Path Category for Free

Open Morphisms from Coalgebras with Non-Deterministic Branching

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FoSSaCS, April 08, 2019

Categorical Approaches to Bisimilarity

Transition type

Functor $F\colon \mathbb{C} \to \mathbb{C}$

↓

Coalgebra

↓

Def. Coalgebra Homomorphism

...

↓

Def. Bisimulation

Span of Coalgebra Homomorphisms

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Paths, Models

(Sub)category $J\colon \mathbb{P} \hookrightarrow \mathbb{M}$

Open Maps [Joyal, Nielsen, Winskel]

Def. Open Map

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lasota02 lasota02

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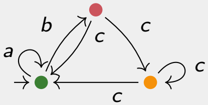
Motivating Example: Category LTS_A

Objects: (X, x_0, Δ)

states X, initial state $x_0 \in X$, transitions $\Delta \subseteq X \times A \times X$.



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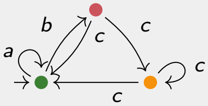

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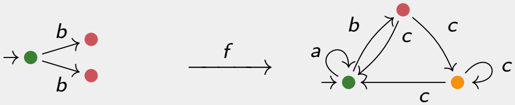

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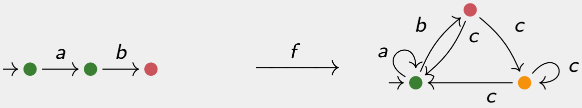

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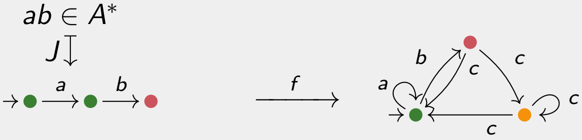

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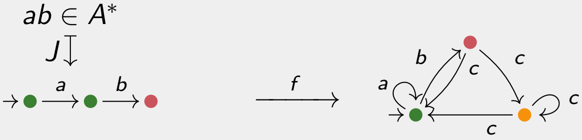

Paths prefix order

Functor $J\colon (A^*, \leq) \longrightarrow \mathsf{LTS}_A$

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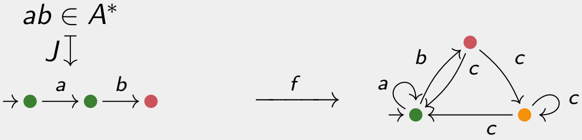

Paths

Functor $J\colon (A^*, \leq) \longrightarrow \mathsf{LTS}_A$

Run of $w \in A^*$ in (X, x_0, Δ)

$f\colon Jw \to (X, x_0, \Delta)$ in LTS_A

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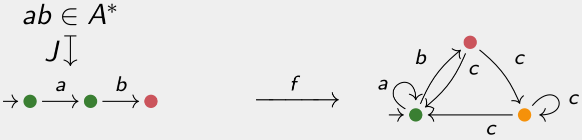


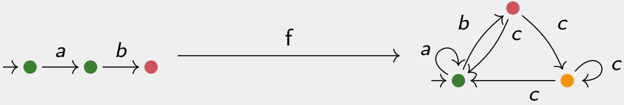
Paths

Functor $J\colon \mathbb{P} \longrightarrow \mathbb{M}$ (prefix order on $\mathbb{P}$)

Run of $w \in \mathbb{P}$ in $M \in \mathbb{M}$

$f\colon Jw \to M$ in $\mathbb{M}$

Open Maps



$$Jw \xrightarrow{f} X$$

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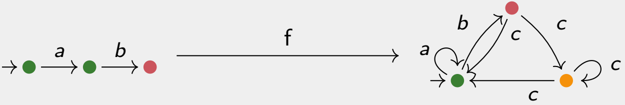
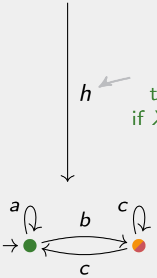
Open Maps



$$\begin{array}{ccc} Jw & \xrightarrow{f} & X \\ & & \downarrow h \\ & & Y \end{array}$$



reflects
transitions
if X reachable

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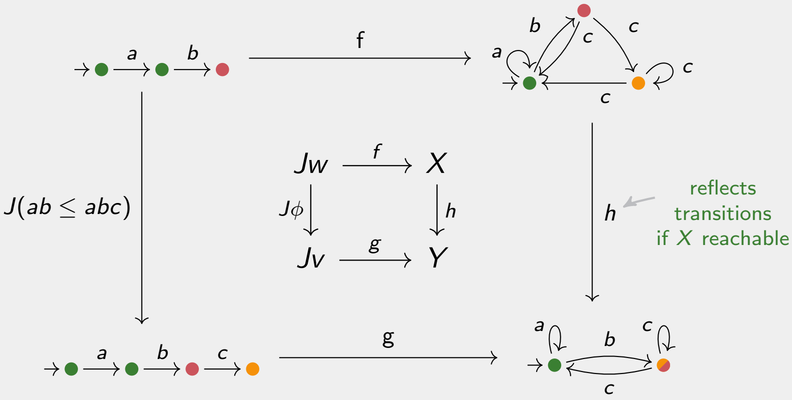
Open Maps



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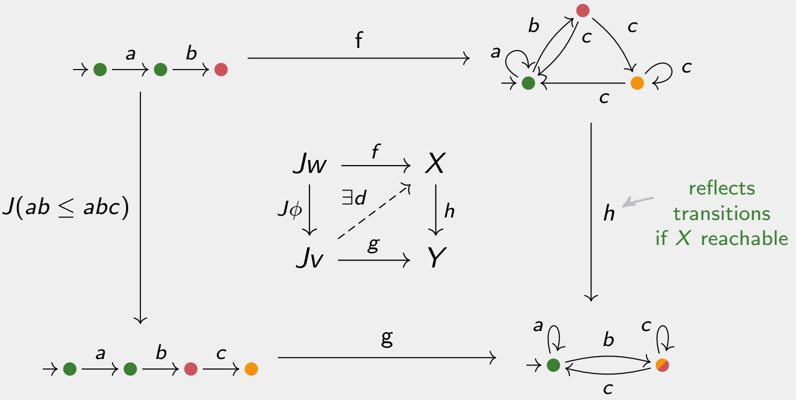

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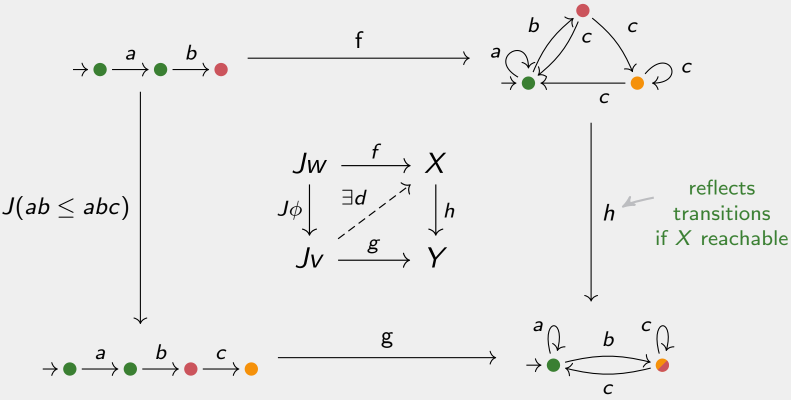

Definition: h open…

…, if for every such square, there exists some dialgonal lifting d.

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Pointed Coalgebras & Lax homomorphisms

$\mathsf{LTS}_A \iff \mathsf{LCoalg}(1, \mathcal{P}(A \times (-)))$

$X, x_0 \in X,$
$\Delta \subseteq X \times A \times X$ $\iff$ 1-pointed $\mathcal{P}(A \times (-))$-coalgebra

$1 \xrightarrow{x_0} X \xrightarrow{\xi} \mathcal{P}(A \times X)$

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$\mathsf{LTS}_A \iff \mathsf{LCoalg}(1, \mathcal{P}(A \times (-)))$

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1-pointed $\mathcal{P}(A\times(-))$-coalgebra

$1 \xrightarrow{x_0} X \xrightarrow{\xi} \mathcal{P}(A\times X)$

$$\begin{array}{c} (X, x_0, \Delta) \\ h \downarrow \\ (Y, y_0, \Delta') \end{array} \iff$$

Lax Coalgebra Homomorphism

Point-wise order $\subseteq$ on $\mathsf{Set}(X, \mathcal{P}Z)$

$$\begin{array}{ccccc} 1 & \xrightarrow{x_0} & X & \xrightarrow{\xi} & \mathcal{P}(A\times X) \\ & \circlearrowleft & h\downarrow & \subseteq & \downarrow \mathcal{P}(A\times h) \\ & \searrow^{y_0} & Y & \xrightarrow[\zeta]{} & \mathcal{P}(A\times Y) \end{array}$$

Pointed Coalgebras & Lax homomorphisms

$\mathsf{LTS}_A \iff \mathsf{LCoalg}(1, \mathcal{P}(A \times (-)))$

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$(X, x_0, \Delta) \xrightarrow{h} (Y, y_0, \Delta')$ $\iff$

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$$\begin{array}{ccccc} 1 & \xrightarrow{x_0} & X & \xrightarrow{\xi} & \mathcal{P}(A\times X) \\ & \circlearrowleft & \downarrow h & \subseteq & \downarrow \mathcal{P}(A\times h) \\ & \xrightarrow{y_0} & Y & \xrightarrow{\zeta} & \mathcal{P}(A\times Y) \end{array}$$

$h\colon X \to Y$ is open $\overset{X \text{ reachable}}{\iff}$ h (proper) coalgebra homomorphism

$$\zeta \cdot h = \mathcal{P}(A\times h) \cdot \xi$$

Pointed Coalgebras & Lax homomorphisms

$\mathsf{LTS}_A \iff \mathsf{LCoalg}(I, T \cdot F)$ e.g. $TX = \mathcal{P}X$, $FX = A \times X$

$X, x_0 \in X,$
$\Delta \subseteq X \times A \times X \iff$

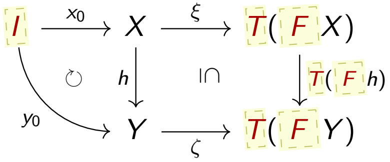
I-pointed $T(F(-))$-coalgebra

$I \xrightarrow{x_0} X \xrightarrow{\xi} T(FX)$

$(X, x_0, \Delta) \xrightarrow{h} (Y, y_0, \Delta') \iff$

Lax Coalgebra Homomorphism

Point-wise order $\subseteq$ on $\mathbb{C}(X, TZ)$



$h \colon X \to Y$ is open $\overset{X \text{ reachable}}{\iff}$ h (proper) coalgebra homomorphism $\zeta \cdot h = T(Fh) \cdot \xi$

Main Result

Theorem

Branching
Input

Given:

- Functors $T, F\colon \mathbb{C} \to \mathbb{C}$ with order $\subseteq$ on $\mathbb{C}(X, TY)$
- F admits precise factorizations w.r.t. $\mathcal{S} \subseteq |\mathbb{C}|$
- $\mathrm{Id} \xrightarrow{\eta} T \xleftarrow{\perp} 1$ plus axioms (T powerset-like)

Then there is a path category $\mathrm{Path}(I, F+1)$
and for every $h\colon X \to Y$ in $\mathrm{LCoalg}(I, TF)$:

- h open & X reachable $\Longrightarrow$ h coalgebra homomorphism
- h coalgebra homomorphism $\Longrightarrow$ h open

Canonical Trace Semantics: $\mathrm{LCoalg}(I, TF) \to \coprod_{n\geq 0} \mathbb{C}(I, F^n 1)$

$\mathrm{trace}(X) = \{[w] \mid Jw \to X\}$ ← preserved by coalgebra hom.

Definition: F-precise morphism

$f\colon X \to FY$ is F-**precise** if

$$
\begin{array}{ccc}
X & \xrightarrow{g} & FW \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Fw} \\
FY & \xrightarrow{Fz} & FZ
\end{array}
\quad \overset{\exists d}{\Longrightarrow} \quad
\begin{array}{ccc}
X & \xrightarrow{g} & FW \\
{\scriptstyle f}\downarrow & \nearrow{\scriptstyle Fd} & \\
FY & &
\end{array}
\quad \& \quad
\begin{array}{ccc}
 & & C \\
 & \overset{d}{\nearrow} & \downarrow{\scriptstyle w} \\
Y & \xrightarrow{z} & D
\end{array}
$$

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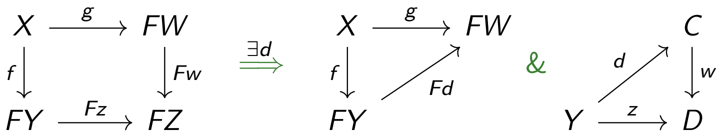
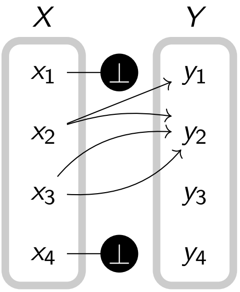


Intuition in Sets

$f\colon X \to FY$ is F-precise $\iff$ every $y \in Y$ is mentioned precisely once in the definition of f

precise = every $y \in Y$ is mentioned precisely once

Example: $FY = Y \times Y + \{\bot\}$ and $f\colon X \to FY$



$f\colon\ X \longrightarrow Y \times Y + \{\bot\}$

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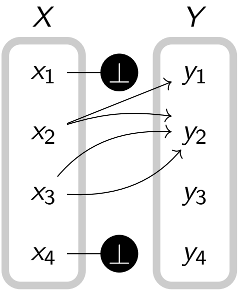

$f\colon\ X \longrightarrow Y \times Y + \{\bot\}$

$$
\begin{aligned}
f(x_1) &= \bot \\
f(x_2) &= (y_1, y_2) \\
f(x_3) &= (y_2, y_2) \\
f(x_4) &= \bot
\end{aligned}
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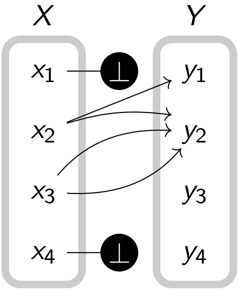

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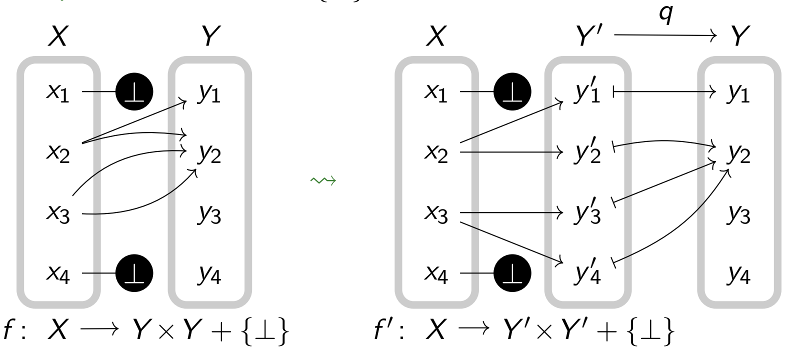

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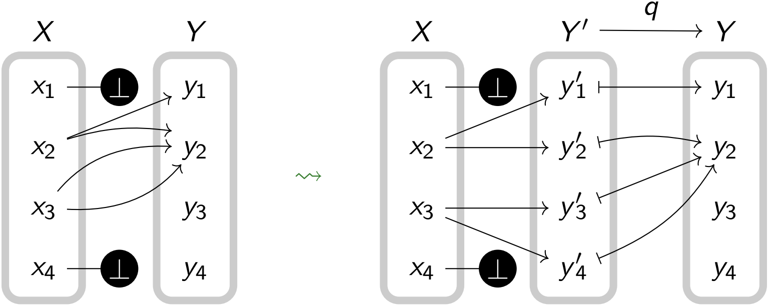
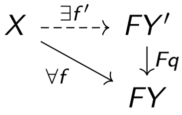


$f\colon\ X \longrightarrow Y \times Y + \{\bot\}$ $\qquad$ $f'\colon\ X \longrightarrow Y' \times Y' + \{\bot\}$

Def.: $F\colon \mathbb{C} \to \mathbb{C}$ admits precise factorizations w.r.t. $\mathcal{S} \subseteq |\mathbb{C}|$

$\forall f,\ X \in \mathcal{S}$,
$\exists Y' \in \mathcal{S}$, f' precise:



Proposition

The following functors admit precise factorizations w.r.t. $\mathcal{S}$:

1. Constant functors if $0 \in S$
2. Products of such functors if $\mathcal{S}$ closed under products
3. Coproducts of such functors if $\mathbb{C}$ extensive and $\mathcal{S}$ closed under coproducts
4. Right adjoints $R \iff$ the left adjoint L preserves $\mathcal{S}$-objects

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1. Polynomial functors
2. Analytic functors, e.g. the bag functor (finite multisets)
3. The binding functor $[\mathbb{A}]$ on Nominal Sets $(\mathbb{A}\#(-) \dashv [\mathbb{A}])$

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Non-Example

Powerset $\mathcal{P}$ because $f(x) = \{y\} = \{y, y\}$

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- h open & X reachable ? $\implies$ h coalgebra homomorphism
- h coalgebra homomorphism $\implies$ h open

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Definition

Path P of length n

$$I = P_0 \xrightarrow{p_0} FP_1 \quad P1 \xrightarrow{p_1} FP_2 \quad \cdots \quad FP_{n+1}$$

F-precise map F-precise map

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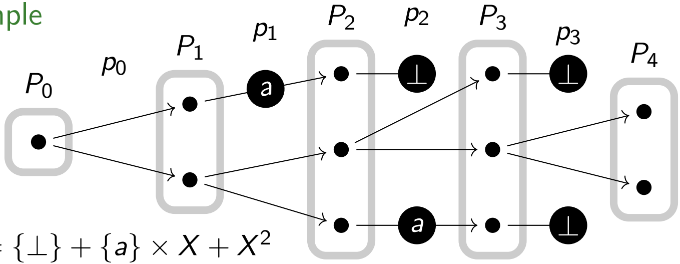


$$FX = \{\perp\} + \{a\} \times X + X^2$$

Definition: Category Path(I, F)

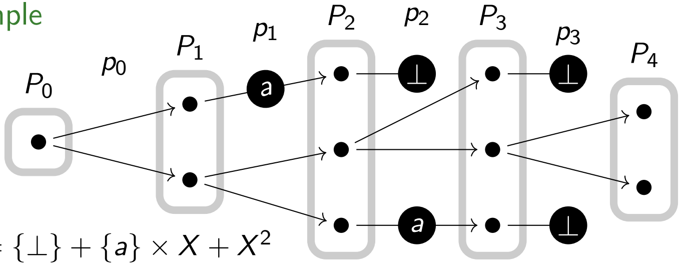
Path P of length n

$$
\begin{array}{ccccccccccc}
I = P_0 & \xrightarrow{p_0} & FP_1 & P1 & \xrightarrow{p_1} & FP_2 & \cdots & FP_{n+1} & & \\
\phi_0 \downarrow \cong & & F\phi_0 \downarrow & \phi_1 \downarrow \cong & & F\phi_1 \downarrow & \cdots & F\phi_{n+1} \downarrow \cong & & \\
I = Q_0 & \xrightarrow{q_0} & FQ_1 & Q1 & \xrightarrow{q_1} & FQ_2 & \cdots & FQ_{n+1} & \cdots & FQ_{m+1}
\end{array}
$$

prefix order

$m \geq n$

Example



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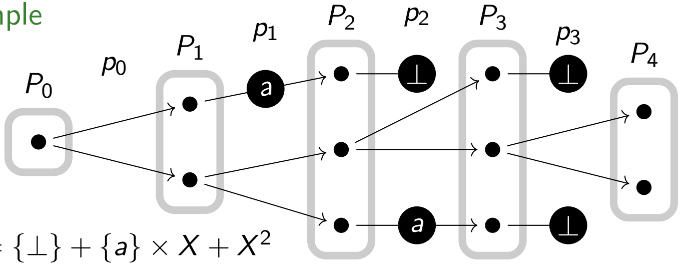

$$FX = \{\bot\} + \{a\} \times X + X^2$$

Relation to final chain of F

Truncation order

Full functor $\quad \text{Path}(I, F) \twoheadrightarrow (\coprod_{n \geq 0} \mathbb{C}(I, F^n 1), \leq)$

Main Result

Theorem

Given:

Branching

Input

- Functors $T, F\colon \mathbb{C} \to \mathbb{C}$ with order $\subseteq$ on $\mathbb{C}(X, TY)$
- F admits precise factorizations w.r.t. $\mathcal{S} \subseteq |\mathbb{C}|$ ✓
- $\mathrm{Id} \xrightarrow{\eta} T \xleftarrow{\perp} 1$ plus axioms (T powerset-like) ?

Then there is a path category $\mathrm{Path}(I, F+1)$?
and for every $h\colon X \to Y$ in $\mathrm{LCoalg}(I, TF)$:

- h open & X reachable ? $\implies$ h coalgebra homomorphism
- h coalgebra homomorphism $\implies$ h open

Canonical Trace Semantics: $\mathrm{LCoalg}(I, TF) \to \coprod_{n\geq 0} \mathbb{C}(I, F^n 1)$

$\mathrm{trace}(X) = \{[w] \mid Jw \to X\}$ ← preserved by coalgebra hom.

Main Result

Theorem

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singletons bottom

Pointings: $\mathrm{Id} \xrightarrow{\eta} T \xleftarrow{\perp} 1$ plus Axioms (powerset-like)

Open map situation $J\colon \mathrm{Path}(I, F+1) \to \mathrm{LCoalg}(I, TF)$

$$P_0 \xrightarrow{p_0} FP_1 + 1 \quad P_1 \xrightarrow{p_1} FP_2 + 1$$

$$\downarrow J$$

$$P_0 + P_1 + P_2 \longrightarrow F(P_1 + P_2) + 1 \xrightarrow{[\eta,\perp]} TF(P_0 + P_1 + P_2)$$

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✓

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Reachability

$$J\colon \mathrm{Path}(I, F+1) \longrightarrow \mathrm{LCoalg}(I, TF)$$

Definition: $(X, x_0, \xi) \in \mathrm{LCoalg}(I, TF)$ is reachable

..., if the runs $f\colon JP \to (X, x_0, \xi)$ are jointly surjective.

Theorem

(X, x_0, ξ) is reachable iff it has no proper subcoalgebra.

Coalgebraic definition of reachability

Main Result

Theorem

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Instances	Tree Automata
$\mathbb{C}$	Set
$\mathcal{S} \subseteq \lvert\mathbb{C}\rvert$	all
I	1
T	$\mathcal{P}$, $\mathcal{P}_{\mathrm{f}}$
$F(X)$	analytic functors: polynomials Σ, finite multisets
$\mathbb{C}(I, F^n 1)$	Trees, height n
trace	Tree language

Instances	Tree Automata	Nominal Automata
$\mathbb{C}$	Set	Nominal Sets
$\mathcal{S} \subseteq \lvert\mathbb{C}\rvert$	all	strong ones
I	1	$\mathbb{A}^{\#k}$
T	$\mathcal{P}$, $\mathcal{P}_{\mathrm{f}}$	$\mathcal{P}_{\mathsf{ufs}}$, $\mathcal{P}_{\mathrm{f}}$
$F(X)$	analytic functors: polynomials Σ, finite multisets	$1 + [\mathbb{A}]X + \mathbb{A} \times X$ (RNNA) [Schröder, Milius Kozen, W, '17]
$\mathbb{C}(I, F^n 1)$	Trees, height n	bar-strings $\lvert\text{support}\rvert \leq k$
trace	Tree language	bar language

Instances	Tree Automata	Nominal Automata	Lasota's construction for $\mathbb{P} \hookrightarrow \mathbb{M}$ [Lasota '02]
$\mathbb{C}$	Set	Nominal Sets	$\mathsf{Set}^{\lvert P \rvert}$
$\mathcal{S} \subseteq \lvert\mathbb{C}\rvert$	all	strong ones	all
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T	$\mathcal{P}$, $\mathcal{P}_\mathrm{f}$	$\mathcal{P}_\mathsf{ufs}$, $\mathcal{P}_\mathrm{f}$	$\mathcal{P}$ (per component)
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What about **weighted systems**?

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...still subsumes all open map situations

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Axioms on T restrict to $\mathcal{P}$..

Need: generalization of open maps

...still subsumes all open map situations

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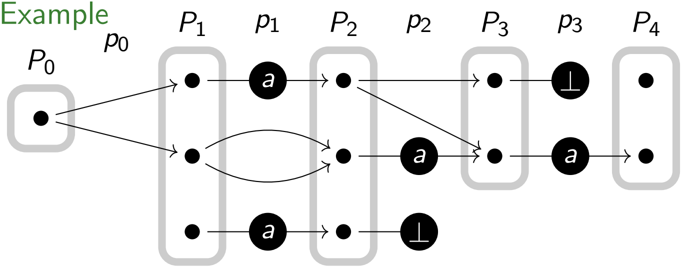
$$FX = \{a\} \times X + X \times X,\ I = \{\bullet\},\ p_k\colon P_k \to FP_{k+1} + \{\bot\}$$

Non-Example



Tree automata in Sets

$I = 1$, T is $\mathcal{P}$ or $\mathcal{P}_{\mathrm{f}}$, F is analytic.

$$FX = \coprod_{\sigma/n \in \Sigma} X^n / G_\sigma$$

Path(I, F)

Path of length n = 'partial' F-tree of height n.

TF-coalgebra homomorphisms

... are open morphisms and thus preserve & reflect tree languages.

RNNA

skmw17 'skmw17

TF-coalgebra for $T = \mathcal{P}_{\mathsf{ufs}}$ $\qquad FX = 1 + \mathbb{A} \times X + [\mathbb{A}]X$

$I := \mathbb{A}^{\#n}$, fixed $n \in \mathbb{N}$ $\qquad \mathcal{S}$ = Strong nominal sets

F-precise maps

- ... don't loose support
- ... don't loose order in the support
- if $f \colon \mathbb{A}^{\#n} \to FY$ is F-precise, then $Y = \mathbb{A}^{\#m}$ with $n \leq m \leq n + 1$.

Path(I, F)

Finite sequence of $F + 1$-precise maps

$\Rightarrow$ essentially bar strings

Lasota's construction for arbitrary $J\colon \mathbb{P} \hookrightarrow \mathbb{M}$ lasota02

'lasota02

Let $J0_{\mathbb{P}} = 0_{\mathbb{M}}$ and the pointing[a] $I = \chi^{0_{\mathbb{P}}}$ elsewhere.

$$\mathbb{F}\colon \mathsf{Set}^{|\mathbb{P}|} \to \mathsf{Set}^{|\mathbb{P}|} \qquad \mathbb{F}\colon (X_P)_{P\in\mathbb{P}} \mapsto \left(\prod_{Q\in\mathbb{P}} \mathcal{P}(X_Q)^{\mathbb{P}(P,Q)}\right)_{P\in\mathbb{P}}$$

Functor $\mathsf{Beh}\colon \mathbb{M} \to \mathsf{LCoalg}(I,\mathbb{F}),\ M \mapsto (\mathbb{M}(P,M))_{P\in\mathbb{P}} \ldots$

[a] No pointing in **[lasota02]**

$\mathbb{F} = T \cdot F$

$$T(X_P)_{P\in\mathbb{P}} = (\mathcal{P}X_P)_{P\in\mathbb{P}} \qquad F(X_P)_{P\in\mathbb{P}} = \Big(\coprod_{Q\in\mathbb{P}} \mathbb{P}(P,Q) \times X_Q\Big)_{P\in\mathbb{P}}$$

Path-category $\mathsf{Path}(I,F)$

$f\colon \chi^P \to FY$ F-precise iff $Y = \chi^Q$ for some $Q \in \mathbb{P}$

$\Rightarrow$ objects in $\mathsf{Path}(I,F)$ are: $0_{\mathbb{P}} \xrightarrow{m_1} P_1 \xrightarrow{m_2} P_2 \cdots \xrightarrow{m_n} P_n$

All the axioms

F	$F+1$ admits precise factorizations, w.r.t. $\mathcal{S}$ and $I \in \mathcal{S}$
T	If $(e_i\colon X_i \to Y)_{i\in I}$ jointly epic, then $f \cdot e_i \sqsubseteq g \cdot e_i$ for all $i \in I \Rightarrow f \sqsubseteq g$. $[\eta, \bot]\colon \mathrm{Id}+1 \to T$, with $\bot_Y \cdot !_X \sqsubseteq f$ for all $f\colon X \to TY$ For every $f\colon X \to TY$, $X \in \mathcal{S}$, $f = \bigsqcup\{[\eta,\bot]_Y \cdot f' \sqsubseteq f \mid f'\colon X \to Y+1\}$ $\forall A \in \mathcal{S}$