Bisimilarity of diagrams

YR-CONCUR 2017

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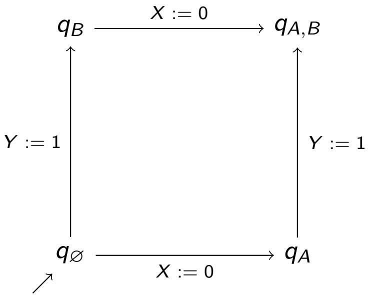
Geometry of true concurrency

Interleaving vs continuity

$$X := 0 \parallel Y := 1$$



Interleaving behaviors : A then B or B then A

Interleaving vs continuity

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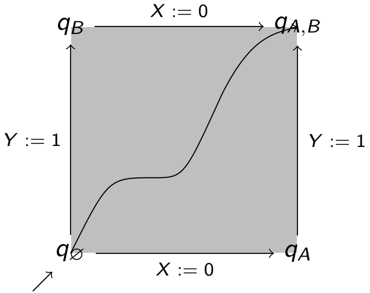


Continuous behaviors : any scheduling of A and B

True concurrency, geometrically

truly concurrent system	directed space (ex : pospace)
states	points
executions	directed paths (ex : monotonic paths)
modulo scheduling of independent actions	modulo directed homotopy

Execution spaces

states = *points* of a partially ordered space X

executions = dipaths = continuous and *monotonic* functions from $[0, 1]$ to X, noted $\overrightarrow{P}(X)$
executions from a to b = $\overrightarrow{P}(X)(a, b) = \{\gamma \in \overrightarrow{P}(X) \mid \gamma(0) = a \wedge \gamma(1) = b\}$

$\overrightarrow{P}(X)(a, b)$ can be equipped with a *topology* that has a **concurrent** meaning
Ex : its *path-connected components* correspond to directed homotopy, i.e., to **equivalence classes of executions modulo the scheduling of independent actions**

the spaces $\overrightarrow{P}(X)(a, b)$ can be finitely presented for geometric models of simple truly concurrent systems
from this finite presentation, it is possible to compute algebraic invariants

Diagram of execution spaces

$\mathcal{E}_X$ = category whose :

- objects are pairs of accessible points (a, b), such that $\exists$ a dipath from a to b
- morphisms are extensions

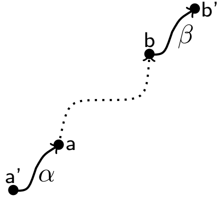
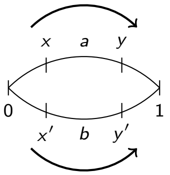


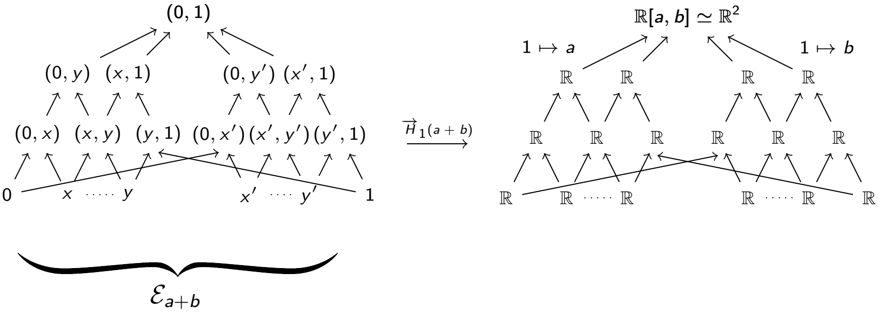
Diagram of execution spaces [D., Goubault×2] :

$$\text{Diagram } \overrightarrow{P}(X) : \mathcal{E}_X \longrightarrow \mathbf{Top}$$
$$(a, b) \longmapsto \overrightarrow{P}(X)(a, b)$$
$$(\alpha, \beta) \longmapsto (\gamma \mapsto \alpha \star \gamma \star \beta)$$

We can apply classical invariants (homology) on this diagram to produce diagrams with values in modules (real or rational vector spaces, Abelian groups)

Example of a produced diagram



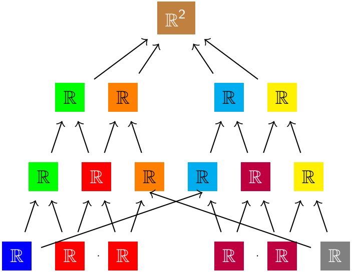


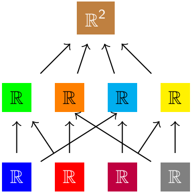
Computability

Those diagrams are not countable, so not computable. But :

Theorem [D., Goubault×2] :

When X is a geometric model of a simple truly concurrent system, we can compute a finitary diagram **equivalent** to $\overrightarrow{H}_n(X)$.





Bisimilarity of diagrams, via open maps

Bisimulations of transition systems

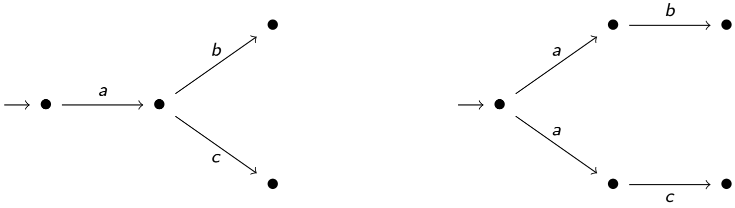
Bisimulations **[Park81]** :

A **bisimulation** between $T_1 = (Q_1, i_1, \Delta_1)$ and $T_2 = (Q_2, i_2, \Delta_2)$ is a relation $R \subseteq Q_1 \times Q_2$ such that :

(i) $(i_1, i_2) \in R$;

(ii) if $(q_1, q_2) \in R$ and $(q_1, a, q_1') \in \Delta_1$ then there is $q_2' \in Q_2$ such that $(q_2, a, q_2') \in \Delta_2$ and $(q_1', q_2') \in R$;

(iii) if $(q_1, q_2) \in R$ and $(q_2, a, q_2') \in \Delta_2$ then there is $q_1' \in Q_1$ such that $(q_1, a, q_1') \in \Delta_1$ and $(q_1', q_2') \in R$.



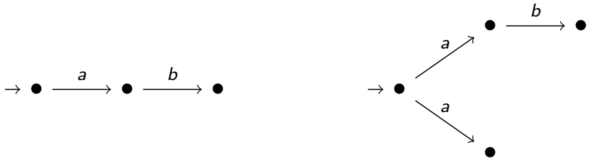
Morphisms and (bi)simulations

Morphism of TS :

A **morphism of TS** $f : T_1 = (Q_1, i_1, \Delta_1) \longrightarrow T_2 = (Q_2, i_2, \Delta_2)$ is a function $f : Q_1 \longrightarrow Q_2$ such that $f(i_1) = i_2$ and for every $(p, a, q) \in \Delta_1$,

$$(f(p), a, f(q)) \in \Delta_2.$$

A morphism always induces a simulation. But bisimulation $\neq$ simulations in both directions.



Are there conditions on morphisms to enforce the existence of a bisimulation ?

Morphisms and (bi)simulations

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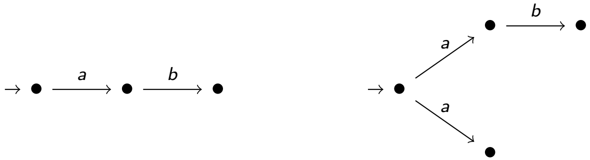
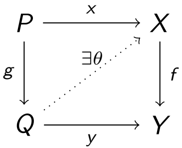


Are there conditions on morphisms to enforce the existence of a bisimulation ?

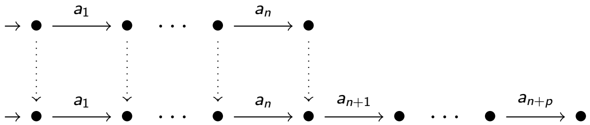
Yes, if they lift transitions **[Joyal et al. 94]**

Lifting properties and open morphisms (in TS)

f has the **right lifting property** with respect to g iff



A morphism is **open [Joyal et al. 94]** if it has the right lifting property with respect to every :



Theorem :

Two systems are bisimilar iff there is a span of open morphisms between them.

Lifting properties and open morphisms (in diagrams)

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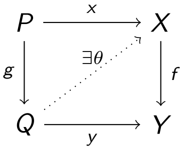

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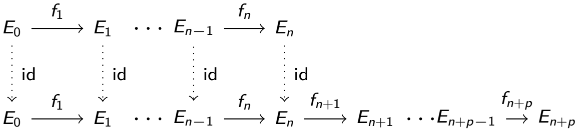


Definition :

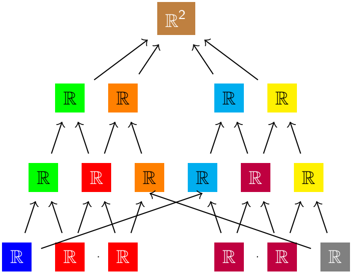
Two diagrams are bisimilar iff there is a span of open morphisms between them.

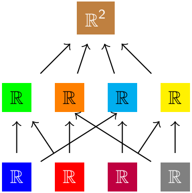
Computability

Those diagrams are not countable, so not computable. But :

Theorem [D., Goubault×2] :

When X is a geometric model of a simple truly concurrent system, we can compute a finitary diagram **bisimilar** to $\overrightarrow{H}_n(X)$.





Two characterizations of bisimilarity

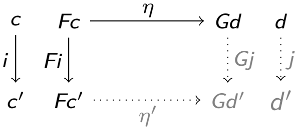
Bisimulation of diagrams

Bisimulation between $F : C \longrightarrow \mathcal{M}$ and $G : D \longrightarrow \mathcal{M}$
$=$ set R of triples (c, η, d) such that :

- c is an object of C,
- d is an object of D,
- $\eta : F(c) \longrightarrow G(d)$ is an isomorphism of modules

satisfying :

- for every object c of C, there exists d and η such that $(c, \eta, d) \in R$
- $(c, \eta, d) \in R$



$(c', \eta', d') \in R$

and symmetrically

Similar to bisimulations of event structures **[Rabinovitch, Trakhtenbrot 88]**.

Bisimilarity and bisimulations

Proposition [D.] :

Two diagrams are bisimilar iff there is a bisimulation between them.

In the case of finitary diagrams with values in finite dimensional real vector spaces, bisimilarity becomes a problem of matrices !

$\rightarrow$ It can be expressed as the existence of invertible matrices satisfying linear conditions which can be encoded in the existential theory of the reals

Theorem [D.] :

Knowing if two finitary diagrams are bisimilar is decidable in EXPSPACE.

Theorem **[D., Goubault×2]** :

When X is a geometric model of a simple truly concurrent system, we can compute a finitary diagram **bisimilar** to $\overrightarrow{H}_n(X)$. It is then decidable wether two such systems have the same diagrams (modulo bisimulation).

Diagrammatic logic

Object formulae : $S ::= [x]P$ $\qquad x \in \mathrm{Ob}(\mathcal{M})$

Morphism formulae : $P ::= \langle f \rangle P \mid ?S \mid \neg P \mid \bigwedge_{i \in I} P_i$ $\qquad f \in \mathrm{Mor}(\mathcal{M})$ and I a set

- $[x]P$ means "at the current states, the value of the diagram is isomorphic to x",
- $\langle g \rangle P$ means "at the current states, there is a outgoing morphism in the diagram that is equivalent to g".

Diagrammatic logic

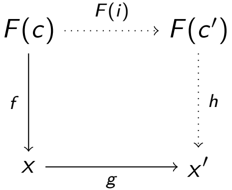
Object formulae : $S ::= [x]P$ $\qquad x \in \mathrm{Ob}(\mathcal{M})$

Morphism formulae : $P ::= \langle f \rangle P \mid ?S \mid \neg P \mid \bigwedge_{i \in I} P_i$ $\qquad f \in \mathrm{Mor}(\mathcal{M})$ and I a set

For $F : \mathcal{C} \longrightarrow \mathcal{M}$:

- $F, c \models [x]P$ iff there exists an isomorphism $f : F(c) \longrightarrow x$ of $\mathcal{M}$ such that $F, f, c \models P$,

- $F, f, c \models \langle g \rangle P$ iff $g : x \longrightarrow x'$ and there exists $i : c \longrightarrow c'$ in $\mathcal{C}$ and an isomorphism $h : F(c') \longrightarrow x'$ such that $h \circ F(i) = g \circ f$ and $F, h, c' \models P$,



Bisimilarity and logic

We say that a diagram $F : \mathcal{C} \longrightarrow \mathcal{M}$ is **logically simulated** by another diagram $G : \mathcal{D} \longrightarrow \mathcal{M}$ if for every object c of $\mathcal{C}$, there exists an object d of $\mathcal{M}$ such that for all object formula S, $F, c \models S$ iff $G, d \models S$. Two diagrams F and G are **logically equivalent** if F is logically simulated by G and G is logically simulated by F.

Proposition [D.] :

Two diagrams are bisimilar iff they are logically equivalent.

Similarly, in the case of finitary diagrams with values in finite dimensional real vector spaces, bisimilarity becomes a problem of matrices !

Theorem [D.] :

Knowing if a finitary diagram satisfies a positive formula in decidable in PSPACE, the full case being in EXPSPACE.

Decidability in the finitary case

Finitary diagrams

- a finite poset $\mathcal{C}, \leq$, the **domain**,
- for every element c of $\mathcal{C}$, a natural number $F(c)$ (which stands for the real vector space $\mathbb{R}^{F(c)}$),
- for every pair $c \leq c'$ of $\mathcal{C}$, a matrix $F(c \leq c')$ of size $F(c) \times F(c')$, with coefficient in rationals,

such that :

- $F(c \leq c)$ is the identity matrix,
- for every triple $c \leq c' \leq c''$, $F(c \leq c'') = F(c' \leq c'').F(c \leq c')$.

In short, a finitary diagram is a functor from a finite poset to the category of matrices in rationals.

Bisimulation of diagrams

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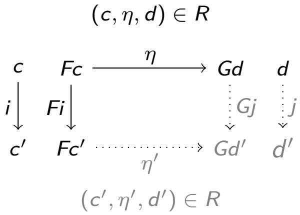
satisfying :

- for every object c of C, there exists d and η such that $(c, \eta, d) \in R$
- 

and symmetrically

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From bisimilarity to a problem of matrices

Given two finitary diagrams F with domain $\mathcal{C}$ and G with domain $\mathcal{D}$:

- guess a bisimulation $R = \{(c, \eta, d)\}$, excepting the isomorphism part η,
- for every "$(c, \eta, d) \in R$", we create a fresh variable X (for the matrix η),
- for every "$(c, \eta, d) \in R$", we check that $F(c) = G(d)$,
- for every $c \in \mathcal{C}$, we check that there is a "$(c, \eta, d) \in R$",
- for every "$(c, \eta, d) \in R$" with variable X, and every $c' > c$, guess a "$(c', \eta', d') \in R$" with variable X' with $d' \geq d$. This produces an equation $G(d \geq d').X = X'.F(c \geq c')$ to be checked.

Result : collection of equations $A.X = X'.B$, for some variables X, X', ...

F and G are bisimilar iff there are non-deterministic choices and values of the variables that satisfies the equations.

How can we check that there is such invertible matrices ?

From a problem of matrices to the existential theory of the reals

Given :

- a set of variables X, which represent invertible matrices (the size is known),
- a set of equations $A.X = X'.B$ with A and B rational matrices.

Produce : a set of equations in **reals**, which has a solution iff the matrix equations have a solution. Check it using the existential theory of the reals (decidable).

X of size $n \longrightarrow$ create n^2 real variables $(x_{i,j})$ representing the coefficients of X

$A.X = X'.B \longrightarrow$ linear equations on $x_{i,j}$ and $x'_{i,j}$ by computing the multiplications

X invertible $\longrightarrow$ create n^2 new variables $(y_{i,j})$ representing the coefficients of a matrix Y, which will be the inverse of X. Produce $2n^2$ polynomial equations in reals by developing $X.Y = \mathrm{Id}$ and $Y.X = \mathrm{Id}$.

Other categories

- **finite sets :** only a finite number of possible bisimulations $\longrightarrow$ **decidable**,
- **presentation of groups and homomorphisms :** isomorphism is already **undecidable**,
- **rational vector spaces :** coincide with real case $\longrightarrow$ **decidable**,
- **Abelian groups of finite type : open question**.

Conclusion

- A notion of bisimilarity of diagrams with applications to directed algebraic topology
- General characterizations for any category :
  - using open morphisms (initial def., nice for the theory),
  - using bisimulations (better for computation),
  - using logic (better for giving certificates)
- Decidability depends mainly on decidability of isomorphism in the category

  $\rightarrow$ using the existential theory of the reals, the finitary case on real vector spaces is decidable in EXPSPACE