Coalgebras and Higher-Order Computation: a GoI Approach

Ichiro Hasuo
National Institute of Informatics
Tokyo, Japan

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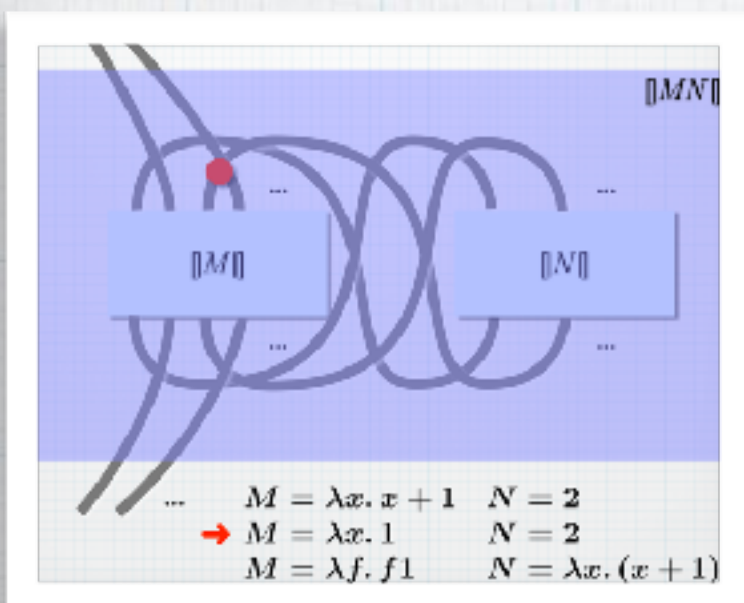
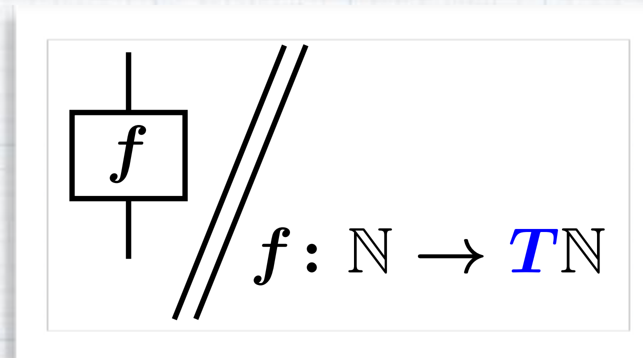
Outline

* Categorical axiomatization
* Compilation to **sequential machines**

Coalgebra meets higher-order computation in Geometry of Interaction [Girard, LC'88]

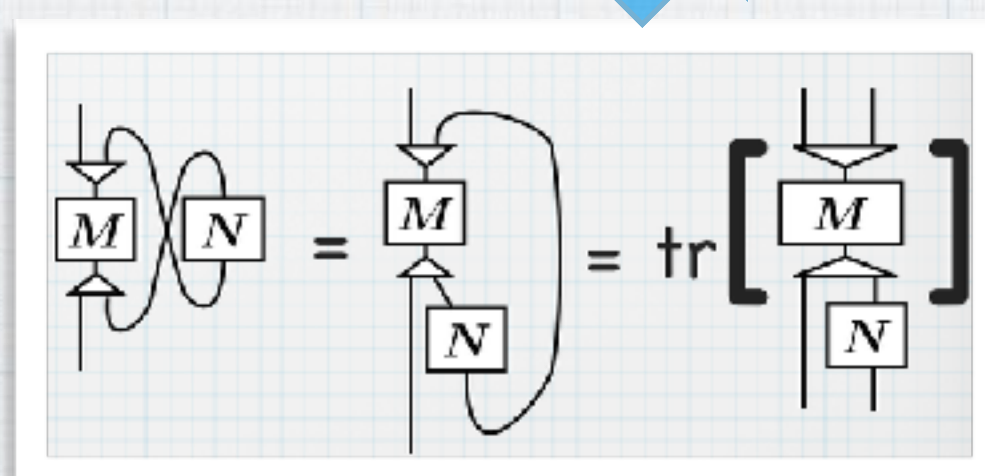
"GoI Animation"





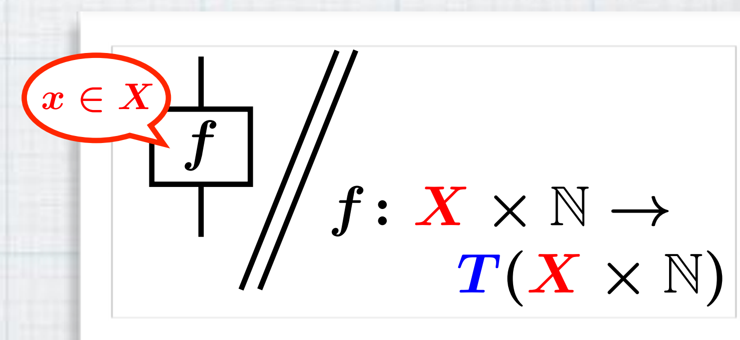
GoI w/ T-branching

[IH & Hoshino, LICS'11]



Categorical GoI

[Abramsky, Haghverdi & Scott, MSCS'02]



Memoryful GoI

[Hoshino, Muroya & IH, CSL-LICS'14 & POPL'16]

References

* [LICS 2011] IH and Naohiko Hoshino. **Semantics of Higher-Order Quantum Computation via Geometry of Interaction.** (Extended ver. Annals Pure & Appl. Logic 2017)
* [CSL-LICS 2014] Naohiko Hoshino, Koko Muroya and IH. **Memoryful Geometry of Interaction: From Coalgebraic Components to Algebraic Effects.**
* [POPL 2016] Koko Muroya, Naohiko Hoshino and IH. **Memoryful Geometry of Interaction II: Recursion and Adequacy.**
* [LOLA 2014] Koko Muroya, Toshiki Kataoka, IH and Naohiko Hoshino. **Compiling Effectful Terms to Transducers: Prototype Implementation of Memoryful Geometry of Interaction (Preliminary Report).**
* [Math. Str. in Comp. Sci. 2011] IH and Bart Jacobs. **Traces for Coalgebraic Components.**

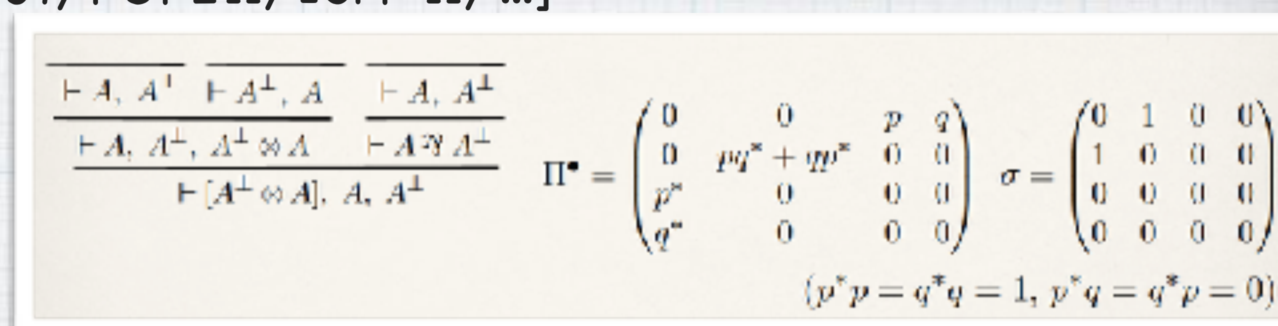
Geometry of Interaction (GoI)

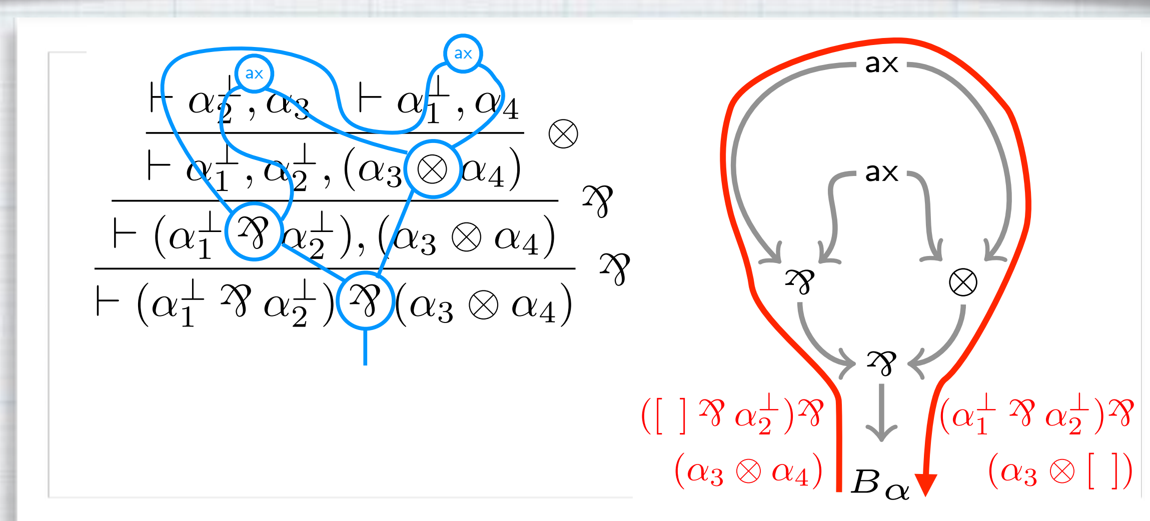
* J.-Y. Girard, at Logic Colloquium '88
* Provides **"denotational" semantics** (w/ operational flavor) for linear λ-term M
* As a **compilation technique**

  [Mackie, POPL'95] [Pinto, TLCA'01] [Ghica et al., POPL'07, POPL'11, ICFP'11, ...]

* Two presentations:
  * (Operator-) Algebraic [Girard]
  * Token machines/ interaction abstract machines

    [Danos & Regnier, TCS'99] [Mackie, POPL'95]

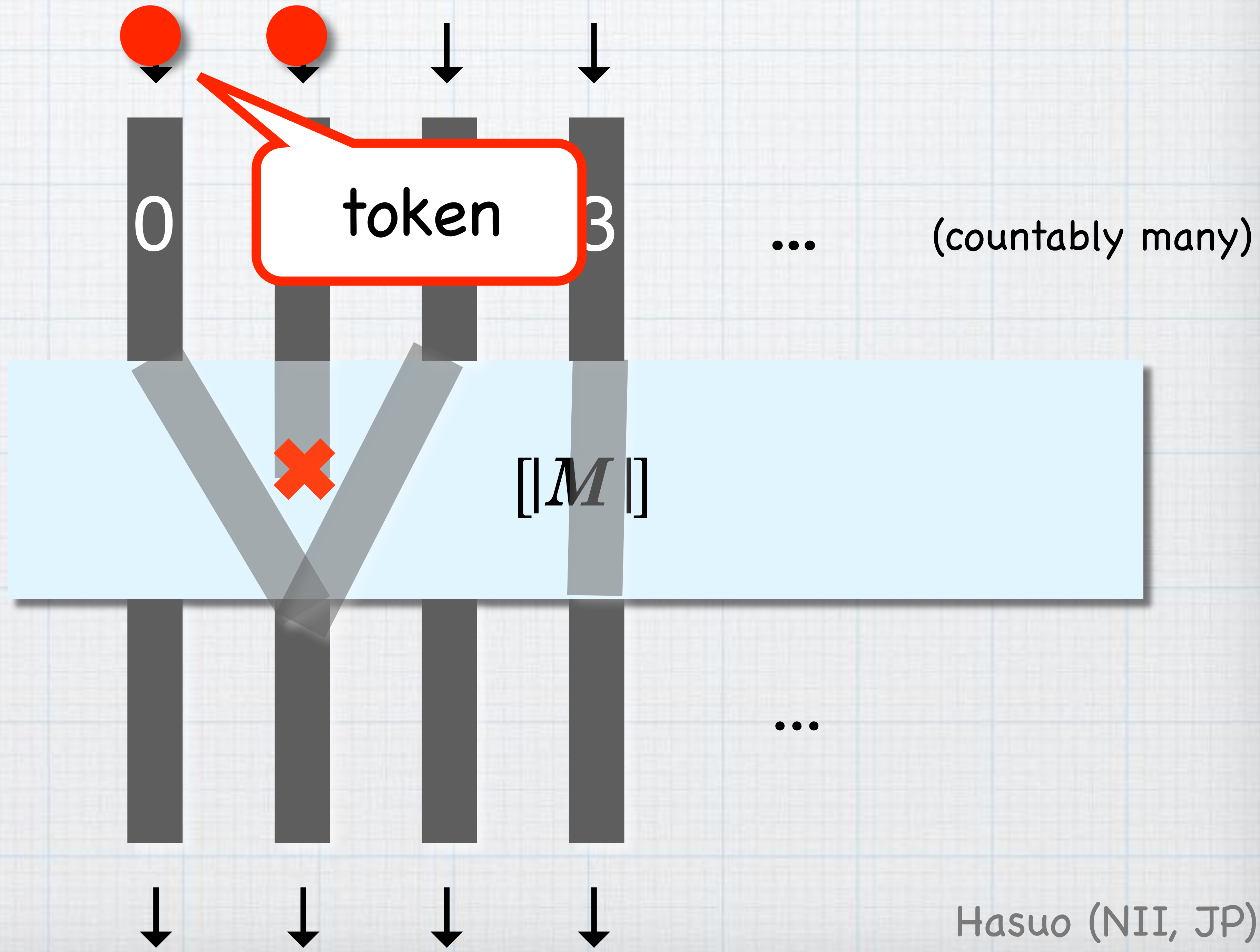




The GoI Animation

$$[\![M]\!] \;=\; (\,\mathbb{N} \rightharpoonup \mathbb{N},\;\text{a partial function}\,)$$

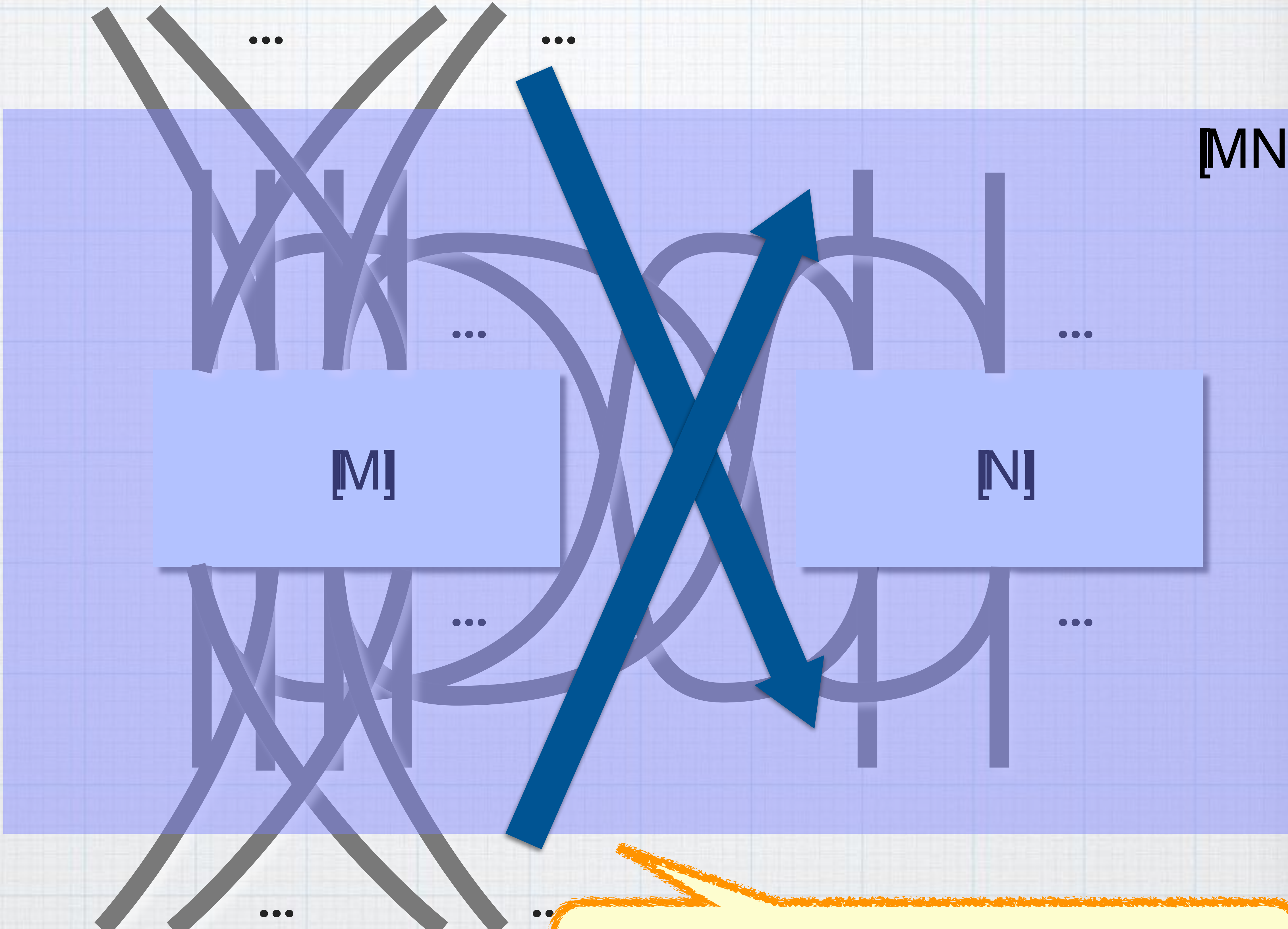
$$=\;\text{“piping”}$$



The GoI Animation

* Function application $[\![MN]\!]$
  * by "parallel composition + hiding"

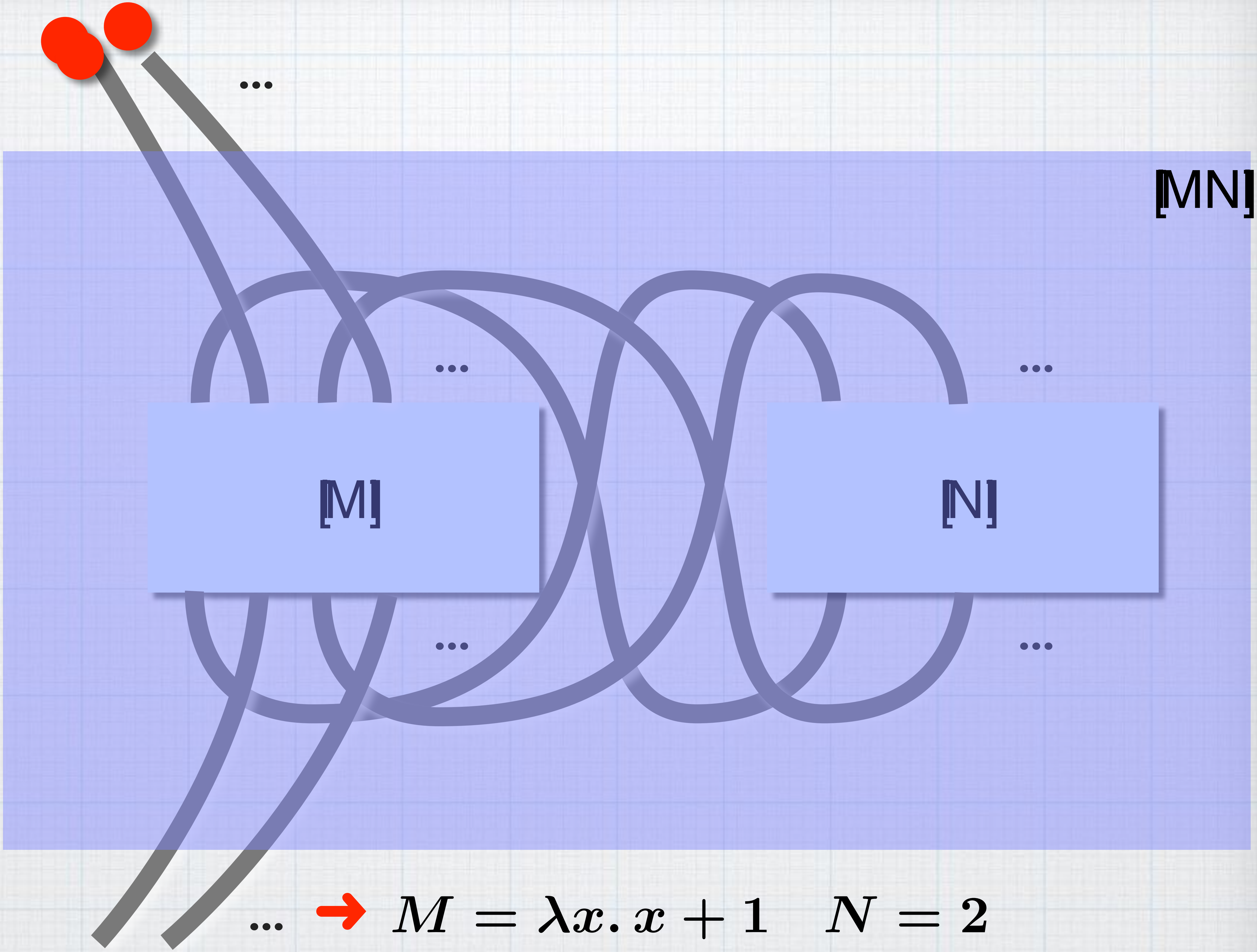
$$[\![MN]\!] =$$



"parallel composition + hiding"
(cf. AJM games)

$[\![MN]\!]$

$=$



$\rightarrow\ M = \lambda x.\, x + 1 \qquad N = 2$

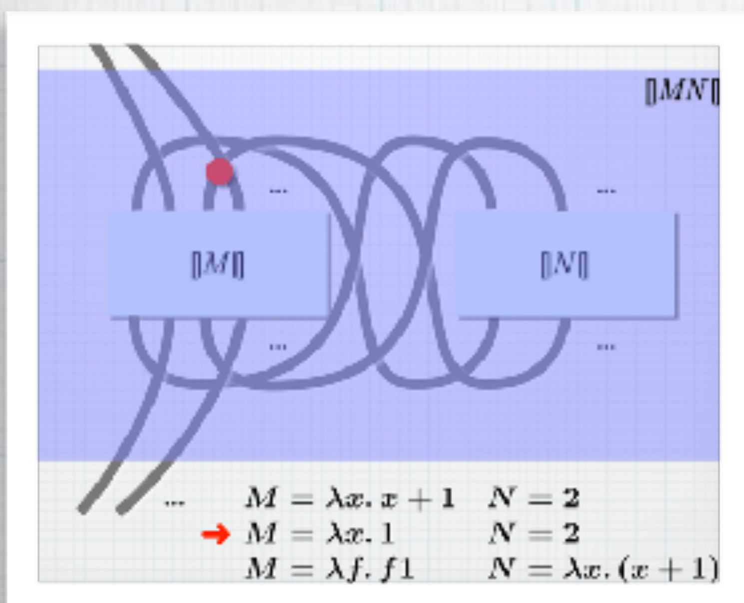
$\rightarrow\ M = \lambda x.\, 1 \qquad N = 2$

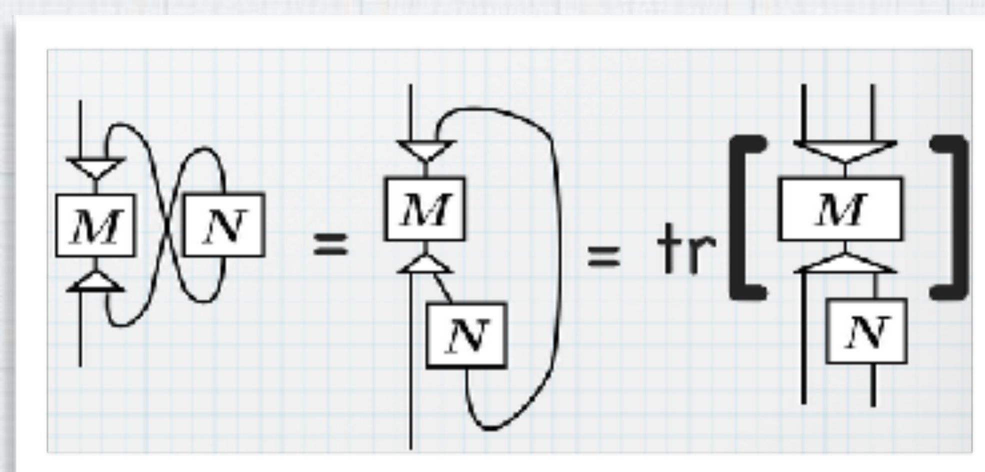
$\rightarrow\ M = \lambda f.\, f 1 \qquad N = \lambda x.\, (x + 1)$

Outline

Coalgebra meets **higher-order computation** in **Geometry of Interaction** [Girard, LC'88]

"GoI Animation"





Categorical GoI

[Abramsky, Haghverdi & Scott, MSCS'02]

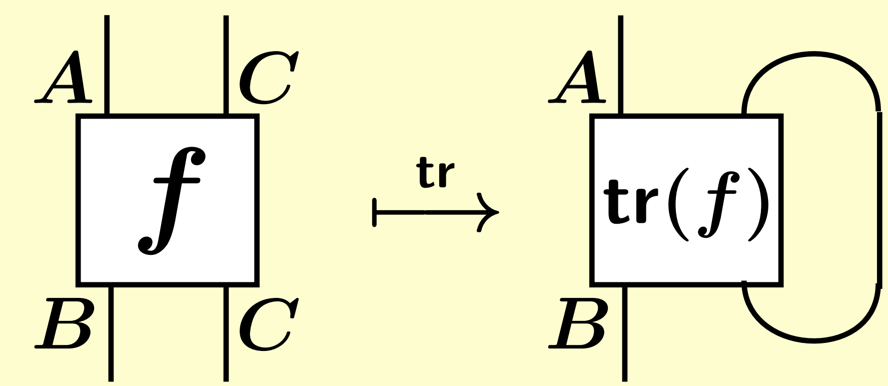
Categorical GoI

* Axiomatics of GoI in the categorical language
* Our main reference:
  * [AHS02] S. Abramsky, E. Haghverdi, and P. Scott, **Geometry of interaction and linear combinatory algebras**, Math. Str. Comp. Sci, 2002
  * Especially its technical report version (Oxford CL), since it's a bit more detailed
* See also:
  * IH and Naohiko Hoshino. **Semantics of Higher-Order Quantum Computation via Geometry of Interaction**. Annals Pure & Applied Logic 2017. arxiv.org/abs/1605.05079

The Categorical GoI Workflow

Traced monoidal category $\mathbb{C}$

+ other constructs ➜ "GoI situation" [AHS02]



⬇ Categorical GoI [AHS02]

Linear combinatory algebra

- * Applicative str. + combinators
- * Model of untyped calculus

⬇ Realizability

- * PER, ω-set, assembly, ...
- * "Programming in untyped λ"

Linear category

Model of typed calculus

What we use (ingredient)

GoI situation

Defn. (GoI situation [AHS02])
A *GoI situation* is a triple $(\mathbb{C}, \boldsymbol{F}, \boldsymbol{U})$ where

- $\mathbb{C} = (\mathbb{C}, \otimes, \boldsymbol{I})$ is a traced symmetric monoidal category (TSMC);
- $\boldsymbol{F} : \mathbb{C} \to \mathbb{C}$ is a traced symmetric monoidal functor, equipped with the following retractions (which are monoidal natural transformations).

$$\begin{array}{rclcll} \boldsymbol{e} & : & \boldsymbol{FF} \lhd \boldsymbol{F} & : & \boldsymbol{e'} & \text{Comultiplication} \\ \boldsymbol{d} & : & \mathbf{id} \lhd \boldsymbol{F} & : & \boldsymbol{d'} & \text{Dereliction} \\ \boldsymbol{c} & : & \boldsymbol{F} \otimes \boldsymbol{F} \lhd \boldsymbol{F} & : & \boldsymbol{c'} & \text{Contraction} \\ \boldsymbol{w} & : & \boldsymbol{K_I} \lhd \boldsymbol{F} & : & \boldsymbol{w'} & \text{Weakening} \end{array}$$

  Here $\boldsymbol{K_I}$ is the constant functor into the monoidal unit $\boldsymbol{I}$;
- $\boldsymbol{U} \in \mathbb{C}$ is an object (called *reflexive object*), equipped with the following retractions.

$$\begin{array}{rclcl} \boldsymbol{j} & : & \boldsymbol{U} \otimes \boldsymbol{U} \lhd \boldsymbol{U} & : & \boldsymbol{k} \\ & & \boldsymbol{I} \lhd \boldsymbol{U} & & \\ \boldsymbol{u} & : & \boldsymbol{FU} \lhd \boldsymbol{U} & : & \boldsymbol{v} \end{array}$$

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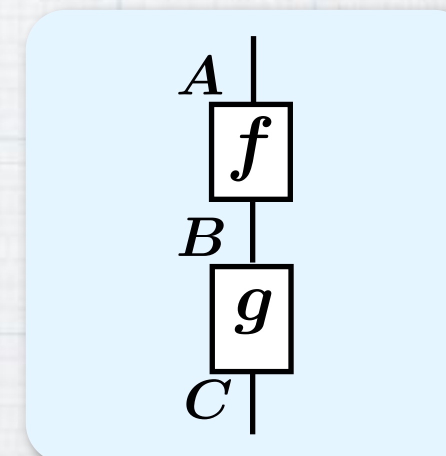
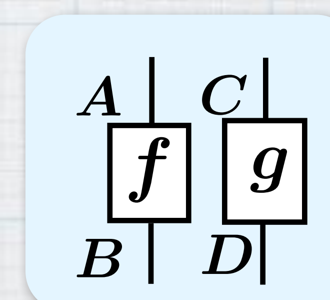
* **Monoidal category** $(\mathbb{C}, \otimes, I)$

* **String diagrams**

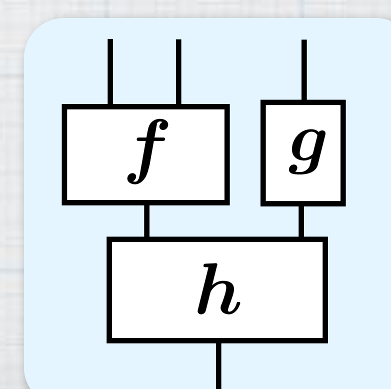
$$\frac{A \xrightarrow{f} B \qquad B \xrightarrow{g} C}{A \xrightarrow{g \circ f} C}$$



$$\frac{A \xrightarrow{f} B \qquad C \xrightarrow{g} D}{A \otimes C \xrightarrow{f \otimes g} B \otimes D}$$



$$h \circ (f \otimes g)$$



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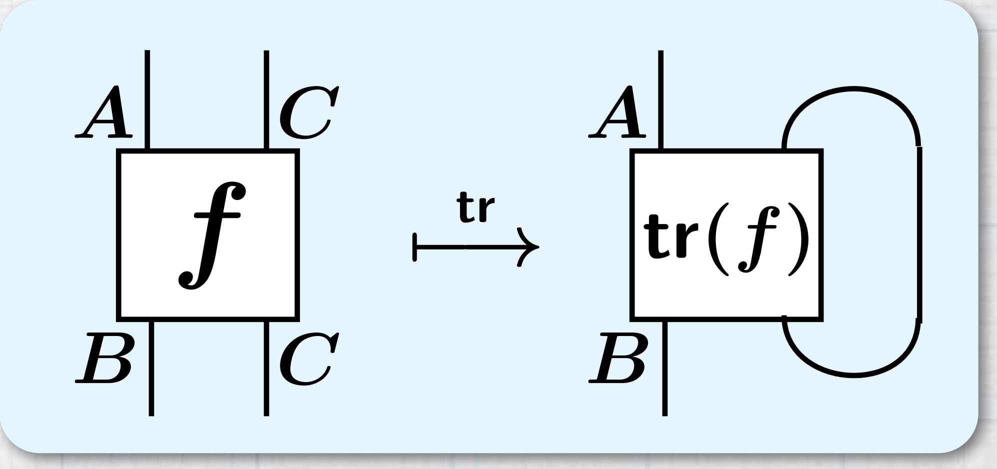
- **Traced** monoidal category
- "feedback"

$$\frac{A \otimes C \xrightarrow{f} B \otimes C}{A \xrightarrow{\mathsf{tr}(f)} B}$$

that is

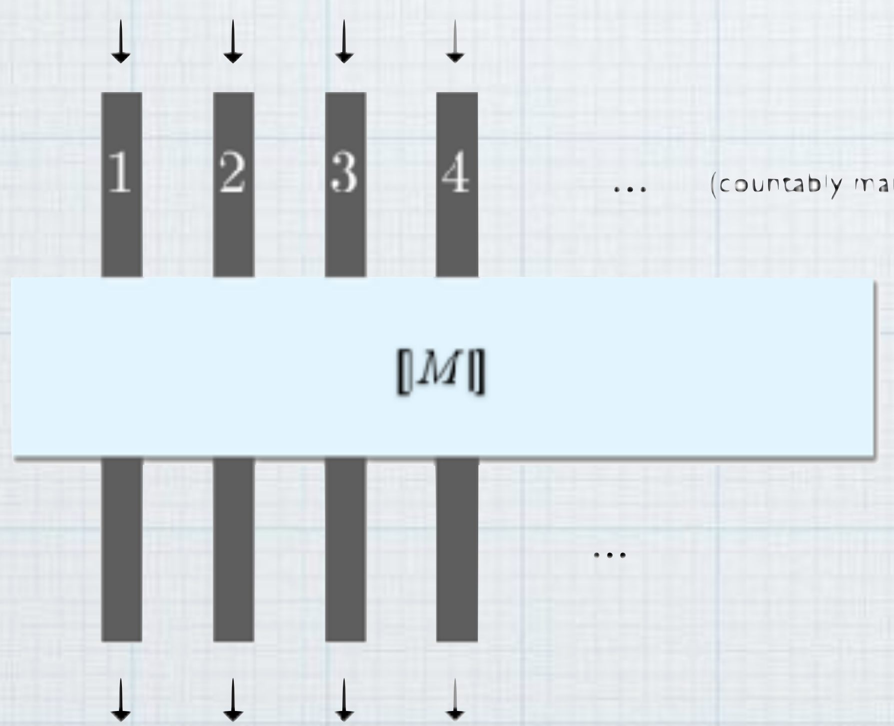


Hasuo (NII, JP)

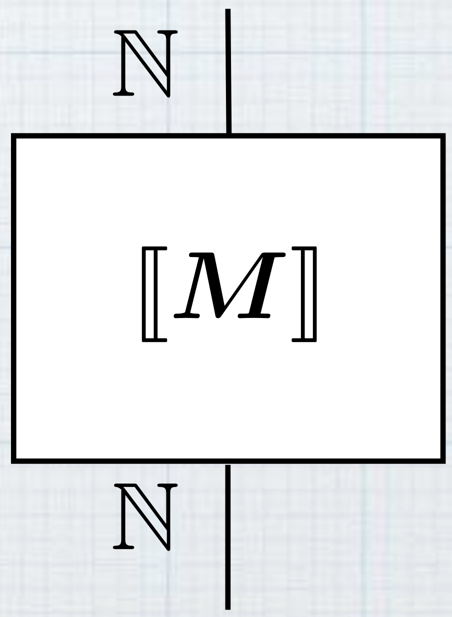
String Diagram vs. "Pipe Diagram"

* I use two ways of depicting partial functions $\mathbb{N} \rightharpoonup \mathbb{N}$

In the monoidal category $(\mathbf{Pfn}, +, \mathbf{0})$



Pipe diagram

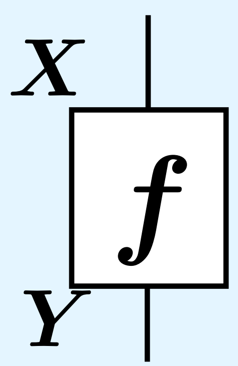


String diagram

Traced Sym. Monoidal Category
$(\mathbf{Pfn}, +, 0)$

* Category **Pfn** of partial functions
  * <u>**Obj.**</u> A set X
  * <u>**Arr.**</u> A partial function

$$\frac{\boldsymbol{X} \to \boldsymbol{Y} \ \text{ in } \mathbf{Pfn}}{\boldsymbol{X} \rightharpoonup \boldsymbol{Y}, \text{ partial function}}$$



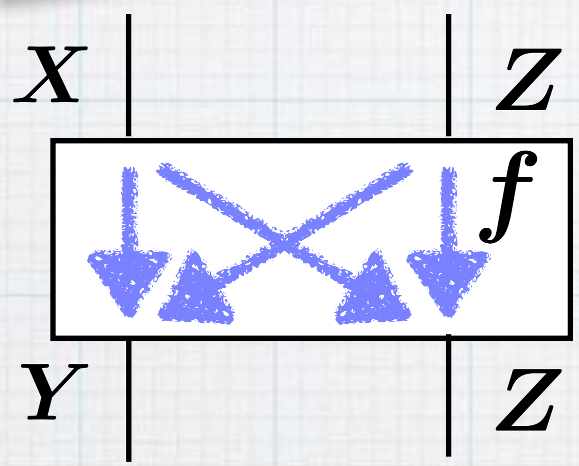
* is traced symmetric monoidal

Traced Sym. Monoidal Category

$(\mathbf{Pfn}, +, 0)$

* $$\frac{X + Z \xrightarrow{f} Y + Z \quad \text{in } \mathbf{Pfn}}{X \xrightarrow{\mathsf{tr}(f)} Y \quad \text{in } \mathbf{Pfn}}$$

How?

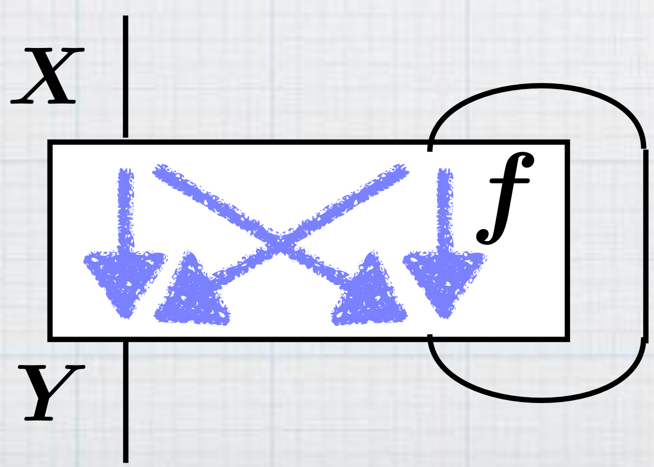
* 

$$f_{XY}(x) := \begin{cases} f(x) & \text{if } f(x) \in Y \\ \bot & \text{o.w.} \end{cases}$$

Similar for f_{XZ}, f_{ZY}, f_{ZZ}

* **Execution formula** (Girard)
* Partiality is essential (infinite

* Trace operator:



$$\mathsf{tr}(f) = f_{XY} \sqcup \left(\coprod_{n \in \mathbb{N}} f_{ZY} \circ (f_{ZZ})^n \circ f_{XZ} \right)$$

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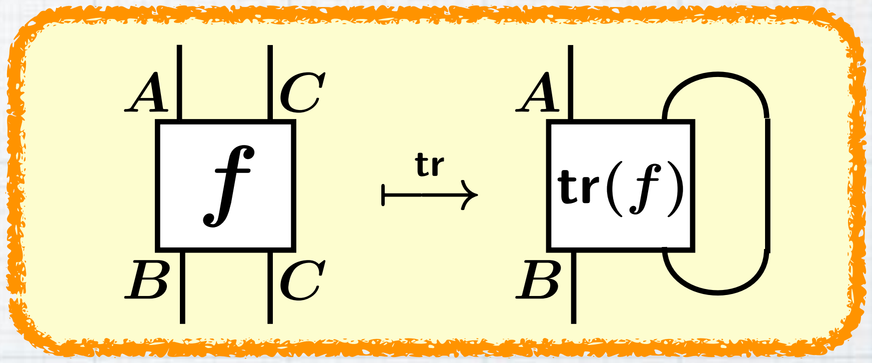
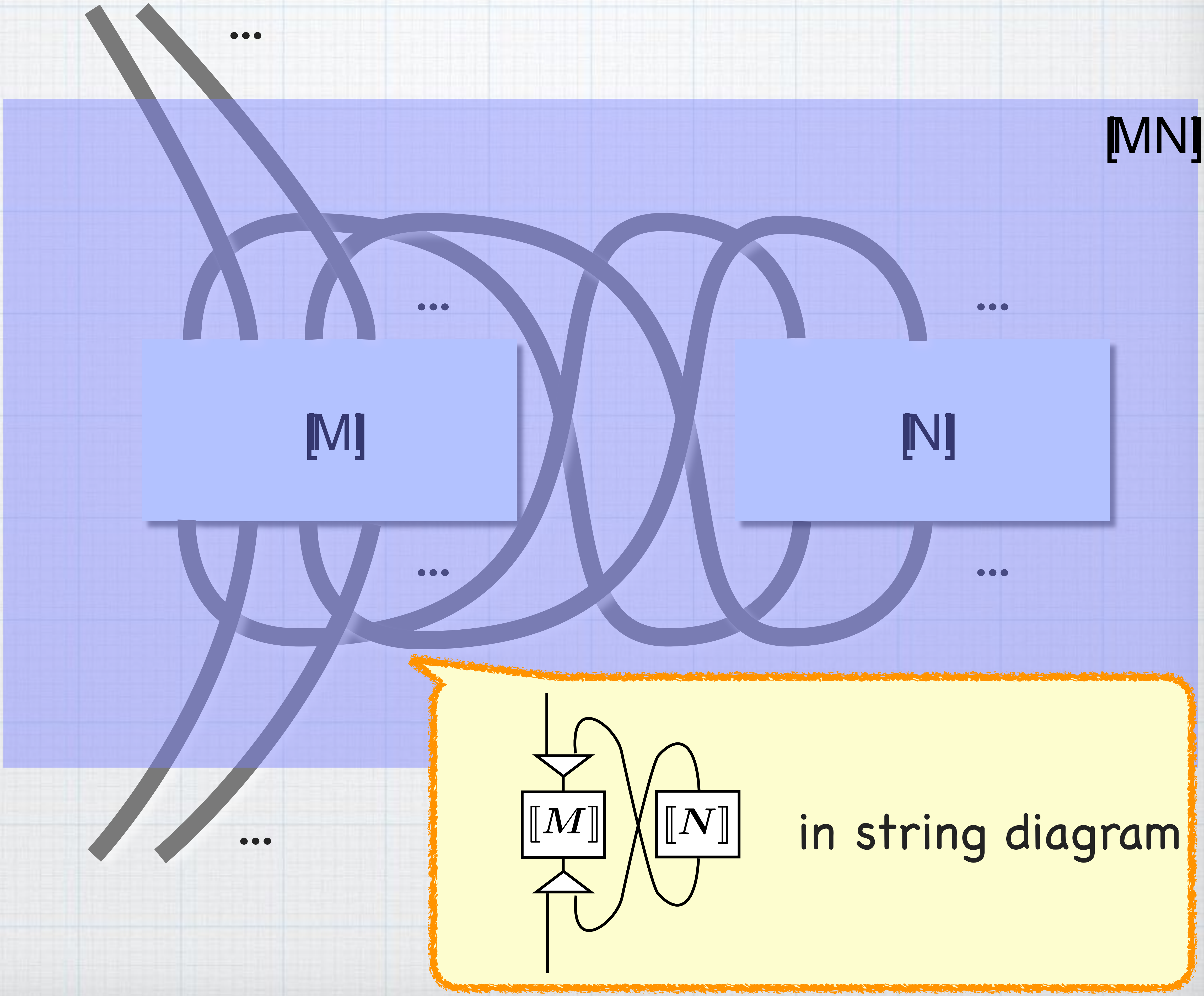
- Traced sym. monoidal cat.
  - Where one can "feedback"



  - Why for GoI?

$[\![MN]\!] =$



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$d : \mathrm{id} \lhd F : d'$	Dereliction
$c : F \otimes F \lhd F : c'$	Contraction
$w : K_I \lhd F : w'$	Weakening

  Here $\boldsymbol{K_I}$ is the constant functor into the monoidal unit $\boldsymbol{I}$;
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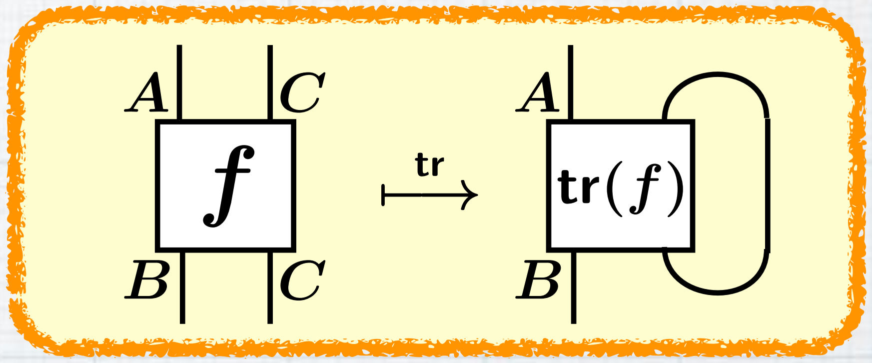
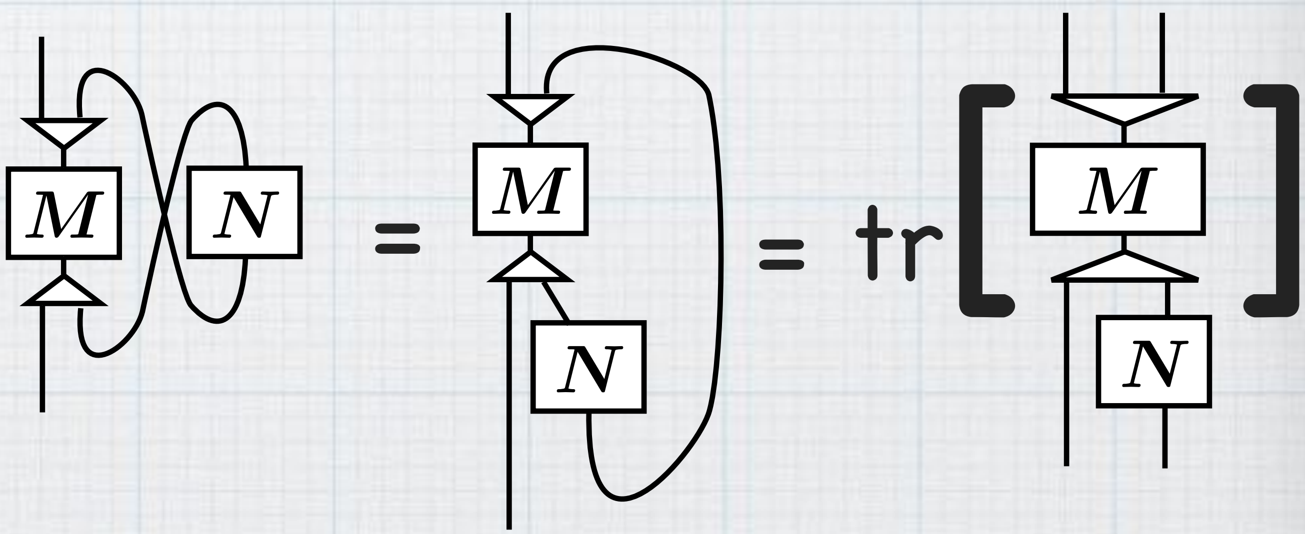
$$\begin{aligned} j : U \otimes U &\lhd U : k \\ I &\lhd U \\ u : FU &\lhd U : v \end{aligned}$$

* Traced sym. monoidal cat.
  * Where one can "feedback"



  * Why for GoI?



  * Leading example: **Pfn**

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\begin{array}{c}
\boldsymbol{j} \;:\; \boldsymbol{U} \otimes \boldsymbol{U} \lhd \boldsymbol{U} \;:\; \boldsymbol{k} \\
\boldsymbol{I} \lhd \boldsymbol{U} \\
\boldsymbol{u} \;:\; \boldsymbol{F}\boldsymbol{U} \lhd \boldsymbol{U} \;:\; \boldsymbol{v}
\end{array}
$$

Defn. (Retraction)
A *retraction* from $\boldsymbol{X}$ to $\boldsymbol{Y}$,

$$\boldsymbol{f} : \boldsymbol{X} \lhd \boldsymbol{Y} : \boldsymbol{g} \;,$$

is a pair of arrows

"embedding"

$$
\begin{tikzcd}
\boldsymbol{X} \arrow[loop left, "\mathbf{id}"] \arrow[r, bend left, "\boldsymbol{f}"] & \boldsymbol{Y} \arrow[l, bend left, "\boldsymbol{g}"]
\end{tikzcd}
$$

"projection"

such that $\boldsymbol{g} \circ \boldsymbol{f} = \mathbf{id}_{\boldsymbol{X}}$.

- Functor F
  - For obtaining $! : A \to A$

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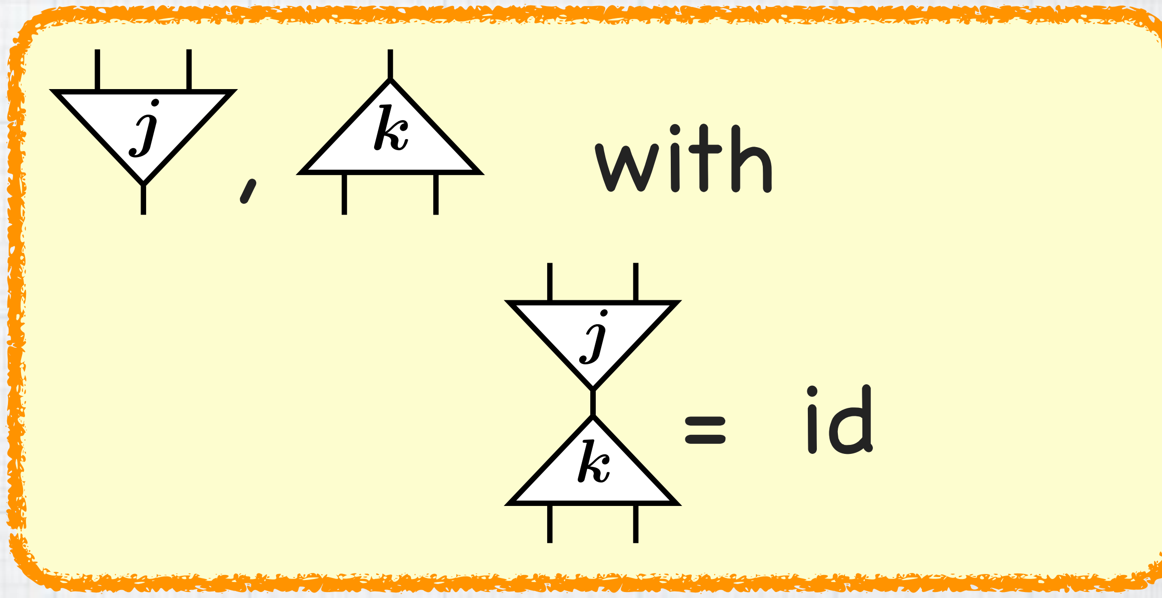
$$\begin{array}{rcl}
j : & U \otimes U \lhd U & : k \\
 & I \lhd U & \\
u : & FU \lhd U & : v
\end{array}$$

* The reflexive object U

* Retr. $\begin{tikzcd} U \otimes U \arrow[r, bend left, "j"] & U \arrow[l, bend left, "k"] \end{tikzcd}$



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  Here $\boldsymbol{K_I}$ is the constant functo[r]

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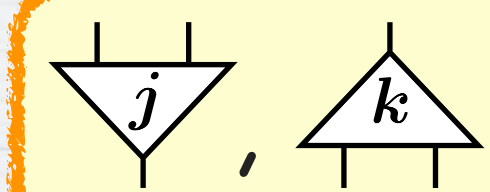
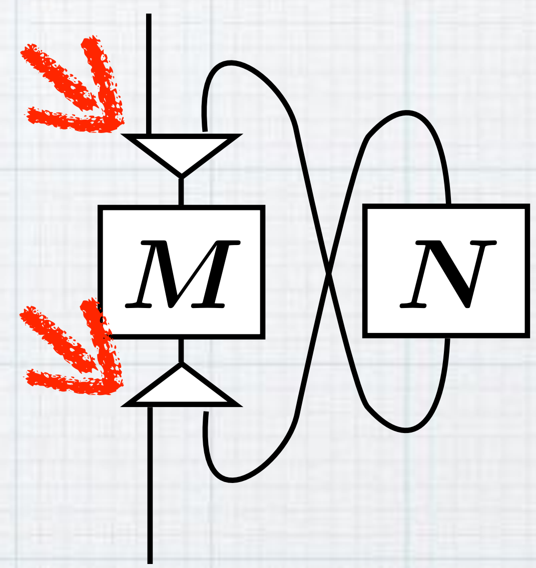
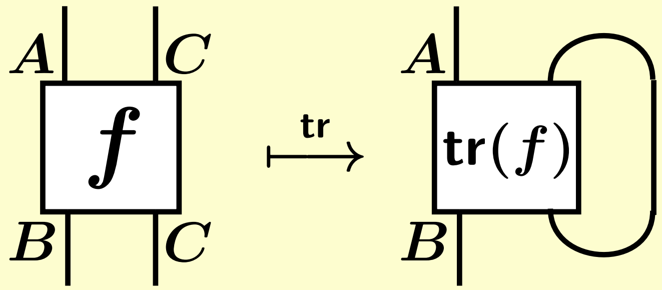


- The reflexive object U
- Why for GoI?



- Example in **Pfn**:

$$\mathbb{N} \in \mathbf{Pfn}, \text{ with}$$
$$\mathbb{N} + \mathbb{N} \cong \mathbb{N},$$
$$\mathbb{N} \cdot \mathbb{N} \cong \mathbb{N}$$



tuation: Summary

Defn. (GoI situation [AHS02])
A *GoI situation* is a triple $(\mathbb{C}, \boldsymbol{F}, \boldsymbol{U})$ wh...e

- $\mathbb{C} = (\mathbb{C}, \otimes, \boldsymbol{I})$ is a traced symmetric monoidal category (TSMC);
- $\boldsymbol{F} : \mathbb{C} \to \mathbb{C}$ is a traced symmetric monoidal functor, equipped with the following retractions (which are monoidal natural transformations):

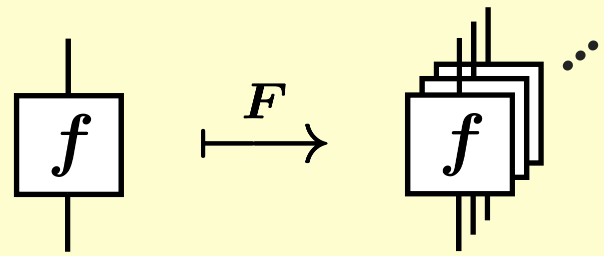
$$\begin{array}{rl} \boldsymbol{e} : \boldsymbol{FF} \lhd \boldsymbol{F} : \boldsymbol{e}' & \\ \boldsymbol{d} : \mathbf{id} \lhd \boldsymbol{F} : \boldsymbol{d}' & \text{De} \\ \boldsymbol{c} : \boldsymbol{F} \otimes \boldsymbol{F} \lhd \boldsymbol{F} : \boldsymbol{c}' & \text{Co} \\ \boldsymbol{w} : \boldsymbol{K_I} \lhd \boldsymbol{F} : \boldsymbol{w}' & \text{We} \end{array}$$

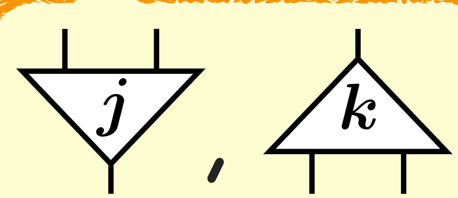
Here $\boldsymbol{K_I}$ is the constant functor into t...

- $\boldsymbol{U} \in \mathbb{C}$ is an object (called *reflexive obj*...) ... the following retractions.

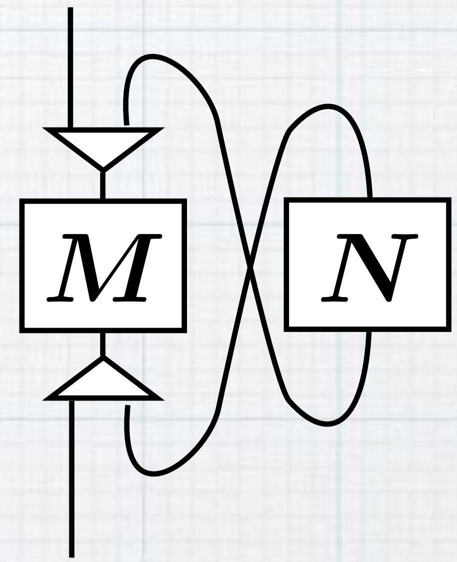
$$\begin{array}{c} \boldsymbol{j} : \boldsymbol{U} \otimes \boldsymbol{U} \lhd \boldsymbol{U} : \boldsymbol{k} \\ \boldsymbol{I} \lhd \boldsymbol{U} \\ \boldsymbol{u} : \boldsymbol{FU} \lhd \boldsymbol{U} : \boldsymbol{v} \end{array}$$

For ! , via





* Categorical axiomatics of the "GoI animation"



* Example:

$$(\mathbf{Pfn}, \mathbb{N} \cdot \_\_, \mathbb{N})$$

Hasuo (NII, JP)

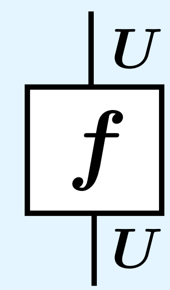
Categorical GoI: Constr. of an LCA

Thm. ([AHS02])
Given a GoI situation $(\mathbb{C}, \boldsymbol{F}, \boldsymbol{U})$, the homset

$$\mathbb{C}(\boldsymbol{U}, \boldsymbol{U})$$

carries a canonical LCA structure.

* Applicative str. $\cdot$
* ! operator
* Combinators B, C, I, ...

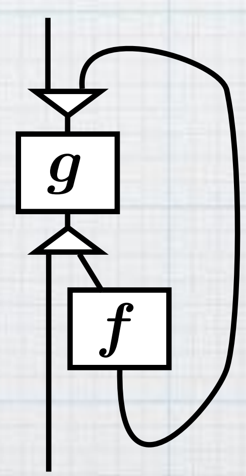


$\in \mathbb{C}(\boldsymbol{U}, \boldsymbol{U})$

* $\boldsymbol{g} \cdot \boldsymbol{f}$

$$:= \mathbf{tr}\big((\boldsymbol{U} \otimes \boldsymbol{f}) \circ \boldsymbol{k} \circ \boldsymbol{g} \circ \boldsymbol{j} \big)$$

=



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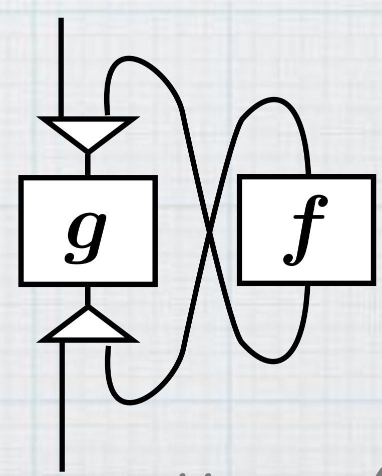


Hasuo (NII, JP)

Summary: Categorical GoI

Defn. (GoI situation [AHS02])
A *GoI situation* is a triple $(\mathbb{C}, F, U)$ where

- $\mathbb{C} = (\mathbb{C}, \otimes, I)$ is a traced symmetric monoidal category (TSMC);
- $F : \mathbb{C} \to \mathbb{C}$ is a traced symmetric monoidal functor, equipped with the following retractions (which are monoidal natural transformations).

$$
\begin{array}{rcccll}
e & : & FF \lhd F & : & e' & \text{Comultiplication} \\
d & : & \mathrm{id} \lhd F & : & d' & \text{Dereliction} \\
c & : & F \otimes F \lhd F & : & c' & \text{Contraction} \\
w & : & K_I \lhd F & : & w' & \text{Weakening}
\end{array}
$$

  Here K_I is the constant functor into the monoidal unit I;
- $U \in \mathbb{C}$ is an object (called *reflexive object*), equipped with the following retractions.

$$
\begin{array}{rcccl}
j & : & U \otimes U \lhd U & : & k \\
 & & I \lhd U & & \\
u & : & FU \lhd U & : & v
\end{array}
$$

Thm. ([AHS02])
Given a GoI situation $(\mathbb{C}, F, U)$, the homset

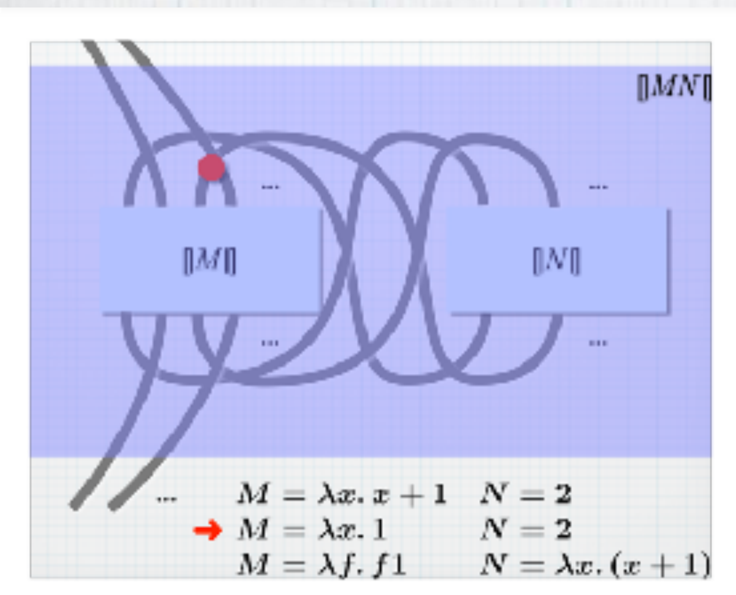
$$\mathbb{C}(U, U)$$

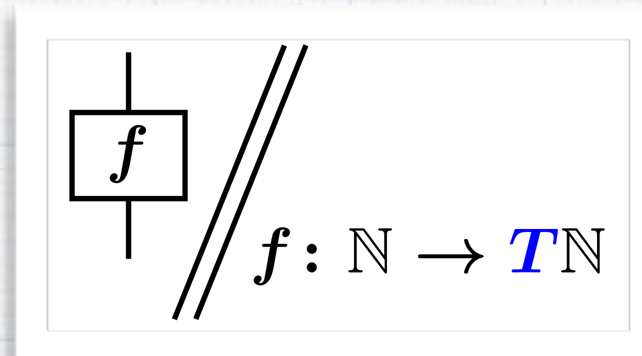
carries a canonical LCA structure.

Outline

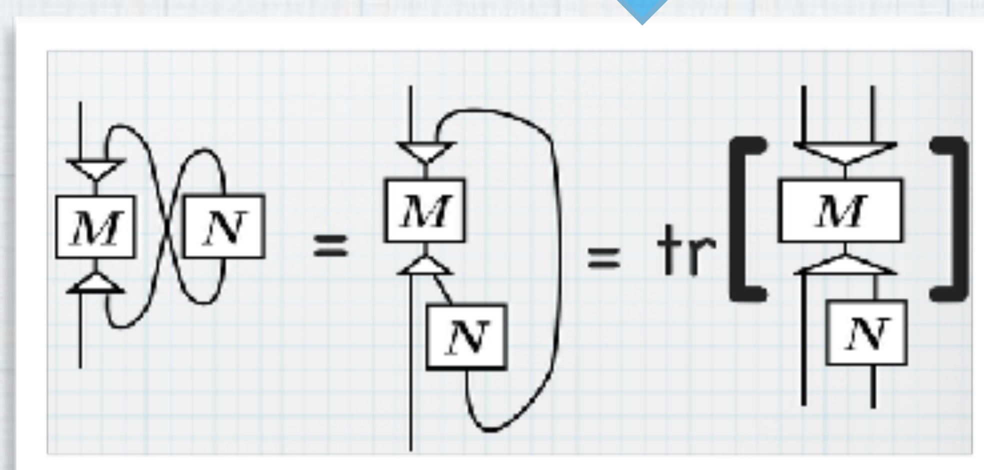
Coalgebra meets **higher-order computation**
in **Geometry of Interaction** [Girard, LC'88]

"GoI Animation"





GoI w/ T-branching
[IH & Hoshino, LICS'11]



Categorical GoI
[Abramsky, Haghverdi & Scott, MSCS'02]

Why Catego
Examples

$Kl(T)$ for different branching monads T

* **Pfn** (partial functions)

$$\frac{\dfrac{X \to Y \ \text{ in } \mathbf{Pfn}}{X \rightharpoonup Y, \text{ partial function}}}{X \to \mathcal{L}Y \ \text{ in } \mathbf{Sets}} \quad \text{where } \mathcal{L}Y = \{\perp\} + Y$$

(Potential) non-termination

* **Rel** (relations)

$$\frac{\dfrac{X \to Y \ \text{ in } \mathbf{Rel}}{R \subseteq X \times Y, \text{ relation}}}{X \to \mathcal{P}Y \ \text{ in } \mathbf{Sets}} \quad \text{where } \mathcal{P} \text{ is the powerset monad}$$

Non-determinism

* **DSRel**

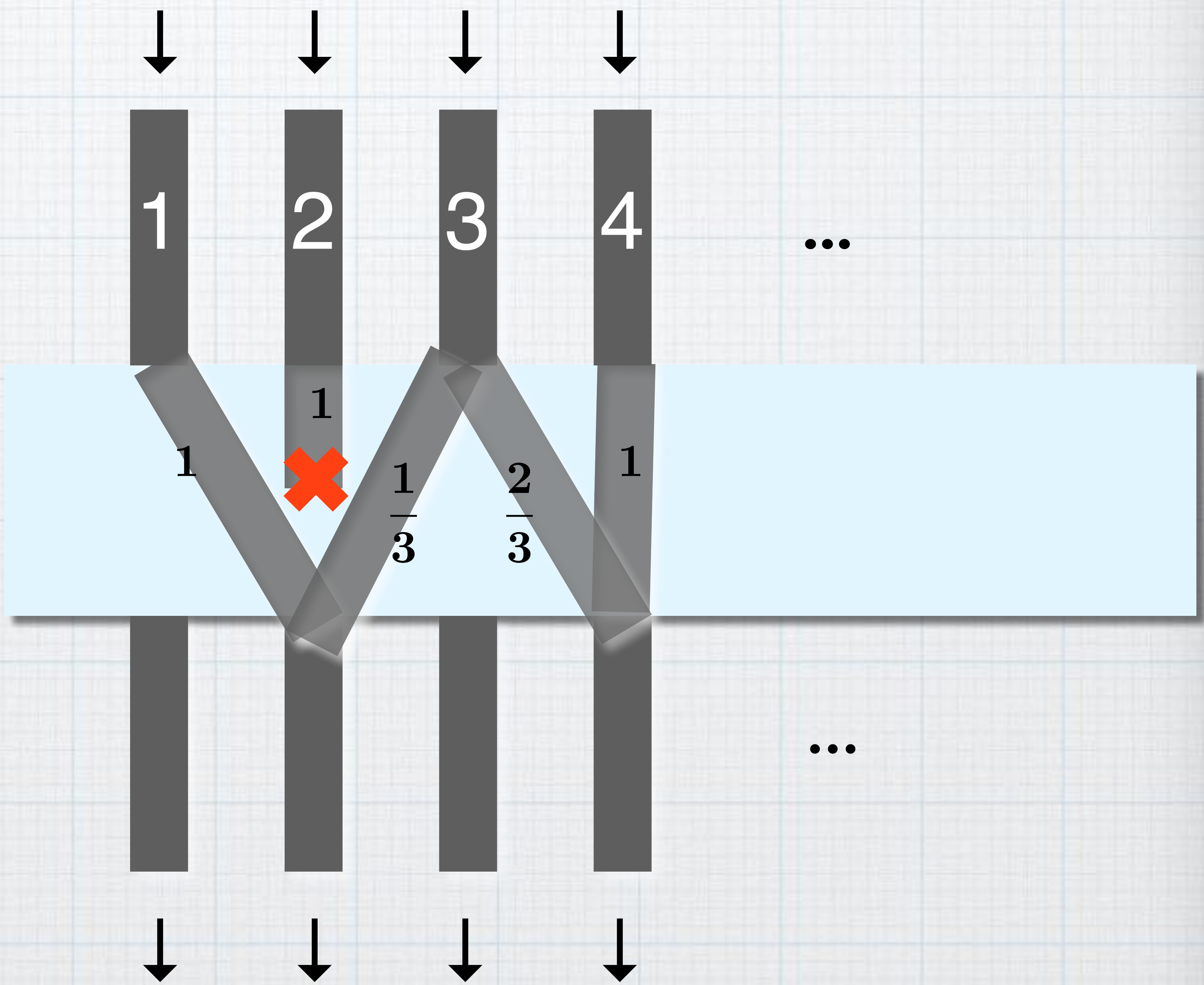
$$\frac{X \to Y \ \text{ in } \mathbf{DSRel}}{X \to \mathcal{D}Y \ \text{ in } \mathbf{Sets}}$$

where $\mathcal{D}Y = \{d : Y \to [0,1] \mid \sum_y d(y) \leq 1\}$

Probabilistic branching

Different Branching in The GoI Animation

- **Pfn** (partial functions)
  - Pipes can be stuck
- **Rel** (relations)
  - Pipes can branch
- **DSRel**
  - Pipes can branch probabilistically



Branching Monad: Source of Particle-Style GoI Situations

Thm. ([Jacobs,CMCS10])
Given a "branching monad" T on **Sets**, the monoidal category

$$(\mathcal{K}\ell(\boldsymbol{T}), +, \mathbf{0})$$

is

- a *unique decomposition category* [Haghverdi,PhD00], hence is
- a traced symmetric monoidal category.

Cor.
$\big(\, (\mathcal{K}\ell(\boldsymbol{T}), +, \mathbf{0}),\ \mathbb{N}\cdot\_\,,\ \mathbb{N}\,\big)$ is a GoI situation.

Monads in [Hasuo,Jacobs&Sokolova07]

- Kl(T) is $\mathrm{Cpo}_\perp$-enriched

Particle-style: trace via the execution formula

$$\mathbf{tr}(f) = f_{XY} \sqcup \left(\coprod_{n\in\mathbb{N}} f_{ZY} \circ (f_{ZZ})^n \circ f_{XZ} \right)$$

The Categorical GoI Workflow

Branching monad B

Coalgebraic trace semantics

Traced monoidal category $\mathbb{C}$

+ other constructs ➜ "GoI situation" [AHS02]

Categorical GoI [AHS02]

Linear combinatory algebra

Realizability

Linear category

Fancy monad

Fancy TSMC

Fancy LCA

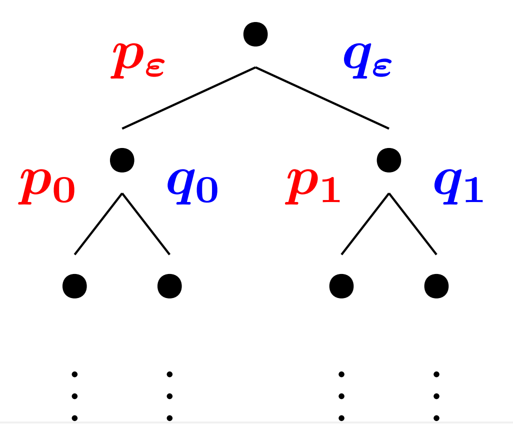
Model of fancy language

* Model for (a variant of) the Selinger-Valiron **quantum λ-calculus**
  (linear λ + prep./Unitary/meas.)
  [Hasuo & Hoshino, LICS'11 & APAL'16]
* via the **quantum branching monad**
* ... with considerable complication :(

$$[\![\Gamma \vdash M : \tau]\!] : \; [\![\Gamma]\!] \longrightarrow ([\![\tau]\!] \multimap R) \multimap R$$

where

$$R = \left\{ \quad \middle| \; p_\alpha, q_\alpha \in [0,1] \right\}$$



* Records measurement outcomes
* R as a suitable **final coalgebra** in the realizability category

Realizability

Linear category

Workflow

Fancy monad

Fancy LCA

Model of fancy language

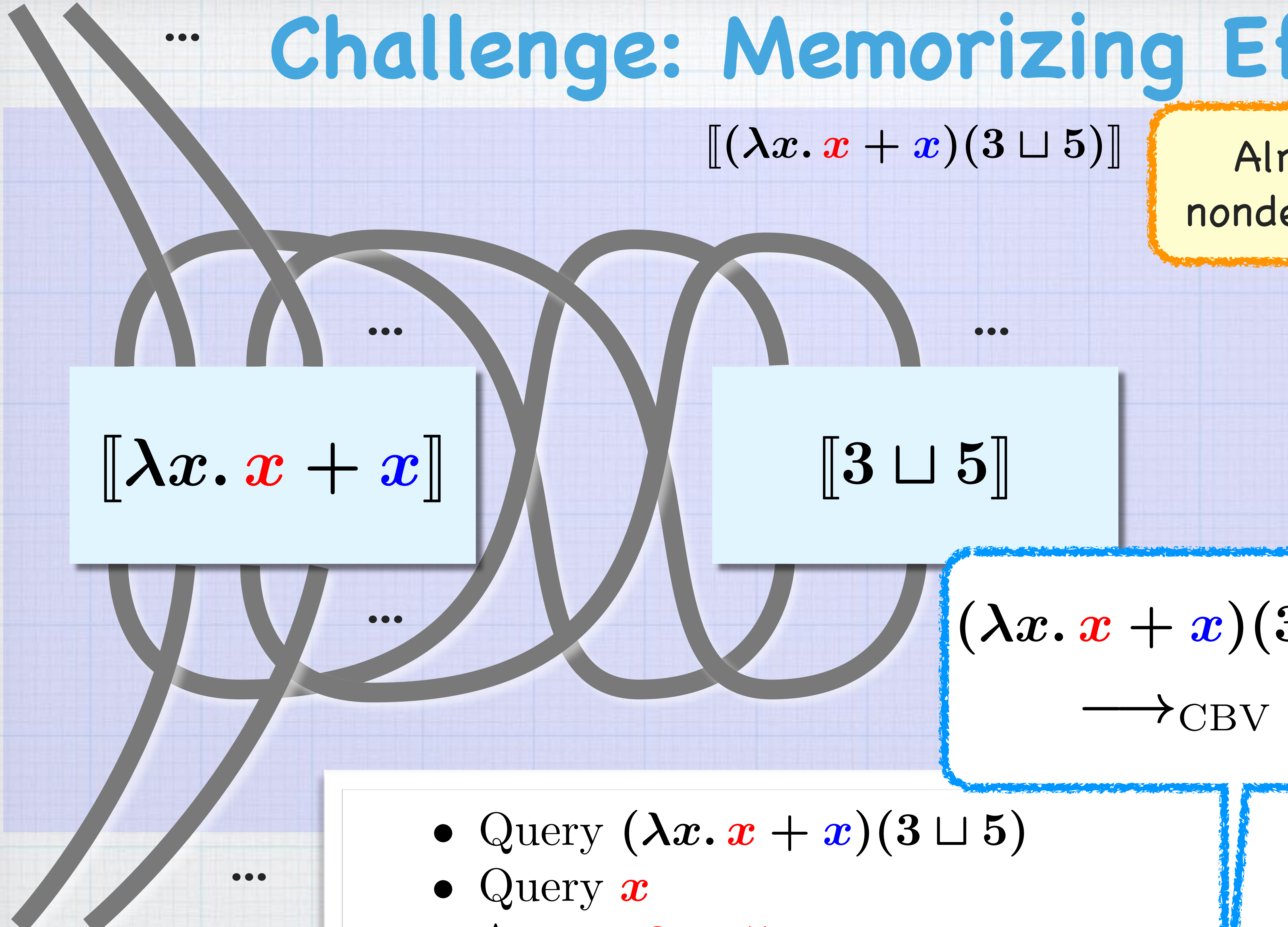
Challenge: Memorizing Effects

Already w/
nondeterminism!

Challenge: Memorizing Effects

$[\![(\lambda x.\, x + x)(3 \sqcup 5)]\!]$

Already w/ nondeterminism!



$(\lambda x.\, x + x)(3 \sqcup 5)$

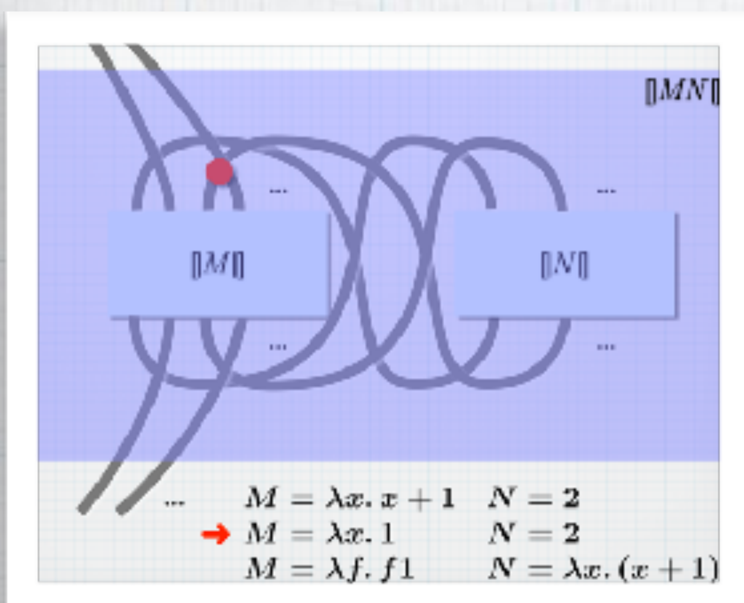
$\longrightarrow_{\mathrm{CBV}}$ **6** or **10** ??

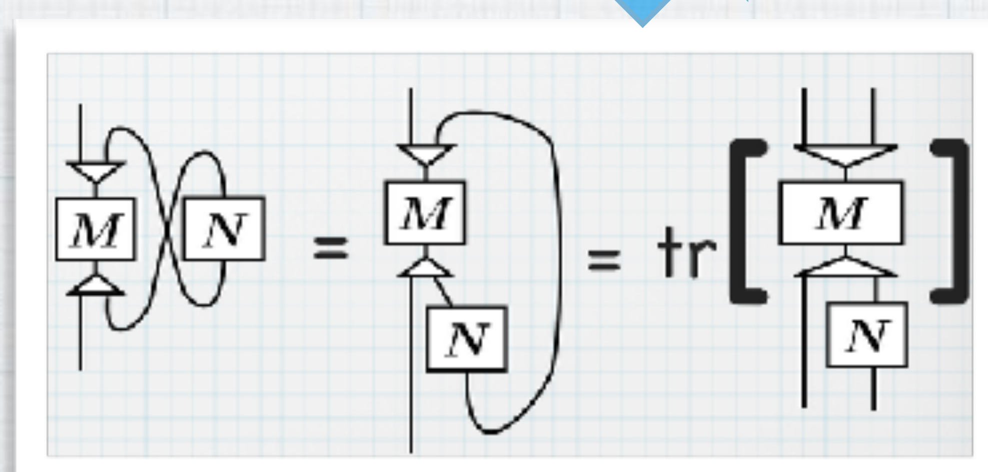
- Query $(\lambda x.\, x + x)(3 \sqcup 5)$
- Query x
- Answer **3** or **5**
- Query x
- Answer **3** or **5**
- Answer $3 + 3$, $3 + 5$, $5 + 3$ or $5 + 5$

Outline

Coalgebra meets **higher-order computation** in **Geometry of Interaction** [Girard, LC'88]

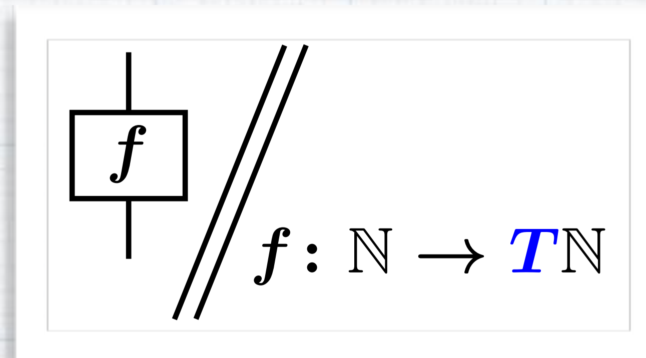
"GoI Animation"





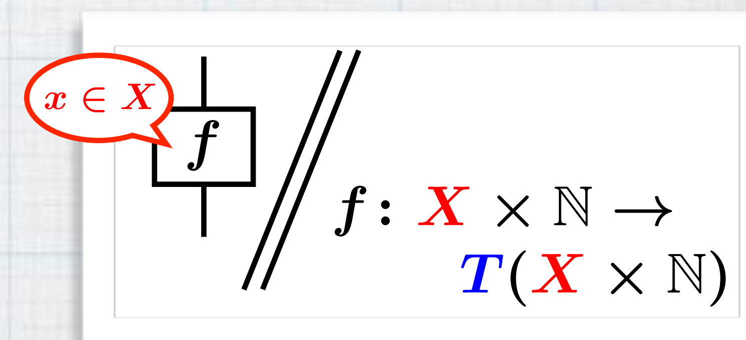
Categorical GoI

[Abramsky, Haghverdi & Scott, MSCS'02]



GoI w/ T-branching

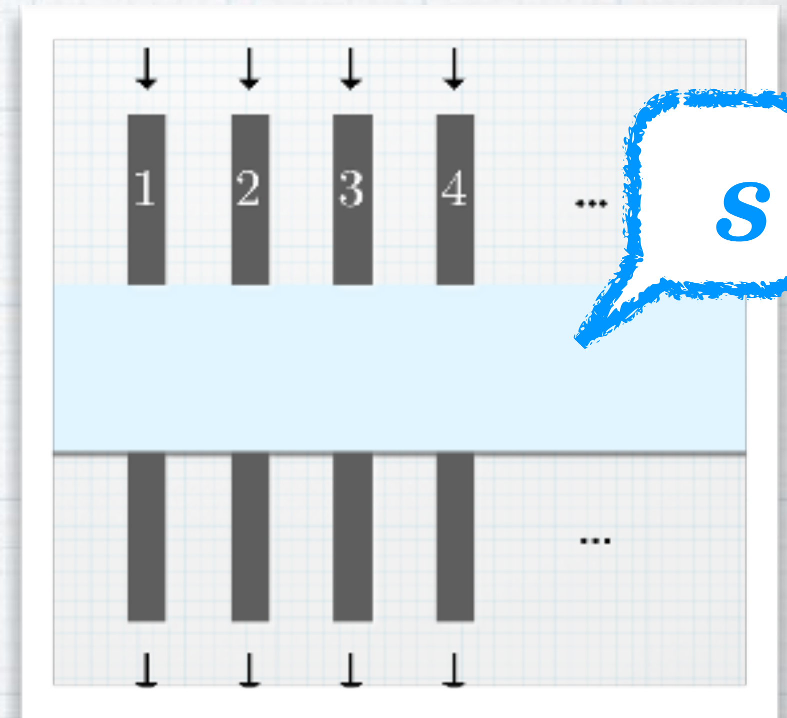
[IH & Hoshino, LICS'11]



Memoryful GoI

[Hoshino, Muroya & IH, CSL-LICS'14 & POPL'16]

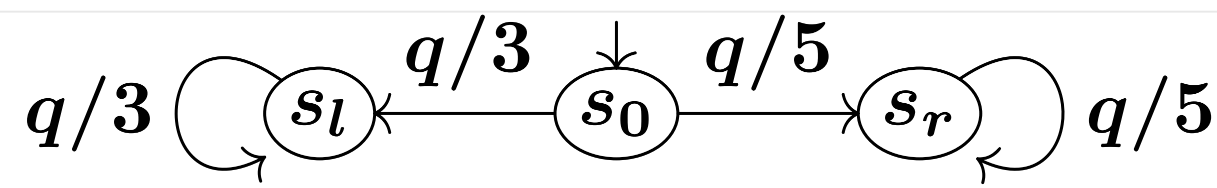
Memoryful GoI



- Equip piping with internal states, or **memory**
- **not** $[\![3 \sqcup 5]\!] : \mathbb{N} \longrightarrow \mathcal{P}\mathbb{N} \ , \quad q \longmapsto \{3, 5\}$

  but a transducer (Mealy machine)

  $[\![3 \sqcup 5]\!] : X \times \mathbb{N} \longrightarrow \mathcal{P}(X \times \mathbb{N}) \ ,$



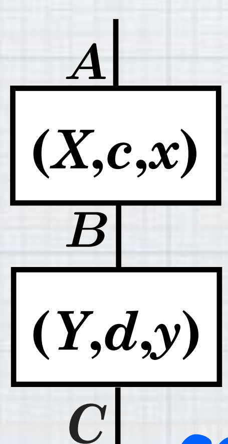
- Not a new idea:
  - **Slices** in GoI for additives [Laurent, TLCA'01]
  - **Resumption GoI** [Abramsky, CONCUR'96]

Memoryful GoI

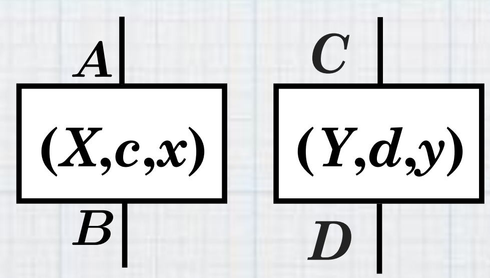
- We introduce memory in a structured manner...
  ➜
  the "traced monoidal category" of transducers

$\boxed{\mathbf{Trans}(T)}$	Objects:	sets $A, B, \ldots$
	Arrows:	$$\frac{A \longrightarrow B \text{ in } \mathbf{Trans}(T)}{\overline{\big(\, X,\ X \times A \xrightarrow{c} T(X \times B),\ x_0 \in X \,\big),\ T\text{-transducer}}}$$

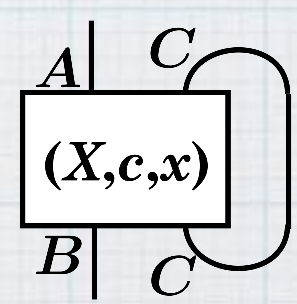
- with operations like



composition $\circ$



tensor $\otimes$



trace

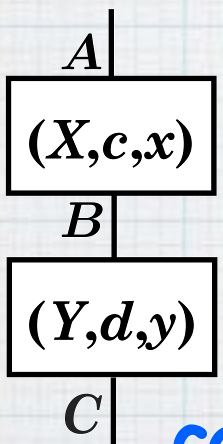
$\mathrm{Trans}(T)$ by Coalgebraic Component Calculus

[Barbosa '03][IH & Jacobs '11]

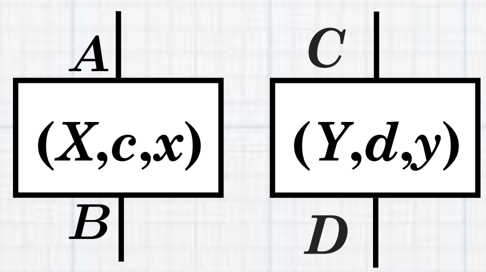
$\boxed{\mathbf{Trans}(T)}$ Objects: sets $A, B, \ldots$

Arrows:

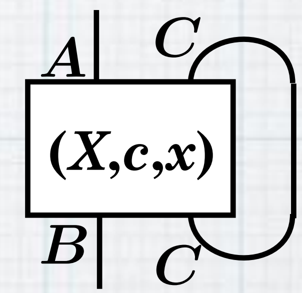
$$\frac{A \longrightarrow B \text{ in } \mathbf{Trans}(T)}{\left(X,\ X \times A \xrightarrow{c} T(X \times B),\ x_0 \in X\right),\ T\text{-transducer}}$$



composition $\circ$



tensor $\otimes$



trace

$$\left(X \times Y,\ \begin{array}{rl} (X \times Y) \times A & \xrightarrow{\cong} (X \times A) \times Y \\ & \xrightarrow{c \times Y} T(X \times B) \times Y \\ & \xrightarrow{\mathrm{str}'} T\big((X \times B) \times Y\big) \\ & \xrightarrow{\cong} T\big(X \times (Y \times B)\big) \\ & \xrightarrow{T(X \times d)} T\big(X \times T(Y \times C)\big) \\ & \xrightarrow{T\mathrm{str}} TT\big(X \times (Y \times C)\big) \\ & \xrightarrow{\mu^T} T\big(X \times (Y \times C)\big) \\ & \xrightarrow{\cong} T\big((X \times Y) \times C\big) \end{array},\ (x, y) \right)$$

The Memoryful GoI Framework

- Exception $\mathbf{1} + E + (\_)$
  - with $\mathbf{0}$-ary opr. $\mathbf{raise}_e$ $(e \in E)$
- Nondeterminism $\mathcal{P}$
  - with binary opr. $\sqcup$
- Probability $\mathcal{D}$, where $\mathcal{D}X = \{d \colon X \to [0,1] \mid \sum_x d(x) \leq 1\}$
  - with binary opr. $\sqcup_p$ $(p \in [0,1])$
- Global state $(\mathbf{1} + S \times \_)^S$
  - with $|V|$-ary $\mathbf{lookup}_l$ and unary $\mathbf{update}_{l,v}$

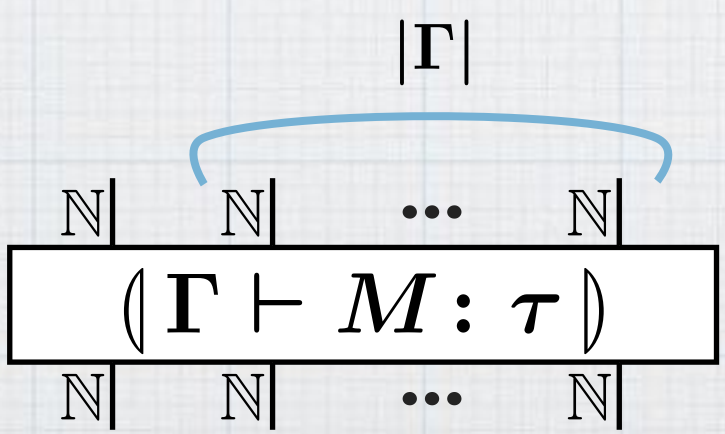
* Given:
  * a **monad** T on $\mathbf{Sets}$, s.t. $\mathbf{Kl}(T)$ is $\mathbf{Cppo}$-enriched
  * an alg. signature Σ with **algebraic operations** on T [Plotkin & Power]

$$\left\{ \alpha_{A,B} \colon (A \Rightarrow TB)^{|\alpha|} \longrightarrow (A \Rightarrow TB) \right\}_{A \in \mathbf{Sets}, B \in \mathcal{K}\ell(T)}$$

* For the calculus: λ_c + (alg. opr. from Σ) + (co)products + arith.

$$\frac{\Gamma \vdash M_1 : \tau \quad \cdots \quad \Gamma \vdash M_{|\alpha|} : \tau}{\Gamma \vdash \alpha(M_1, \ldots, M_{|\alpha|}) : \tau}\ \alpha \in \Sigma$$

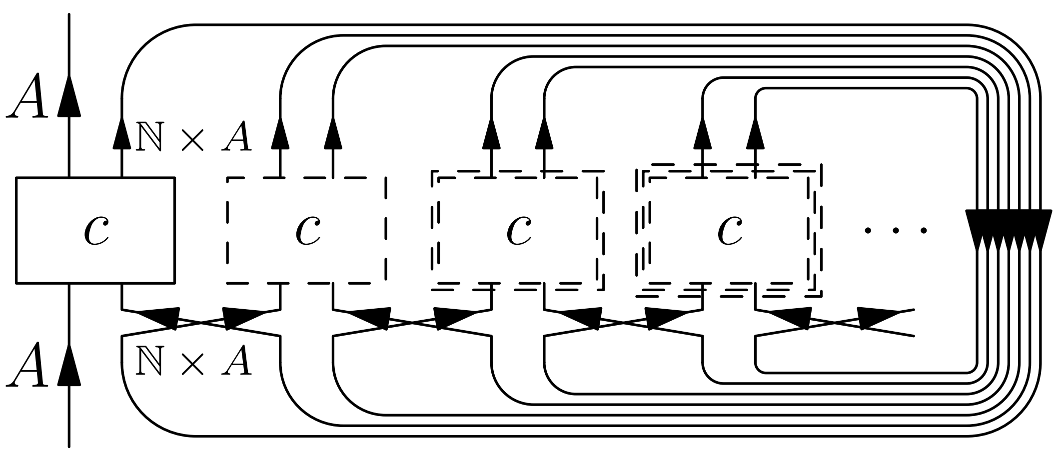
* We give



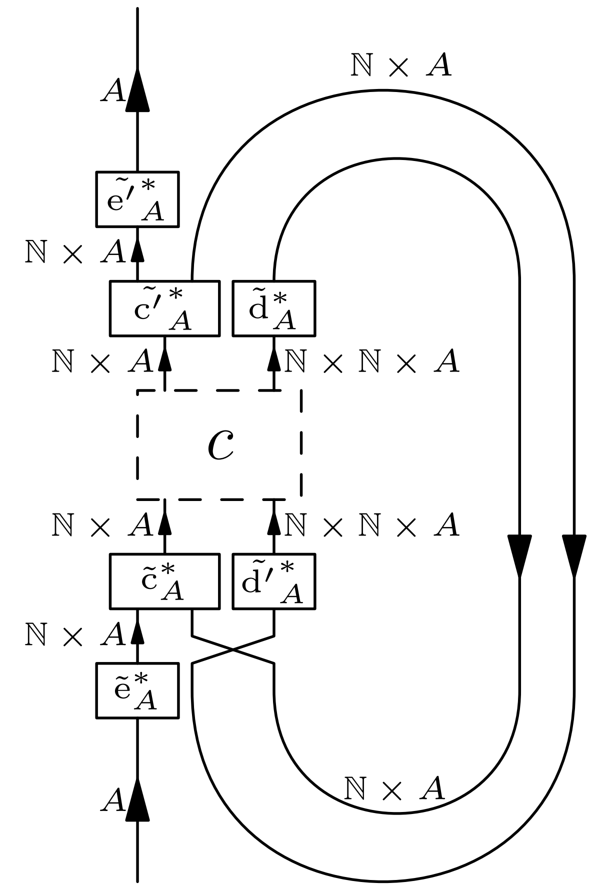
in $\mathbf{Trans}(T)$

Missing Ingredient II: Recursion

Girard style
fixed point operator



Mackie style
fixed point operator



- Obviously a fixed point
- Fixed-point induction
- Finitary string diagram

Theorem The two coincide. (for any suitable T!)

Theorem (Adequacy)

Let $\vdash M : \mathtt{nat}$. Then, as elem. of $T(\mathbb{N})$,

$$\left(\overset{\mathbb{N}\,|}{\underset{\mathbb{N}\,|}{\boxed{(\!|\vdash M : \mathtt{nat}\,|\!)}}} \right)^{\dagger} = [\![\, |M| \,]\!] \;.$$

Interpretation

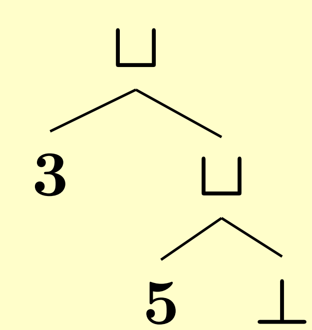
$[\![\_]\!] : \mathbf{EffVal}^{\Sigma}_{\mathbb{N}} \longrightarrow T(\mathbb{N})$

(exploiting free conti. Σ-alg.)

feeding a query and observing the outcome

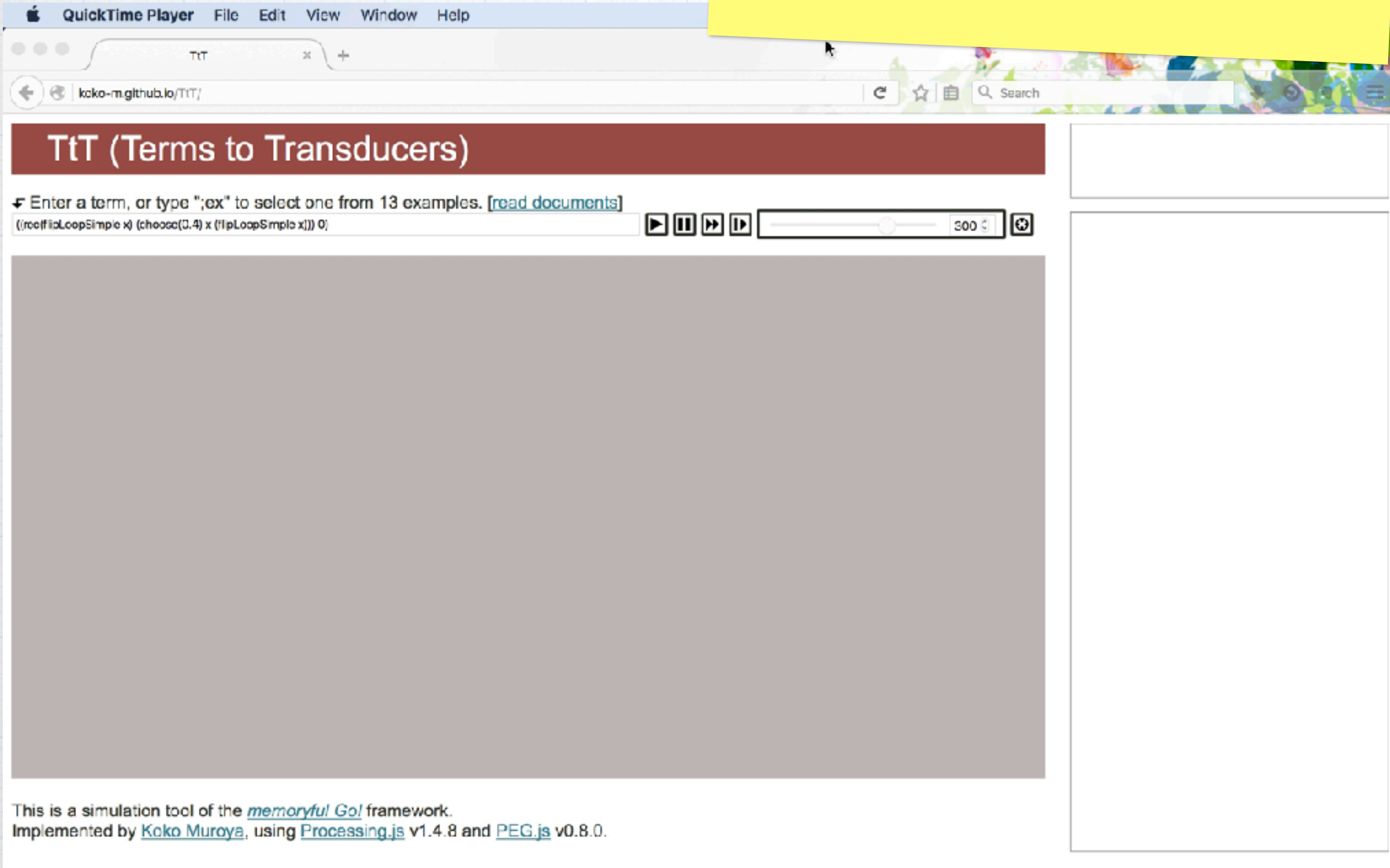
Opr. sem.:
Plotkin-Power **effect-value**. E.g.

$$\big|\, \mathbf{3} \sqcup (\mathbf{5} \sqcup \mathtt{div}) \,\big| =$$



Our Tool TtT

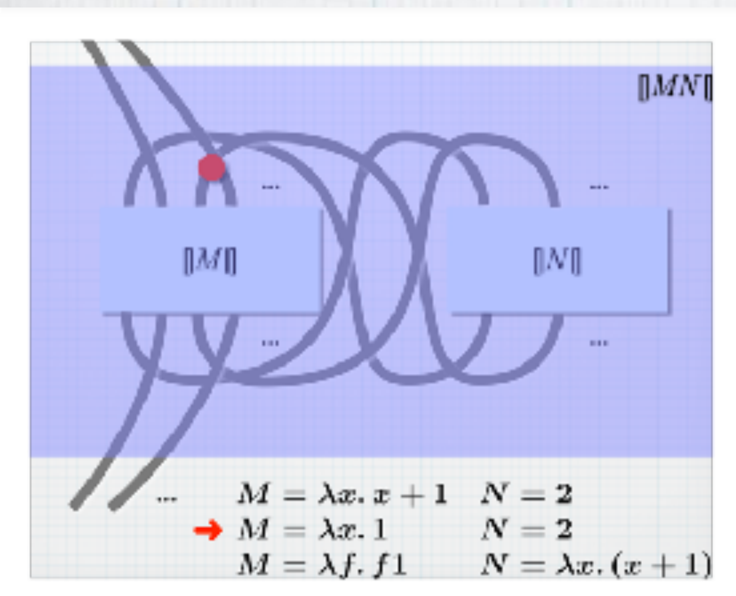
Developed by Koko Muroya
http://koko-m.github.io/TtT/

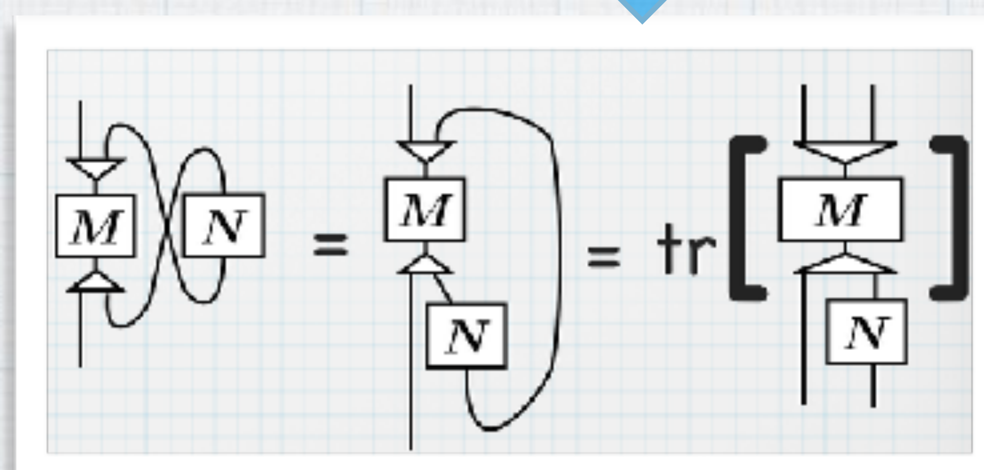


Summary

Coalgebra meets **higher-order computation** in **Geometry of Interaction** [Girard, LC'88]

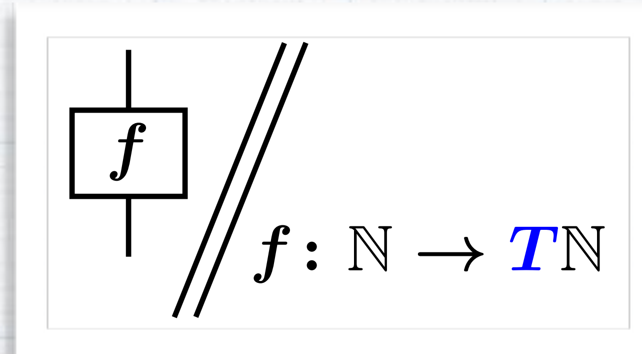
"GoI Animation"





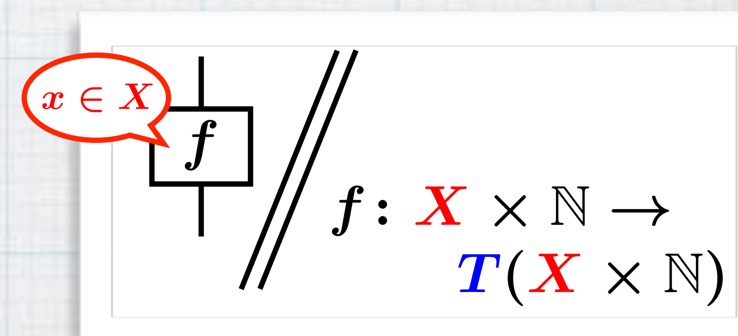
Categorical GoI

[Abramsky, Haghverdi & Scott, MSCS'02]



GoI w/ T-branching

[IH & Hoshino, LICS'11]



Memoryful GoI

[Hoshino, Muroya & IH, CSL-LICS'14 & POPL'16]

Retracing some paths in Process Algebra

Samson Abramsky
Laboratory for the Foundations of Computer Science
University of Edinburgh

1 Introduction

The very existence of the CONCUR conference bears witness to the fact that "concurrency theory" has developed into a subject unto itself, with substantially different emphases and techniques to those prominent elsewhere in the semantics of computation.

Whatever the past merits of this separate development, it seems timely to look for some convergence and unification. In addressing these issues, I have found it instructive to trace some of the received ideas in concurrency back to their origins in the early 1970's. In particular, I want to focus on a seminal paper by Robin Milner [Mil75][1], which led in a fairly dir to his enormously influential work on CCS [Mil80, Mil89]. I will tak extreme) the liberty of of applying hindsight, and show how some paths could have been taken, which, it can be argued, lead to a mor approach to the semantics of computation, and moreover one wh be better suited to modelling today's concurrent, object-oriented la and the type systems and logics required to support such languages.

2 The semantic universe: transducers

Milner's starting point was the classical automata-theoretic notion of *transducers*, *i.e.* structures

$$(Q, X, Y, q_0, \delta)$$

Thank you for your attention!
Ichiro Hasuo (NII, Japan)
http://group-mmm.org/~ichiro/

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CONCUR'96