Mathematical Semantics of Computer Systems, *MSCS* (4810-1168)
Handout for Lecture 13 (2015/02/09)

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1 Today's Agenda

We continue the last lecture and follow slides by Samson Abramsky (Oxford) found at `www.math.helsinki.fi/logic/sellc-2010/course/LectureIII.pdf`. See [2] for further details. There is a big body of literature on the λ-calculus, including [1, 3, 4].

- Categorical Semantics, as a typed λ-calculus and Cartesian closed categories as examples.

2 Cartesian Closed Categories as Models of Typed λ-Calculus

2.1 In the Previous Lecture

Definition. Type judgment. Type derivation tree.

NB: we use the term calculus *a la Church* (where bound variables have explicit types).

Lemma. *Each derivable type judgment has a unique derivation tree.*

Definition. Cartesian closed category: a category with finite products and exponentials.

2.2 In This Lecture

Definition. Interpretation $[\![\_]\!]$ of typed λ-calculus. Interpreting types, type derivation trees, type judgments, and terms.

Definition. Substitution lemma: interpretation of $s[t/x]$ is given by composition of arrows.

Definition. Conversion rules, including congruence rules.

Theorem. *Soundness of categorical semantics: if $s =_{\beta\eta} t$, then $[\![s]\!] = [\![t]\!]$.*

If we have time:

- The Curry-Howard correspondence; terms as proofs; conversion as proof normalization
- Subject reduction, strong normalization, recursion and observational equivalence
- Bisimulation and coalgebra.

3 Final Report Assignment

Due: 14:59, Thursday 19 February, 2015

Submit to the lecturer's mailbox (on the corridor), to the (official) report box of the department, or by email to `ichiro@is.s.u-tokyo.ac.jp`

You can choose questions to answer. Each question is assigned points; and you are expected to answer 100 points worth. However, in case you have not submitted some of the previous report assignments, you can make it up by answering more.

1. (40) Prove the substitution lemma.

2. (30) Let $f\colon A \to B$ be an arrow in a Cartesian closed category $\mathbb{C}$. It induces natural transformations: $A \times (\_) \Rightarrow B \times (\_)$, and $(\_)^B \Rightarrow (\_)^A$. Show that the adjunctions $A \times (\_) \dashv (\_)^A$ and $B \times (\_) \dashv (\_)^B$ are "compatible" with those natural transformations.

   - You first have to formalize what "compatibility" means.

3. We are interested in the CCC structure of a presheaf category $[\mathbb{C}^{\mathrm{op}}, \mathbf{Sets}]$. Here $\mathbb{C}$ is a small category.

   (a) (30) Prove that $[\mathbb{C}^{\mathrm{op}}, \mathbf{Sets}]$ has all small limits and colimits.
   (Hint: they are computed "pointwise.")

   (b) (40) The category $[\mathbb{C}^{\mathrm{op}}, \mathbf{Sets}]$ has exponentials, too. Let $P, Q\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Sets}$ be presheaves; Q^P be an exponential; and $C \in \mathbb{C}$. Describe the set $(Q^P)(C)$.
   (Hint: use the Yoneda lemma!)

   (c) (30) Specialize the above answer to the case when $\mathbb{C}$ is the following category with two objects and four arrows.

$$\begin{tikzcd} V \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & E \end{tikzcd}$$

   Here the identity arrows are implicit. Discuss the relationship with the notion of graph homomorphism.

4. *End* is a notion that generalizes that of limit.

   (a) (30) Describe its definition, and show that limits are special cases of ends.

   (b) (40) Let $\mathbb{C}$ be a small category; and $F, G\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Sets}$ be presheaves. Describe the set $\mathbf{Nat}(F, G)$ of natural transformations as a limit.

5. (50) Let $\mathbb{C}$ be a small category and consider the presheaf category $[\mathbb{C}^{\mathrm{op}}, \mathbf{Sets}]$. Prove that any presheaf $P\colon \mathbb{C}^{\mathrm{op}} \to \mathbf{Sets}$ is a colimit of a certain diagram that consists solely of representable presheaves (i.e. those presheaves of the form $\mathbf{y}C = \mathbb{C}(\_, C)$).

   (Hint: You have to find a suitable diagram. It is given by so-called the *category of elements.*)

References

[1] H.P. Barendregt. *The Lambda Calculus. Its Syntax and Semantics.* North-Holland, Amsterdam, 2^{nd} rev. edn., 1984.

[2] J. Lambek and P.J. Scott. *Introduction to higher order Categorical Logic.* No. 7 in Cambridge Studies in Advanced Mathematics. Cambridge Univ. Press, 1986.

[3] M.H. Sørensen and P. Urzyczyn. *Lectures on the Curry-Howard Isomorphism*, vol. 149 of *Studies in Logic and the Foundations of Mathematics.* Elsevier Science Inc., New York, NY, USA, 2006.

[4] G. Winskel. *The Formal Semantics of Programming Languages.* MIT Press, 1993.