Categorical approaches to bisimilarity

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Jérémy Dubut

National Institute of Informatics
Japanese-French Laboratory for Informatics

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Bisimilarity of Transition Systems

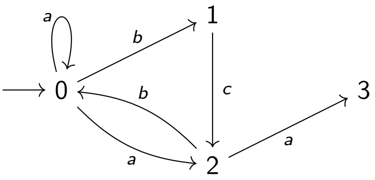
Transition Systems

Transition system :

A **TS** $T = (Q, i, \Delta)$ on the alphabet Σ is the following data:

- a set Q (of **states**);
- an **initial state** $i \in Q$;
- a set of **transitions** $\Delta \subseteq Q \times \Sigma \times Q$.

- $\Sigma = \{a, b, c\}$,
- $Q = \{0, 1, 2, 3\}$,
- $i = 0$,
- $\Delta = \{(0, a, 0), (0, b, 1), (0, a, 2), (1, c, 2), (2, b, 0), (2, a, 3)\}$.



Bisimulations of Transition Systems

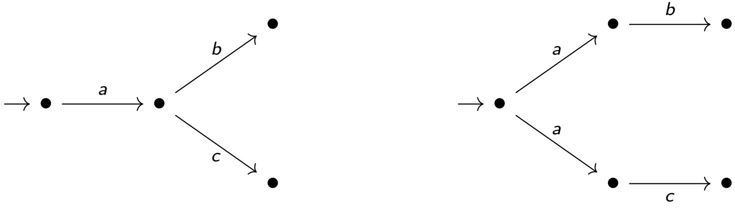
Bisimulations [Park81] :

A **bisimulation** between $T_1 = (Q_1, i_1, \Delta_1)$ and $T_2 = (Q_2, i_2, \Delta_2)$ is a relation $R \subseteq Q_1 \times Q_2$ such that:

(i) $(i_1, i_2) \in R$;

(ii) if $(q_1, q_2) \in R$ and $(q_1, a, q_1') \in \Delta_1$ then there is $q_2' \in Q_2$ such that $(q_2, a, q_2') \in \Delta_2$ and $(q_1', q_2') \in R$;

(iii) if $(q_1, q_2) \in R$ and $(q_2, a, q_2') \in \Delta_2$ then there is $q_1' \in Q_1$ such that $(q_1, a, q_1') \in \Delta_1$ and $(q_1', q_2') \in R$.



Several Characterisations of Bisimilarity

Bisimilarity:

Given two TS T and T', the following are equivalent:

- **[Park81]** There is a bisimulation between T and T'.
- **[Stirling96]** Defender has a strategy to never loose in a 2-player game on T and T'.
- **[Hennessy80]** T and T' satisfy the same formulae of the Hennessy-Milner logic.

In this case, we say that T and T' are **bisimilar**.

Morphisms of Transition Systems

Morphism of TS:

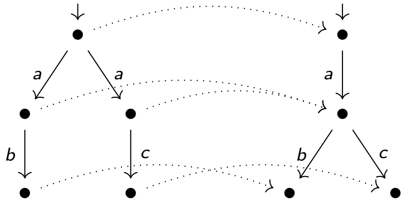
A **morphism of TS** $f : T_1 = (Q_1, i_1, \Delta_1) \longrightarrow T_2 = (Q_2, i_2, \Delta_2)$ is a function

$$f : Q_1 \longrightarrow Q_2$$

such that:

- **preserving the initial state:** $f(i_1) = i_2$,
- **preserving the transitions:** for every $(p, a, q) \in \Delta_1$, $(f(p), a, f(q)) \in \Delta_2$.

$\mathbf{TS}(\Sigma)$ = category of transition systems and morphisms



Morphisms are functional simulations:

Morphisms are precisely functions f between states whose graph $\{(q, f(q)) \mid q \in Q_1\}$ is a simulation.

Categorical Characterisations

Bisimilarity, using morphisms:

Two TS T and T' are bisimilar iff there is a span of functional bisimulations between them.

$$T \xleftarrow{\ f\ } T'' \xrightarrow{\ g\ } T'$$

Bisimilarity from Coalgebra

J. Rutten. *Universal coalgebra: a theory of systems.* Theoretical Computer Science **249**(1), 3–80 (2000)

Transition systems, as pointed coalgebras

Set of transitions, as functions:

There is a bijection between sets of transitions $\Delta \subset Q \times \Sigma \times Q$ and functions of type:

$$\delta : Q \longrightarrow \mathcal{P}(\Sigma \times Q)$$

where $\mathcal{P}(X)$ is the powerset $\{U \mid U \subset X\}$.

Initial states, as functions:

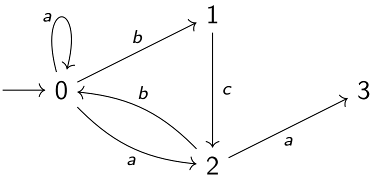
There is a bijection between initial states $i \in Q$ and functions of type:

$$\iota : * \longrightarrow Q$$

where $*$ is a singleton.

Example

- $\Sigma = \{a, b, c\}$,
- $Q = \{0, 1, 2, 3\}$,
- $i = 0$,
- $\Delta = \{(0, a, 0), (0, b, 1), (0, a, 2), (1, c, 2), (2, b, 0), (2, a, 3)\}$.



$$\begin{array}{rcccl} \iota & : & * & \longrightarrow & Q \\ & & * & \mapsto & 0 \end{array}$$

$$\begin{array}{rcccl} \delta & : & Q & \longrightarrow & \mathcal{P}(\Sigma \times Q) \\ & & 0 & \mapsto & \{(a, 0), (b, 1), (a, 2)\} \\ & & 1 & \mapsto & \{(c, 2)\} \\ & & 2 & \mapsto & \{(b, 0)\} \\ & & 3 & \mapsto & \varnothing \end{array}$$

Pointed coalgebras

Pointed coalgebras:

Given an endofunctor $G : \mathcal{C} \longrightarrow \mathcal{C}$ and an object $I \in \mathcal{C}$, a **pointed coalgebra** is the following data:

- an object $Q \in \mathcal{C}$,
- a morphism $\iota : I \longrightarrow Q$ of $\mathcal{C}$,
- a morphism $\sigma : Q \longrightarrow G(Q)$ of $\mathcal{C}$.

G is often decomposed as $T \circ F$, where:

- T: "branching type", e.g, non-deterministic, probabilistic, weighted.
  For TS: $T = \mathcal{P}$.
- F: "transition type".
  For TS: $F = \Sigma \times \_$.

I is often the final object, but we will see other examples.
For TS: $I = *$, the final object.

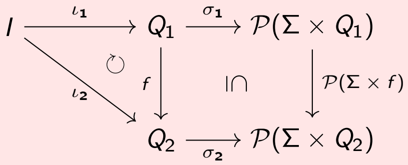
Morphisms of TS, using Pointed Coalgebras

Morphisms of TS are lax morphisms of pointed coalgebras

A morphism of TS, seen as pointed coalgebras $T = (Q_1, \iota_1, \delta_1)$ and $T' = (Q_2, \iota_2, \delta_2)$ is the same as a function

$$f : Q_1 \longrightarrow Q_2$$

satisfying



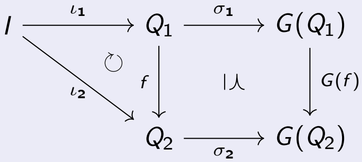
Lax Morphisms of Pointed Coalgebras

Lax Morphisms:

Assume there is an order $\preceq$ on every Hom-set of the form $\mathcal{C}(X, G(Y))$. A **lax morphism** from (Q_1, ι_1, δ_1) to (Q_2, ι_2, δ_2) is a morphism

$$f : Q_1 \longrightarrow Q_2$$

of $\mathcal{C}$ satisfying



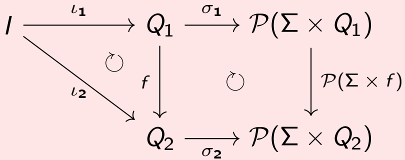
$\mathbf{Coal}_{\mathrm{lax}}(G, I)$ = category of pointed coalgebras and lax morphisms.

What about functional bisimulations?

Functional bisimulations are homomorphisms of pointed coalgebras

For two TS, seen as pointed coalgebras $T = (Q_1, \iota_1, \delta_1)$ and $T' = (Q_2, \iota_2, \delta_2)$, and for a function of the form $f : Q_1 \longrightarrow Q_2$, the following are equivalent:

- The graph $\{(q, f(q)) \mid q \in Q_1\}$ of f is a bisimulation.
- f is a homomorphism of pointed coalgebras, that is, the following diagram commutes:



Bisimilarity, using homomorphisms of pointed coalgebras

For two TS T and T', the following are equivalent:

- T and T' are bisimilar.
- There is a span of homomorphisms of pointed coalgebras between T and T'.

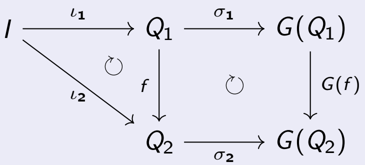
Homomorphisms of Pointed Coalgebras

Morphisms:

A **homomorphism** from (Q_1, ι_1, δ_1) to (Q_2, ι_2, δ_2) is a morphism

$$f : Q_1 \longrightarrow Q_2$$

of $\mathcal{C}$ satisfying



Coal(G, I) = category of pointed coalgebras and homomorphisms.

Summary

	coalgebra	
data type	$G : \mathcal{C} \to \mathcal{C}$, $I \in \mathcal{C}$ $\preceq$ on $\mathcal{C}(X, G(Y))$	
systems	pointed coalgebras	
functional simulations	lax morphisms	
functional bisimulations	homomorphisms	
bisimilarity	existence of a span of functional bisimulations	

Bisimilarity from Open Maps

A. Joyal, M. Nielsen, G. Winskel. *Bisimulation from Open Maps.* Information and Computation **127**, 164–185 (1996)

Runs in a Transition System

Run

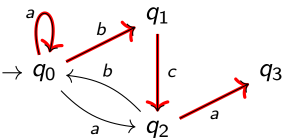
A **run** in a transition system (Q, i, Δ) is sequence written as:

$$q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} q_n$$

with:

- $q_j \in Q$ and $a_j \in \Sigma$
- $q_0 = i$
- for every j, $(q_j, a_{j+1}, q_{j+1}) \in \Delta$

$$q_0 \xrightarrow{a} q_0 \xrightarrow{b} q_1 \xrightarrow{c} q_2 \xrightarrow{a} q_3$$



Runs, Categorically

Finite Linear Systems:

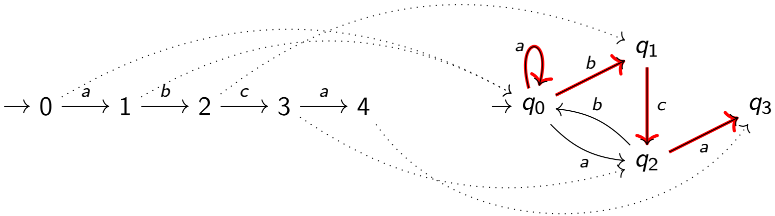
A **finite linear system** is a TS of the form $\langle a_1, \ldots, a_n \rangle = ([n], 0, \Delta)$ where:

- $[n]$ is the set $\{0, \ldots, n\}$;
- Δ is of the form $\{(i, a_{i+1}, i+1) \mid i \in [n-1]\}$ for some a_1, ..., a_n in Σ.

$$\to 0 \xrightarrow{a_1} 1 \xrightarrow{a_2} 2 \quad \cdots \quad n-1 \xrightarrow{a_n} n$$

Runs are morphisms

There is a bijection between runs of T and morphisms of TS between a finite linear system to T.

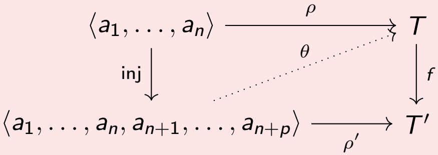


Functional Bisimulations, from Lifting Properties of Paths

Functional bisimulations are open maps:

For a morphism f of TS from T to T', the following are equivalent:

- The reachable graph of f, that is, $\{(q, f(q)) \mid q \text{ reachable}\}$ is a bisimulation.
- f has the right lifting property w.r.t. path extensions, that for every commutative square (in plain):



  there is a lifting (in dot), making the two triangles commute.

Bisimilarity, using open maps

For two TS T and T', the following are equivalent:

- T and T' are bisimilar.
- There is a span of open maps between T and T'.

Open maps

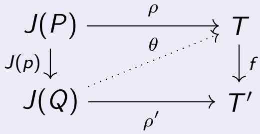
Open map situation:

An **open map situation** is a category $\mathcal{M}$ (of **systems**) together with a subcategory $J : \mathbb{P} \hookrightarrow \mathcal{M}$ (of **paths**).

- $\mathcal{M}$ = category of systems (Ex : $\mathbf{TS}(\Sigma)$),
- $\mathbb{P}$ = sub-category of finite linear systems.

Open maps:

A morphism $f : T \longrightarrow T'$ of $\mathcal{M}$ is said to be **open** if for every commutative square (in plain):



where $p : P \longrightarrow Q$ is a morphism of $\mathbb{P}$, there is a lifting (in dot) making the two triangles commute.

Summary

	coalgebra	open maps
data type	$G : \mathcal{C} \to \mathcal{C}$, $I \in \mathcal{C}$ $\preceq$ on $\mathcal{C}(X, G(Y))$	$J : \mathbb{P} \hookrightarrow \mathcal{M}$
systems	pointed coalgebras	objects of $\mathcal{M}$
functional simulations	lax morphisms	morphisms of $\mathcal{M}$
functional bisimulations	homomorphisms	open maps
bisimilarity	existence of a span of functional bisimulations	

From Open Maps to Coalgebra

S. Lasota. *Coalgebra morphisms subsume open maps*. Theor. Comput. Sci. **280**(1–2): 123–135 (2002)

Problem Setting

Input: An open map situation $\mathbb{P} \hookrightarrow \mathcal{M}$ such that:

- $\mathbb{P}$ is small,
- $\mathbb{P}$ and $\mathcal{M}$ has a common initial object 0

Problem: Construct

- a coalgebra situation:
  - $G : \mathcal{C} \longrightarrow \mathcal{C}$,
  - $I \in \mathcal{C}$,
  - $\preceq$ on $\mathcal{C}(X, G(Y))$.
- a functor Beh : $\mathcal{M} \longrightarrow \mathbf{Coal}_{\mathrm{lax}}(G, I)$

such that

$$f \text{ is an open map iff } Beh(f) \text{ is a homomorphism.}$$

Solution

- $\mathcal{C} = \mathbf{Set}^{\mathbf{ob}(\mathbb{P})}$,
- $G((X_P)_{P\in\mathbf{ob}(\mathbb{P})}) = \left(\prod_{Q\in\mathbb{P}} \left(\mathcal{P}(X_Q)\right)^{\mathbb{P}(P,Q)}\right)_{P\in\mathbb{P}}$
- $I_0 = *$, $I_P = \varnothing$ otherwise,
- $\preceq$ point-wise inclusion,
- $\mathrm{Beh}(X)$:
  - X_P = set of runs labelled by P, i.e., the set $\mathcal{M}(P, X)$,
  - $\iota : (I_P) \longrightarrow (X_P)$ maps $*$ to unique morphism from 0 to X,
  - $\sigma_P = (\sigma_{P,Q})_Q : X_P \longrightarrow \prod_{Q\in\mathbb{P}} \left(\mathcal{P}(X_Q)\right)^{\mathbb{P}(P,Q)}$, where $\sigma_{P,Q}$ maps a run ρ labelled by P to the set of runs labelled by Q that extend ρ.

Theorem [Lasota02]:

f is an open map iff $Beh(f)$ is a homomorphism.

From Coalgebra to Open Maps

T. Wißman, J. Dubut, S. Katsumata, I. Hasuo. Path Category For Free – Open Morphisms From Coalgebras With Non-Deterministic Branching. FoSSaCS'19

Problem Setting

Input: A coalgebra situation:

- $G = T \circ F : \mathcal{C} \longrightarrow \mathcal{C}$,
- $I \in \mathcal{C}$,
- $\preceq$ on $\mathcal{C}(X, G(Y))$.

satisfying some axioms.

Problem: Construct an open map situation $J : \mathbb{P} \hookrightarrow \mathbf{Coal}_{\mathrm{lax}}(G, I)$ such that lax homomorphism $f : c_1 \longrightarrow c_2$:

if f is a homomorphism then f is open
if f is open and c_2 is reachable, then f is a homomorphism

THE key notion: F-precise morphisms

F-precise morphisms:

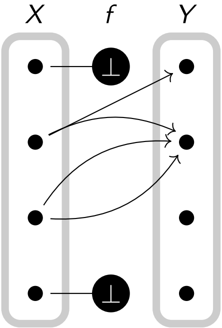
A morphism $s\colon S \to FR$ of $\mathcal{C}$ is F-**precise** if for all f, g, h:

$$\begin{array}{ccc} S & \xrightarrow{f} & FC \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle Fh} \\ FR & \xrightarrow{Fg} & FD \end{array} \quad \overset{\exists d}{\Longrightarrow} \quad \begin{array}{ccc} S & \xrightarrow{f} & FC \\ {\scriptstyle s}\downarrow & \nearrow{\scriptstyle Fd} & \\ FR & & \end{array} \quad \& \quad \begin{array}{ccc} & & C \\ & \nearrow{\scriptstyle d} & \downarrow{\scriptstyle h} \\ R & \xrightarrow{g} & D \end{array}$$

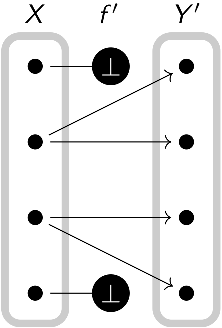
Intuition: a morphism $s\colon S \to FR$ is precise iff every element of R is used exactly once in the definition of s.

Examples

$FX = X \times X + \bot$



not precise



precise

The Path Category $\mathbb{P} = \mathrm{Path}(I, F)$

A path consists in:

- a finite sequence $P_0, \ldots, P_n$ of objects of $\mathcal{C}$ with $P_0 = I$,
- a finite sequence of $F + 1$-precise maps:

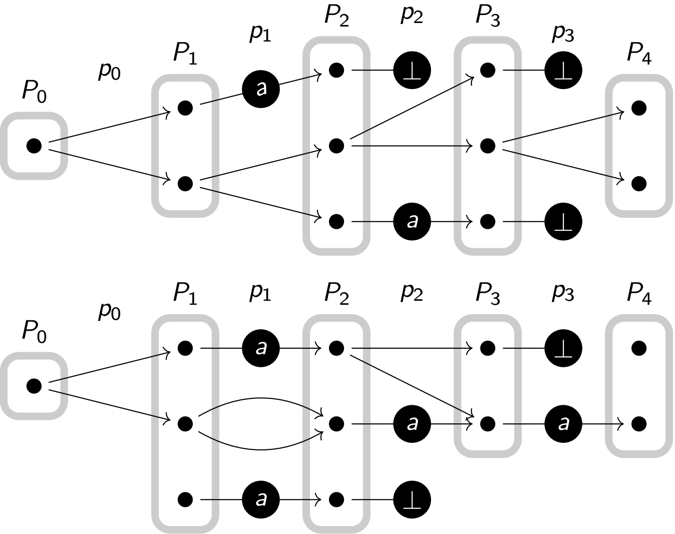
$$f_k : P_k \longrightarrow FP_{k+1} + 1$$

A morphism between paths, from $(P_k, p_k)_{k \leq n}$ to $(Q_j, q_j)_{j \leq m}$ consists in a sequence of isomorphisms $\phi_k : P_k \longrightarrow Q_k$ such that:

$$\begin{array}{ccc} P_k & \xrightarrow{p_k} & FP_{k+1} + 1 \\ {\scriptstyle \phi_k}\downarrow & & \downarrow{\scriptstyle F\phi_{k+1}+1} \\ Q_k & \xrightarrow{q_k} & FQ_{k+1} + 1 \end{array}$$

Examples

$$FX = \{a\} \times X + X \times X,\ I = *,\ p_k : P_k \to FP_{k+1} + \{\bot\}$$



The Functor J

Assumptions on $\mathcal{C}$ and T:

1. $\mathcal{C}$ has finite coproducts,
2. $\eta : \mathrm{Id}_{\mathcal{C}} \longrightarrow T$,
3. $\perp : 1 \longrightarrow T$ such that $\perp_X \in \mathcal{C}(1, T(X))$ is the least element for $\preceq$,
4. some others.

Typical example: the powerset functor $\mathcal{P}$

- **Set** has disjoint unions and empty set,
- η is the unit $\eta_X(x) = \{x\}$,
- $\perp$ is given by the empty subset $\perp_X(*) = \varnothing$,
- ...

Theorem:

There is a functor $J : \mathrm{Path}(I, F) \longrightarrow \mathbf{Coal}_{\mathrm{lax}}(TF, I)$ given by $J(P_k, p_k) :=$

$$I \xrightarrow{\mathrm{in}_0} \coprod_{k \leq n} P_k \xrightarrow{\left[\mathrm{inl} \cdot [F\mathrm{in}_{k+1} \cdot p_k]_{k<n}, \mathrm{inr} \cdot !\right]} F \coprod_{k \leq n} P_k + 1 \overset{[\eta, \perp]}{\rightarrowtail} TF \coprod_{k \leq n} P_k$$

Wrapping up

Theorem:

A homomorphism of pointed coalgebras in $\mathbf{Coal}_{\mathrm{lax}}(TF, I)$ is open.

Proposition-Definition:

For a pointed coalgebra $c = (Q, \iota, \sigma)$, the following are equivalent:

- c has no proper subcoalgebra,
- the set of all morphisms of the form $J(P_k, p_k) \longrightarrow c$ is jointly epic.

In this case, we say that c is **reachable**.

Theorem:

An open map $h : c \longrightarrow c'$ in $\mathbf{Coal}_{\mathrm{lax}}(TF, I)$ where c is reachable is a homomorphism.

Instances

Labelled Transition Systems

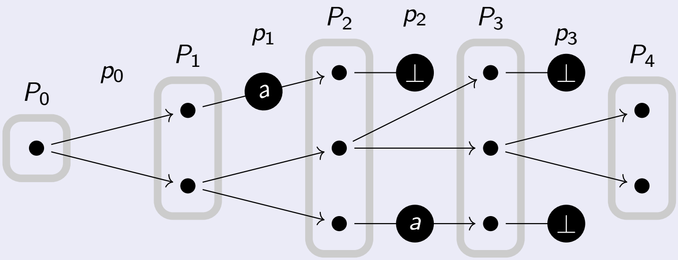
- $\mathcal{C} = \mathbf{Set}$,
- $F(X) = \Sigma \times X$,
- $T = \mathcal{P}$,
- $I = *$.

Paths are given by words.

Various Tree-like Automata

- $\mathcal{C} = \mathbf{Set}$,
- F analytic, i.e., $F(X) = \coprod_{\sigma/n \in \Sigma} X^n/G_\sigma$,
- $T = \mathcal{P}$,
- $I = *$.

Paths are given by "partial" trees.



Multi-Sorted Transition Systems **[Lasota'02]**

- $\mathcal{C} = \mathbf{Set}^{\mathbf{ob}(\mathbb{P})}$,
- $F((X_P)_{P\in\mathbb{P}}) = \left(\coprod_{Q\in\mathbb{P}} \mathbb{P}(P,Q) \times X_Q\right)_{P\in\mathbb{P}}$,
- $T((X_P)_{P\in\mathbb{P}}) = (\mathcal{P}(X_P))_{P\in\mathbb{P}}$,
- $I_0 = *,\ I_P = \varnothing$.

Paths are given by sequences of path extensions from the initial path category:

$$0 \xrightarrow{m_1} P_1 \xrightarrow{m_2} P_2 \cdots \xrightarrow{m_n} P_n$$

Consequence:

We cannot expect a more general translation from coalgebra to open maps.

Regular Nondeterministic Nominal Automata **[Schröder et al.'17]**

- $\mathcal{C} = \mathbf{Nom}$,
- $F(X) = 1 + \mathbb{A} \times X + [\mathbb{A}]X$, where $[\mathbb{A}]\_$ is a *binding* operator,
- $T = \mathcal{P}_{ufs}$, the set of *uniformly* finitely supported,
- $I = \mathbb{A}^{\#n}$, the set of n-tuples of distincts atoms.

General Kripke Frames [Kupke et al.'04]

- $\mathcal{C} = \textbf{Stone}$, the category of Stone spaces
- $F = \mathsf{Id}$,
- $T = \mathcal{V}$, the Vietoris topology on the set of compact subsets,
- $I = *$.

Conclusion

[Wißman, D., Katsumata, Hasuo – FoSSaCS'19] $\longrightarrow$ Non-deterministic branching

	coalgebra	open maps
data type	$G : \mathcal{C} \to \mathcal{C}$, $I \in \mathcal{C}$ $\preceq$ on $\mathcal{C}(X, G(Y))$	$J : \mathbb{P} \hookrightarrow \mathcal{M}$
systems	pointed coalgebras	objects of $\mathcal{M}$
functional simulations	lax morphisms	morphisms of $\mathcal{M}$
functional bisimulations	homomorphisms	open maps
bisimilarity	existence of a span of functional bisimulations	

[Lasota'02] $\longleftarrow$ Small category of paths