The directed homotopy hypothesis

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I. Directed algebraic topology
Objective

Compare spaces with a notion of direction of time up to continuous deformation that preserves this direction

Problem coming from:

- geometric semantics of truly concurrent systems
  - PV-programs [Dijkstra 68]
  - scan/update [Afek et al. 90]
  - higher dimensional automata [Pratt 91]
- theory of relativity [Dodson, Poston 97]
Non directed case : algebraic topology

Compare spaces with a notion of direction of time up to continuous deformation that preserves this direction.
Dihomotopies

**Directed space** = topological space $X$ with a collection of specified paths (continuous functions from $[0, 1]$ to $X$), called **dipaths**

2 dipaths are **dihomotopic** = you can deform continuously one into the other while staying a dipath

![Diagram showing dihomotopic and non-dihomotopic dipaths](image)

(di)homotopic  non (di)homotopic
Homotopy vs dihomotopy

Fahrenberg’s matchbox [Fahrenberg 04]
Homotopy vs dihomotopy

homotopic...
Homotopy vs dihomotopy

... but not dihomotopic
Purposes of our paper

- give algebraic representatives of directed spaces up to continuous deformation that preserves direction
- explicit what we mean by continuous deformation that preserves direction (through the notion of directed deformation retract)
- define a algebraic gadget (via a notion of “weak” enriched categories) that reflects directed phenomena

Theorem:
If two directed spaces are dihomotopy equivalent then their induced partially enriched categories are weakly equivalent.
II.

Grothendieck’s homotopy hypothesis
Homotopy hypothesis: the motto

« Topological spaces are the same as $\infty$-groupoids. »
Topological spaces as ∞-groupoids

∞-category = objects + 1-cells (= morphisms between objects) + 2-cells (= morphisms between 1-cells) + ... 

objects = points
1-cells = paths (= 0-homotopies)
2-cells = (1-)homotopies
... 
n-cells = (n-1)-homotopies

∞-groupoid = ∞-category whose n-cells are invertible up-to (n+1)-cells

Here: n-homotopies are invertible up-to (n+1)-homotopies

Ex: a path γ has \( t \mapsto \gamma(1 - t) \) as inverse up-to homotopy
But what are exactly $\infty$-groupoids?

Many ways to « model » $\infty$-groupoids

$\infty$-groupoids $\equiv$ Kan complexes

n-cells $\equiv$ n-simplices

n-cells have inverse up-to (n+1)-cells $\equiv$ n-horns have (n+1)-fillers

Singular simplicial complex $\text{Sing} : \text{Top} \longrightarrow \text{Kan} (\subseteq \text{Simp})$

\[
\begin{array}{c}
\text{x} \\
\gamma \\
\text{c_x} \\
\text{x}
\end{array}
\]
But what are exactly $\infty$-groupoids?

Many ways to « model » $\infty$-groupoids

$\infty$-groupoids $=$ Kan complexes
n-cells $=$ n-simplices
n-cells have inverse up-to (n+1)-cells $=$ n-horns have (n+1)-fillers

Singular simplicial complex $Sing : Top \longrightarrow Kan \ (\subseteq Simp)$

![Diagram of a singular simplicial complex]
A formal statement of the homotopy hypothesis

Theorem [Quillen 67]:

The Quillen-Serre model structure on topological spaces is Quillen-equivalent to the Kan-Quillen model structure on simplicial sets.

A few consequences:

- A topological space is weakly homotopy equivalent to the geometric realization of its singular simplicial complex (and so to a CW-complex).
- Two topological spaces are weakly homotopy equivalent iff the geometric realization of their singular simplicial complex are weakly homotopy equivalent.
A formal statement of the homotopy hypothesis

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- two topological spaces are weakly homotopy equivalent iff the geometric realization of their singular simplicial complex are weakly homotopy equivalent

« If two topological spaces are equivalent up-continuous deformation then their induced ∞-groupoids are equivalent (up-to weak equivalence in the suitable model structure) »
III.

A first proposal of directed homotopy hypothesis
Topological spaces as $\infty$-groupoids

$\infty$-category $=$ objects $+$ 1-cells ($=$ morphisms between objects) $+$ 2-cells ($=$ morphisms between 1-cells) $+$ $\ldots$

- objects $=$ points
- 1-cells $=$ paths ($=$ 0-homotopies)
- 2-cells $=$ (1-)homotopies
  $\vdots$
- n-cells $=$ (n-1)-homotopies

$\infty$-groupoid $=$ $\infty$-category whose n-cells are invertible up-to (n+1)-cells

Here $:$ n-homotopies are invertible up-to (n+1)-homotopies

Ex $:$ a path $\gamma$ has $t \mapsto \gamma(1 - t)$ as inverse up-to homotopy
Directed topological spaces as $\infty$-groupoids

$\infty$-category = objects + 1-cells (= morphisms between objects) + 2-cells (= morphisms between 1-cells) + ... 

objects = points 
1-cells = dipaths (= 0-dihomotopies) 
2-cells = (1-)dihomotopies 
... 
n-cells = (n-1)-dihomotopies

$\infty$-groupoid = $\infty$-category whose n-cells are invertible up-to (n+1)-cells

Here: n-dihomotopies are invertible up-to (n+1)-dihomotopies
Directed topological spaces as ∞-groupoids

∞-category = objects + 1-cells (= morphisms between objects) + 2-cells (= morphisms between 1-cells) + ... 

- objects = points 
- 1-cells = dipaths (= 0-dihomotopies) 
- 2-cells = (1-)dihomotopies 
- ... 
- n-cells = (n-1)-dihomotopies 

∞-groupoid = ∞-category whose n-cells are invertible up-to (n+1)-cells

Here: n-dihomotopies are invertible up-to (n+1)-dihomotopies

True for \( n \geq 1 \), but dipaths are not invertible up-to dihomotopy!
Directed topological spaces as \((\infty,1)\)-categories

\(\infty\)-category = objects + 1-cells (= morphisms between objects) + 2-cells (= morphisms between 1-cells) + \ldots

- objects = points
- 1-cells = dipaths (= 0-dihomotopies)
- 2-cells = (1-)dihomotopies
  \[ \vdots \]
- n-cells = (n-1)-dihomotopies

\((\infty,1)\)-category = \(\infty\)-category whose n-cells are invertible up-to (n+1)-cells for \(n \geq 1\)

Here : n-dihomotopies are invertible up-to (n+1)-dihomotopies for \(n \geq 1\)
Directed homotopy hypothesis: the motto?

« Directed topological spaces are the same as \((\infty, 1)\)-categories. »
But what are exactly \((\infty, 1)\)-categories?

Many ways to « model » \((\infty, 1)\)-categories:
- quasi-categories \((=\) weak Kan complexes) [Joyal]
- enriched categories in Kan complexes [Bergner]
- ...

\((\infty, 1)\)-categories
\begin{align*}
\text{objects} & = \text{objects} \\
\text{n-cells} & = (n-1)\text{-simplices of Hom-objects} \\
\text{n-cells have inverse} & = (n-1)\text{-horns of Hom-objects} \\
\text{up-to (n+1)-cells for } n \geq 1 & = \text{have n-fillers for } n \geq 1
\end{align*}
One direction of a directed homotopy hypothesis?

Singular trace category $\mathbb{T} : dTop \longrightarrow KanCat \subseteq SimpCat$ [Porter]

$\mathbb{T}(X)$ = simplicially enriched category such that:

- objects = points of $X$

- Hom-object from $x$ to $y$ = singular simplicial complex of $\mathbb{T}(X)(x, y)$ (space of dipaths from $x$ to $y$ up-to increasing reparametrization)

« Can we compare (weak) dihomotopy types of directed spaces by their singular trace categories (up-to weak equivalence) ? »
Yes and no : the case of the directed segment

In many equivalences, $\overrightarrow{I}$ is equivalent to a point $\ast$

$\mathbb{T}(\overrightarrow{I})$ and $\mathbb{T}(\ast)$ are not weakly equivalent.
Two problems

- Specify what we mean by equivalence directed spaces up-to continuous deformations which preserves directedness.
  - the match box not equivalent to a point
  - the directed segment equivalent to a point
  - few algebraic constructions are invariant (directed components [Goubault, Haucourt 07], natural homology [DGG 15])

- Fix the directed homotopy hypothesis.
III.

The need for equivalences in directed algebraic topology
Reminder on classical algebraic topology

A (strong) deformation retract of $X$ on a subspace $A$ is a continuous map

$$H : X \longrightarrow P(X) = [[0, 1] \rightarrow X]$$

such that:

- for every $x \in X$, $H(x)(0) = x$;
- for every $a \in A$, $t \in [0, 1]$, $H(a)(t) = a$;
- for every $x \in X$, $H(x)(1) \in A$.

**Theorem:**

Two topological spaces are homotopy equivalent iff there is a span of deformation retracts between them.
Definition in directed algebraic topology

A future deformation retract of $X$ on a sub-dspace $A$ is a continuous map

$$H : X \longrightarrow \overrightarrow{P}(X)$$

such that:

- for every $x \in X$, $H(x)(0) = x$;
- for every $a \in A$, $t \in [0, 1]$, $H(a)(t) = a$;
- for every $x \in X$, $H(x)(1) \in A$;
- for every $t \in [0, 1]$, the map $H_t : x \mapsto H(x)(t)$ is a dmap;
- for every $\delta$ of $A$ from $z$ to $H_1(x)$ there is a dipath $\gamma$ of $X$ from $y$ to $x$ with $H_1(y) = z$ and $H_1 \circ \gamma$ dihomotopic to $\delta$.

Definition:

Two dspaces are dihomotopy equivalent iff there is a zigzag of future and past deformation retracts between them.
Something’s wrong, isn’t it?

There is a future deformation retract from the matchbox to its upper face (and so to its upper corner)!
Something’s wrong, isn’t it?

There is a future deformation retract from the matchbox to its upper face (and so to its upper corner)!

Problem: the dipaths along which we deform do not preserve the fact that dipaths are not dihomotopic.
Inessential dipaths

Idea from [Fajstrup, Goubault, Haucourt, Raussen] for category of components.

The set $\mathcal{I}(X)$ of inessential dipaths of $X$ is the largest set of dipaths such that:

- it is closed under concatenation and dihomotopy;
- for every $\gamma \in \mathcal{I}(X)$ from $x$ to $y$, for every $z \in X$ such that $\overrightarrow{P}(X)(z, x)$, the map $\gamma \star \_ : \overrightarrow{P}(X)(z, x) \rightarrow \overrightarrow{P}(X)(z, y)$ $\delta \mapsto \gamma \star \delta$ is a homotopy equivalence;
- symmetrically for $\_ \star \gamma$;
- $\mathcal{I}(X)$ has the right and left Ore condition modulo dihomotopy:

Ex : $\epsilon$ is not inessential in the matchbox
Better definition in directed algebraic topology

A future deformation retract of $X$ on a sub-dspace $A$ is a continuous map

$$H : X \rightarrow J(X)$$

such that:

- for every $x \in X$, $H(x)(0) = x$;
- for every $a \in A$, $t \in [0, 1]$, $H(a)(t) = a$;
- for every $x \in X$, $H(x)(1) \in A$;
- for every $t \in [0, 1]$, the map $H_t : x \mapsto H(x)(t)$ is a dmap;
- for every $\delta$ of $A$ from $z$ to $H_1(x)$ there is a dipath $\gamma$ of $X$ from $y$ to $x$ with $H_1(y) = z$ and $H_1 \circ \gamma$ dihomotopic to $\delta$.

Definition:

Two dspaces are dihomotopy equivalent iff there is a zigzag of future and past deformation retracts between them.
First results

- the directed segment is dihomotopy equivalent to a point
- the matchbox is not dihomotopy equivalent to a point
- if two dspaces are dihomotopy equivalent then they have the same directed components and their natural homology are bisimilar
IV.

A new proposal of directed homotopy hypothesis
Fixation of the directed homotopy hypothesis

- replacing enriched categories by partially enriched categories (which encode accessibility)
- changing weak equivalences
- proving the following:

**Theorem:**

If two directed spaces are dihomotopy equivalent then their induced partially enriched categories are weakly equivalent.
Fixation of the directed homotopy hypothesis

- replacing enriched categories by partially enriched categories (which encode accessibility)
- changing weak equivalences
- proving the following:

**Theorem:**
If two directed spaces are dihomotopy equivalent then their induced partially enriched categories are weakly equivalent.

« One can compare directed spaces by comparing their partially enriched category (up-to weak equivalence). »
Conclusion

Summary:
- We have defined a dihomotopy equivalence, which behaves well on examples and for which natural homology is an invariant.
- We have defined a new structure, closed to $(\infty, 1)$-categories, and designed its weak equivalence, for which it is an invariant of dihomotopy equivalence.

Many open questions:
- Are there two weakly equivalent dspaces that are not dihomotopy equivalent?
- Are there model structures on dspaces (or partially enriched categories) for which the weak equivalence is dihomotopy equivalence (or weak equivalence)?
- Do we have a kind of geometric realization from partially enriched categories to dspaces in order to formulate a complete directed homotopy equivalence?
- Are the partially enriched categories (in Top or Simp) a nice model of $(\infty, 1)$-categories?
A new proposal of directed homotopy hypothesis
The symptomatic case of the directed segment

\[ \begin{array}{c}
0 \\
\downarrow \\
1
\end{array} \quad \xrightarrow{\rightarrow}
\]

In any reasonable equivalence, \( \rightarrow \) is equivalent to a point \( * \)

\( \mathbb{T}(\rightarrow) \) and \( \mathbb{T}(*) \) are not weakly equivalent:

- for \( x < y \), \( \mathbb{T}(\rightarrow)(y, x) \) is empty while \( \mathbb{T}(*)(*,*) \) is not
- their category of components are not equivalent (one has empty Hom-sets while the other has not)
The symptomatic case of the directed segment

\[ \xymatrix{ 0 \ar[r] & 1 \ar@<0.5ex>[r]^{\vec{1}} } \]

In any reasonable equivalence, \( \vec{1} \) is equivalent to a point \( * \)

\( \mathbb{T}(\vec{1}) \) and \( \mathbb{T}(*) \) are not weakly equivalent:

- for \( x < y \), \( \overrightarrow{T}(\vec{1})(y, x) \) is empty while \( \overrightarrow{T}(*)(*, *) \) is not
- their category of components are not equivalent (one has empty Hom-sets while the other has not)

Empty path spaces have a particular behavior that must be studied with care
Reminder on enriched categories and functors

Let \(( V, U, \otimes )\) be a monoidal category.
A (small) enriched category \( C \) on \( V \) consists in the following data :

- a set of objects \( Ob(C) \)
- for every pair of objects \( A, B \), an object \( C(A, B) \) of \( V \)
- for every triple of objects \( A, B, C \), a morphism in \( V \)

\[ \circ_{A,B,C} : C(A, B) \otimes C(B, C) \rightarrow C(A, C) \]

- for every object \( A \), a morphism in \( V \)

\[ u_A : U \rightarrow C(A, A) \]

satisfying some coherence diagrams (associativity, unity).

An enriched functor \( F : C \rightarrow D \) on \( V \) consists in the following data :

- a function \( F : Ob(C) \rightarrow Ob(D) \);
- for every pair of objects \( A, B \) of \( C \), a morphism in \( V \)

\[ F_{A,B} : C(A, B) \rightarrow D(F(A), F(B)) \]

satisfying some coherence diagrams (composition, unity).
A better definition to handle emptiness

Let \((V, U, \otimes)\) be a monoidal category.

A (small) partially enriched category \(\mathcal{C}\) on \(V\) consists in the following data:

- a preordered set of objects \(\text{Ob}(\mathcal{C}), \leq\)
- for every pair of objects \(A \leq B\), an object \(\mathcal{C}(A, B)\) of \(V\)
- for every triple of objects \(A \leq B \leq C\), a morphism in \(V\)

\[\circ_{A, B, C} : \mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \to \mathcal{C}(A, C)\]

- for every object \(A\), a morphism in \(V\)

\[u_A : U \to \mathcal{C}(A, A)\]

satisfying some coherence diagrams (associativity, unity), compatible with \(\leq\).

An enriched functor \(F : \mathcal{C} \to \mathcal{D}\) on \(V\) consists in the following data:

- a monotonic function \(F : \text{Ob}(\mathcal{C}) \to \text{Ob}(\mathcal{D})\);
- for every pair of objects \(A \leq B\) of \(\mathcal{C}\), a morphism in \(V\)

\[F_{A, B} : \mathcal{C}(A, B) \to \mathcal{D}(F(A), F(B))\]

satisfying some coherence diagrams (composition, unity), compatible with \(\leq\).
From dTop to PeCat(HoTop) : the dipath category

\( \mathbb{P}(X) = \text{partially enriched category on } HoTop : \)
- objects = points of \( X \);
- \( x \leq y \) iff \( \overrightarrow{P}(X)(x, y) \neq \emptyset \);
- for \( x \leq y \), \( \mathbb{P}(X)(x, y) = \overrightarrow{P}(X)(x, y) \);
- composition = concatenation up-to homotopy;
- unit = constant path.

We can have defined it with value in \( HoSimp \) or \( Ab \) by composing with singular simplicial complex or homology.

We recover the fundamental category \( \pi_1(X) \) by composing with the connected components functor.
What about the category of components?

For \textbf{[Bergner]}, it is just $\pi_1(X)$.
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No good for directed segment.
Already known since [Fajstrup, Goubault, Haucourt, Raussen].
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We have to define a category of « directed » components.
Yoneda morphisms, category of directed components

A slight modification of [Fajstrup, Goubault, Haucourt, Raussen]

The set $\mathcal{Y}(C)$ of Yoneda morphisms of a category $C$ is the largest set of morphisms such that:

- it is closed under concatenation;
- for every $f : c \to c' \in \mathcal{Y}(C)$, for every object $c''$ of $C$ such that $C(c', c'') \neq \emptyset$, the function $\_ \circ f : C(c', c'') \to C(c, c'')$ $g \mapsto g \circ f$ is a bijection;
- symmetrically for $f \circ \_$;
- it has right and left Ore conditions

\[\pi_0(C) = C[\mathcal{Y}(C)^{-1}] = C\] in which we inverse the morphisms in $\mathcal{Y}(C)$

\[\pi_0(X) = \pi_0(\pi_1(X))\]
Example: the directed segment

\[ \begin{array}{c}
\vdots \\
0 \\
\end{array} \longrightarrow \quad \begin{array}{c}
\vdots \\
1 \\
\end{array} \]

\( \mathbb{P}(I) \) is such that:
- \( x \leq y \) is the usual ordering on \( I \);
- for every \( x \leq y \), \( \mathbb{P}(I)(x, y) \) is contractible.

The fundamental category \( \pi_1(I) \) is the poset \( (I, \leq) \).

The category of components \( \pi_0(I) \) is the preordered set \( (I, I \times I) \), which is equivalent to the category with one object and one morphism.
Weak dihomotopy equivalence

We say that a dmap \( f : X \to Y \) is a weak dihomotopy equivalence iff

- it induces an equivalence between the categories of \textit{directed} components
- it induces a fully-faithful enriched functor between dipath categories i.e. for \( x \leq_X x' \), the map

\[
P(f)_{x,x'} : P(X)(x,x') \to P(Y)(f(x), f(x'))
\]

which maps \( \gamma \) to \( f \circ \gamma \) is a homotopy equivalence.

We say that two dspaces are weakly dihomotopy equivalent iff there is zigzag of weak dihomotopy equivalence between them.
Examples

$\vec{1}$ is weakly equivalent to a point.

$\mathbb{P}(s, t)$ is homotopy equivalent to a two point space, so the match box cannot be weakly equivalent to a point.
Invariance

**Theorem:**

If two dspaces are dihomotopy equivalent, then they are weakly dihomotopy equivalent.
Invariance

Theorem:
If two dspaces are dihomotopy equivalent, then they are weakly dihomotopy equivalent.

« One can compare dspaces by comparing their dipath category (up-to weak equivalence). »
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If two dspaces are dihomotopy equivalent, then they are weakly dihomotopy equivalent.

« One can compare dspaces by comparing their dipath category (up-to weak equivalence). »

« Are dspaces the same as partially enriched categories in HoTop (or HoSimp)? »