Hybrid System Falsification and Reinforcement Learning
Formal Method for Cyber-Physical Systems

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SOKENDAI lesson, July 1, 8, and 22
Lecture structure

Lectures

- 1st: falsification (problem, framework, logics...), by me
- 2nd: deep learning for falsification (learning, reinforcement learning, application to falsification...), by David
- 3rd: advanced techniques in falsification and reinforcement learning, by David and me

Evaluation

Easy practical assignment (in Python).

Questions?

- Ask them during the lesson.
- Find me at my desk (Palaceside building).
- clovis.eberhart@gmail.com
First lecture overview

1. Formal methods landscape
2. Framework
3. Hybrid systems
4. Formulas
5. Optimisation
Formal method landscape

Kapinski, Deshmukh, Jin, Ito, Butts, *Simulation-Based Approaches for Verification of Embedded Control Systems*, IEEE Control Magazine, 2010
Testing

Given: a system $S$, a property $\varphi$.
Goal: generate a test suite $\{t_i\}_{i \in I}$. 
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Goal: generate a test suite $\{t_i\}_{i \in I}$.

Characteristics

- **simple** (run tests)
- **black-box** (unknown systems)
- **versatile** (guarantees, explainable failures...)
- no formal guarantee
- too general
Verification

Given: a model $\mathcal{M}$, a property $\varphi$.
Goal: automatically prove that $\mathcal{M} \models \varphi$. 
## Verification

**Given:** a model $\mathcal{M}$, a property $\varphi$.

**Goal:** automatically prove that $\mathcal{M} \models \varphi$.

### Characteristics

- **complex** (design model, use specific techniques, so typically not used by engineers)
- **white-box** (known systems only)
- **formal proof** (strong guarantee)
- **ill-suited to CPS** (continuous systems)
### Verification

**Given**: a model $\mathcal{M}$, a property $\varphi$.

**Goal**: automatically prove that $\mathcal{M} \vDash \varphi$.

### Characteristics

- **complex**: (design model, use specific techniques, so typically not used by engineers)
- **white-box**: (known systems only)
- **formal proof**: (strong guarantee)
- **ill-suited to CPS**: (continuous systems)

### Theorem proving

**Given**: a model $\mathcal{M}$, a property $\varphi$.

**Goal**: prove that $\mathcal{M} \vDash \varphi$. 
Falsification

Given: a system $S$, a property $\varphi$.
Goal: generate a counterexample to $S \models \varphi$. 
Falsification

**Falsification**

Given: a system $S$, a property $\varphi$.
Goal: generate a counterexample to $S \vDash \varphi$.

**Characteristics**
- particular case of testing
- black-box (unknown systems)
- relatively simple
- no proof (no formal guarantee)
Verification versus falsification

- Verification:
  - finds a proof: system verifies property,
  - finds nothing:

- Falsification:
  - finds a counterexample: system violates property,
  - finds nothing:
Verification versus falsification

- **Verification:**
  - finds a proof: system verifies property,
  - finds nothing: nothing can be said.

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  - finds a counterexample: system violates property,
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Verification versus falsification

- **Verification:**
  - finds a proof: *system verifies property*,
  - finds nothing: *nothing can be said*.

- **Falsification:**
  - finds a counterexample: *system violates property*,
  - finds nothing: *nothing can be said*.

**Interaction:**

- verification for falsification: constraining state space by reachability analysis,
- falsification for verification: coverage-based falsification techniques.
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General framework

Reminder

Given: a system $S$, a property $\varphi$.
Goal: generate a counterexample to $S \models \varphi$. 
General framework

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Questions

- What is a system?
General framework

Reminder
Given: a system $S$, a property $\varphi$.
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Questions
- What is a system? $\leadsto$ hybrid system
Reminder

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Questions

- What is a system? $\leadsto$ hybrid system
- What is a property?
Reminder

Given: a system $S$, a property $\varphi$.
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Questions

- What is a system? $\leadsto$ hybrid system
- What is a property? $\leadsto$ logical formula
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Questions
- What is a system? $\leadsto$ hybrid system
- What is a property? $\leadsto$ logical formula
- What is a counterexample?
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Questions
- What is a system? $\leadsto$ hybrid system
- What is a property? $\leadsto$ logical formula
- What is a counterexample? $\leadsto$ an input (and output) signal to the system that violates the property
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Goal: generate a counterexample to $S \models \varphi$.

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- What is a system? $\leadsto$ hybrid system
- What is a property? $\leadsto$ logical formula
- What is a counterexample? $\leadsto$ an input (and output) signal to the system that violates the property

Challenges
- infinite (and high-dimensional) search space
- non-linear dynamics
The falsification loop

\[ \sigma_{\text{in}} \xrightarrow{\mathcal{S}} \varphi \xrightarrow{\varphi} \sigma_{\text{in}} \backslash (\sigma_{\text{in}},\sigma_{\text{out}}) \]
The falsification loop

\[ S \]

\[ \sigma_{\text{in}} \]

\[ \varphi \]

\[ \sigma_{\text{out}} \]

\[ \sigma_{\text{out}} \models \varphi \]

\[ \sigma_{\text{out}} \not\models \varphi \]

\[ \sigma_{\text{in}} / (\sigma_{\text{in}}, \sigma_{\text{out}}) \]
The falsification algorithm

Input: A system $S$, a formula $\varphi$, a satisfaction predicate $\models$, and a timeout $t_{\text{max}}$
Output: A signal $\sigma_{\text{in}}$ such that $S(\sigma_{\text{in}}) \not\models \varphi$

found $= \text{false}$;
while not(found) and $t < t_{\text{max}}$ do
    $\sigma_{\text{in}} = \text{generate}()$;
    $\sigma_{\text{out}} = S(\sigma_{\text{in}})$;
    found $= \sigma_{\text{out}} \not\models \varphi$;
end
if found then
    return $\sigma_{\text{in}}$
else
    return "timeout"
end
Optimisation-based falsification

\[ \sigma_{\text{in}} \rightarrow S \rightarrow \varphi \rightarrow \text{falsification} \]

\[ \frac{\sigma_{\text{in}}}{(\sigma_{\text{in}}, \sigma_{\text{out}})} \]
Optimisation-based falsification

\[ \sigma_{in} \]

\[ \mathcal{S} \]

\[ \sigma_{out} \]

\[ \varphi \]

\[ \rho(\sigma_{out}, \varphi) \geq 0 \]

\[ \rho(\sigma_{out}, \varphi) < 0 \]

\[ \sigma_{in}/(\sigma_{in}, \sigma_{out}) \]

\[ \text{falsification} \]

\[ \text{generator} \]
The optimisation-based falsification algorithm

**Input:** A system $S$, a formula $\varphi$, a robustness function $\rho$, and a timeout $t_{\text{max}}$

**Output:** A signal $\sigma_{\text{in}}$ such that $S(\sigma_{\text{in}}) \not\models \varphi$

```plaintext
found = false;
while not(found) and $t < t_{\text{max}}$ do
    $\sigma_{\text{in}} = \text{search}\_\text{minimum}(\rho);$  
    $\sigma_{\text{out}} = S(\sigma_{\text{in}});$  
    found = $\rho(\sigma_{\text{out}}, \varphi) < 0;$
end
if found then
    return $\sigma_{\text{in}}$
else
    return “timeout”
end
```

Required: $\rho(\sigma, \varphi) \geq 0 \iff \sigma \models \varphi$
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Hybrid systems

Definition

A hybrid system is a dynamical system that exhibits both continuous and discrete dynamic behavior – a system that can both flow (described by a differential equation) and jump (described by a state machine or automaton).

Hybrid system: definition

A **Hybrid system** is a tuple \( \mathcal{H} = (Q, X, \text{GUARD}, \text{JUMP}, U, \text{FLOW}) \) of:

- a finite set of **modes** \( Q \),
- a family of continuous state spaces \( X = \{ X_q \subseteq \mathbb{R}^{n_q} \mid q \in Q \} \),
- \( \text{GUARD}_{q, q'} \subseteq X_q \) is the set of states in \( X_q \) that can transition to mode \( q' \),
- \( \text{JUMP}_{q, q'} : X_q \to X_{q'} \) describes the transition from \( q \) to \( q' \),
- \( U \) is the **input space**,  
- \( \text{FLOW}_q \) is a set of differential equations in \( X_q \) and \( U \), seen as a function \( X_q \times U \times \mathbb{R}_{\geq 0} \to X_q \).
Example of hybrid system: thermostat

\[ \frac{dT}{dt} = -KT \quad T \geq T_0 \]

\[ \frac{dT}{dt} = K(h - T) \quad T \leq T_1 \]

\[ T = T_0 \| \cdot \]

\[ T = T_1 \| \cdot \]
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\[ Q = \{ q_0, q_1 \}, \]

\[ X_{q_0} = \{ T \in \mathbb{R} \mid T \geq T_0 \}, \]
Example of hybrid system: thermostat

\[
\begin{align*}
  q_0 & \quad \frac{dT}{dt} = -KT \\
  T & \geq T_0
\end{align*}
\]

\[
\begin{align*}
  q_1 & \quad \frac{dT}{dt} = K(h - T) \\
  T & \leq T_1
\end{align*}
\]

- \( Q = \{ q_0, q_1 \} \),
- \( X_{q_0} = \{ T \in \mathbb{R} \mid T \geq T_0 \} \),
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\end{align*} \]

- \( Q = \{ q_0, q_1 \} \),
- \( X_{q_0} = \{ T \in \mathbb{R} | T \geq T_0 \} \),
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- \( U = \emptyset \)
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- \( \text{FLOW}_{q_0} = (\frac{dT}{dt} = -KT) \)
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q_0 \quad &\frac{dT}{dt} = -KT \quad T \geq T_0 \\
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Example of hybrid system: timed automata

![Diagram of timed automata](image)
Example of hybrid system: timed automata

\[ Q = \{ q_0, q_1, q_2 \}, \]
Example of hybrid system: timed automata

\[ Q = \{ q_0, q_1, q_2 \}, \]

\[ X_{q_0} = \mathbb{R}^2, \]
Example of hybrid system: timed automata

\[ Q = \{ q_0, q_1, q_2 \}, \]
\[ X_{q_0} = \mathbb{R}^2, \]
\[ X_{q_1} = \left\{ (x, y) \in \mathbb{R}^2 \mid y \leq 5 \right\}, \]
Example of hybrid system: timed automata

- $Q = \{q_0, q_1, q_2\}$,
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- $\text{GUARD}_{q_0,q_1} = \mathbb{R}^2$
Example of hybrid system: timed automata

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- $\text{JUMP}_{q_1, q_1}(x, y) = (0, y)$
- $U = \emptyset$
- $\text{FLOW}_{q} = \left(\frac{dx}{dt} = \frac{dy}{dt} = 1\right)$
Example of hybrid system: navigation

- \( Q = \{(i, j) \mid i, j \in [0..4]\} \)
- \( X: \)
  - \( X_{(i,j)} = [0, 1] \times [0, 1] \times \mathbb{R} \times \mathbb{R} \) \((i, j \in [1..3])\)
  - \( X_{(0,j)} = (-\infty, 1] \times [0, 1] \times \mathbb{R} \times \mathbb{R} \) \((j \in [1..3])\)
  - \( X_{(4,j)} = [0, \infty) \times [0, 1] \times \mathbb{R} \times \mathbb{R} \) \((j \in [1..3])\)
  - \( X_{(i,0)} = [0, 1] \times (-\infty, 1] \times \mathbb{R} \times \mathbb{R} \) \((i \in [1..3])\)
  - \( X_{(i,4)} = [0, 1] \times [0, \infty) \times \mathbb{R} \times \mathbb{R} \) \((i \in [1..3])\)
  - \( X_{(0,0)} = (-\infty, 1] \times (-\infty, 1] \times \mathbb{R} \times \mathbb{R} \)
  - \( \ldots \)
Example of hybrid system: navigation

- **GUARD:**
  - $\text{GUARD}_{(i,j),(i-1,j)} = \{(0, y, v_x, v_y)\}$
  - $\text{GUARD}_{(i,j),(i+1,j)} = \{(1, y, v_x, v_y)\}$
  - $\text{GUARD}_{(i,j),(i,j-1)} = \{(x, 0, v_x, v_y)\}$
  - $\text{GUARD}_{(i,j),(i,j+1)} = \{(x, 1, v_x, v_y)\}$

- **JUMP:**
  - $\text{JUMP}_{(i,j),(i-1,j)}(x, y, v_x, v_y) = (x + 1, y, v_x, v_y)$
  - $\text{JUMP}_{(i,j),(i+1,j)}(x, y, v_x, v_y) = (x - 1, y, v_x, v_y)$
  - $\text{JUMP}_{(i,j),(i,j-1)}(x, y, v_x, v_y) = (x, y + 1, v_x, v_y)$
  - $\text{JUMP}_{(i,j),(i,j+1)}(x, y, v_x, v_y) = (x, y - 1, v_x, v_y)$
Example of hybrid system: navigation

- \( U = ([−0.1, 0.1]^{\mathbb{R}_{\geq 0}})^2 \)
- FLOW\(_{i,j}\):
  - \( \frac{dx}{dt} = v_x + u_x(t) \)
  - \( \frac{dy}{dt} = v_y + u_y(t) \)
  - \( \frac{dv_x}{dt} = 0.1(v_y - v_y^{(i,j)}) - 1.2(v_x - v_x^{(i,j)}) \)
  - \( \frac{dv_x}{dt} = 0.1(v_x - v_x^{(i,j)}) - 1.2(v_y - v_y^{(i,j)}) \)

where
- \((u_x, u_y) \in U\)
- \(v_x^{(i,j)}\) and \(v_y^{(i,j)}\) are constants.
Run of a hybrid automaton

Run

Finite or infinite sequence \((q_0, x_0) \rightarrow_{t_0} (q_1, x_1) \rightarrow_{t_1} \ldots\) such that, for each \(i \in [1..n - 1]\):

- \(y_i \in \text{GUARD}(q_i, q_{i+1})\),
- \(\text{JUMP}_{q_i, q_{i+1}}(y_i) = x_{i+1}\),

where \(y_i = \text{FLOW}_{q_i}(x_i, u, t_i - t_{i-1})\).

\[
(1, 2, x_0) \rightarrow_{t_0} (2, 2, x_1) \rightarrow_{t_1} (2, 1, x_2) \rightarrow_{t_2} (2, 0, x_3)
\]

\[
((0, 1), x'_0) \rightarrow_{t'_0} (0, 0, x'_1) \rightarrow_{t'_1} (1, 0, x'_2) \rightarrow_{t'_2} (2, 0, x'_3)
\]
**Runs as signals**

It is often more practical to consider runs of a hybrid system as timed signals.

**Signal**

A signal on a set of variables \( \{ x_i \mid i \in [1..n] \} \), where each \( x_i \) takes values in \( X_i \), is a function \( \mathbb{R}_{\geq 0} \rightarrow \prod_{i=1}^{n} X_i \).

**Translation**

In the case where all \( X_q \)’s are subspaces of a given \( X \), the following signal corresponds to the run \( \rho = (q_0, x_0) \rightarrow_{t_0} (q_1, x_1) \rightarrow_{t_1} \ldots \) of a hybrid automaton under input signal \( u \):

\[
\sigma_\rho : \left[ 0, \sum_{i=1}^{n} t_i \right) \rightarrow Q \times X
\]

\[
\left( \sum_{i=1}^{k} t_i \right) + t \mapsto (q_k, \text{FLOW}_{q_k}(x_k, u, t)),
\]

where \( t < t_{k+1} \).
Signals in practice

**Time-boundedness**

Since simulation cannot be run for an infinite amount of time, so all considered signals are time-bounded: they are not functions $\mathbb{R} \geq 0 \rightarrow U$ but $[0, T) \rightarrow U$ for some $T \in \mathbb{R}$.

**Finite representation of input signals**

We must also stick to classes of signals that can be represented by finite means, e.g., piecewise constant signals, piecewise affine signals, spline (piecewise polynomial) signals...  

Control points (those points between which the function is interpolated) can be chosen equidistant, or according to other policies.
Discretisation of signals

Discretisation of output signals
Output signals may not be finitely representable, so they are discretised.

Attention
Discretisation of signals can lead to:
- false positives: e.g., for formula $F \varphi$, if $\sigma(t_i) \not\models \varphi$, but $\sigma(t) \models \varphi$ for some $t_i < t < t_{i+1}$, the algorithm will return $\sigma$ as a (wrong) counterexample,
- false negatives: for dual reasons.
Initial position and signals

Two kinds of falsifications:
- falsification on initial position,
- falsification on signals.

Differences:
- An initial position can be seen as a constant signal, so falsification on initial positions is easier.
- Many falsification methods incrementally modify the shape of the signal (learning approaches, Monte-Carlo Tree Search...), so they are ill-suited to falsification on initial positions.
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Signal temporal logic

Syntax

\[ \varphi ::= T \mid f \sim 0 \mid \neg \varphi \mid \varphi \land \varphi \mid \varphi \mathbf{U}_I \varphi \]

where \( I = [a, \infty) \) or \([a, b]\) for \( a < b \), \( f : S \to \mathbb{R} \), and \( \sim \in \{>, =\} \) is a comparison operator.

Syntactic sugar

- \( \bot = \neg T \), \( \varphi \lor \psi = \neg (\neg \varphi \land \neg \psi) \ldots \)
- \( f \geq 0 = f > 0 \lor f = 0 \), \( f \leq 0 = \neg f > 0 \ldots \)
- \( \mathbf{F}_I \varphi = \top \mathbf{U}_I \varphi \), \( \mathbf{G}_I \varphi = \neg \mathbf{F}_I \neg \varphi \)
- \( \mathbf{U} \varphi = \mathbf{U}_{[0, \infty)} \varphi \ldots \)
STL examples

To avoid awkward notations, $f \sim 0$ is never written as a function, but as a computation on states. Thus, if $f : (q, x, y, v_x, v_y) \mapsto x^2 + y^2 - 1$, $f < 0$ would be written $x^2 + y^2 - 1 < 0$ or $x^2 + y^2 < 1$.

Examples

- $q \not\equiv (3, 1) \cup q = (1, 3)$
- $G \ (\text{gear} = 2 \rightarrow \neg F_{[0, \varepsilon]} (\text{gear} \not\equiv 2 \land F_{[0, \tau]} \text{gear} = 2))$
- $(q = q_i \land x \in [x_i^-, x_i^+]) \land F_{[0, \tau]} (q = q_f \land x \in [x_f^-, x_f^+])$
- $(q = q_i \land x \in [x_i^-, x_i^+]) \land G_{[0, \tau]} \neg (q = q_f \land x \in [x_f^-, x_f^+])$
- $G \ (\text{danger} \rightarrow F_{[0, t]} \neg \text{danger})$ with danger $= (q = q_d \land x \in [x_d^-, x_d^+])$
### Boolean semantics of STL

The semantics of STL formulas is defined over timed signals:

#### Boolean semantics

\[
\begin{align*}
\sigma, t \models \top & \iff \top \\
\sigma, t \models f \sim 0 & \iff f(\sigma(t)) \sim 0 \\
\sigma, t \models \neg \varphi & \iff \sigma, t \not\models \varphi \\
\sigma, t \models \varphi \land \psi & \iff \sigma, t \models \varphi \text{ and } \sigma, t \models \psi \\
\sigma, t \models \varphi \mathsf{U}_{[a,b]} \psi & \iff \text{there is } t' \in [a, b] \text{ such that } \sigma, t' \models \psi \\
& \quad \text{and for all } t'' \in [a, t'), \sigma, t'' \models \varphi
\end{align*}
\]

\[\sigma \models \varphi \] stands for \(\sigma, 0 \models \varphi\).
Robustness semantics of STL

We can define a robustness semantics, whose value is not a boolean, but a real:

\[
\begin{align*}
\rho(\sigma, \top, t) &= \infty \\
\rho(\sigma, f > 0, t) &= f(\sigma(t)) \\
\rho(\sigma, f = 0, t) &= -|f(\sigma(t))| \\
\rho(\sigma, \neg \varphi, t) &= -\rho(\sigma, \varphi, t) \\
\rho(\sigma, \varphi \land \psi, t) &= \min(\rho(\sigma, \varphi, t), \rho(\sigma, \psi, t)) \\
\rho(\sigma, \varphi \text{ U}_{[a,b]} \psi, t) &= \sup_{t' \in [a,b]} \min \left( \rho(\sigma, \psi, t'), \inf_{t'' \in [a, t')} \rho(\sigma, \varphi, t'') \right) \\
\rho(\sigma, \varphi) \text{ stands for } \rho(\sigma, \varphi, 0).
\end{align*}
\]
Soundness of robustness semantics

**Theorem**

If $\rho(\sigma, \varphi) > 0$, then $\sigma \models \varphi$.

If $\rho(\sigma, \varphi) < 0$, then $\sigma \not\models \varphi$.

**Proof.**

Structural induction.

**Remark**

If $\rho(\sigma, \varphi) = 0$, nothing can be said: if $\sigma(x)(0) = x_0$ for $\varphi = (x = x_0)$:
- $\rho(\sigma, \varphi) = \rho(\sigma, \neg \varphi) = 0$, but
- $\sigma \models \varphi$, $\sigma \not\models \neg \varphi$.

Advantage: if functions $f$ are continuous or smooth, we can use optimisation techniques to find the minimal robustness.
Computing the robustness semantics

Computing $\rho$

Given: piecewise-constant $\sigma$ (because discretised).
Goal: compute $\rho(\sigma, \varphi)$.

Naive idea: compute $\rho(\sigma, \varphi, t)$ inductively.

Problem

Complexity to compute $\rho(\sigma, \varphi \cup_{[a,b]} \psi, -)$ is $\mathcal{O}(\text{number of control points} \times \text{number of control points in } [a, b])$
Efficiently computing $\rho(\sigma, \varphi, -)$

First step: get rid of $U_I$:

- $\varphi U_{[a,b]} \psi \equiv F_{[a,b]} \varphi \land G_{[0,a]} (\varphi U \psi)$
- $\varphi U_{[a,\infty)} \psi \equiv G_{[0,a]} (\varphi U \psi)$

Goal

The function $t \mapsto \rho(\sigma, \varphi, t)$ is a signal $\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ denoted $\rho(\sigma, \varphi, -)$. Our goal is to compute it from $\rho(\sigma, \psi, -)$ for subformulas $\psi$ of $\varphi$.

Need to know how to recursively compute signals for $\top$, $f \sim 0$, $\neg \varphi$, $\varphi \lor \psi$, $\varphi U \psi$, and $F_{[a,b]} \varphi$ in $O$(number of control points).
Computing $\rho(\sigma, \varphi \cup \psi, -)$

Without loss of generality, signals $y$ and $y'$ representing $\rho(\sigma, \varphi, -)$ and $\rho(\sigma, \psi, -)$ have the same control points $t_i$ (otherwise, take the union of control points).

Let $z$ be the signal corresponding to $\rho(\sigma, \varphi \cup \psi, -)$, then:

$$z(t_i) = \sup_{t \in [t_i, \infty)} \min\{y'(t), \inf_{[t_i, t]} y\}$$

$$= \max\{\min\{y'(t_i), y(t_i)\}, \sup_{t \in [t_{i+1}, \infty)} \min\{y'(t), \inf_{[t_i, t]} y\}\}$$

$$= \max\{\min\{y'(t_i), y(t_i)\}, \min\{y(t_i), \sup_{t \in [t_{i+1}, \infty)} \min\{y'(t), \inf_{[t_i, t]} y\}\}\}$$

$$= \min\{y(t_i), \max\{y'(t_i), z(t_{i+1})\}\}$$

Thus, there is an algorithm to compute $\rho(\sigma, \varphi \cup \psi, -)$ from $\rho(\sigma, \varphi, -)$ and $\rho(\sigma, \varphi, -)$ whose complexity is linear in the number of control points.
Computing $\rho(\sigma, F_{[a,b]} \varphi, -)$

We need to compute $z(t) = \sup_{t+[a,b]} y = \max_{t_i \in t+[a,b]} \{y(t_i)\}$.

**Idea**

Compute $M$ such that $i \in M$ iff $t_i \in t + [a, b]$ and for all $t_j \in t + [a, b]$, $y(t_j) < y(t_i)$.

Thus: $y(t_{\min M}) = \max_{t_i \in t+[a,b]} \{y(t_i)\}$.

Analysis

- $M$ can be implemented to have all operations in $\mathcal{O}(1)$ (doubly-linked list),
- all control points are popped at most once,
- in total, the number of comparisons (done when searching for elements to pop) is at most $2n$,

therefore, computing $\rho(\sigma, F_{[a,b]} \varphi, -)$ can be done in time linear with respect to the number of control points.

**Overall complexity**

The number of control points is at most $d_h(\varphi)|\sigma|$, so the whole complexity is $\mathcal{O}(|\varphi|d_h(\varphi)|\sigma|)$.

**Generalisation**

The same argument (but more complex) applies to more general signals (say, piecewise affine).
Table of Contents

1. Formal methods landscape
2. Framework
3. Hybrid systems
4. Formulas
5. Optimisation
Falsification is based on a number of optimisation techniques:

- ant colony,
- CMA-ES,
- cross-entropy,
- gradient descent,
- hill-climbing,
- Nelder-Mead,
- simulated annealing,
- ...
Gradient descent

Goal: find a local minimum to a function $f$.

1 $i = 0$;
2 while continue do
3 $x_{i+1} = x_i - \gamma_i \cdot \nabla(f)(x_i)$;
4 $\text{best} = x_{i+1}$;
5 if $f(x_{i+1}) > f(x_i)$ then
6     continue = false;
7     $\text{best} = x_i$;
8 end
9 $i++$;
end
10 return $\text{best}$

Finds a local minimum: may be useful in certain cases.
Hill climbing

Goal: find a local maximum to a function $f$.

1 $i = 0$;
2 while continue do
3     for $x' \in \text{neighbours}(x_i)$ do
4         if $f(x') > f(x_i)$ then
5             $x_{i+1} = x'$;
6             break;
7         end
8     end
9     continue = $x_{i+1} \neq x_i$;
10    $i++$;
11 end
12 return $x_{i-1}$

- Finds a local maximum: may be useful in certain cases.
- Simpler than gradient descent (no derivatives to compute), but less efficient.
CMA-ES

CMA-ES: Covariance Matrix Adaptation Evolution Strategy
Goal: find the maximum of a function $f$ on a space $X$.

1 while continue do
2 \[ x_1, \ldots, x_n = \text{sample-multivariate-normal}(m, \sigma^2 C); \]
3 \[ x_1, \ldots, x_n = \text{sort}(x_1, \ldots, x_n, f); \]
4 \[ m, \sigma, C = \text{update}(x_1, \ldots, x_n, m, \sigma, C); \]
end
5 return $m$

Wikipedia
CMA-ES for falsification

**Characteristics**
- evolutionary algorithm (the parameters evolve towards a better sampler)
- sampling-based $\implies$ no need to compute derivatives

**Adaptation**
Take $X$ to be the space of signals $\sigma$ and $f$ to be $-\rho(\sigma, \varphi)$.

**Remark**
This is a way to adapt sampling-based optimisation to falsification, so it works for other such algorithms, such as the cross-entropy method.
Nelder-Mead method

Goal: find the maximum of a function $f$ on a space $X$ of dimension $n$.

1  $x_1, \ldots, x_{n+1} = \text{simplex}();$
2  while continue do
3      $x_i = \text{worst-vertex}(x_1, \ldots, x_{n+1}, f);$\n4      $x'_i = \text{reflect}(x_i, (x_j)_{j \neq i});$
end
5  return best-vertex($f$)

- 1: the structure is just a simplex.
- 3: when reflecting a vertex, if the new value is much better than the previous one, we keep stretching, otherwise, we shrink.

Adaptation to falsification

Take $X$ a subspace of signals of finite dimension (e.g., fix a shape and control points), and $f = -\rho(\sigma, \varphi)$. 
Ant colony

Goal: find a path in a graph that maximises performance.

1 while continue do
2   foreach “ant” i in colony do
3       $p_i = \text{construct-solution}(i)$;
4       local-update-pheromones($i, f$);
5   end
6   global-update-pheromones();
7   $p = \text{argmax}(p_i, f)$
end
8 return $p$

- 3: ants walk randomly on the graph, choosing neighbours with more pheromone more often.
- 4: ants put pheromone on their chosen edge, making it more attractive; the better the ant’s solution (i.e., the smaller $f(p_i)$), the more pheromone she puts.
- 5: pheromone gets put on all edges taken by the best solution.
Ant colony for falsification

Adaptation

Take $G$ to be the graph:

- whose vertices are input signals $\sigma$,
- there is an edge between $\sigma$ and $\tau$ when they are “close enough”, say, $\|\sigma - \tau\|_\infty < C$, i.e., for all $t < t_{\text{max}}$ and component $x$ of $U$, $|\sigma(t)(x) - \tau(t)(x)| < C$,

and $f(\sigma_1 \ldots \sigma_n)$ to be $-\rho(\sigma_n, \phi)$.

The ant colony algorithm tries to find a path that maximises $f$, i.e., minimises $\rho(\sigma, \phi)$.

Remark

This is a way to adapt optimisation of a function on a graph to falsification, so it works for other optimisation algorithms that work on graphs, such as simulated annealing.
Simulated annealing

Goal: find a global maximum of a function $f$ on a space $X$.

$p = p_0$;

$i = 0$;

while $T(i) > 0$ do

\[ x_{i+1} = \text{neighbour}(x_i); \]

if $E(x_{i+1}) < E(x_i)$ and $\text{rand}() < P(E(x_{i+1}), E(x_i), T))$

\[ \text{then} \]

\[ x_{i+1} = x_i; \]

\[ \text{end} \]

\[ i += 1; \]

\[ \text{end} \]

return $x_{i-1}$

- Avoids local minima by “cooling” the system down progressively.
- Many parameters $\Rightarrow$ adaptative, but tricky.
Falsification:
- method to find counterexamples to a property,
- useful in the world of formal methods,
- black-box method,
- relies on optimisation algorithms.

Hybrid system:
- continuous and discrete parameters,
- non-linear behaviour,
- very expressive.

Formulas:
- expressed in a temporal logic,
- boolean and robustness semantics.