Trees in partial Higher Dimensional Automata

Methods and Tools for Distributed Hybrid Systems

Jérémy Dubut

National Institute of Informatics
Japanese-French Laboratory for Informatics

July 4th

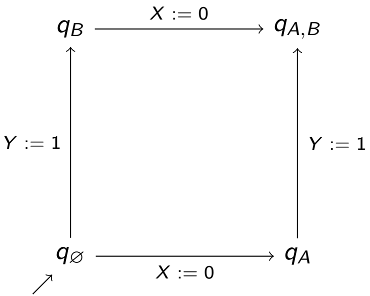
Concurrency vs. true concurrency

True concurrency has the flavor of
a directed homotopy theory

(directed algebraic topology)

Concurrency has the flavor of a homotopy theory

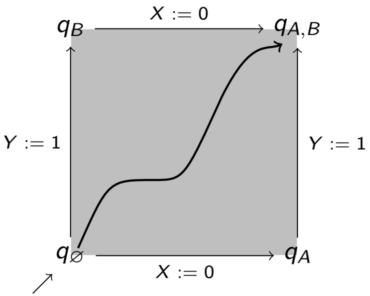
Independent actions



Concurrency

Interleaving behaviors: A then B or B then A

Independent actions



True concurrency

Continuous behaviors: any scheduling of A and B
Refinement **[van Glabbeek, Goltz]**: in reality $X := 0$ and $Y := 1$ are not atomic

Goals:

- Presenting a model of true concurrency: Higher Dimensional Automata,
- Extending it to nicely encode paths and homotopies: partial Higher Dimensional Automata,
- Making a parallel between constructions in concurrency and constructions in Quillen's model structures, on the example of partial HDA.

Higher Dimensional Automata

Extending graphs

Precubical sets

A **precubical set** is:

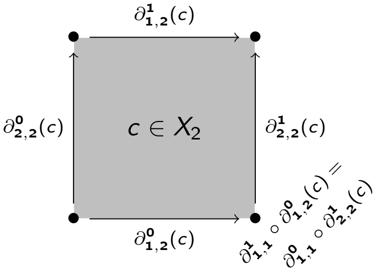
- a collection of sets $(X_n)_{n\geq 0}$,
- a collection of function $(\partial^{\alpha}_{i,n} : X_n \longrightarrow X_{n-1})_{n>0,\, 1\leq i\leq n,\, \alpha\in\{0,1\}}$.

satisfying for $i > j$:

$$\partial^{\beta}_{j,n} \circ \partial^{\alpha}_{i,n+1} = \partial^{\alpha}_{i-1,n} \circ \partial^{\beta}_{j,n+1}$$

Graph:

- $X_0 = V$, $X_1 = E$ and $X_{n>1} = \varnothing$,
- $s = \partial^0_{1,1}$ and $t = \partial^1_{1,1}$,
- equations are trivial.



Extending systems

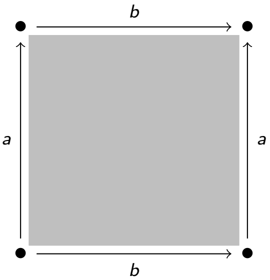
Higher Dimensional Automata **[Pratt]**

An **HDA** on the alphabet on Σ is:

- a precubical set (X, ∂),
- an initial state $i_0 \in X_0$,
- a labelling function $\lambda : X_1 \longrightarrow \Sigma$.

satisfying for every $c \in X_2$:

$$\lambda(\partial_i^1(c)) = \lambda(\partial_i^0(c))$$



Category of HDA

Morphisms

A morphism of precubical sets from (X, ∂) to (Y, δ) is a collection

$$f_n : X_n \longrightarrow Y_n$$

of functions such that:

$$f_{n-1} \circ \partial_{i,n}^{\alpha} = \delta_{i,n}^{\alpha} \circ f_n$$

Category of HDA

The category $\mathbf{HDA}_{\Sigma}$ of HDA has as morphisms from $(X, \partial, i_0, \lambda)$ to (Y, δ, j_0, η) the morphisms of precubical sets f from (X, ∂) to (Y, δ) such that:

- $f_0(i_0) = j_0$
- $\lambda = \eta \circ f_1$

Paths and homotopies in Higher Dimensional Automata

Runs in HDA

Path [van Glabbeek]

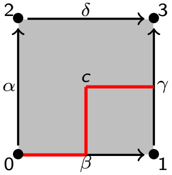
A **path** in a HDA is sequence written as:

$$q_0 \xrightarrow{j_1,\alpha_1} q_1 \xrightarrow{j_2,\alpha_2} \dots \xrightarrow{j_n,\alpha_n} q_n$$

with:

- $q_i \in X$, $j_i \in \mathbb{N}$, $\alpha_i \in \{0,1\}$
- $q_0 = i_0$
- for every i,
  - if $\alpha_i = 0$, $q_{i-1} = \partial_{j_i}^{\alpha_i}(q_i)$
  - if $\alpha_i = 1$, $q_i = \partial_{j_i}^{\alpha_i}(q_{i-1})$

$$0 \xrightarrow{1,0} \beta \xrightarrow{1,0} c \xrightarrow{2,1} \gamma$$

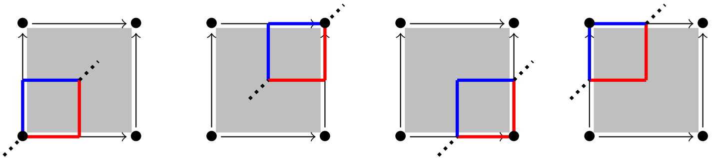


Homotopies

Elementary homotopies [van Glabbeek]

A path $i_0 = q_0 \xrightarrow{j_1, \alpha_1} \dots \xrightarrow{j_n, \alpha_n} q_n$ is elementary homotopic to $i_0 = q'_0 \xrightarrow{k_1, \beta_1} \dots \xrightarrow{k_n, \beta_n} q'_n$ if there is $1 \leq l \leq n-1$ such that:

- for every $p \neq l$ $q_p = q'_p$
- for every $r \notin \{l, l+1\}$ $j_r = k_r$, $\alpha_r = \beta_r$
- $\alpha_l = \beta_{l+1}$ and $\alpha_{l+1} = \beta_l$
- $k_l > j_l$, $j_l = k_{l+1}$ and $k_l = j_{l+1} - 1$



$\alpha_l = 0,\ \beta_l = 0$ $\quad$ $\alpha_l = 1,\ \beta_l = 1$ $\quad$ $\alpha_l = 1,\ \beta_l = 0$ $\quad$ $\alpha_l = 0,\ \beta_l = 1$

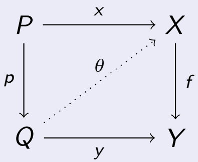
Open maps **[Joyal, Nielsen, Winskel]**

Given:

- a category $\mathcal{M}$ (category of systems and functional simulations)
- a subcategory $\mathcal{P} \subseteq \mathcal{M}$ (execution shapes)

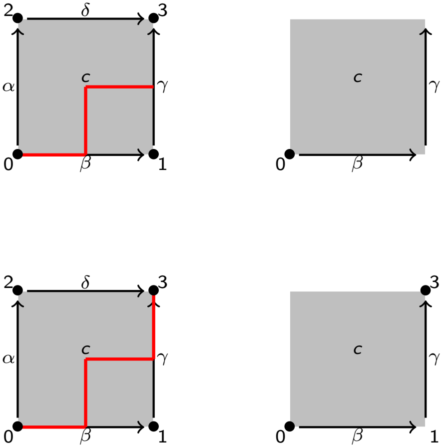
Open maps

We say that a morphism $f : X \longrightarrow Y$ of $\mathcal{M}$ is $\mathcal{P}$-**open** if for every such commutative square (in plain):



with $p \in \mathcal{P}$, there is a diagonal filler (dotted).

Internalizing paths and homotopies in HDA?



Partial HDA

Partial precubical sets, concretely

Partial precubical sets

A **partial precubical set** is:

- a collection of sets $(X_n)_{n \geq 0}$,
- a collection of partial functions $\partial^{\alpha_1, \ldots, \alpha_k}_{i_1 < \ldots < i_k} : X_n \longrightarrow X_{n-k}$.

satisfying:

$$\partial^{\beta_1, \ldots, \beta_l}_{j_1 < \ldots < j_l} \circ \partial^{\alpha_1, \ldots, \alpha_k}_{i_1 < \ldots < i_k} \subseteq \partial^{\gamma_1, \ldots, \gamma_{n+p}}_{h_1 < \ldots < h_{k+l}}$$

Ex: for $i > j$, $\partial^{\beta}_{j} \circ \partial^{\alpha}_{i} \subseteq \partial^{\beta, \alpha}_{j < i}$ and $\partial^{\alpha}_{i-1} \circ \partial^{\beta}_{j} \subseteq \partial^{\beta, \alpha}_{j < i}$

Morphisms of partial precubical sets

Morphisms of partial precubical sets

A morphism of partial precubical set is a collection of *total* functions $f_n : X_n \longrightarrow Y_n$ such that:

$$f_{n-k} \circ \partial^{\alpha_1,\ldots,\alpha_k}_{i_1<\ldots<i_k} \subseteq \delta^{\alpha_1,\ldots,\alpha_k}_{i_1<\ldots<i_k} \circ f_n$$

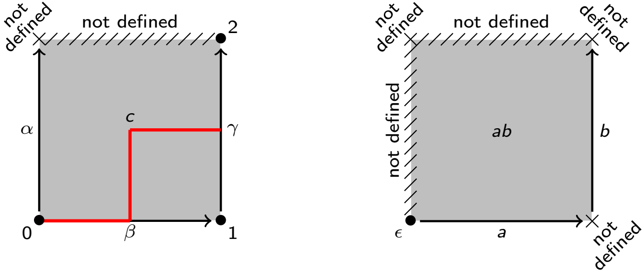
Categorically: partial precubical sets are lax functors **[Niefield]**

Completing a pHDA

[Dubut]

This process forms a functor χ : pHDA$_\Sigma \longrightarrow$ HDA$_\Sigma$ which is the left adjoint of the embedding of HDA$_\Sigma$ in pHDA$_\Sigma$.

Internalizing paths



Path (right) – path shape representing it (left)

A run in a pHDA X is the same as a morphism from a path shape to X.

We can do something similar for homotopies.

Premisses of a homotopy theory for the concurrency of pHDA

Concurrency vs. Homotopy theory

Homotopy	Concurrency
cofibration generators (basic constructions of the theory)	path shapes and extensions
trivial fibration (rlp w.r.t. cofibration generators)	open maps w.r.t. path shapes
cofibrant objects (obtained from basic constructions)	trees
cofibrant replacement (process to obtain a cofibrant object)	unfolding

What is a tree?

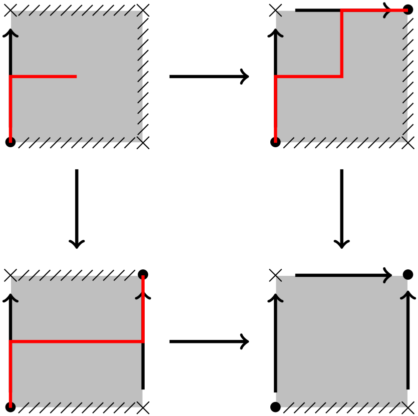
Intuitively, a tree is equivalently:

- a system with exactly one path from the initial state to any state
- a system obtained by unfolding another system
- a system obtained from one state and recursively extending paths
- a system obtained by glueing together paths

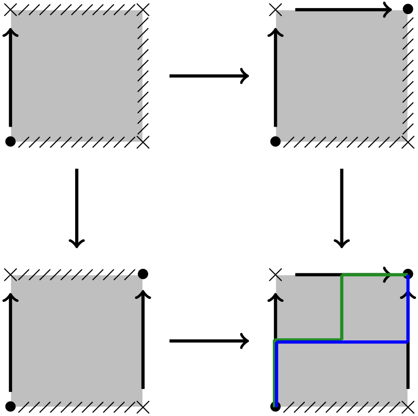
Trees, as colimits of paths

Proposition ([Dubut]):

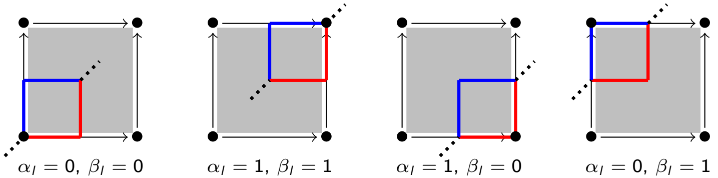
Every small digram with values in the category of path shapes has a colimit in $\mathbf{pHDA}_\Sigma$. We denote by $\mathbf{Tr}_\Sigma$ the full sucategory of such colimits.



Unique path property?



Unique path property modulo homotopy

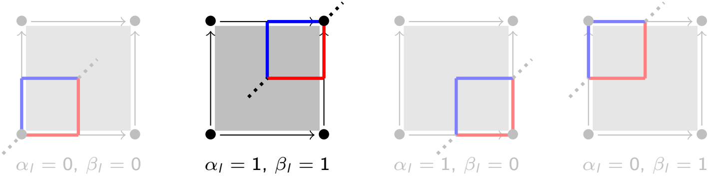


Homotopy

Proposition:

Trees have a unique path modulo homotopy from the initial state to any state.

Unique path property modulo confluent homotopy

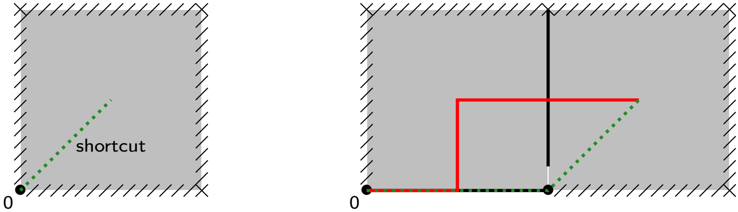


Confluent homotopy

Proposition:

Trees have a unique path modulo **confluent** homotopy from the initial state to any state.

Is it enough?



Proposition:

Trees does not have any shortcuts.

Unfolding

Definition:

The unfolding $U(X)$ of a pHDA X is a pHDA whose states are confluent homotopy classes of paths.

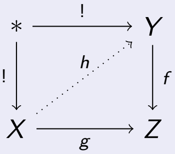
Proposition:

- The unfolding of a pHDA is a tree.
- There is an open map $\mathrm{unf}_X : U(X) \longrightarrow X$.
- When X has the unique path property modulo confluent homotopy and is without shortcuts, unf_X is an isomorphism.
- U extends to a functor $U : \mathbf{pHDA}_\Sigma \longrightarrow \mathbf{Tr}_\Sigma$, which is the right adjoint of the embedding of $\mathbf{Tr}_\Sigma$ in $\mathbf{pHDA}_\Sigma$.

Cofibrant objects

Definition:

We say that a pHDA X is a cofibrant object if the unique morphism $! : * \longrightarrow X$ has the left lifting property w.r.t. every open maps. That is:



Main result

Theorem:

The following are equivalent for a pHDA:

- being the colimit in $\mathbf{pHDA}_\Sigma$ of a diagram with values in path shapes.
- having a unique path modulo confluent homotopy from the initial state to any state and being without any shortcuts.
- being isomorphic to the unfolding of another pHDA.
- being a cofibrant object.

Conclusion and futur work

Conclusion and futur work

Done:

- A nicer categorical definition of partial HDA, using lax functors.
- Suitable descriptions of the notion of trees (colimit of paths, unique path property, unfolding)
- Premisses of a homotopy theory of the concurrency of pHDA;

To do:

- Understanding what hold and what fail in Quillen's axioms.
- Adding homotopies to get a homotopy theory of the true concurrency of pHDA.
- Allowing shortcuts as runs.
- Generalizing to more general open maps framework.