forall C. \omega$ _chain $C \rightarrow$

$\forall s. \text{extreme_bound } (\sqsubseteq) C s \rightarrow \text{extreme_bound } (\sqsubseteq) (f ` C) (f s)$ "

ω -continuity implies "near" monotonicity

- lemma

assumes " ω _continuous f" **and** " $x \sqsubseteq y$ " **and** " $x \sqsubseteq x$ " **and** " $y \sqsubseteq y$ "
shows " $f\ x \sqsubseteq f\ y$ "

proof-

have " ω _chain {x, y}" ...

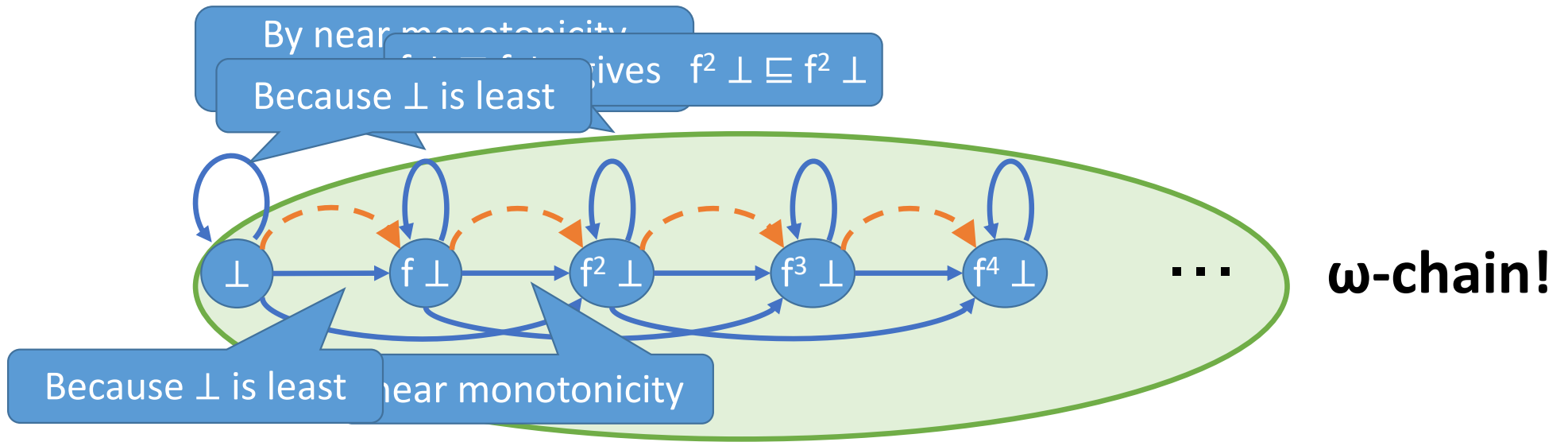
have "extreme_bound {x, y} y" ...

have "extreme_bound (f ` {x, y}) (f y) **using** ω _continuity..."

then show " $f\ x \sqsubseteq f\ y$ " **by** auto

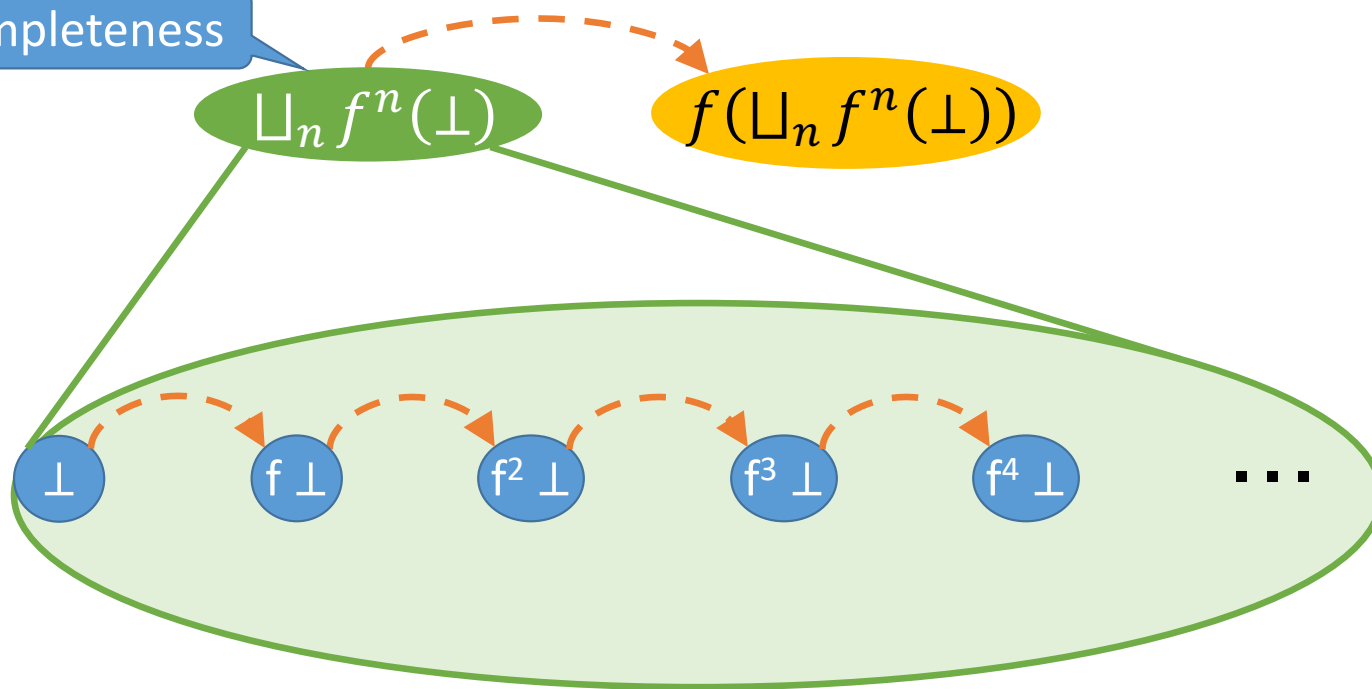
qed

$\{f^n(\perp) \mid n \in \mathbb{N}\}$ is an ω -chain

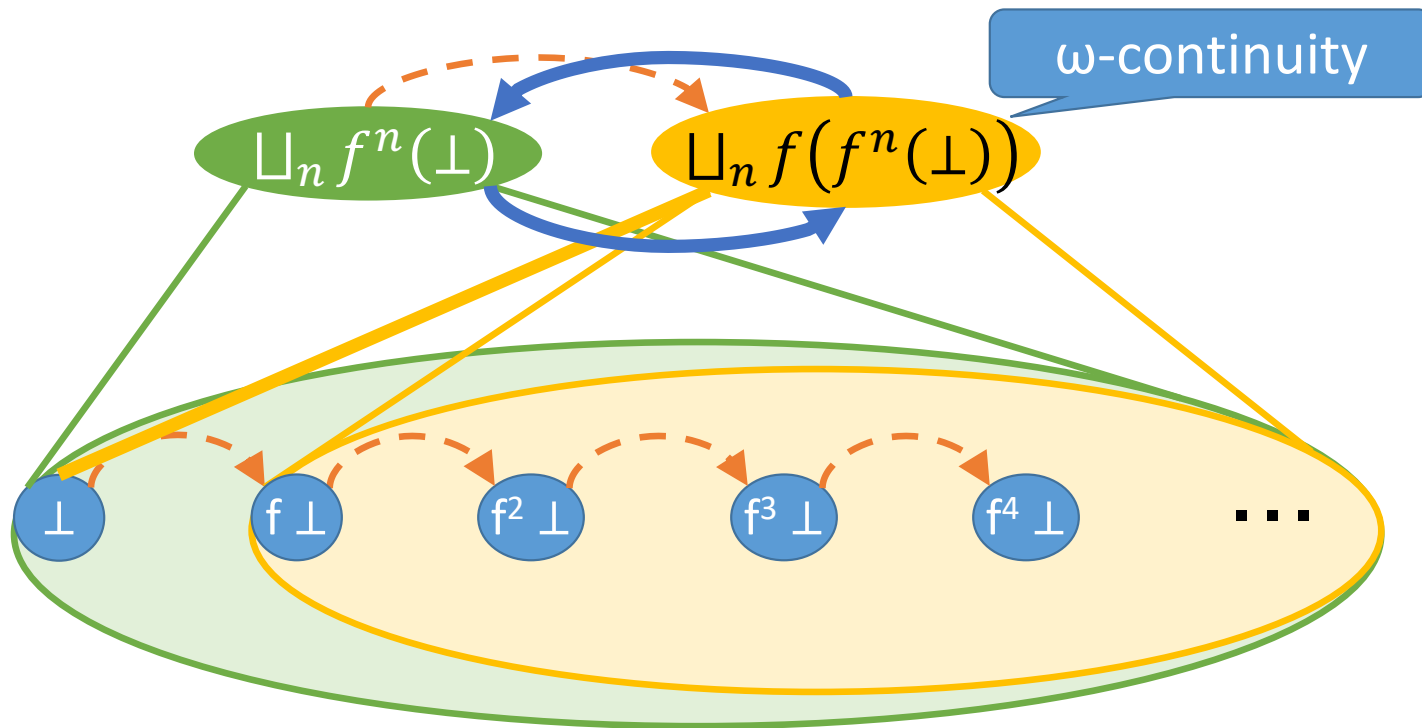


$\bigsqcup_n f^n(\perp)$ is quasi-fixed; as usual

by ω -completeness



$\sqcup_n f^n(\perp)$ is quasi-fixed; as usual



Kleene fixed point, part 1: Construction

- Main result 4:

theorem

shows " $\exists p. \text{extreme_bound } (\sqsubseteq) \{f^n(\perp) \mid n \in \mathbb{N}\} p$ "

and " $\text{extreme_bound } (\sqsubseteq) \{f^n(\perp) \mid n \in \mathbb{N}\} p \Rightarrow f p \sim p$ "

there is a supremum for $\{f^n(\perp) \mid n \in \mathbb{N}\}$

and any such is a quasi-fixed point

Kleene fixed point, part 2: Leastness

- **Theorem** (Kleene, part 2)

Let f be a Scott-continuous map on a directed-complete order.
Then $\bigsqcup_n f^n(\perp)$ is the least fixed point

- **Theorem** [Mashburn 1983]

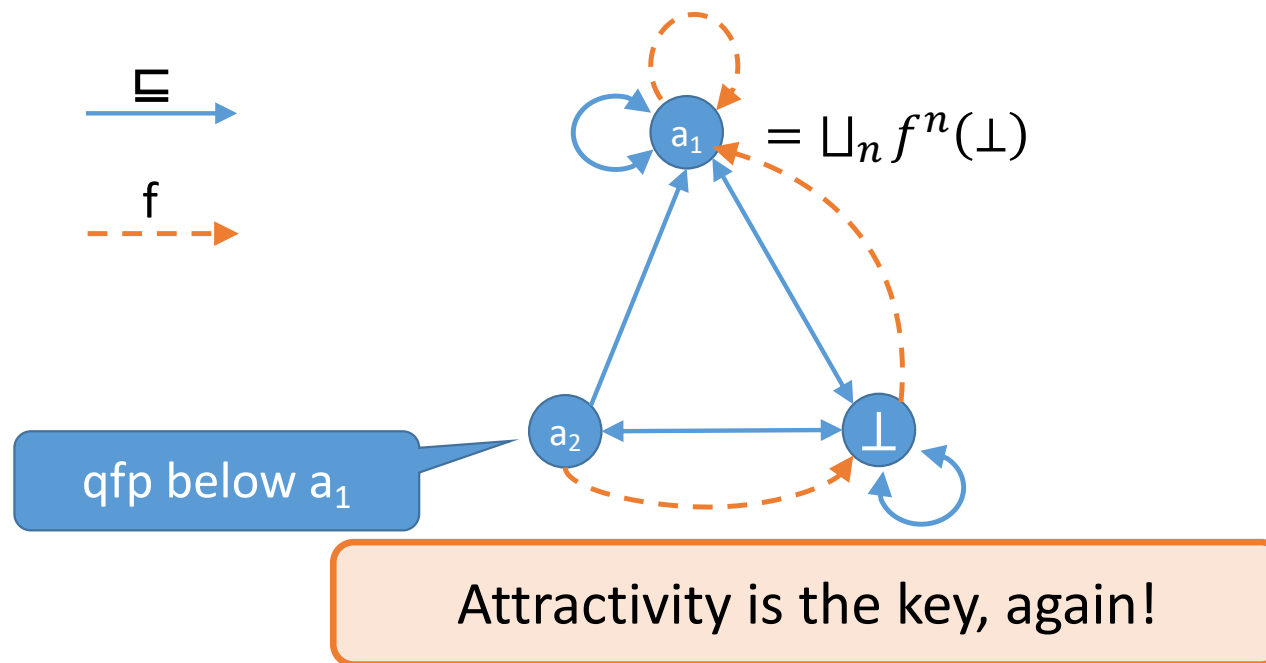
Let f be an ω -continuous map on a ω -complete order.
Then $\bigsqcup_n f^n(\perp)$ is the least fixed point.

- **Conjecture**

Let f be an ω -continuous map on a ω -complete **non-order**.
Are suprema of $\{f^n(\perp) \mid n \in \mathbb{N}\}$ least quasi-fixed points?

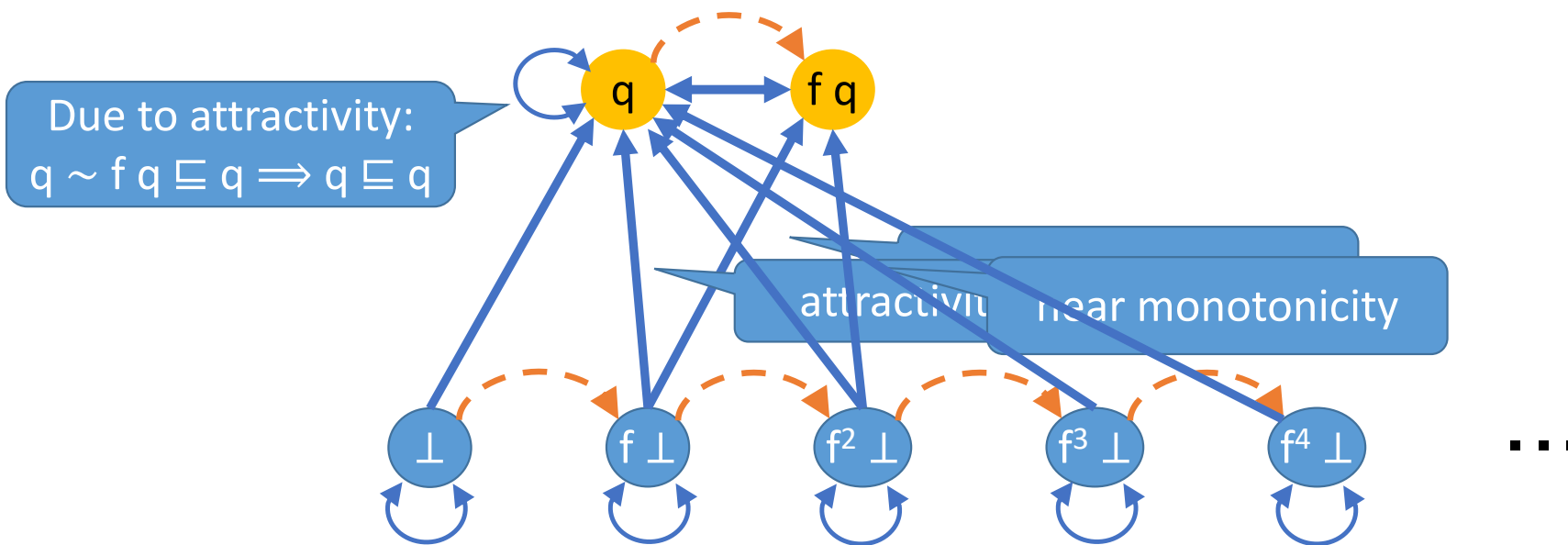
Is $\sqcup_n f^n(\perp)$ least?

- Counterexample [Nitpick]



$\bigsqcup_n f^n(\perp)$ is least under attractivity

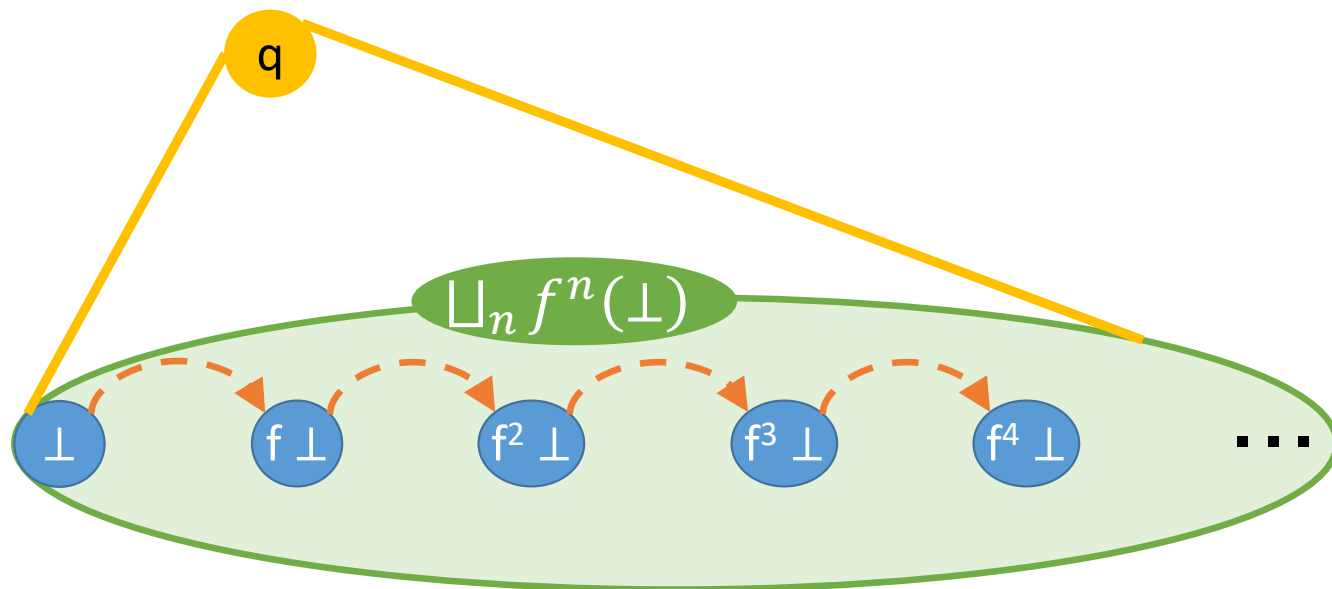
fix q assume " $q \sim f q$ " have " $f^n \perp \sqsubseteq q$ "



$\bigsqcup_n f^n(\perp)$ is least under attractivity

fix q **assume** " $q \sim f q$ " **have** " $f^n \perp \sqsubseteq q$ " **by** ...

then show " $\bigsqcup_n f^n(\perp) \sqsubseteq q$ "...



Kleene fixed point, part 2

Main result 5 (last):

corollary (in attractive)

"extreme_bound (\sqsubseteq) $\{f^n(\perp) \mid n \in \mathbb{N}\}$ s \leftrightarrow extreme (\exists) $\{q. f q \sim q\}$ s"

suprema of $\{f^n(\perp) \mid n \in \mathbb{N}\}$ are the least quasi-fixed points

Conclusion

- An Isabelle/HOL library for non-orders
- Generalized some (folklore) results on completeness
- Generalized Knaster—Tarski fixed-point theorem
 - monotone map on complete non-order has a quasi-fixed point
 - if attractive, the set of quasi-fixed points is complete
- Generalized Kleene fixed-point theorem
 - for an ω -continuous map on ω -complete non-order, suprema of $\{f^n \perp \mid n \in \mathbb{N}\}$ is a quasi-fixed point
 - if attractive, they are the least quasi-fixed points