Constructing Fibred Double Categories

Towards New Sheaf Models of Programming Languages

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Playgrounds

Game semantics: semantics = interaction.

Playgrounds

$$\text{pseudo double category} \longmapsto \text{notion of strategy.}$$

- Pseudo double category:
  - with extra structure
  - and properties.
- Strategies:
  - sheaf semantics $\approx$ presheaf semantics $+$ innocence
  - or concurrent game strategies.

Building playgrounds from signatures

- Until now:
    - automatic: playground $\mapsto$ notion of strategy,
    - playground $=$ pseudo double category $+$ properties,
    - problem: prove properties by hand for each language.
- This work:
    - signatures $\to$ playgrounds $\to$ strategies,
    - prove properties abstractly,
    - $\rightsquigarrow$ new playground for HO games.

Overview

1. Multi-party HO
2. A playground for HO
3. Fibredness

HO games: notations

Arena = finite forest of "moves".

- Polarity (O or P) of a move = parity of its depth.
- Roots have polarity O.
- Notation:
  - A an arena,
  - m a root of A,
  - $A \cdot m$ is the forest below m.

Play = P-visible, alternated, justified sequence of even length.
View = non-empty play such that $\lceil s \rceil = s$.

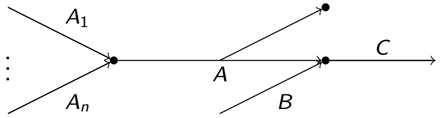
A framework for HO

First: construct a multi-party framework for HO games.
Then: organise it into a pseudo double category.
Overview:

- Positions
- Generator moves
- Moves
- Plays

Positions

Position: (kind of) graph:



Positions

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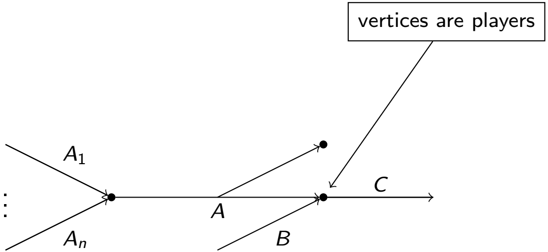

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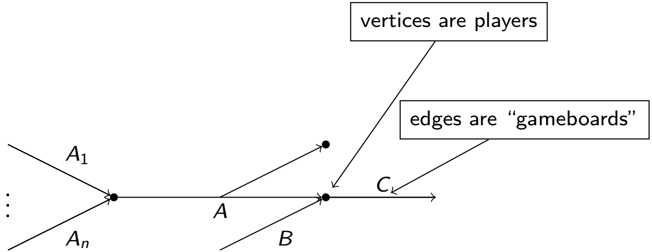

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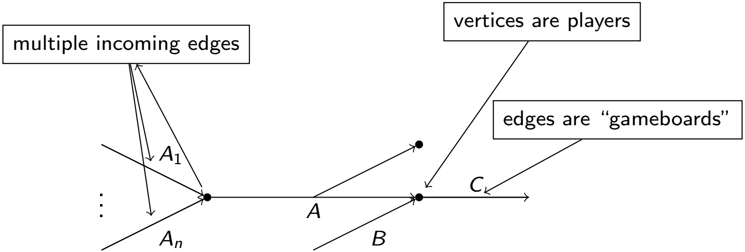

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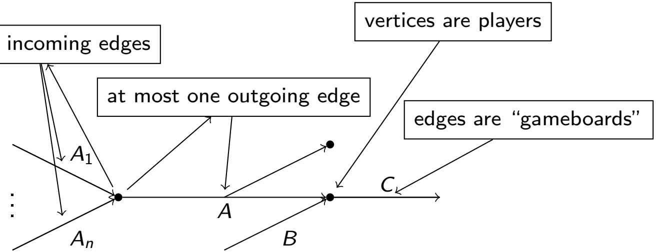
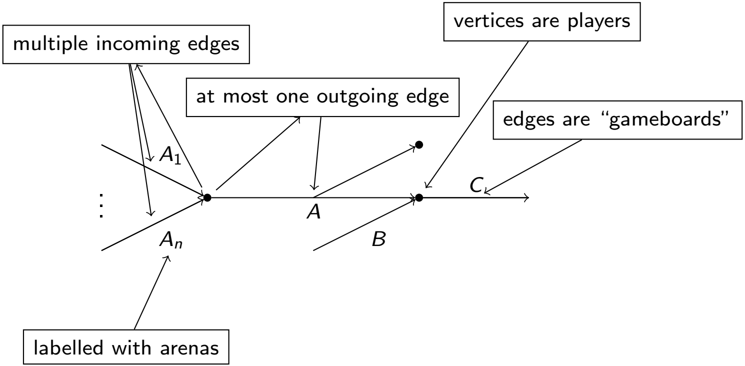


Positions

Position: (kind of) graph:



Positions

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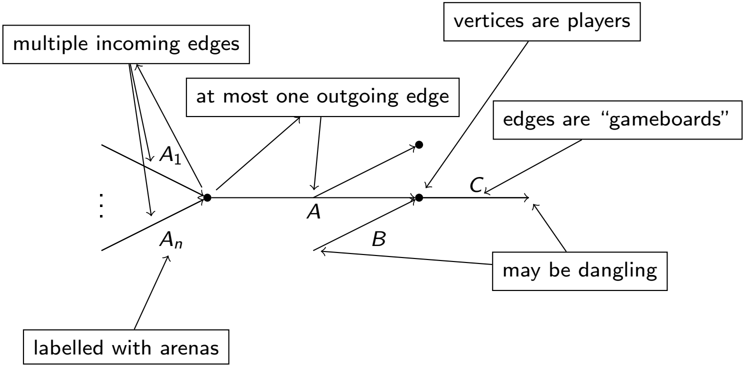

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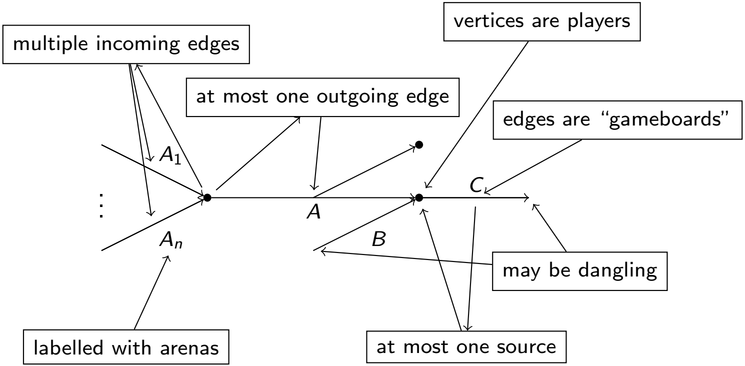

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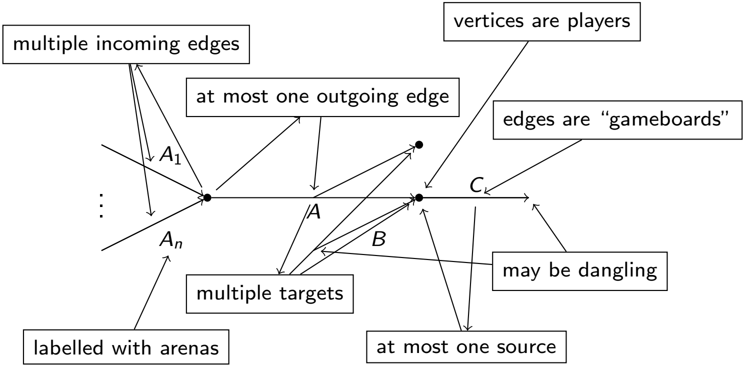

Positions

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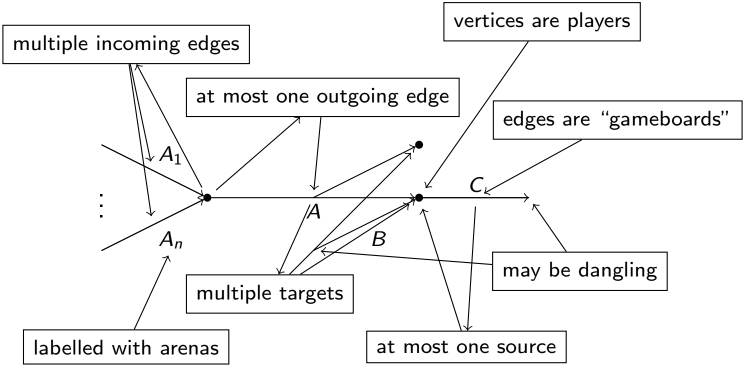


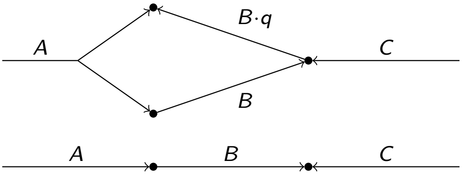
$\rightsquigarrow$ each player has a sequent $A_1, \ldots, A_n \vdash B$ or $A_1, \ldots, A_n \vdash$ of arenas.

Moves (1)

Typical move:

final position:

initial position:



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Typical move:

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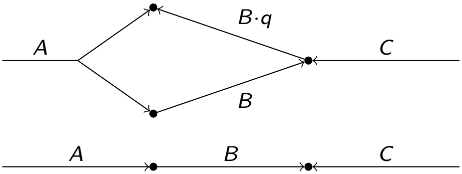
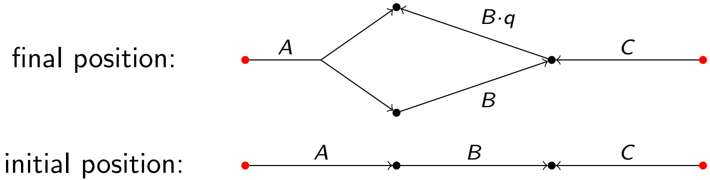


Analogy with sequent calculus:

$$\frac{A, B \cdot q \vdash}{A \vdash B} \qquad \frac{B, C \vdash B \cdot q}{B, C \vdash}$$

Moves (2)

More generally, embed into any context.
Example:



Plays

- Sequences of moves?
- Too naive.
- Need expressive morphisms between plays (e.g., Ong-Tsukada).

Solution: retain only causal dependencies between moves.
Problem: how to represent this formally?

Representing positions

Consider the following category $\mathbb{C}_1$:

- objects = sequents and arenas,
- morphisms = occurrences of arenas in sequents.

$$\begin{tikzcd}
A_1, \ldots, A_n \vdash A & A_1, \ldots, A_n \vdash & \ldots \\
A_i \arrow[u, "s_i"] \arrow[ur, "s_i"'] & A \arrow[ul, "t"] & \ldots
\end{tikzcd}$$

Any position yields a presheaf on $\mathbb{C}_1$!

Representing positions: examples

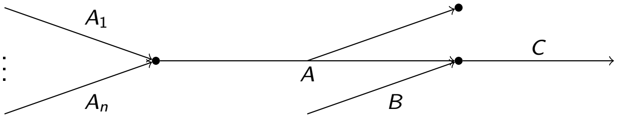
$$
\begin{tikzcd}[column sep=small]
A_1, \ldots, A_n \vdash A & & A_1, \ldots, A_n \vdash & & \ldots \\
A_i \arrow[u, "s_i"] \arrow[urr, "s_i"] & & A \arrow[ull, "t"'] & & \ldots
\end{tikzcd}
$$

- A dangling edge labelled with A: the representable presheaf y_A.
- A player on any sequent: the representable presheaf on that sequent.
- The position $\xrightarrow{A} \bullet \xrightarrow{B} \bullet \xleftarrow{C}$:

$$
\begin{tikzcd}
\mathrm{y}_B \arrow[r, "s_1"] \arrow[d, "t"'] & \mathrm{y}_{B,C\vdash} \arrow[d] \\
\mathrm{y}_{A\vdash B} \arrow[r] & \cdot \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

Diagram associated to a position

The presheaf representing



is a colimit of

$$
\begin{tikzcd}
A_1 \arrow[dr] & & & (A\vdash) & \\
\vdots & (A_1,\ldots,A_n\vdash A) & A \arrow[l] \arrow[ur] \arrow[r] & (A,B\vdash C) & C. \arrow[l] \\
A_n \arrow[ur] & & B \arrow[ur] & &
\end{tikzcd}
$$

Representing moves

- How to represent plays?
- How to represent moves?
- Moves are more than just relations.
- Add objects to $\mathbb{C}_1$ to represent moves!
- Glue them together to form plays.

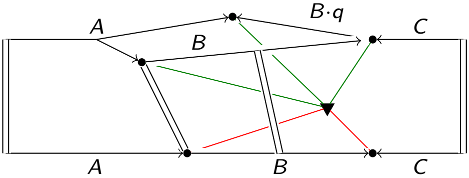
Which objects?

Adding objects to $\mathbb{C}_1$: fill the shell!

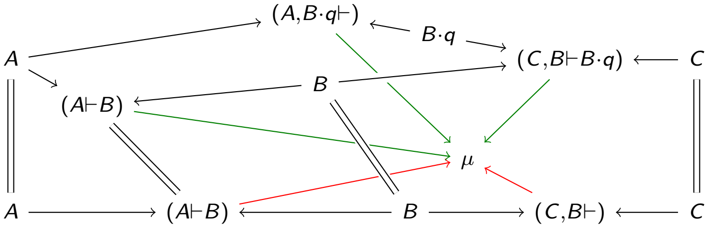
New object μ for move:

final position:

initial position:



Add green and red arrows making the diagram commute:



The augmented base category

Add an object for each instance

$$\frac{\Gamma, B \cdot q \vdash}{\Gamma \vdash B} \qquad \frac{\Delta_1, B, \Delta_2 \vdash B \cdot q}{\Delta_1, B, \Delta_2 \vdash}$$

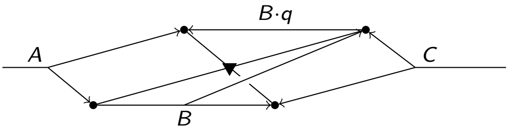
with associated morphisms.

Definition

Let $\mathbb{C}$ denote the augmented category.

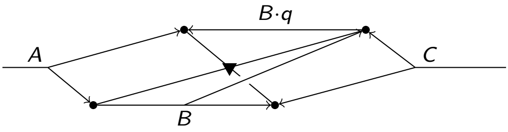
Initial and final positions

Representable presheaf y_μ: shapeless "blob".



Initial and final positions

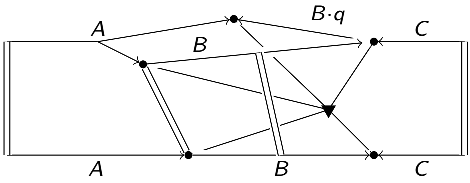
Representable presheaf y_μ: shapeless "blob".



Organise it as a cospan $Y \to \mathsf{y}_\mu \leftarrow X$!

final position:

initial position:



Summary

For each intended move:

- new object μ in $\mathbb{C}$,
- cospan $Y \to \mathrm{y}_\mu \leftarrow X$, the generator move,
- X initial position,
- Y final position.

Next step: want moves to occur inside larger context:

- glue the generator move
- along some stable part
- to some position (the context).

Stable part of a generator move

Any position I equipped with morphisms making

$$\begin{tikzcd} & Y \arrow[d] \\ I \arrow[ur] \arrow[r] \arrow[dr] & \mathrm{y}_{\mu} \\ & X \arrow[u] \end{tikzcd}$$

commute.

Move

Gluing of a generator move to some context Z:

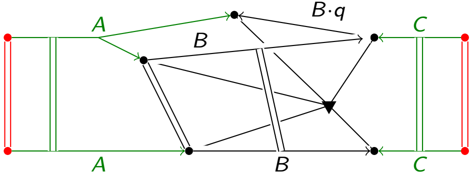
$$\begin{tikzcd}
 & Y \arrow[rr] \arrow[d] & & Y' \arrow[d, dashed] \\
 & y_\mu \arrow[rr] & & M' \\
I \arrow[uur] \arrow[ur] \arrow[rr] \arrow[dr] & & Z \arrow[uur] \arrow[ur] \arrow[dr] & \\
 & X \arrow[uu] \arrow[rr] & & X' \arrow[uu, dashed]
\end{tikzcd}$$

(levelwise pushout).

Example move

final position:

initial position:

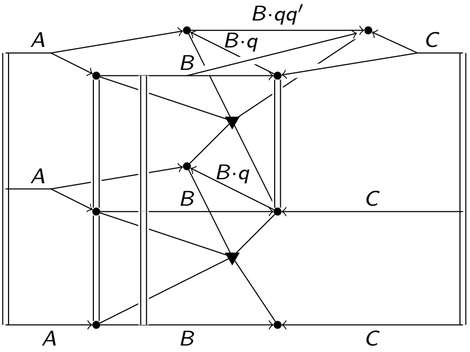


$$
\begin{tikzcd}
 & Y \arrow[rr] \arrow[d] & & Y' \arrow[d, dashed] \\
 & \mathrm{y}_{\mu} \arrow[rr] & & M' \\
\color{green}{\mathrm{y}_A + \mathrm{y}_C} \arrow[uur] \arrow[ur] \arrow[rr] \arrow[dr] & & \color{red}{\mathrm{y}_{(A\vdash)} + \mathrm{y}_{(C\vdash)}} \arrow[uur] \arrow[ur] \arrow[dr] & \\
 & X \arrow[uu] \arrow[rr] & & X' \arrow[uu, dashed]
\end{tikzcd}
$$

Play

Definition

Play = finite composite of moves in $\mathrm{Cospan}(\widehat{\mathbb{C}})$.



A (pseudo) double category

- Objects:
- Horizontal morphisms:
- Vertical morphisms:
- Cells:

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d, "U"'] \arrow[r, phantom, ""] & Y' \arrow[d, "U'"] \\ X \arrow[r, "h"'] \arrow[ur, phantom, "\overset{\alpha}{\Longrightarrow}"] & X' \end{tikzcd}$$

A (pseudo) double category

- Objects: positions.
- Horizontal morphisms: morphisms between positions.
- Vertical morphisms: plays.
- Cells: morphisms l making the diagram commute.

$$
\begin{tikzcd}
Y \arrow[r, "k"] \arrow[dd, "U\bullet"'] & Y' \arrow[dd, "\bullet U'"] \\
\arrow[r, phantom, "\overset{\alpha}{\Longrightarrow}"] & \\
X \arrow[r, "h"'] & X'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Y \arrow[r, "k"] \arrow[d] & Y' \arrow[d] \\
U \arrow[r, "l"] & U' \\
X \arrow[r, "h"'] \arrow[u] & X' \arrow[u]
\end{tikzcd}
$$

Composition

- Horizontal compositions: straightforward.
- Vertical composition: by pushout.
- Vertical composition of cells: by universal property of pushout.

Interchange law:

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, "\bullet" marking] \arrow[dr, phantom, "\xRightarrow{\alpha}"] & Z' \arrow[r] \arrow[d, "\bullet" marking] \arrow[dr, phantom, "\xRightarrow{\alpha'}"] & Z'' \arrow[d, "\bullet" marking] \\
Y \arrow[r] \arrow[d, "\bullet" marking] \arrow[dr, phantom, "\xRightarrow{\beta}"] & Y' \arrow[r] \arrow[d, "\bullet" marking] \arrow[dr, phantom, "\xRightarrow{\beta'}"] & Y'' \arrow[d, "\bullet" marking] \\
X \arrow[r] & X' \arrow[r] & X''
\end{tikzcd}$$

Comparison with Ong-Tsukada

Ong-Tsukada		playgrounds
views	$=$	views
plays	$\leftrightarrow$	plays
no players, no positions	$\not\leftrightarrow$	players, positions
time, sequential	$\not\leftrightarrow$	no time, causal
P-view	$\not\leftrightarrow$	$\emptyset$
P-visibility	$\not\leftrightarrow$	$\emptyset$
composition, hiding	$\not\leftrightarrow$	hopeless

Views

- For Ong-Tsukada: $\mathbb{V}_{A,B}$ = category of views on $A \to B$ (ordered by prefix).
- In our sense: views are some (ad-hoc) non-empty plays on $A \vdash B$, $\mathbb{E}^{\mathbb{V}}_{A \vdash B}$ = category of views on $A \vdash B$, where morphisms are "temporal inclusion".

Lemma

$\mathbb{V}_{A,B}$ and $\mathbb{E}^{\mathbb{V}}_{A \vdash B}$ are equivalent.

Plays

Wilder notion of play in our setting.
However:

Conjecture

Innocent strategies in our sense are equivalent to innocent strategies in the sense of Ong-Tsukada.

Idea: prove that

$$\begin{tikzcd} \mathbb{V}_{A,B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\xRightarrow{id}"] & \mathbb{P}_{A,B} \arrow[d] \\ \mathbb{E}^{\mathbb{V}}_{A\vdash B} \arrow[r] & \mathbb{E}_{A\vdash B} \end{tikzcd}$$

is an exact square.
Enough to prove: $\mathbb{P}_{A,B} \to \mathbb{E}_{A\vdash B}$ fully faithful.

Behaviours

Important construction of playgrounds: $\mathbb{E}$.
Innocent strategies $\approx$ sheaves over $\mathbb{E}$.

$\mathbb{E}$

- objects: $U \colon Y \to X$,
- morphisms $(U \colon Y \to X) \to (U' \colon Y' \to X')$:

$$\begin{tikzcd} Z \arrow[r, "l"] \arrow[d, "W"'] & Y' \arrow[dd, "U'"] \\ Y \arrow[r, phantom, "\xRightarrow{\alpha}"] \arrow[d, "U"'] & \\ X \arrow[r, "h"'] & X'. \end{tikzcd}$$

Composition in $\mathbb{E}$

By pasting

$$
\begin{tikzcd}
\widetilde{Z} \arrow[r, dashed, "\tilde{l}"] \arrow[d, dashed, "\widetilde{W}"'] \arrow[dr, phantom, "\overset{\gamma}{\Rightarrow}"] & Z' \arrow[r, "l'"] \arrow[d, "W'"'] & Y'' \arrow[ddd, "U''"] \\
Z \arrow[r, "l"] \arrow[d, "W"'] & Y' \arrow[dd, "U'"] \arrow[r, phantom, "\overset{\beta}{\Longrightarrow}"] & \\
Y \arrow[d, "U"'] \arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] & & \\
X \arrow[r, "h"'] & X' \arrow[r, "h'"'] & X'',
\end{tikzcd}
$$

where γ is a cartesian lifting of W' along l.

Crucial property: fibredness

Fibredness

A pseudo double category $\mathbb{D}$ is fibred if its "vertical codomain" functor cod is a fibration.

$$
\begin{tikzcd}
Y'' \arrow[dr, dashed] \arrow[drr, bend left] \arrow[dd, "U''"'] & & \\
& Y' \arrow[r] \arrow[dd, "U'"'] \arrow[r, phantom, shift right=6ex, "\Rightarrow"] & Y \arrow[dd, "U"] \\
X'' \arrow[dr] \arrow[drr, bend left] \arrow[ur, phantom, "\Rightarrow"] & & \\
& X' \arrow[r] & X
\end{tikzcd}
$$

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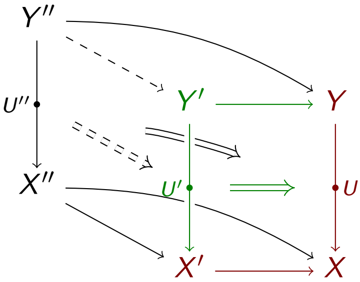

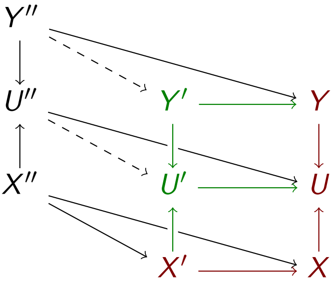

Factorisation systems

$(\mathcal{L}, \mathcal{R})$ two classes of morphisms of $\mathbb{C}$ such that:

- every morphism of $\mathbb{C}$ can be decomposed as $r \circ l$, for some $l \in \mathcal{L}$ and $r \in \mathcal{R}$,
- $\mathcal{L} \perp \mathcal{R}$, i.e., for all $l \in \mathcal{L}$, $r \in \mathcal{R}$:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "l"'] & \cdot \arrow[d, "r"] \\ \cdot \arrow[r] \arrow[ur, dashed, "\exists!"] & \cdot \end{tikzcd}$$

Properties:

- $\mathcal{L}$ and $\mathcal{R}$ contain all isos, are stable under composition,
- $\mathcal{L}$ is stable under pushouts,
- ...

Cofibrant generation

$\mathcal{M}^{\perp}$ = class of morphisms g such that for all f of $\mathcal{M}$, $f \perp g$.
${}^{\perp}\mathcal{M}$ = idem.

Small object argument (Bousfield)

For any set J of morphisms of $\mathbb{C}$, $\left(J^{\perp}, {}^{\perp}(J^{\perp})\right)$ is a factorisation system.

Cofibrantly generated factorisation system on $\mathrm{Cospan}(\widehat{\mathbb{C}})$: J set of $X \to \mathrm{y}_{\mu}$ for $Y \to \mathrm{y}_{\mu} \leftarrow X$ generator move.

Construction of the candidate lifting

$$
\begin{tikzcd}
 & Y \arrow[d] \\
 & U \\
X' \arrow[r] & X \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Y \arrow[d] \\
U' \arrow[r, "r"] & U \\
X' \arrow[u, "l"] \arrow[r] & X \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\
U' \arrow[r] & U \\
X' \arrow[u] \arrow[r] & X \arrow[u]
\end{tikzcd}
$$

Proof of fibration

$$\begin{tikzcd}
Y'' \arrow[d] \arrow[dr, dashed] \arrow[drr] & & \\
U'' \arrow[dr, dashed] \arrow[drr] & Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\
X'' \arrow[u] \arrow[dr] \arrow[drr] & U' \arrow[r] & U \\
& X' \arrow[u] \arrow[r] & X \arrow[u]
\end{tikzcd}$$

- $U'' \to U'$ by the lifting property of factorisation systems,
- $Y'' \to Y'$ by universal property of pullback.

Conclusion

Done:

- general construction of playgrounds,
- better understanding of the link between playgrounds and classical game semantics.

Work to do:

- prove full faithfulness of $\mathbb{P}_{A,B} \to \mathbb{E}_{A \vdash B}$,
- influence of space,
- influence of time,
- get the "right" views back,
- make hiding work.