Trace Semantics for Coalgebras: a Generic Theory

Ichiro Hasuo

Radboud University Nijmegen

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Introduction

- **Trace semantics** is defined for various non-det. systems:
  - different input/output types,
  - different "nondeterminism": e.g. classical non-det. vs. probability.
- They are instances of one categorical construction:

  coinduction in a Kleisli category

- Demonstrates the abstraction power of **category theory**, **coalgebras** in particular in computer science!
  - Same mathematical principle hidden behind
  - apparently different constructions

- Coalgebraic modelling of a non-det. system, which is suitable for trace semantics:

$$\begin{array}{c} TFX \\ c\uparrow \\ X \end{array} \quad \text{in } \mathbf{Sets},$$

  - A monad T specifies the type of non-det.;
  - An endofunctor F specifies the input/output type.

- Here

  - the **monad structure** of T and
  - a **distributive law** $\pi : FT \Rightarrow TF$

  play central roles.

- Coalgebraic modelling of a non-det. system, which is suitable for trace semantics:

$$\begin{array}{c} TFX \\ c\uparrow \\ X \end{array} \text{ in } \mathbf{Sets}, \quad \text{i.e.} \quad \begin{array}{c} FX \\ c\uparrow \\ X \end{array} \text{ in } \mathcal{K}\ell(T).$$

  - A monad T specifies the type of non-det.;
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- Here
  - the **monad structure** of T and
  - a **distributive law** $\pi : FT \Rightarrow TF$

  play central roles.

- **Main theorem**
  An initial algebra in $\mathbf{Sets}$ gives rise to
  - an initial algebra, and also
  - a final coalgebra,

  in a Kleisli category $\mathcal{K}\ell(T)$.
  [Under some order-theoretic assumptions]
- Finality yields the **finite trace map**: in $\mathcal{K}\ell(T)$,

$$\begin{array}{ccc} FX & \overset{\mathcal{K}\ell(F)(\mathbf{tr}_c)}{\dashrightarrow} & FA \\ c \uparrow & & \cong \uparrow J\alpha^{-1} \\ X & \underset{\mathbf{tr}_c}{\dashrightarrow} & A \end{array} .$$

- The proof of main result
  (**initial algebra-final coalgebra coincidence**)
  uses:
  - a classic result of limit-colimit coincidence in a suitably order-enriched setting
    [Smyth & Plotkin, Siam J. Comput., '82]

- IH, Bart Jacobs and Ana Sokolova.
  Generic Trace Theory.
  To appear in CMCS'06.

- The result covers:
  - **I/O types** almost all polynomial F
  - **type of "nondeterminism"** :
    - **lift monad** $\mathcal{L} = 1 + \_$
      systems with **non-termination, exception**
    - **powerset monad** $\mathcal{P}$
      (classical) non-deterministic systems
    - **subdistribution monad** $\mathcal{D}$
      probabilistic systems

- The result is **generic**:
  generalizing our previous papers
  - [IH & Jacobs, CALCO'05] $T = \mathcal{P}$
  - [IH & Jacobs, CALCO-jnr] $T = \mathcal{D}$

- Order-enriched structure is explicitly used for the first time.

Overview

We'd rather spend all time for preliminaries...

- various examples of trace semantics
- monads, distributive laws, Kleisli categories
- construction of
  - initial algebra via initial sequence
  - final coalgebra via final sequence
- Smyth & Plotkin's limit-colimit coincidence

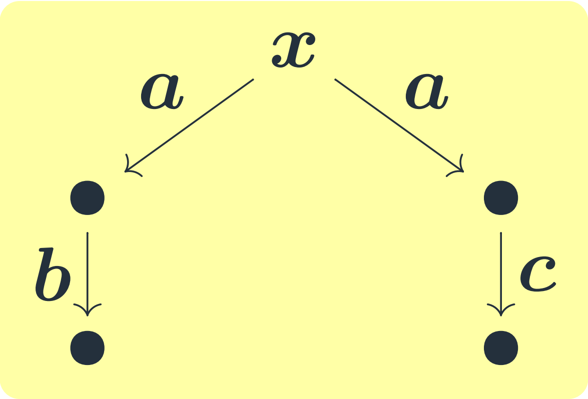
We go slowly, very slowly, ...

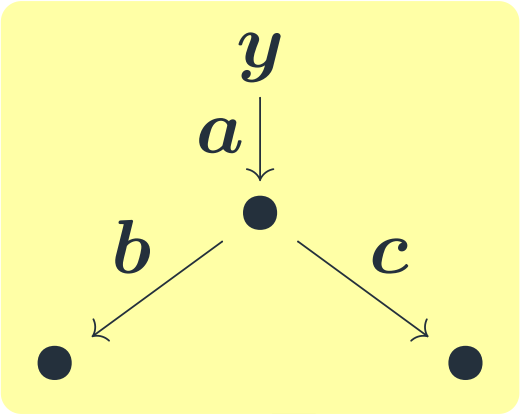
Preliminaries I: trace semantics

Linear time-branching time spectrum

Various semantics for non-det. systems...

Compare two non-deterministic systems.





x and y are

- different wrt. **bisimilarity**, but

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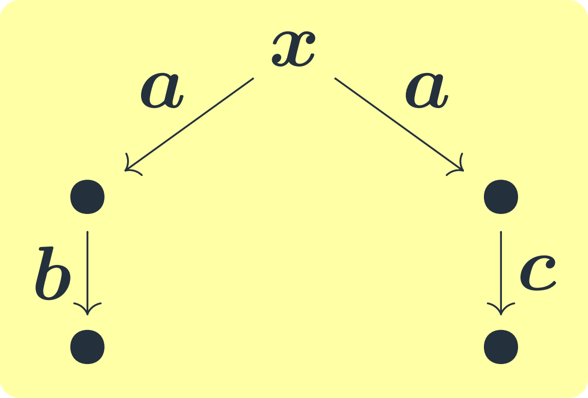
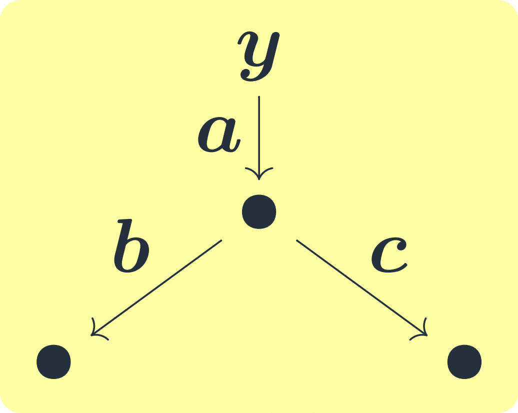




x and y are

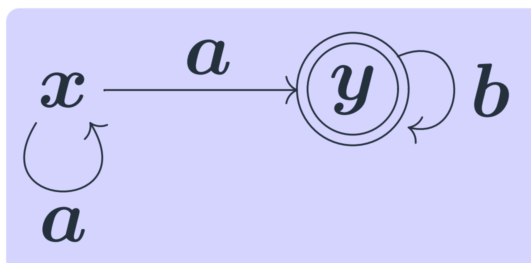
- different wrt. **bisimilarity**, but
- equivalent wrt. **trace semantics**!
  $\mathbf{tr}(x) = \mathbf{tr}(y) = \{ab, ac\}$.

Trace semantics

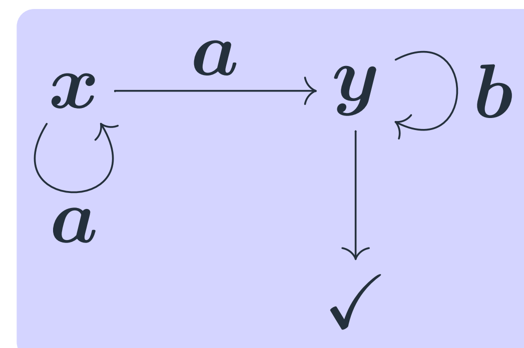
For (classical) non-deterministic systems,

trace = the set of all possible linear-time behavior

For



that is



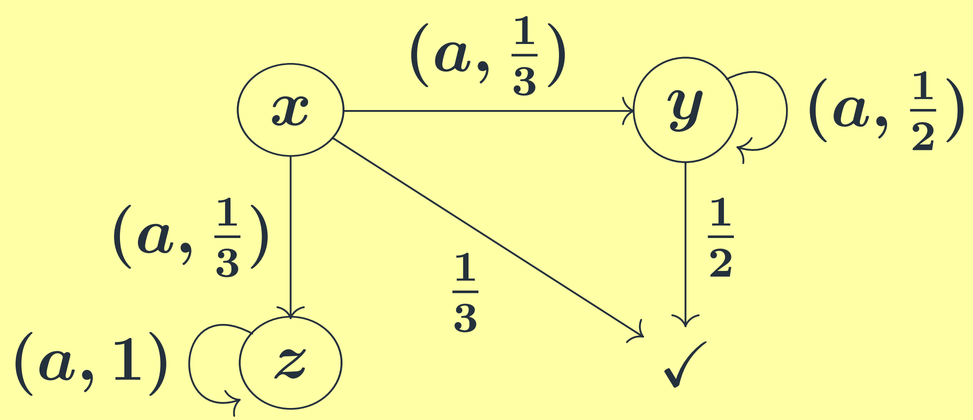
,

$$\mathbf{tr}(y) = b^* = \{\langle\rangle, b, bb, bbb, \ldots\}$$

$$\mathbf{tr}(x) = (a + a^2 + a^3 + \cdots) \cdot \mathbf{tr}(y)$$

$$= \{a^{n+1}b^m \mid n, m \in \mathbb{N}\}$$

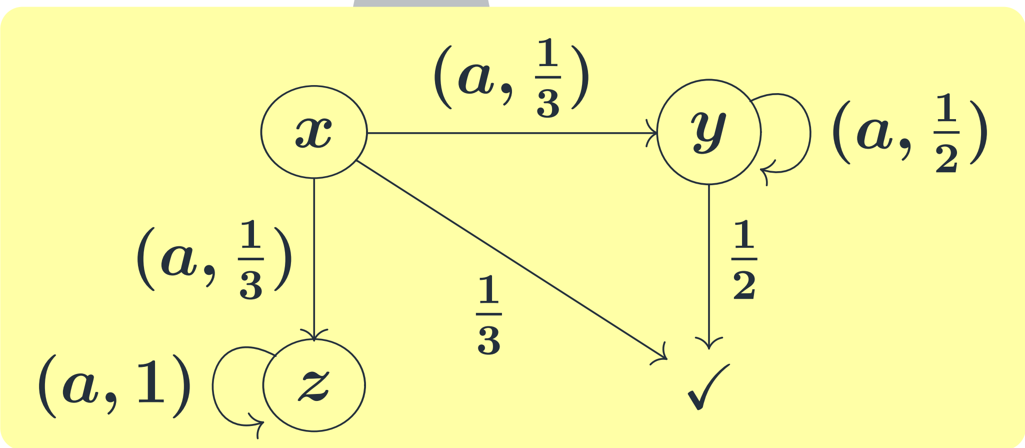
Another type of nondeterminism: **probabilistic systems**



Question : What is the “trace” of x?

Another “nondeterminism”

Another type of nondeterminism: **probabilistic systems**



Question: What is the “trace” of x?

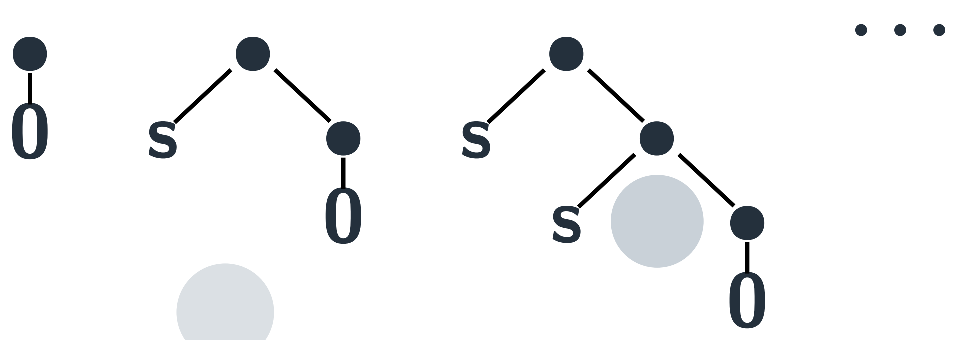
Answer: the **probability distribution** over possible linear-time behavior

$$\langle\rangle \mapsto \frac{1}{3} \qquad a \mapsto \frac{1}{3}\cdot\frac{1}{2} \qquad a^2 \mapsto \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2} \qquad \cdots$$

Consider a **context-free grammar**.

- Terminal symbols: $\mathbf{0}$, **s**
- Non-terminal symbol: T
- Generation rules: $\mathrm{T} \blacktriangleright \mathbf{0}$, $\mathrm{T} \blacktriangleright \mathbf{s}\mathrm{T}$

From T, the following parse trees can be generated:



This is the "**trace**" of T.

A **trace** of (a state of) a non-det. system is:

- For **(classical) non-deterministic** systems,

  the **set** of possible linear-time behavior

- For **probabilistic** systems,

  the **probability distribution** over possible linear-time behavior

- The **input/output type** specifies what is a "linear-time behavior".

Preliminaries II: monads, distributive laws, Kleisli categories

Non-Det. systems as coalgebras

A non-det. system is modelled as a coalgebra

$$\begin{array}{c} TFX \\ {\scriptstyle c}\uparrow \\ X \end{array} \quad \text{in } \mathbf{Sets}$$

- A monad T specifies the type of nondeterminism;
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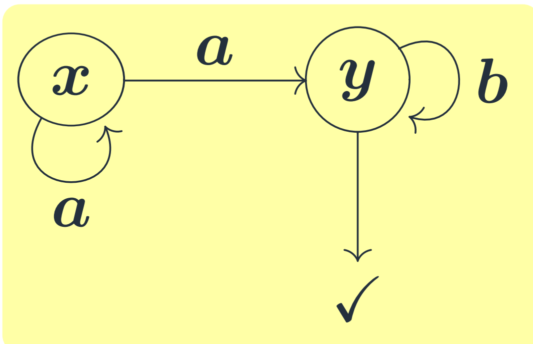
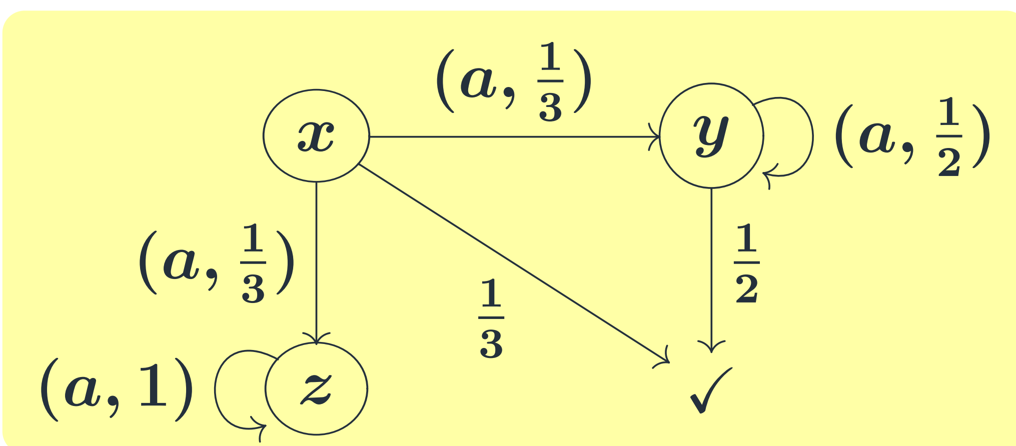
Examples (Details on blackboard...)

- 

  $$\begin{array}{ccc} TFX & \{(a,x),\ (a,y)\} & \{(b,y),\ \checkmark\} \\ c\uparrow & \uparrow & \uparrow \\ X & x & y \end{array}$$

  - I/O type: $F = 1 + \Sigma \times \_$
  - Type of nondeterminism: $T = \mathcal{P}$ (classical non-det.)

- 

  $$\begin{array}{cc} TFX & \left\{ \begin{array}{ccc} (a,y) & \mapsto & 1/2 \\ \checkmark & \mapsto & 1/2 \end{array} \right. \\ c\uparrow & \uparrow \\ X & y \end{array}$$

  - I/O type: $F = 1 + \Sigma \times \_$
  - Type of nondeterminism: $T = \mathcal{D}$ (probability)

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"Nondeterminism" is modelled due to

- the monad structure of T, and
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Monads

A **monad**

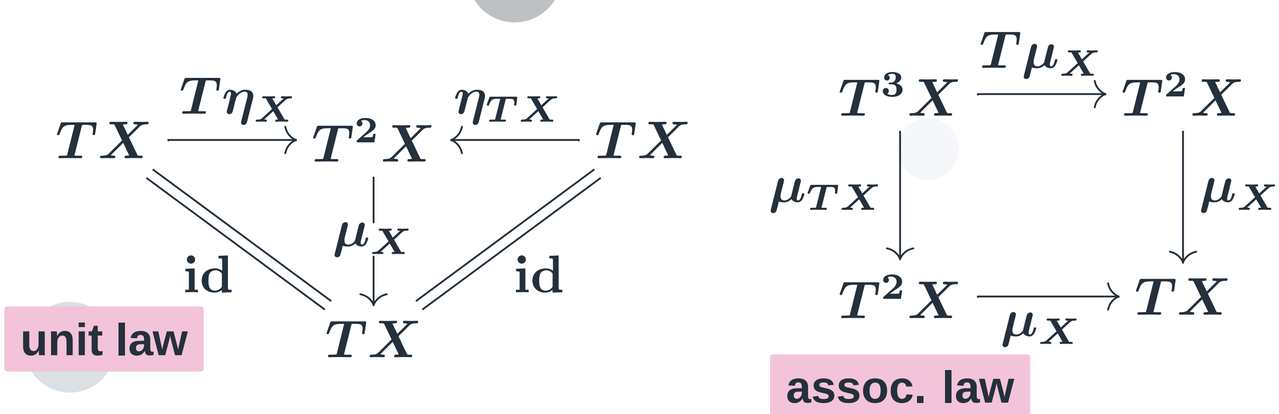
$$T : \mathbb{C} \longrightarrow \mathbb{C}$$

is an endofunctor with additional structures: for each object X,

- $X \xrightarrow{\eta_X} TX$ **unit**
- $T^2X \xrightarrow{\mu_X} TX$ **multiplication**

such that:

- $\eta : \mathrm{id} \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ are natural transformations;
- they are compatible in the sense:



- Generalization of notion of **monoids**.
- Examples of our interest: (details on blackboard)
  - **Lift monad** $\mathcal{L}X = 1 + X$
  - **Powerset monad**

$$\mathcal{P}X = \{X' \mid X' \subseteq X\}$$

  - **Subdistribution monad**

$$\mathcal{D}X = \{d : X \to [0,1] \mid \sum_{x \in X} d(x) \leq 1\}$$

Monads

- More generally, an adjunction $\mathbb{A} \; L \left(\dashv \right) R \; \mathbb{C}$ yields a monad $RL : \mathbb{C} \longrightarrow \mathbb{C}$.
- Hence a functor

$$X \quad \longmapsto \quad \left[\begin{array}{l} \text{Free ("term") algebra} \\ \text{with variables from } X \end{array} \right]$$

comes with a monad structure.

- More generally, an adjunction $L \left(\dashv \right) R$ between $\mathbb{A}$ and $\mathbb{C}$ yields a monad $RL : \mathbb{C} \longrightarrow \mathbb{C}$.
- Hence a functor

$$X \quad \longmapsto \quad \left[\begin{array}{c} \text{Free (“term”) algebra} \\ \text{with variables from } X \end{array} \right]$$

comes with a monad structure.
- The converse is also true: every monad arises from an adjunction
  - **Eilenberg-Moore** construction (biggest, final)
  - **Kleisli** construction (smallest, initial)

Kleisli categories

Kleisli category $\mathcal{K}\ell(T)$ for $T : \mathbb{C} \longrightarrow \mathbb{C}$, a monad.

- **Object**

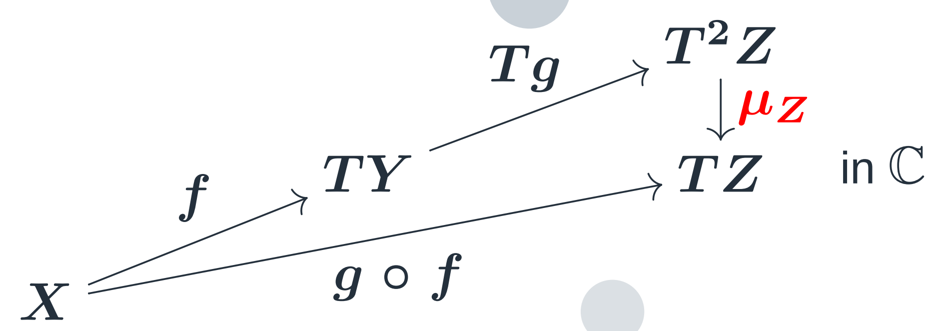
$$\frac{X \in \mathcal{K}\ell(T)}{X \in \mathbb{C}}$$

- **Arrow**

$$\frac{X \xrightarrow{\ f\ } Y \text{ in } \mathcal{K}\ell(T)}{X \xrightarrow{\ f\ } TY \text{ in } \mathbb{C}}$$

- **Composition**

$$X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z \text{ in } \mathcal{K}\ell(T)$$

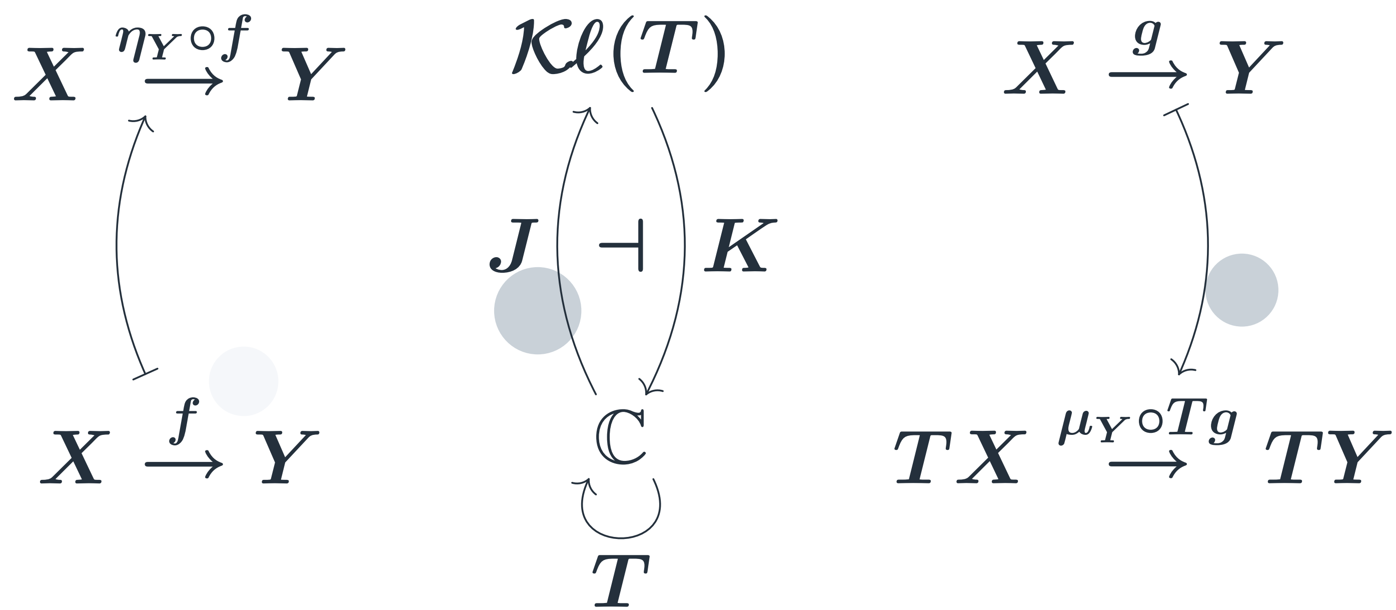


- **Id. arrow**

$$\frac{X \xrightarrow{\ \mathrm{id}\ } X \text{ in } \mathcal{K}\ell(T)}{X \xrightarrow{\ \eta_X\ } TX \text{ in } \mathbb{C}}$$

Kleisli categories

- Examples: $T = \mathcal{L}, \mathcal{P}, \mathcal{D}$. On the blackboard.
- There is an **adjunction**:



which yields the monad T.

- Moreover, this Kleisli adjunction is the initial one among those which yield T.

Motivation

- A system of the form $\begin{array}{c} TX \\ \uparrow c \\ X \end{array}$ can be **iterated**:

In **Sets**

$$\begin{array}{ccccc} X & \xrightarrow{c^n} & TX & \xrightarrow{Tc} & T^2X \\ & & & \searrow^{c^{n+1}} & \downarrow \mu_X \\ & & & & TX \end{array}$$

In $\mathcal{K}\ell(T)$

$$\begin{array}{ccc} X & \xrightarrow{c^n} & X \\ & \searrow^{c^{n+1}} & \downarrow c \\ & & X \end{array}$$

- How about $\begin{array}{c} TFX \\ c \uparrow \\ X \end{array}$ which is of our interest?

Distributive laws

- A **distributive law** is a natural transformation

$$\pi : FT \Rightarrow TF$$

  which is compatible with the monad structure of T.

- It **swaps** T over F.

- The direction is opposite in [Bartels, PhD thesis], since:
  - Here the base category is Kleisli,
  - In [Bartels, PhD thesis] the base category is Eilenberg-Moore.
  - Duality in a suitable 2-categorical sense.

Distributive laws

View in $\mathbf{Sets}$

If a system $\begin{array}{c} TFX \\ c \uparrow \\ X \end{array}$ comes with

a distributive law $\pi : FT \Rightarrow TF$,

we can define n**-th iteration** of c:

$$\begin{array}{c} TF^nX \\ c^n \uparrow \\ X \end{array}$$

- Construction on the blackboard.
- Example: $T = \mathcal{P}$ and $F = 1 + \Sigma \times \_\_$.

Another view, in $\mathcal{K}\ell(T)$

- A distr. law $FT \Rightarrow TF$ **lifts** F

$$\mathcal{K}\ell(T) \underset{J}{\overset{K}{\leftrightarrows}} \mathbf{Sets}, \quad J \dashv K$$

$$\begin{array}{ccc} \mathcal{K}\ell(T) & \xrightarrow{\mathcal{K}\ell(F)} & \mathcal{K}\ell(T) \\ {\scriptstyle J}\uparrow & & \uparrow{\scriptstyle J} \\ \mathbf{Sets} & \xrightarrow[F]{} & \mathbf{Sets} \end{array}$$

Construction on the blackboard.

- A system is now in the Kleisli category

$$\begin{array}{c} TFX \\ {\scriptstyle c}\uparrow \\ X \end{array} \text{ in } \mathbf{Sets} \quad \text{is:} \quad \begin{array}{c} \mathcal{K}\ell(F)X \\ {\scriptstyle c}\uparrow \\ X \end{array} \text{ in } \mathcal{K}\ell(T).$$

Summary

- A non-det. system is as a coalgebra $\begin{array}{c} TFX \\ c \uparrow \\ X \end{array}$
- Its "nondeterminism" (suitable for trace semantics) is due to
  - ☐ **unit η** of T ("singleton")
  - ☐ **multiplication μ** of T ("union")
  - ☐ **distr. law $FT \Rightarrow TF$**

  allowing for iteration of the system $\begin{array}{c} TF^nX \\ c^n \uparrow \\ X \end{array}$
- We move to Kleisli category where the system is $\begin{array}{c} \mathcal{K}\ell(F)X \\ c \uparrow \\ X \end{array}$.

Preliminaries III: Initial/final sequences

Overview

- We sketch: generic construction of
  - initial F-algebra via **initial sequence**
  - final F-coalgebra via **final sequence**

  for $F : \mathbb{C} \longrightarrow \mathbb{C}$.

- Assumptions are categorical.
  For initial sequence construction,
  - existence of initial object $0 \in \mathbb{C}$;
  - existence of certain colimits in $\mathbb{C}$;
  - F preserves such colimits.

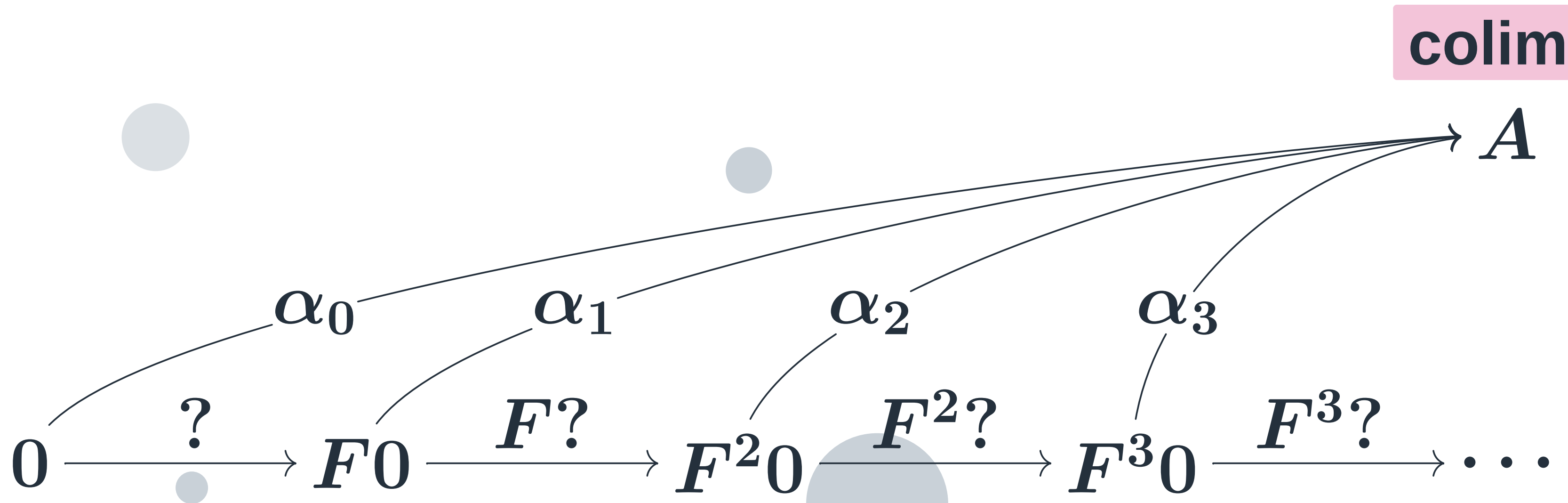
- For illustration the example is $\mathbb{C} = \mathbf{Sets}$.
  Later applied to $\mathbb{C} = \mathcal{K}\ell(T)$.

Initial sequence

initial obj.

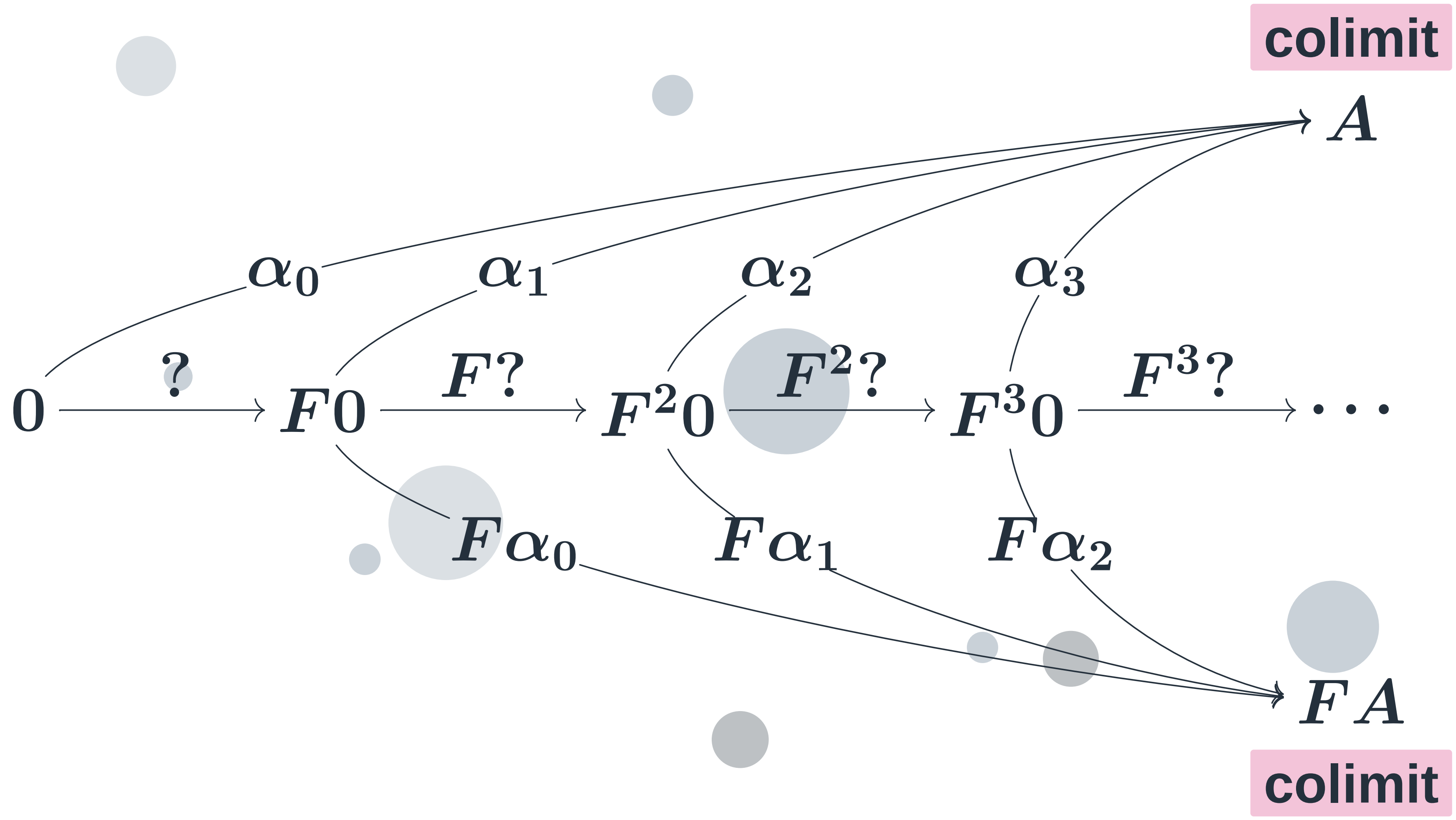
$$0 \xrightarrow{?} F0 \xrightarrow{F?} F^2 0 \xrightarrow{F^2 ?} F^3 0 \xrightarrow{F^3 ?} \cdots$$

Initial sequence



Initial sequence

Assume: F preserves the upper colimit.



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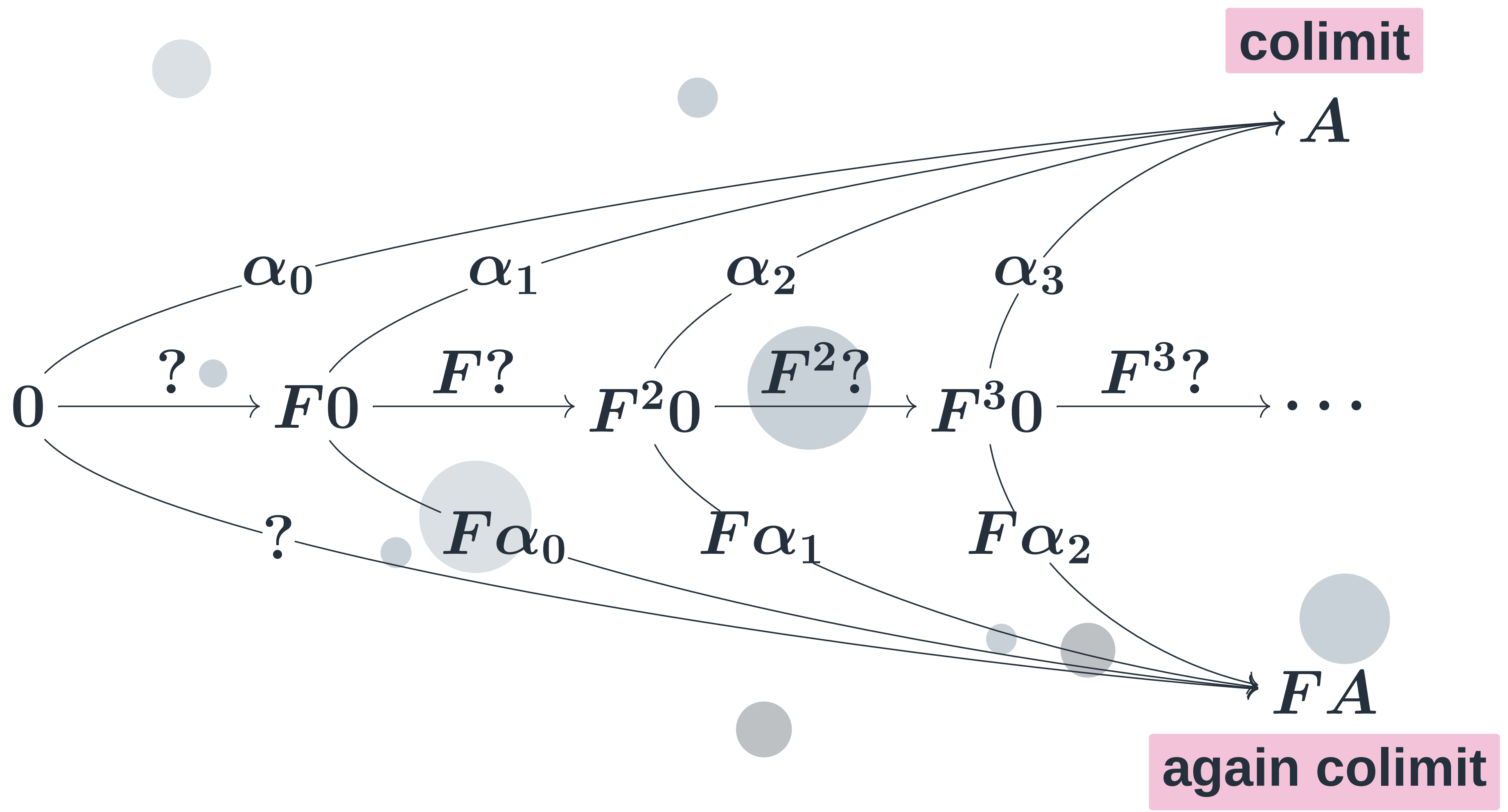
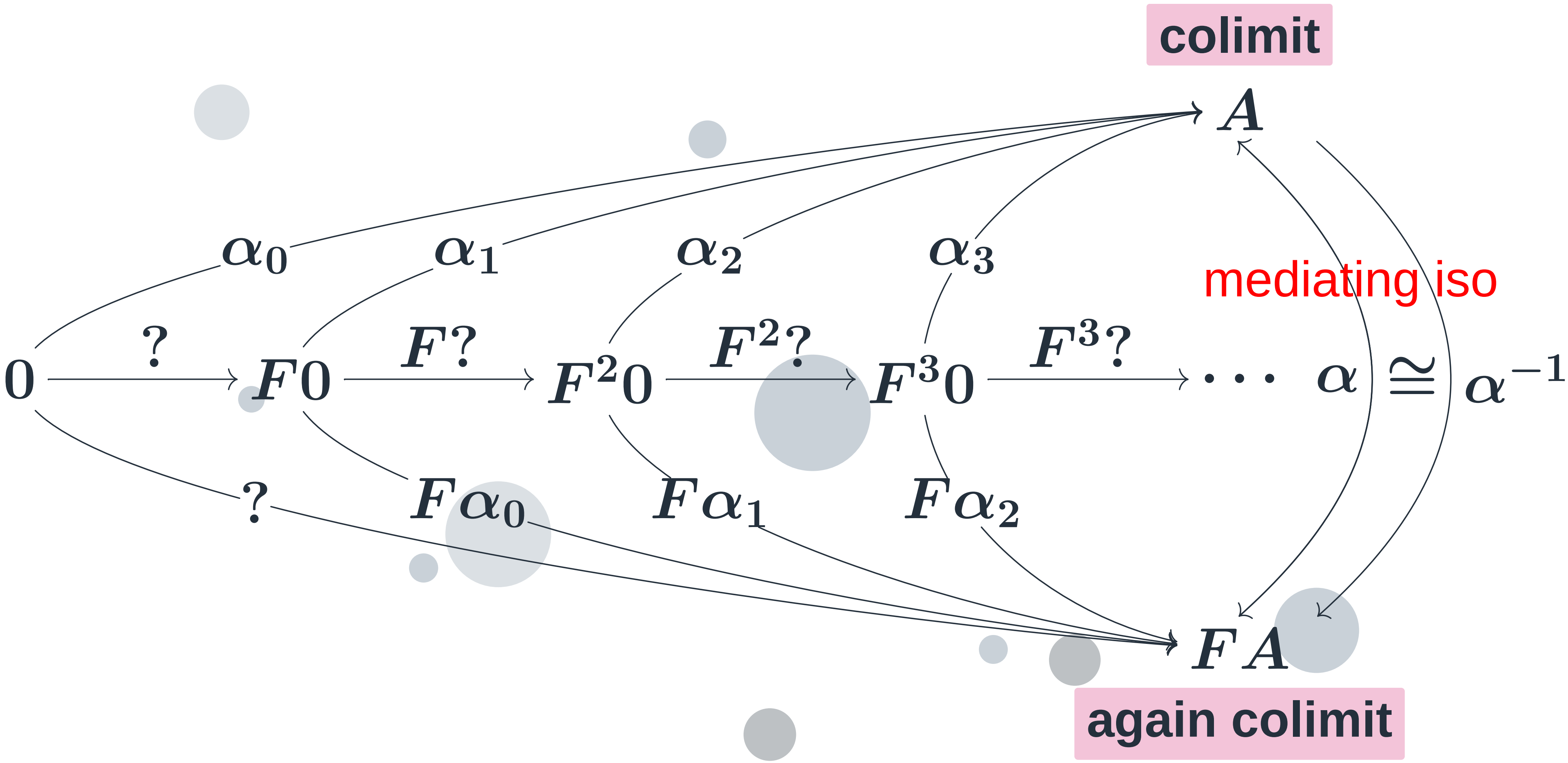


Initial sequence



$\alpha : FA \xrightarrow{\cong} A$ is an initial algebra.

Initial sequence

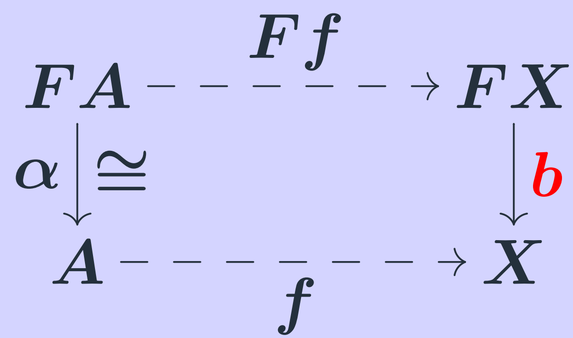
Construction of f in
$$\begin{array}{ccc} FA & \overset{Ff}{\dashrightarrow} & FX \\ {\scriptstyle\alpha}\downarrow{\scriptstyle\cong} & & \downarrow{\color{red}b} \\ A & \underset{f}{\dashrightarrow} & X \end{array}$$

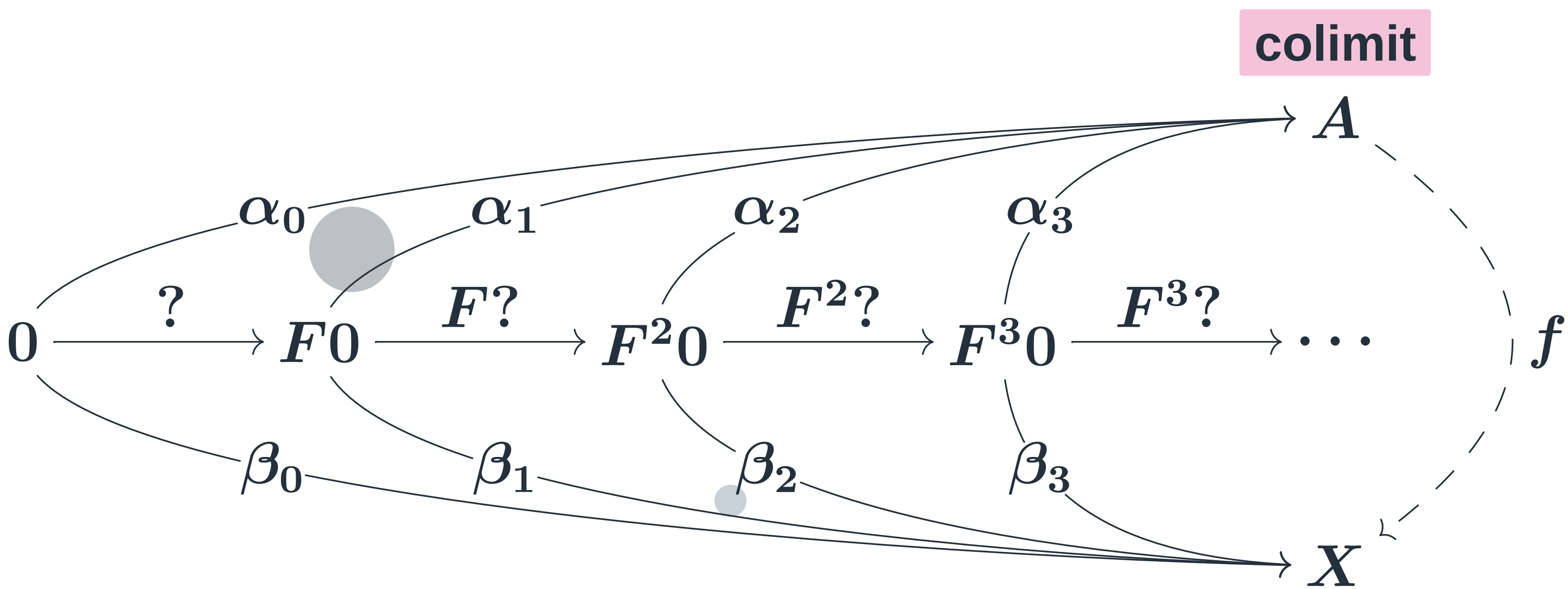
- $\begin{array}{c} FX \\ \downarrow{\color{red}b} \\ X \end{array}$ induces a cocone over initial sequence:
$$\begin{array}{ccc} F^{n+1}0 & \xrightarrow{F\beta_n} & FX \\ & \searrow^{\beta_{n+1}} & \downarrow{\color{red}b} \\ & & X \end{array}$$

$$0 \xrightarrow{?} F0 \xrightarrow{F?} F^2 0 \xrightarrow{F^2?} F^3 0 \xrightarrow{F^3?} \cdots$$
$$\beta_0 : 0 \to X, \quad \beta_1 : F0 \to X, \quad \beta_2 : F^2 0 \to X, \quad \beta_3 : F^3 0 \to X$$

Initial sequence

Construction of f in





Initial sequence, in $\mathbf{Sets}$

$F = 1 + \Sigma \times \_\,,$ where $1 = \{\checkmark\}$ and $\Sigma = \{a\}$.

Question What is an initial algebra?

initial obj.

$$0 \xrightarrow{?} F0 \xrightarrow{F?} F^2 0 \xrightarrow{F^2 ?} F^3 0 \xrightarrow{F^3 ?} \cdots$$

Initial sequence, in $\mathbf{Sets}$

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initial obj.

$$\begin{array}{ccccccccc}
0 & \xrightarrow{?} & F0 & \xrightarrow{F?} & F^2 0 & \xrightarrow{F^2 ?} & F^3 0 & \xrightarrow{F^3 ?} & \cdots \\
 & & \| & & \| & & \| & & \\
 & & 1 & & 1 + \Sigma & & 1 + \Sigma + \Sigma^2 & &
\end{array}$$

Initial sequence, in $\mathbf{Sets}$

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initial obj.

$$\begin{array}{ccccccccc} 0 & \xrightarrow{?} & F0 & \xrightarrow{F?} & F^2 0 & \xrightarrow{F^2?} & F^3 0 & \xrightarrow{F^3?} & \cdots \\ & & \| & & \| & & \| & & \\ & & 1 & & 1+\Sigma & & 1+\Sigma+\Sigma^2 & & \\ & & \checkmark & \longmapsto & \checkmark & \longmapsto & \checkmark & \longmapsto & \cdots \\ & & & & a & \longmapsto & a & \longmapsto & \cdots \\ & & & & & & aa & \longmapsto & \cdots \end{array}$$

Initial sequence, in Sets

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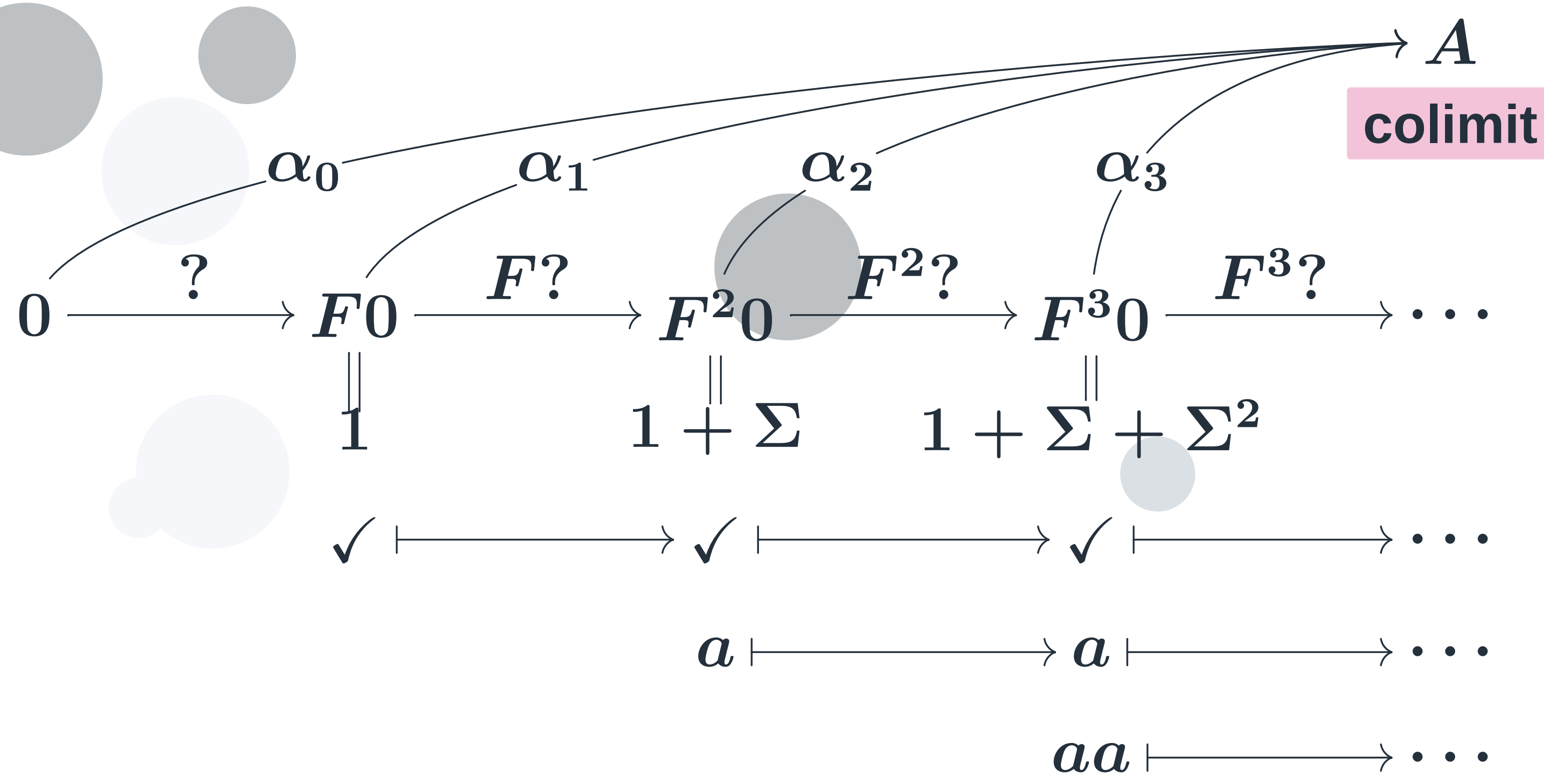

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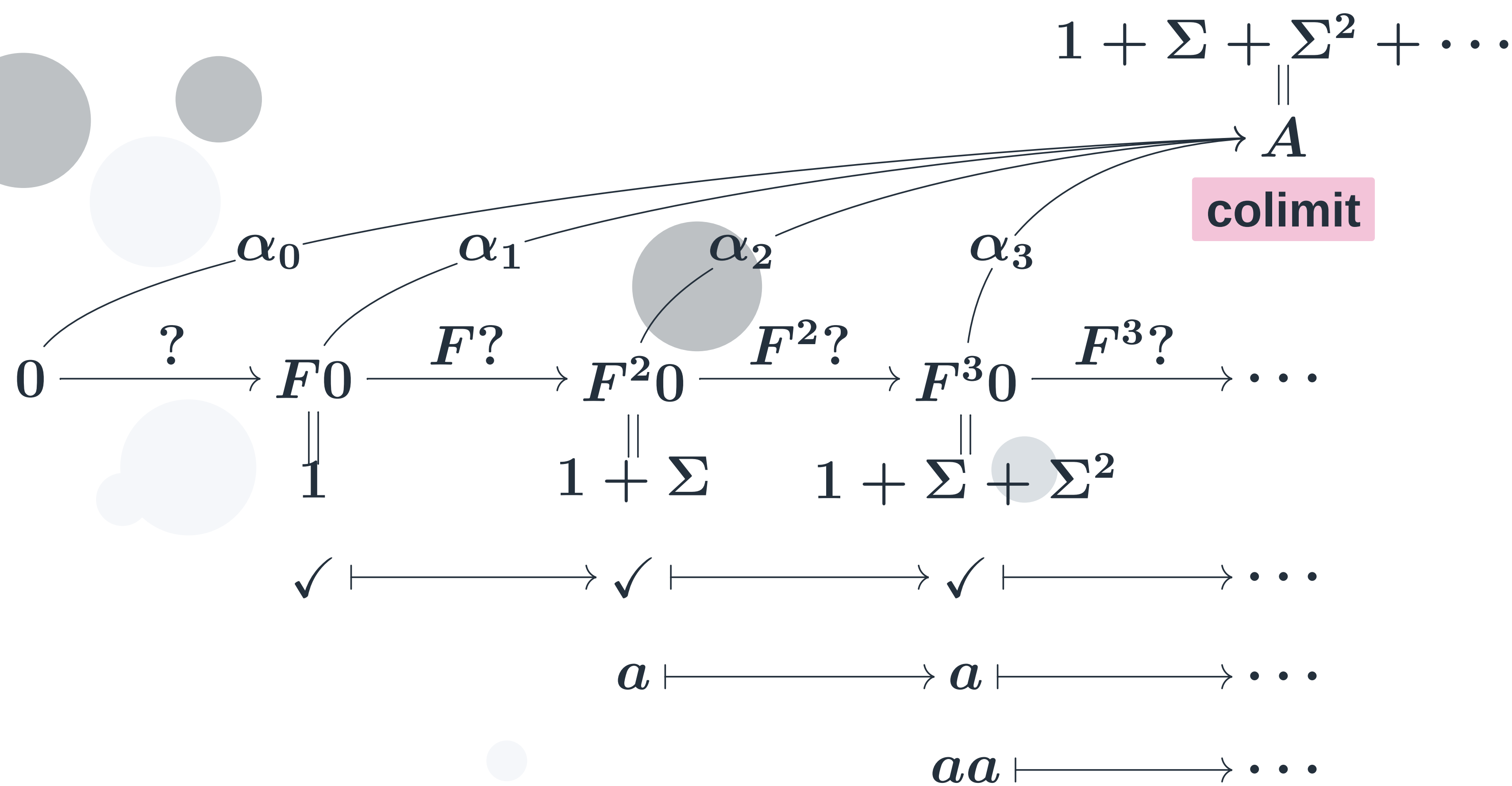

colimit $=$ $\left\{ \begin{array}{l} \text{coproduct, then} \\ \text{coequalizer} \end{array} \right\}$ $\overset{\text{in } \mathbf{Sets}}{=}$ union

Initial sequence, in $\mathbf{Sets}$

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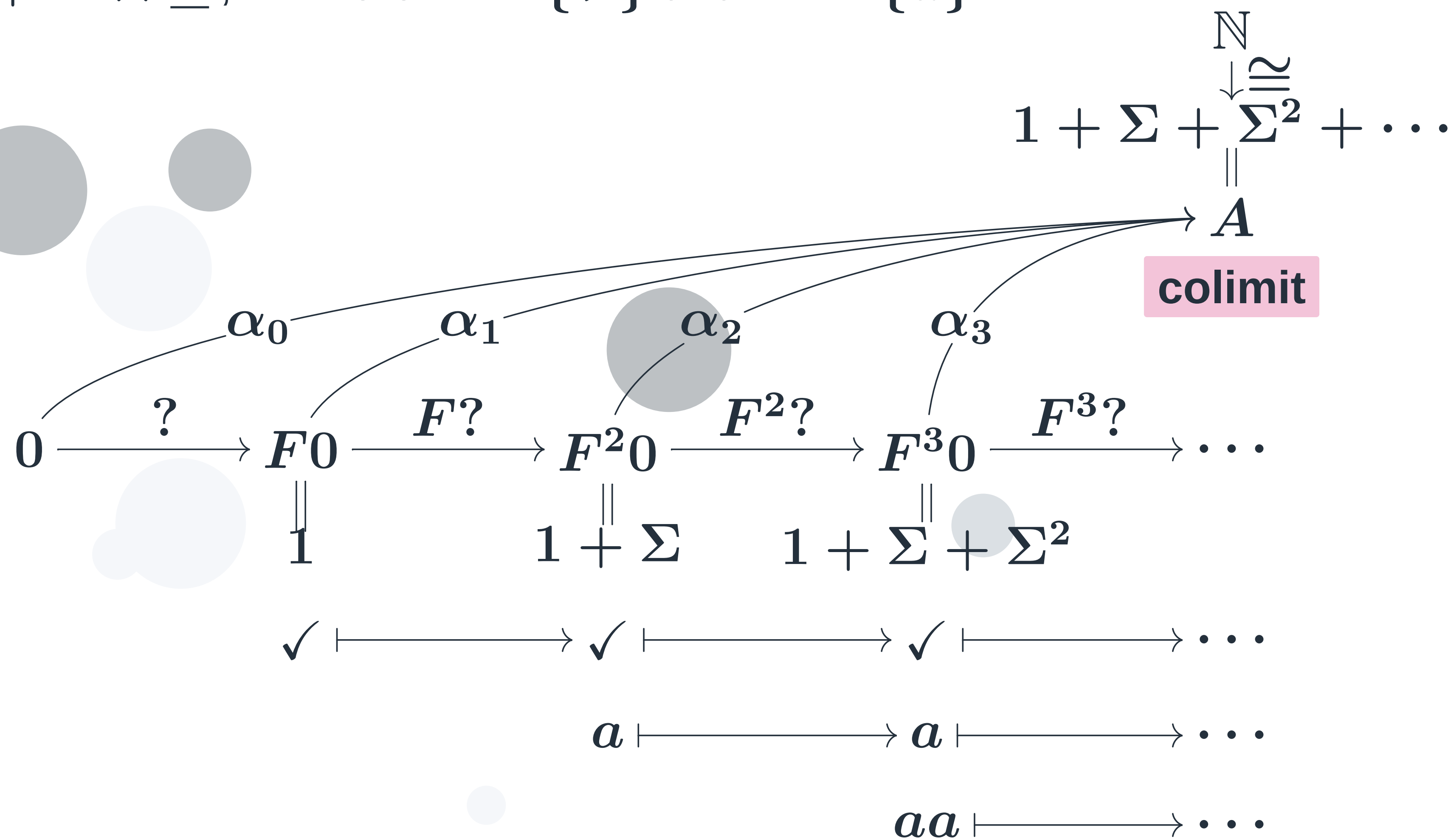


$F^n 0 = \{\text{terms with depth} \leq n\}$

Final sequence

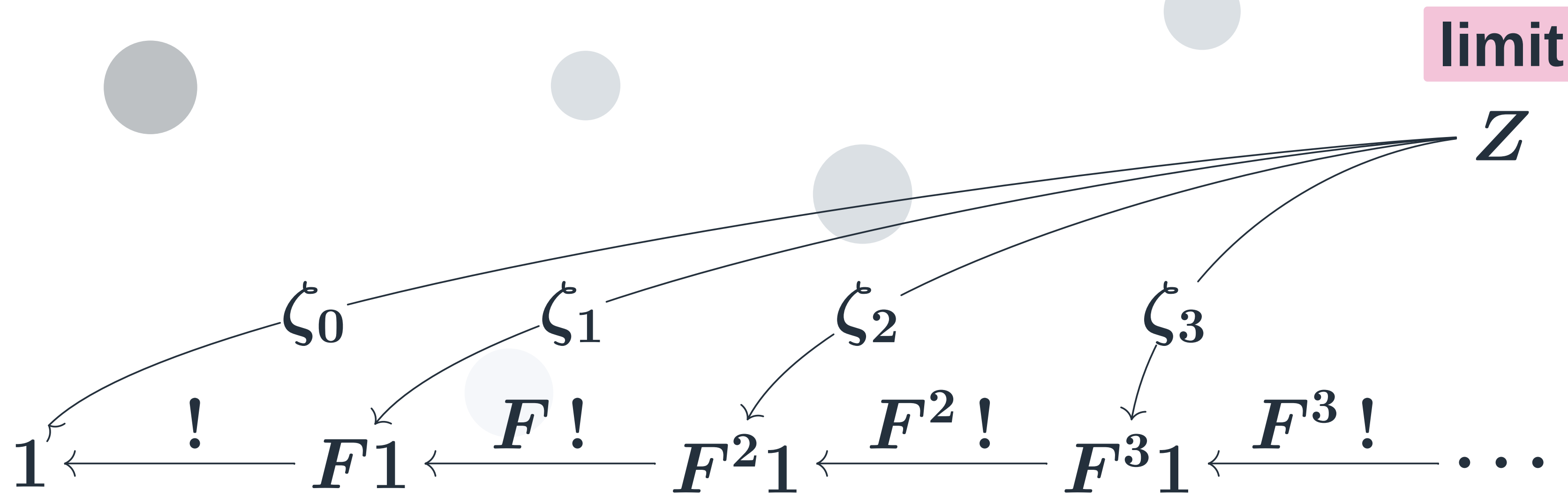
Dual of initial sequence...

Final obj.

$$1 \xleftarrow{\;!\;} F1 \xleftarrow{\;F\,!\;} F^2 1 \xleftarrow{\;F^2\,!\;} F^3 1 \xleftarrow{\;F^3\,!\;} \cdots$$

Final sequence

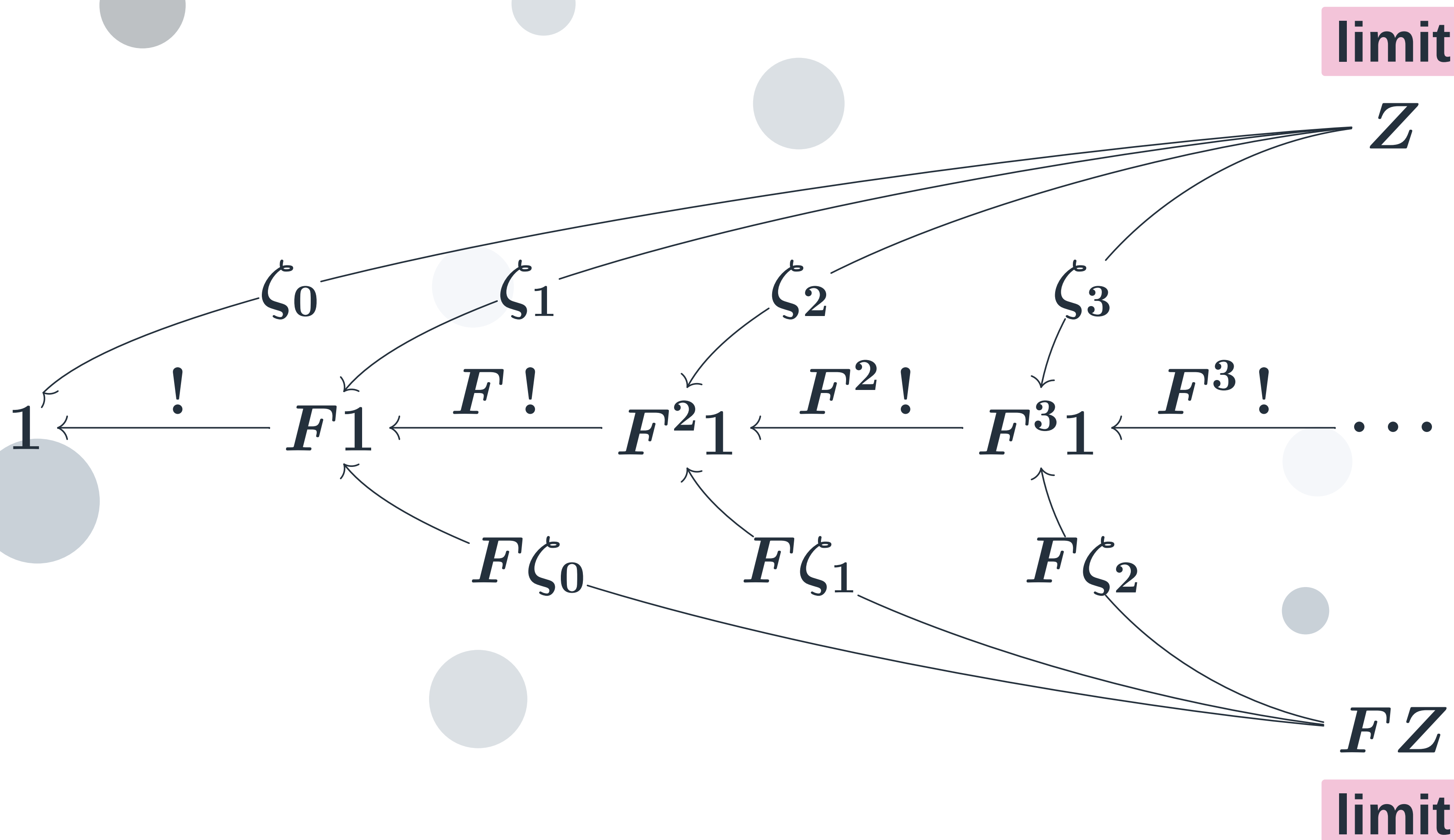
Dual of initial sequence...



Final sequence

Dual of initial sequence...

Assume: F preserves the upper limit.



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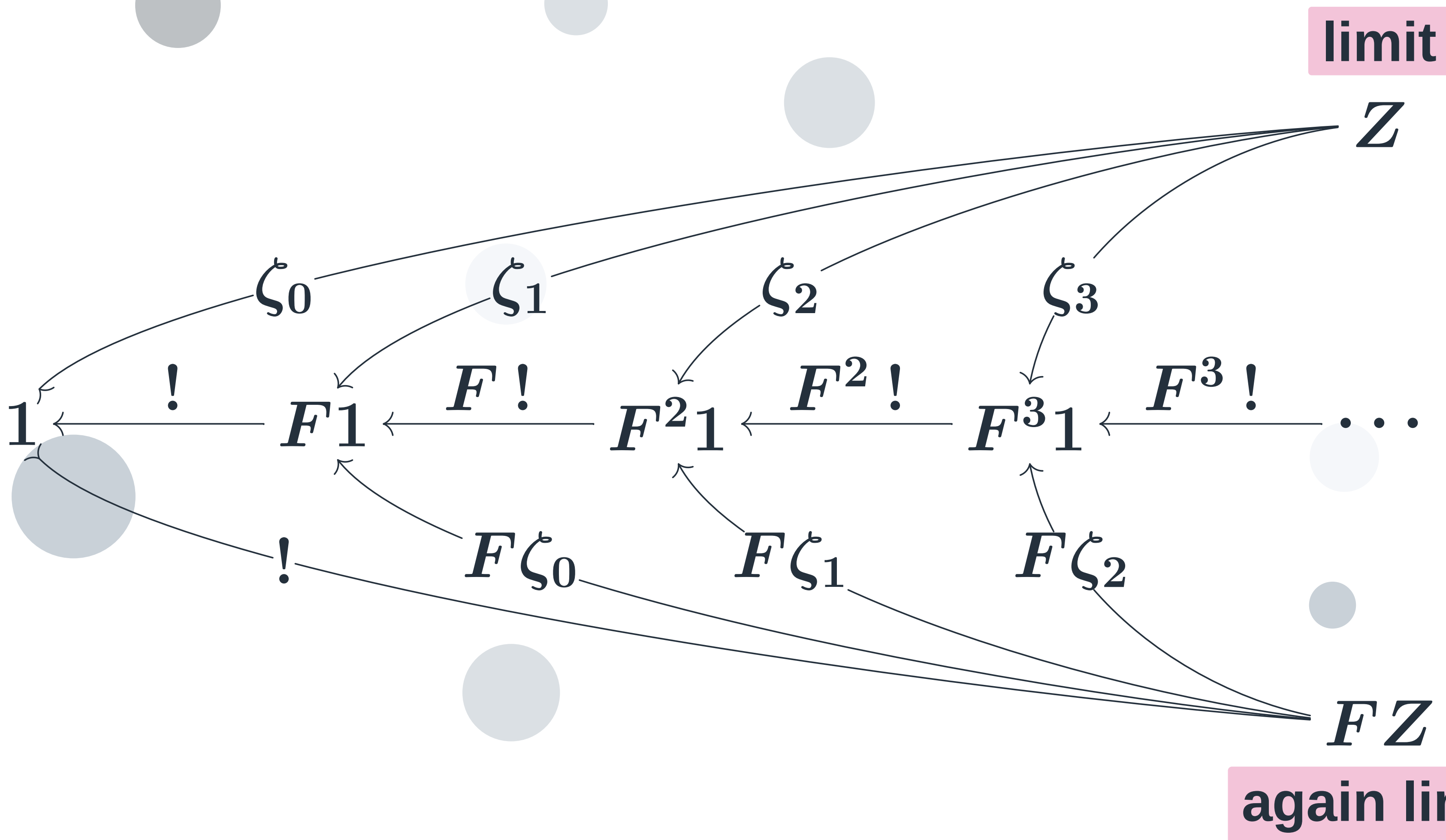
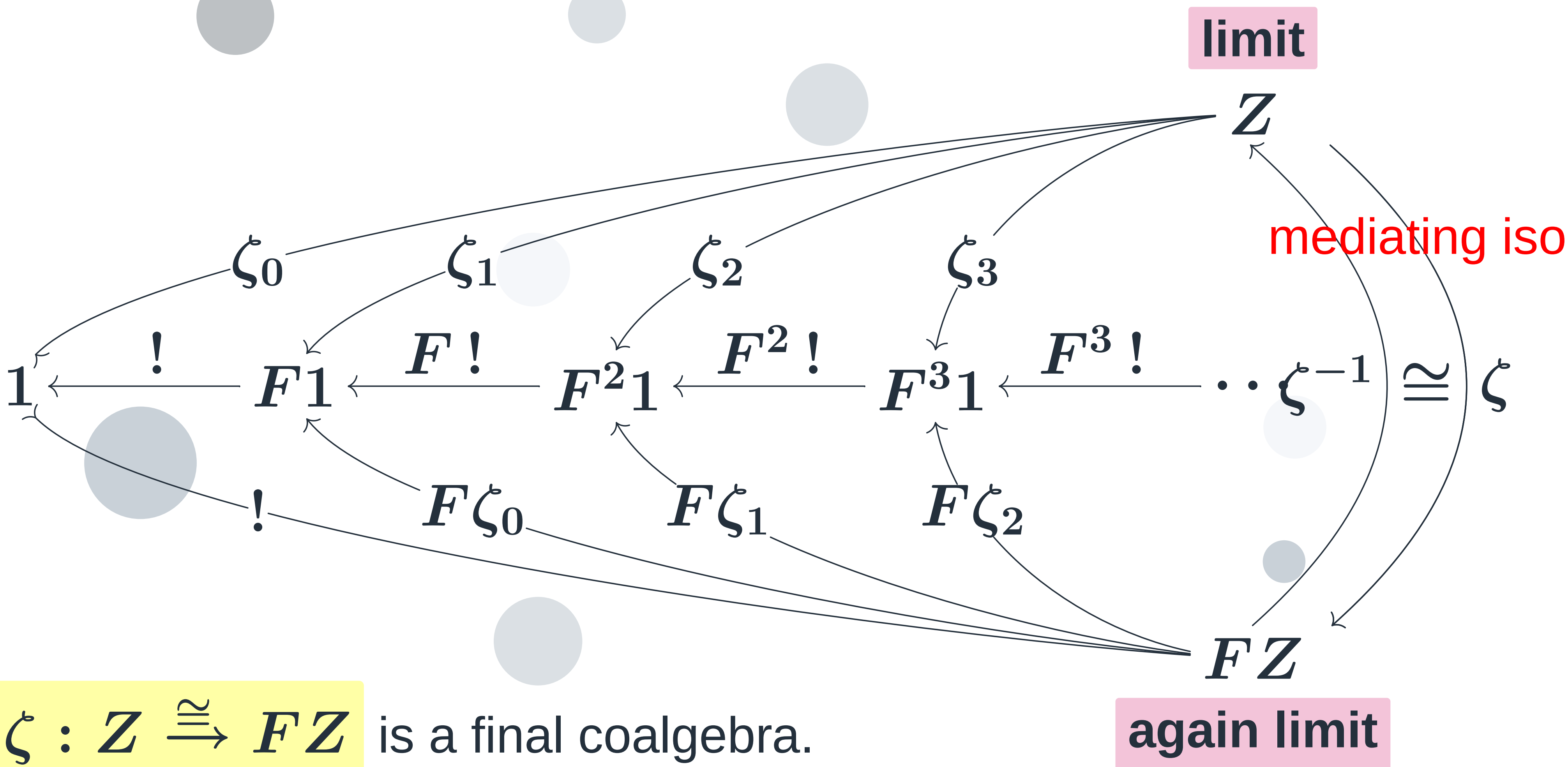
Dual of initial sequence...

Assume: F preserves the upper limit.



Final sequence

Dual of initial sequence...



$\zeta : Z \xrightarrow{\cong} FZ$ is a final coalgebra.

Summary

- Initial sequence and final sequence.
- In $\mathbf{Sets}$ the constructions coincide with familiar structural (co)induction.
- However, the constructions are purely categorical.
  - They work also in other categories!
  - Later applied in $\mathcal{K}\ell(T)$.
- Too much time left? Final sequence in $\mathbf{Sets}$.

Preliminaries IV: Limit-colimit coincidence

- Taking colimit of initial sequence seems

taking **union** of an **increasing chain**

- Taking colimit of initial sequence seems

  taking **union** of an **increasing chain**

- In a certain setting it is!
  - $\mathbf{O}$-limits (order-theoretic notion) coincide with limits;
  - $\mathbf{O}$-colimits coincide with colimits.

- $\mathbf{O}$-limit —— obvious duality, coincidence —— $\mathbf{O}$-colimit

  $\mathbf{O}$-limit $\Updownarrow$ limit; $\mathbf{O}$-colimit $\Updownarrow$ colimit

-

Overview

- Taking colimit of initial sequence seems

  taking **union** of an **increasing chain**

- In a certain setting it is!
  - $\mathbf{O}$-limits (order-theoretic notion) coincide with limits;
  - $\mathbf{O}$-colimits coincide with colimits.

- $\mathbf{O}$-limit —— obvious duality, coincidence —— $\mathbf{O}$-colimit

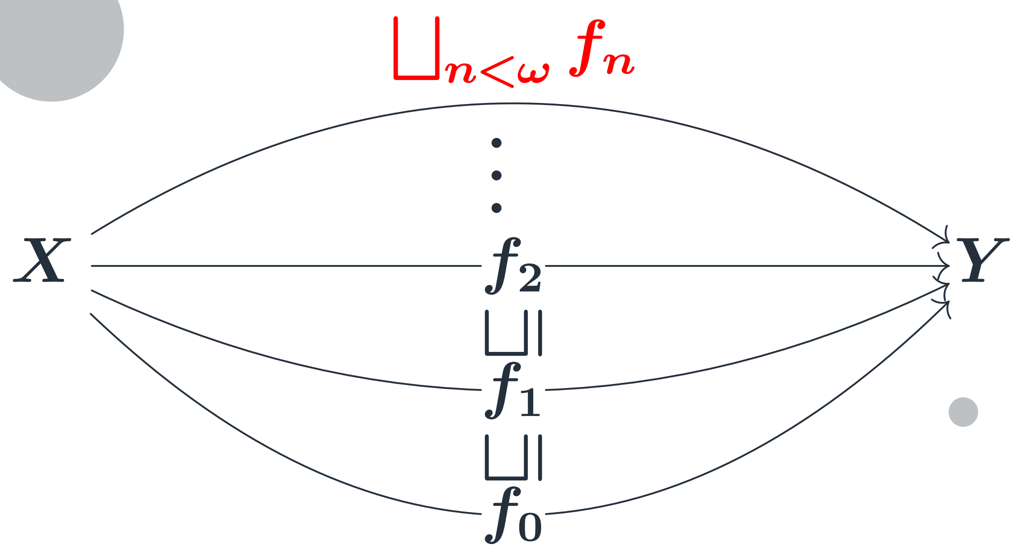
  $\Updownarrow$ $\qquad\qquad\qquad\qquad$ $\Updownarrow$

  limit —— limit-colimit coincidence —— colimit

- [Smyth & Plotkin, SIAM J. Comp., 1982]

DCpo-enriched categories

- Each homset is a dcpo:
  - ☐ order between arrows $X \rightrightarrows Y$ ($\sqsubseteq$) and
  - ☐ supremum of increasing ω-chain:



- Composition preserves supremums:

$$\left(X \xrightarrow[f_0,\ \vdots]{\bigsqcup f_n} Y \xrightarrow[g_0,\ \vdots]{\bigsqcup g_n} Z\right) = \left(X \xrightarrow[g_0 \circ f_0,\ \vdots]{\bigsqcup (g_n \circ f_n)} Z\right)$$

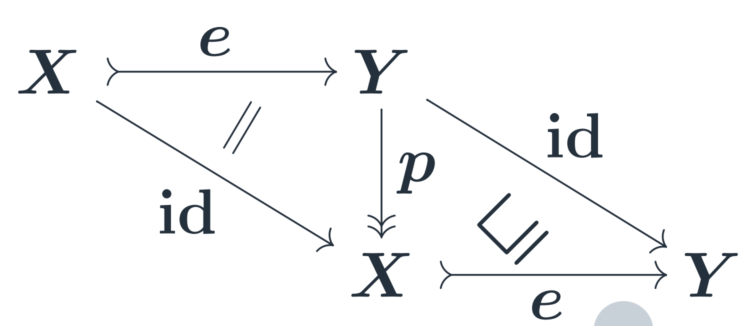
- Examples: $\mathcal{K}\ell(T)$ for $T = \mathcal{L}, \mathcal{P}, \mathcal{D}$ (on blackboard)

Embedding-projection pairs

In a $\mathbf{DCpo}$-enriched category,

$$X \underset{p}{\overset{e}{\rightleftarrows}} Y \quad \text{s.t.} \quad \begin{cases} p \circ e = \mathrm{id} & \text{and} \\ e \circ p \sqsubseteq \mathrm{id} \end{cases}$$

- Diagramatically,

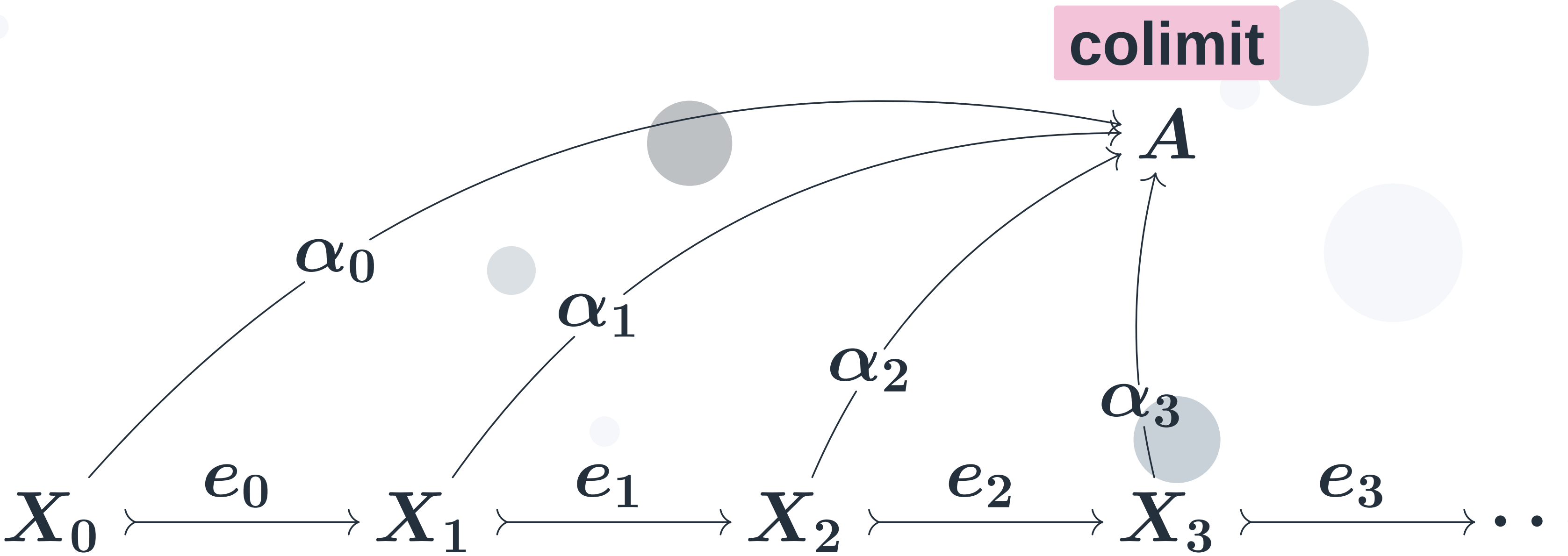


- e is mono and p is epi. Both are split.
- $\begin{cases} p \text{ is the } \textbf{smallest left-inverse} \text{ of } e \\ e \text{ is the } \textbf{smallest right-inverse} \text{ of } p \end{cases}$

  Hence corresponding emb./proj. is unique: (e, e^P) and (p^E, p).
- Intuition?

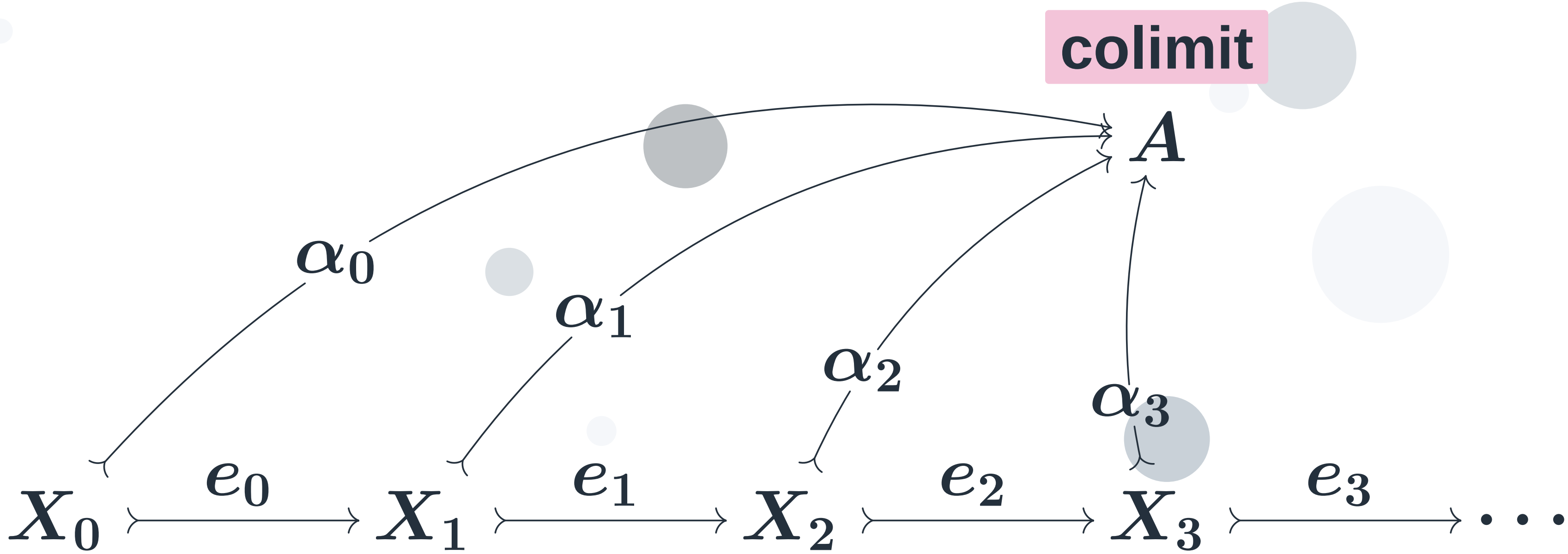
$\mathbf{O}$-colimits

$\mathbf{DCpo}$-enriched. Each e_n is an embedding.



O-colimits

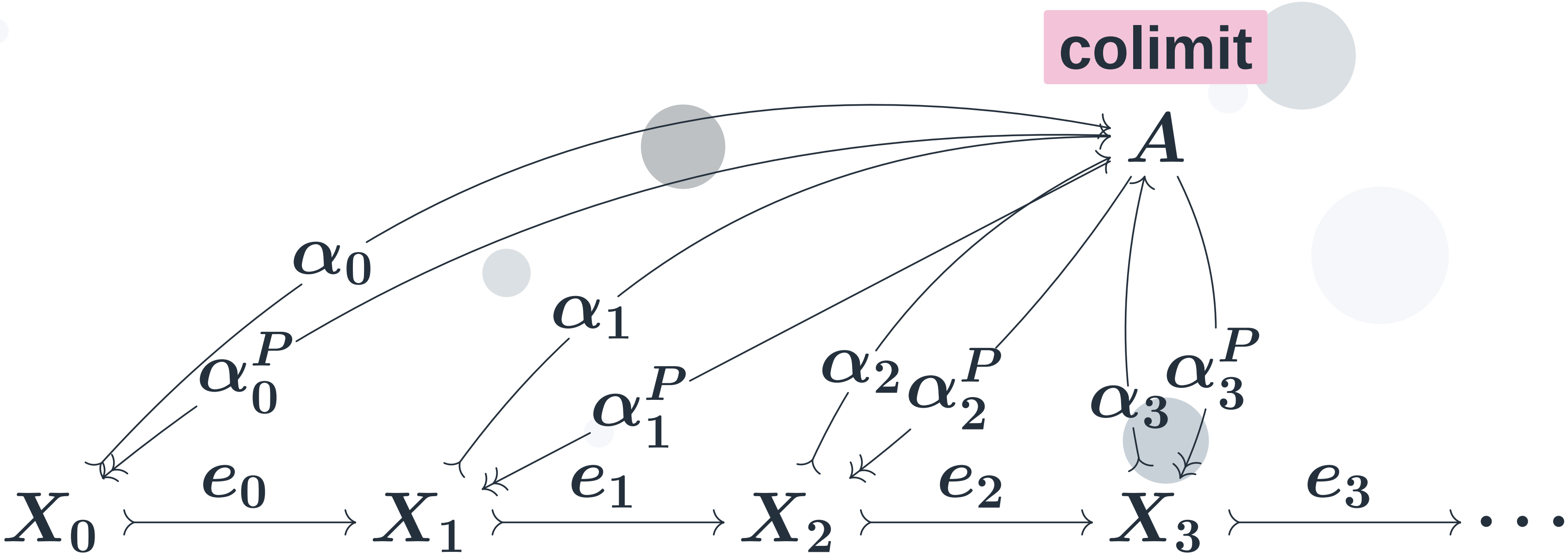
$\mathbf{DCpo}$-enriched. Each e_n is an embedding.



- Each α_n is also an embedding.

O-colimits

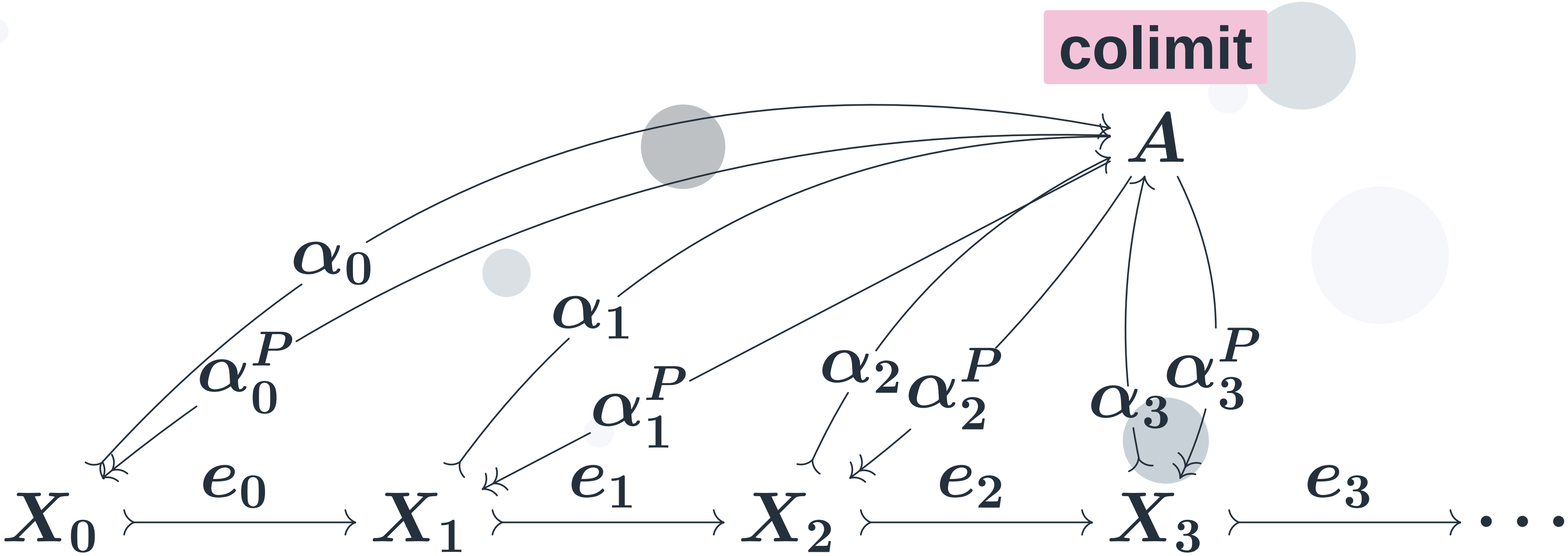
DCpo-enriched. Each e_n is an embedding.



- Each α_n is also an embedding.
- $\{\, A \xrightarrow{\alpha_n^P} X_n \xrightarrow{\alpha_n} A \,\}_{n<\omega}$ is increasing.

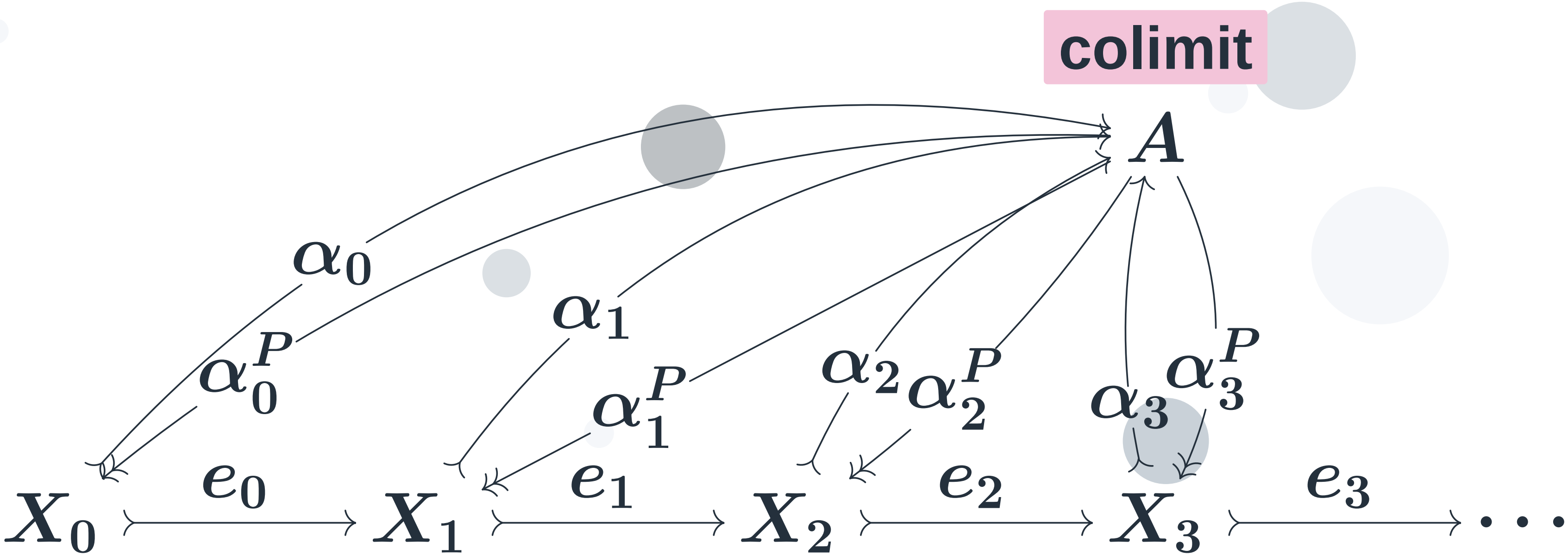
$\mathbf{O}$-colimits

$\mathbf{DCpo}$-enriched. Each e_n is an embedding.



- Each α_n is also an embedding.
- $\{ A \xrightarrow{\alpha_n^P} X_n \xrightarrow{\alpha_n} A \}_{n<\omega}$ is increasing.
- Its supremum is $A \xrightarrow{\mathrm{id}} A$.

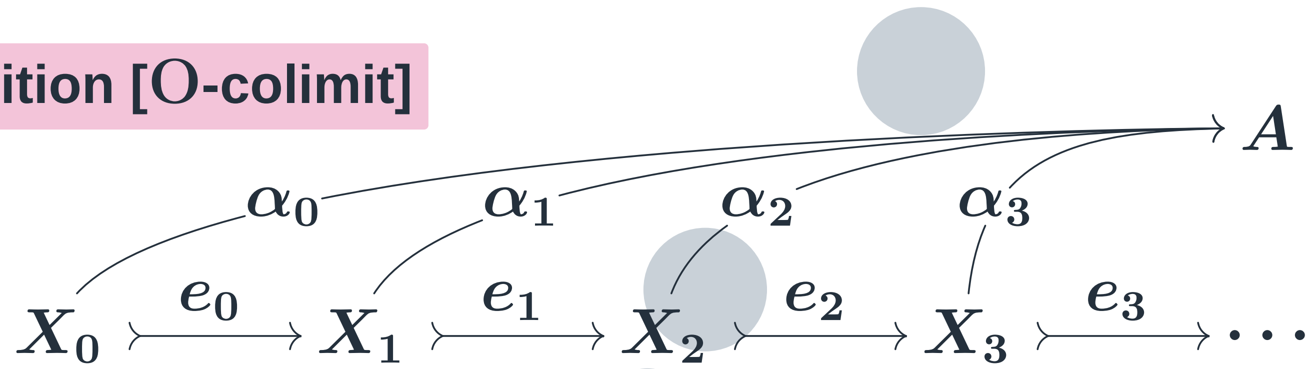
$\mathbf{DCpo}$-enriched. Each e_n is an embedding.



- Each α_n is also an embedding. **Notion of O-colimit!**
- $\{ A \xrightarrow{\alpha_n^P} X_n \xrightarrow{\alpha_n} A \}_{n<\omega}$ is increasing.
- Its supremum is $A \xrightarrow{\mathrm{id}} A$.

$\mathbf{O}$-colimits

Definition [$\mathbf{O}$-colimit]

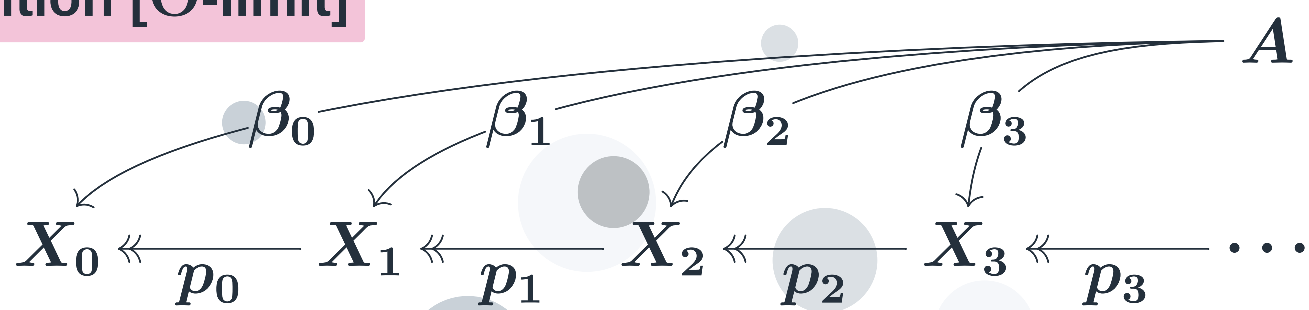


- Each α_n is an embedding.
- $\{ A \xrightarrow{\alpha_n^P} X_n \xrightarrow{\alpha_n} A \}_{n<\omega}$ is increasing.
- Its supremum is $A \xrightarrow{\mathrm{id}} A$.

Theorem [Smyth & Plotkin]

- An $\mathbf{O}$-colimit is a colimit.
- Conversely, a colimit of $X_0 \xrightarrow{e_0} X_1 \xrightarrow{e_1} \cdots$ is an $\mathbf{O}$-colimit.

Definition [$\mathbf{O}$-limit]



- Each β_n is a projection.
- $\{\, A \xrightarrow{\beta_n} X_n \xrightarrow{\beta_n^E} A \,\}_{n<\omega}$ is increasing.
- Its supremum is $A \xrightarrow{\mathrm{id}} A$.

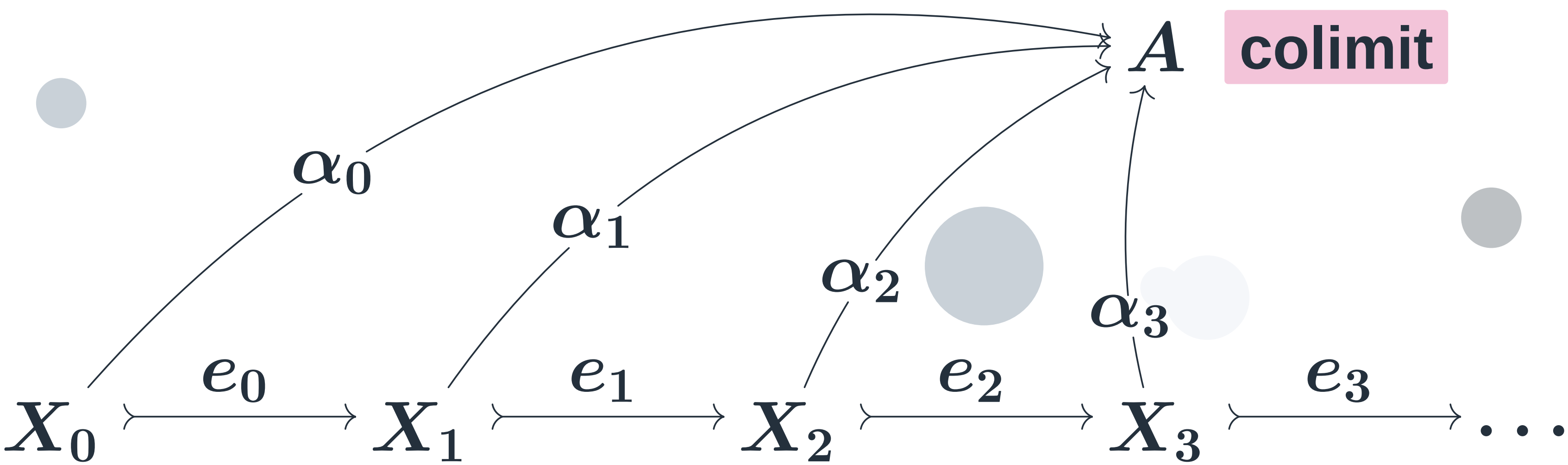
Theorem [Smyth & Plotkin]

- An $\mathbf{O}$-limit is a limit.
- Conversely, a limit of $X_0 \overset{p_0}{\twoheadleftarrow} X_1 \overset{p_1}{\twoheadleftarrow} \cdots$ is an $\mathbf{O}$-limit.

Limit-colimit coincidence

Theorem [Smyth & Plotkin]

$\mathbf{DCpo}$-enriched. Each e_n is an embedding.

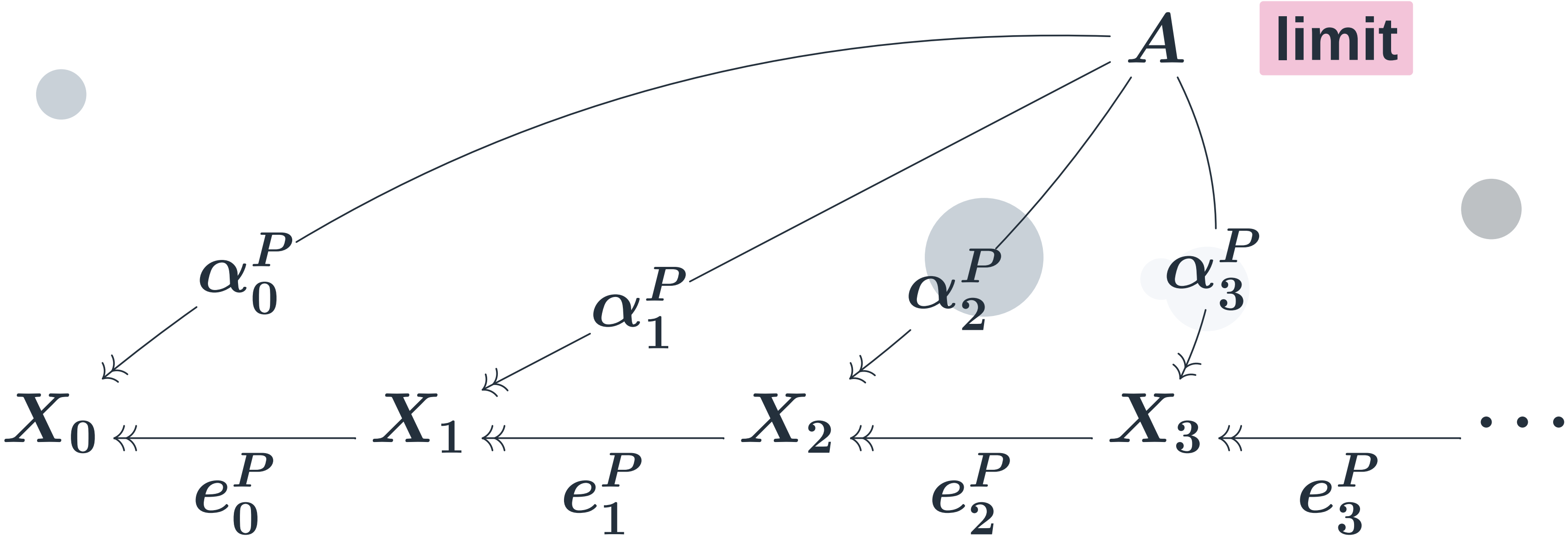


if and only if...

Limit-colimit coincidence

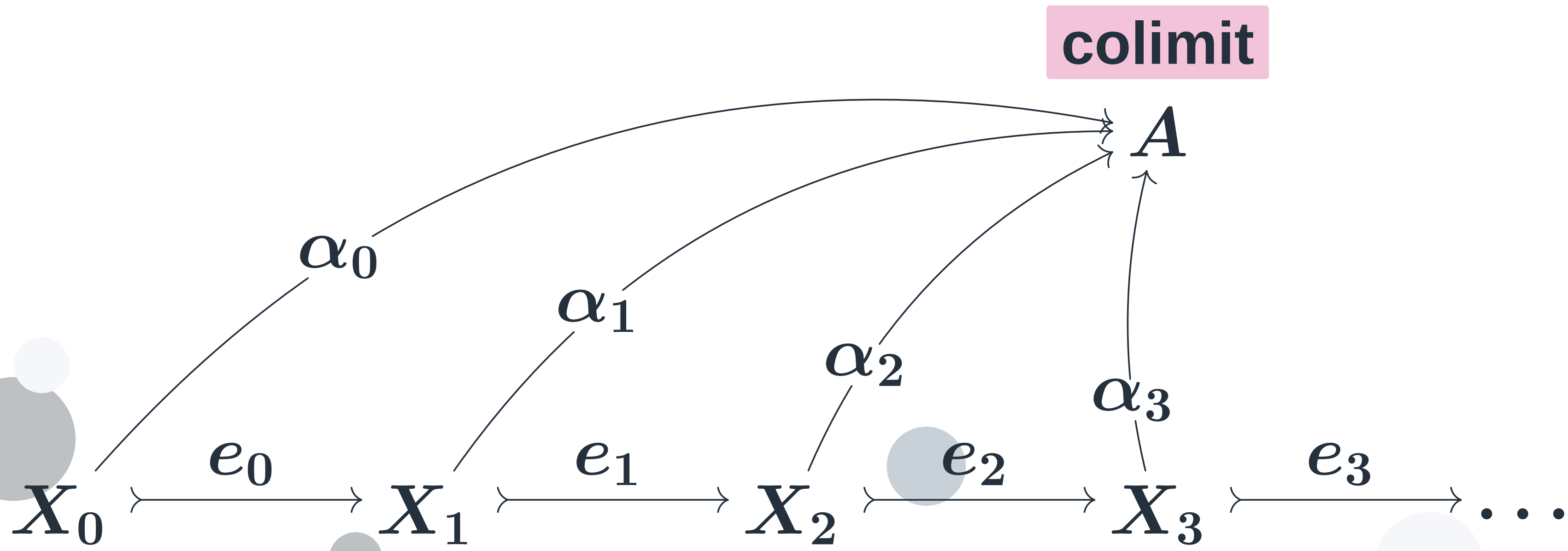
Theorem [Smyth & Plotkin]

$\mathbf{DCpo}$-enriched. Each e_n is an embedding.



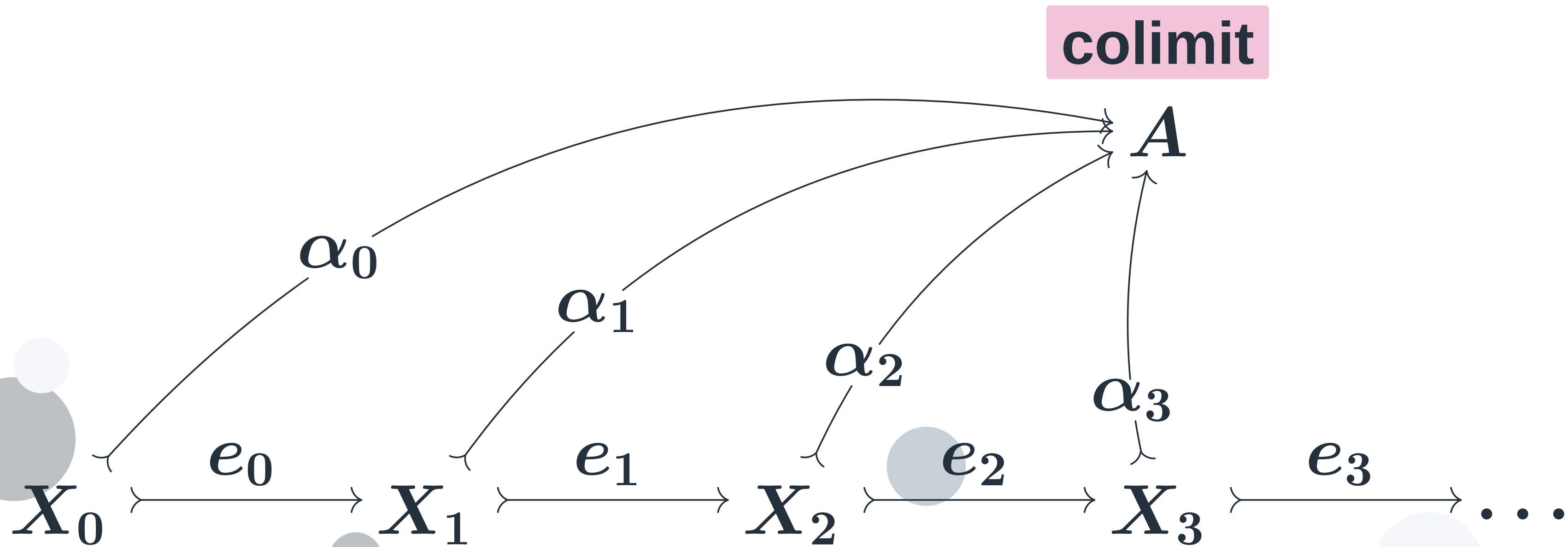
Proof: Limit-colimit coincidence

$\mathbf{DCpo}$-enriched. Each e_n is an embedding.



Proof: Limit-colimit coincidence

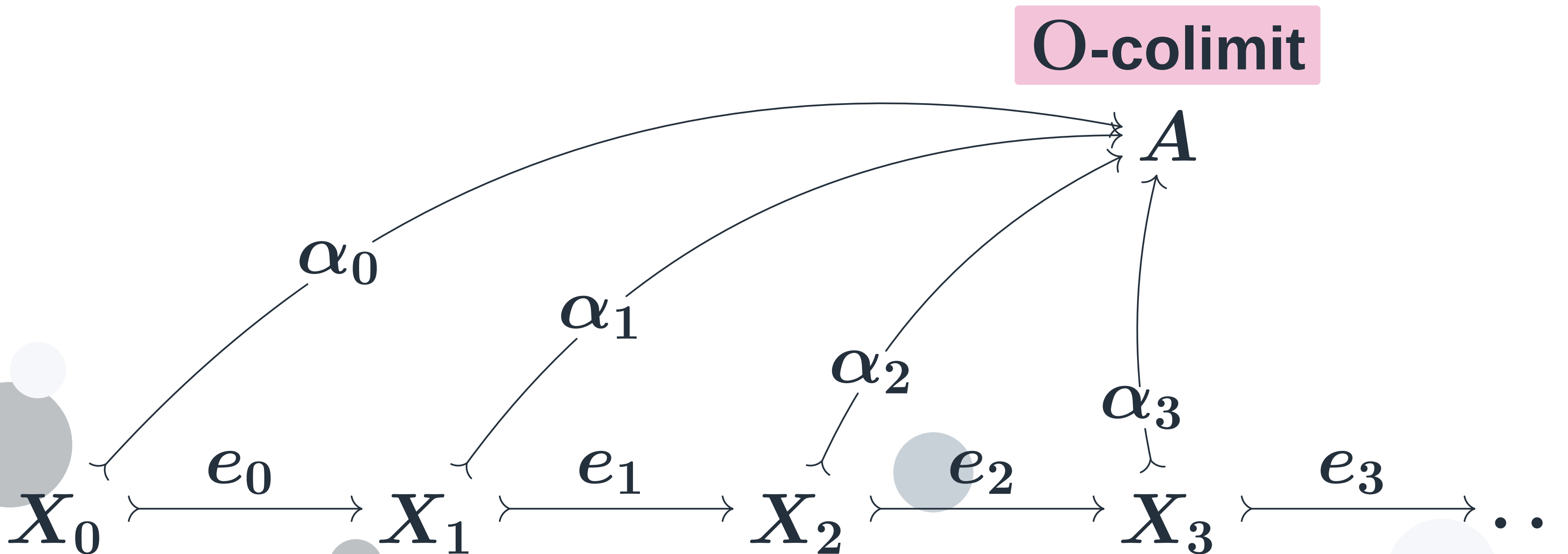
$\mathbf{DCpo}$-enriched. Each e_n is an embedding.



- Colimit of ω-chain of embeddings consists of embeddings.

Proof: Limit-colimit coincidence

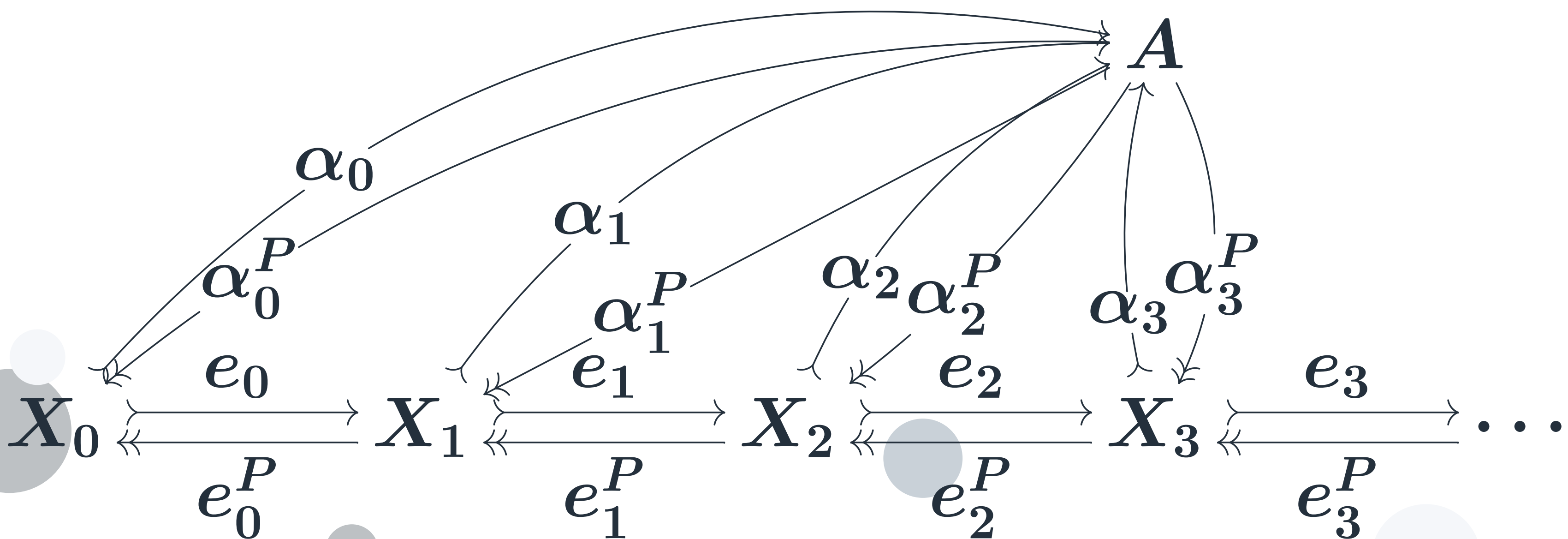
$\mathbf{DCpo}$-enriched. Each e_n is an embedding.



- $\{\, A \xrightarrow{\alpha_n^P} X_n \xrightarrow{\alpha_n} A \,\}_{n<\omega}$ is increasing and its supremum is $A \xrightarrow{\mathrm{id}} A$.
- Colimit $\Longleftrightarrow$ $\mathbf{O}$-colimit.

Proof: Limit-colimit coincidence

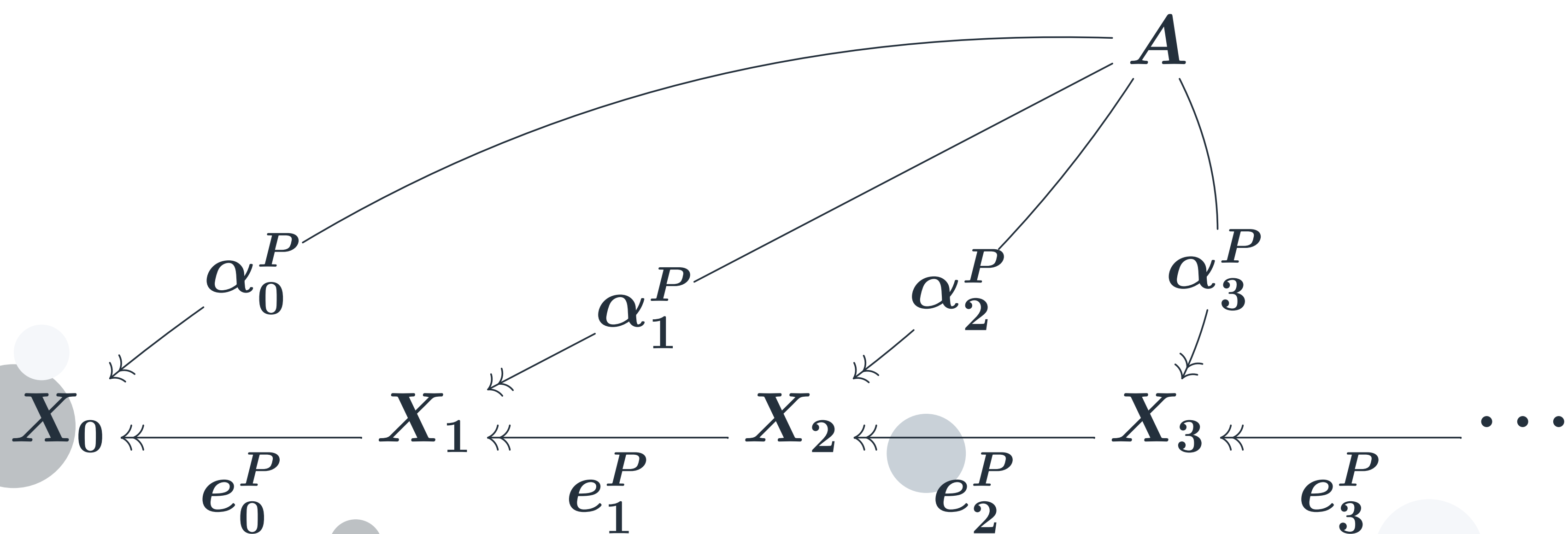
$\mathbf{DCpo}$-enriched. Each e_n is an embedding.



- $\{\, A \xrightarrow{\alpha_n^P} X_n \xrightarrow{\alpha_n} A \,\}_{n<\omega}$ is increasing and its supremum is $A \xrightarrow{\mathrm{id}} A$.

Proof: Limit-colimit coincidence

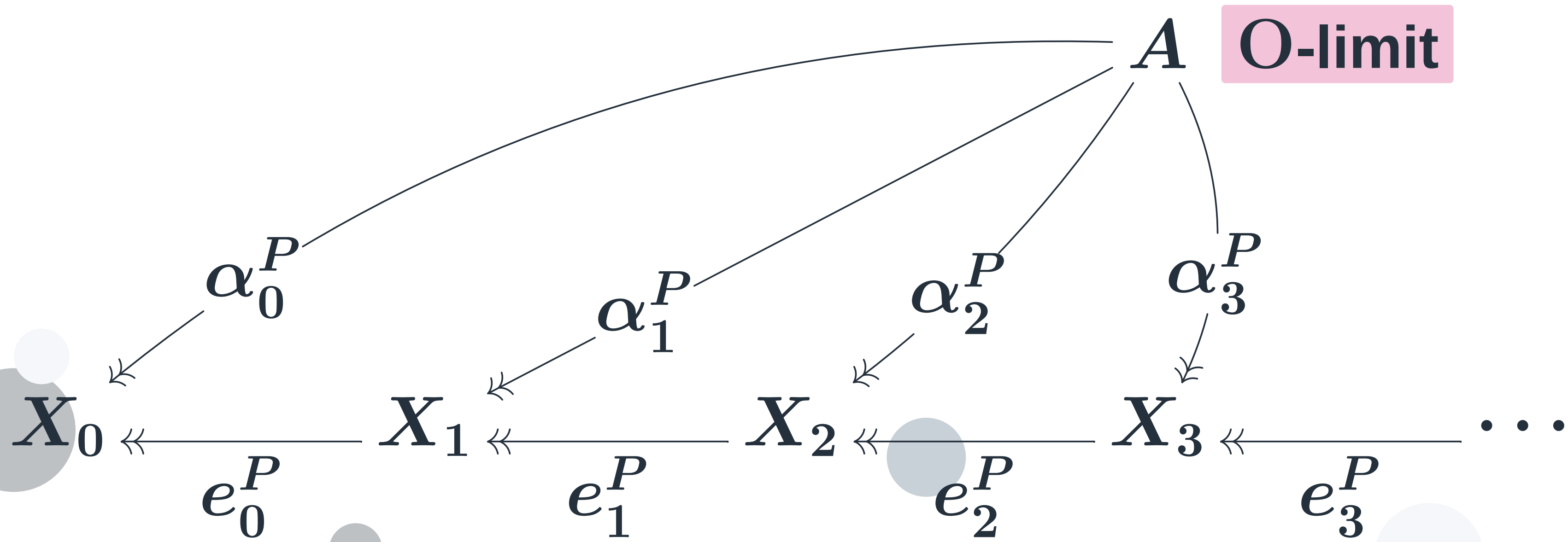
$\mathbf{DCpo}$-enriched. Each e_n is an embedding.



- $\{ A \xrightarrow{\alpha_n^P} X_n \xrightarrow{(\alpha_n^P)^E} A \}_{n<\omega}$ is increasing and its supremum is $A \xrightarrow{\mathrm{id}} A$.
- $\alpha_n = \left(\alpha_n^P\right)^E$.

Proof: Limit-colimit coincidence

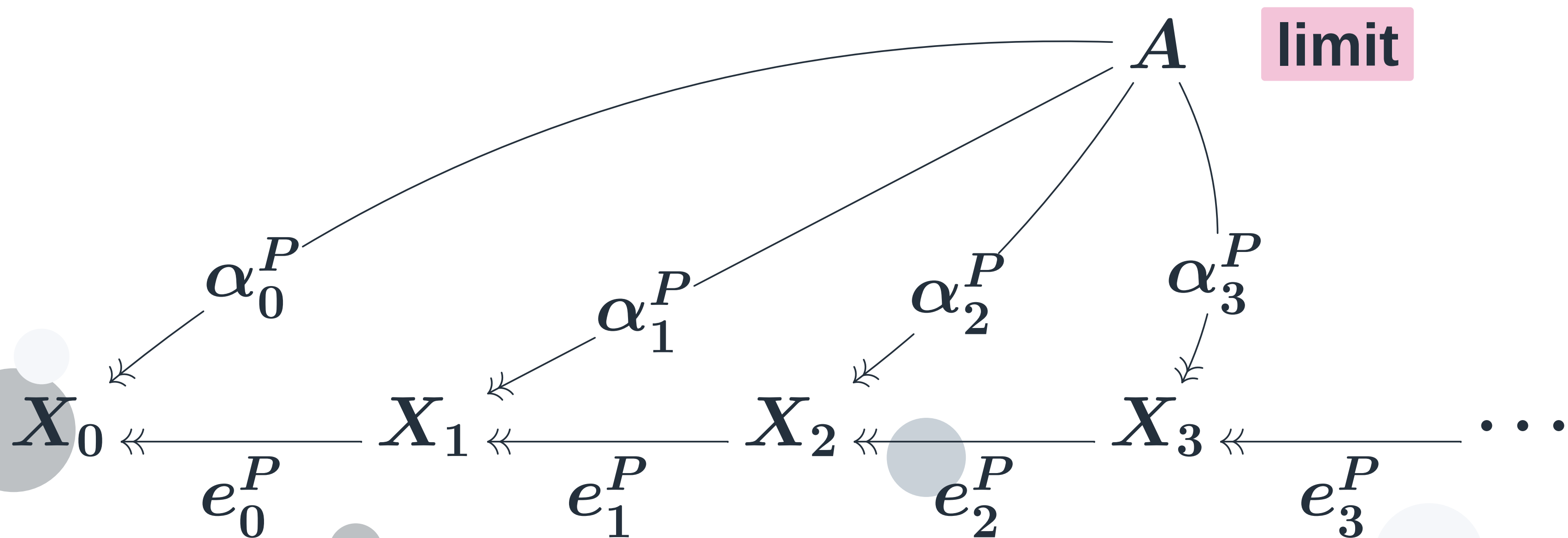
$\mathbf{DCpo}$-enriched. Each e_n is an embedding.



- $\{ A \xrightarrow{\alpha_n^P} X_n \xrightarrow{(\alpha_n^P)^E} A \}_{n<\omega}$ is increasing and its supremum is $A \xrightarrow{\mathrm{id}} A$.
- Obvious duality between $\mathbf{O}$-colimits and $\mathbf{O}$-limits!

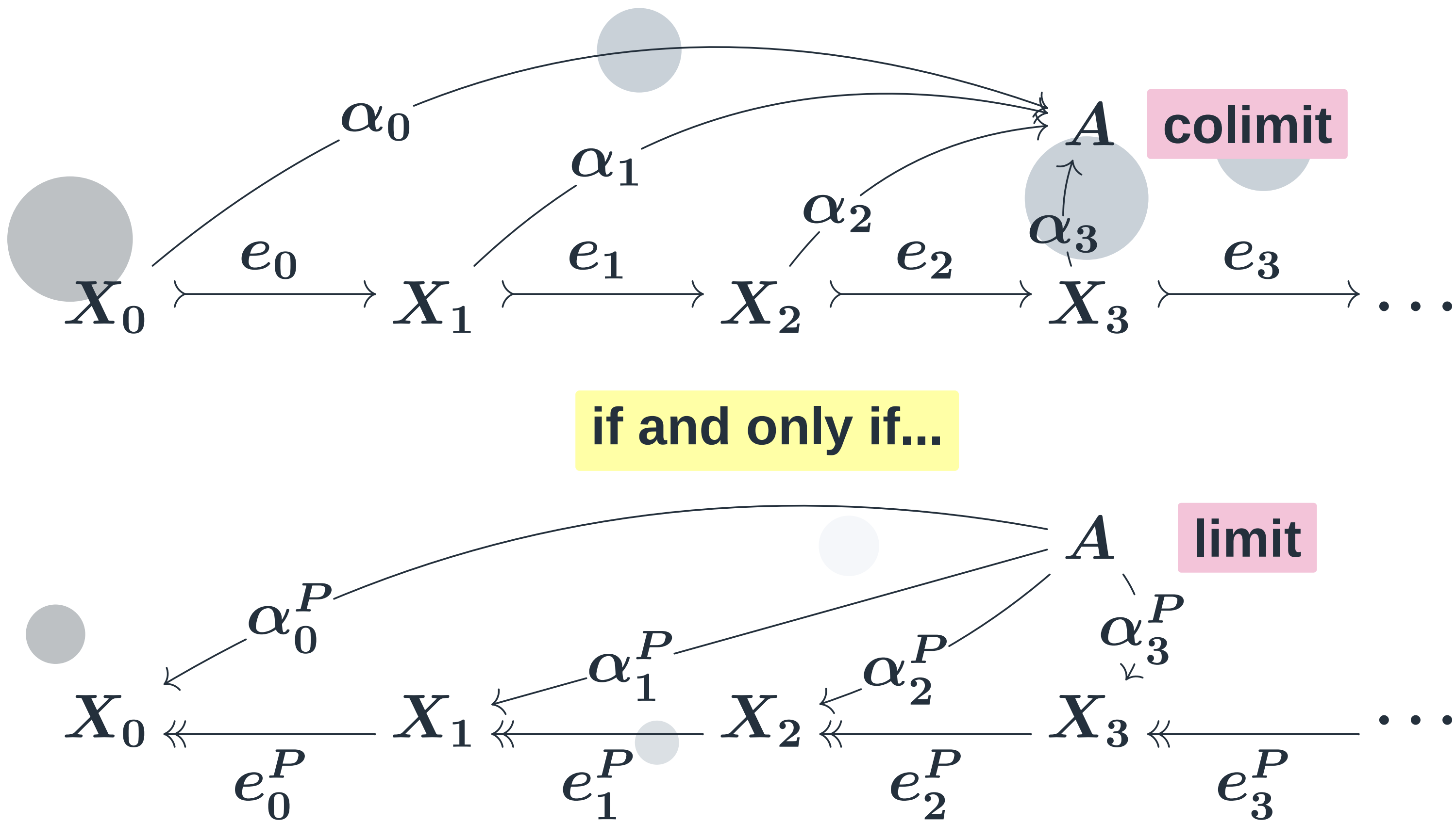
Proof: Limit-colimit coincidence

$\mathbf{DCpo}$-enriched. Each e_n is an embedding.



- Limit $\Longleftrightarrow$ $\mathbf{O}$-limit.
- Q.E.D.

Summary



- Base category will be $\mathcal{K}\ell(T)$.
- The chain will be initial/final sequences.
- Implies **initial alg.-final coalg. coincidence!**

Main technical result

Initial algebra-final coalgebra coincidence

- A system is $\begin{array}{c} TFX \\ c\uparrow \\ X \end{array}$ in $\mathbf{Sets}$, i.e. $\begin{array}{c} FX \\ c\uparrow \\ X \end{array}$ in $\mathcal{K}\ell(T)$.
- **Main theorem**

  $\begin{array}{c} FA \\ \alpha\downarrow\cong \\ A \end{array}$ in $\mathbf{Sets}$: initial algebra. Then

  - $\begin{array}{c} FA \\ J\alpha\downarrow\cong \\ A \end{array}$ in $\mathcal{K}\ell(T)$: initial $\mathcal{K}\ell(F)$-algebra;
  - $\begin{array}{c} FA \\ J\alpha^{-1}\uparrow\cong \\ A \end{array}$ in $\mathcal{K}\ell(T)$: final $\mathcal{K}\ell(F)$-coalgebra.

Initial algebra-final coalgebra coincidence

- A system is $\begin{matrix} TFX \\ c\uparrow \\ X \end{matrix}$ in $\mathbf{Sets}$, i.e. $\begin{matrix} FX \\ c\uparrow \\ X \end{matrix}$ in $\mathcal{K}\ell(T)$.

  [monads, distributive laws, Kleisli categories]

- **Main theorem**

  $\begin{matrix} FA \\ \alpha\downarrow\cong \\ A \end{matrix}$ in $\mathbf{Sets}$: initial algebra. Then

  - $\begin{matrix} FA \\ J\alpha\downarrow\cong \\ A \end{matrix}$ in $\mathcal{K}\ell(T)$: initial $\mathcal{K}\ell(F)$-algebra;
  - $\begin{matrix} FA \\ J\alpha^{-1}\uparrow\cong \\ A \end{matrix}$ in $\mathcal{K}\ell(T)$: final $\mathcal{K}\ell(F)$-coalgebra.

  [initial/final sequence
  limit-colimit coincidence]

Assumptions

- **Distributive law $FT \Rightarrow TF$.**
  - Available for
    - "shapely" functors F,

$$F, G, F_i ::= \mathrm{id} \mid \Sigma \mid F \times G \mid \coprod_{i \in I} F_i \ ,$$

and

    - **commutative** monads T.

Assumptions

- Kleisli category $\mathcal{K}\ell(T)$ is $\mathbf{DCpo}_{\bot}$-enriched.
  - Each homset has the minimum: $X \underset{\bot_{X,Y}}{\overset{f}{\rightrightarrows}} Y$, with $\bot_{X,Y} \sqsubseteq f$
- Composition in $\mathcal{K}\ell(T)$ is **left-strict**:

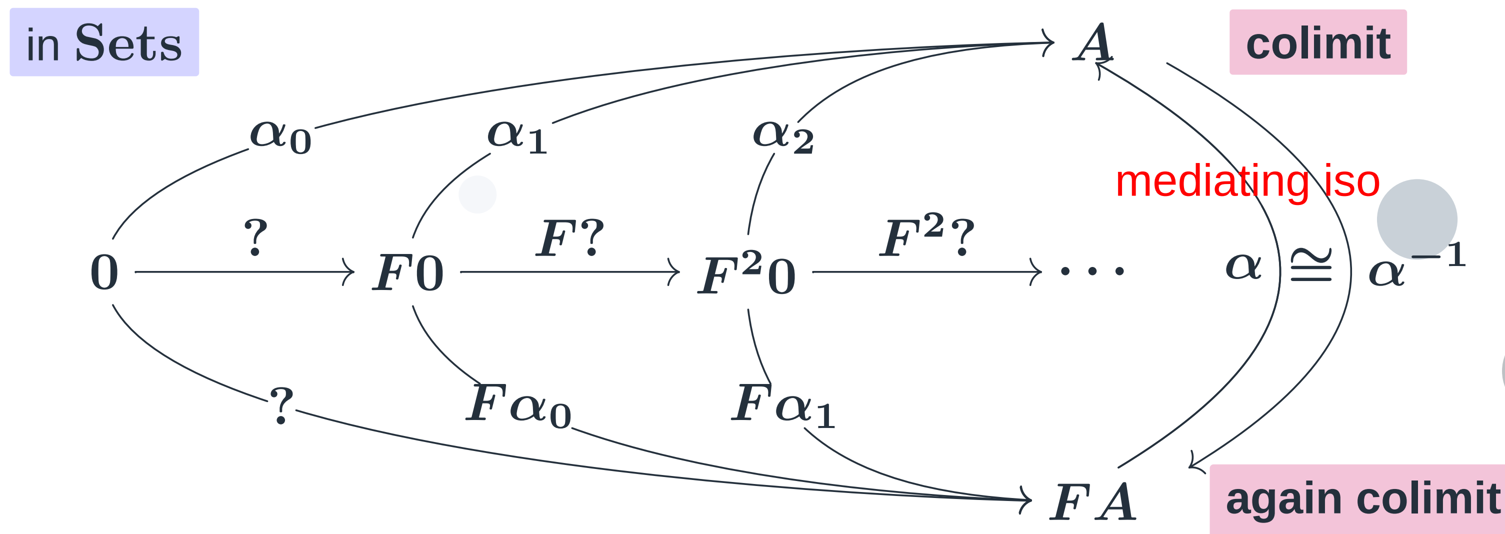
$$X \xrightarrow{f} Y \xrightarrow{\bot_{Y,Z}} Z \;=\; X \xrightarrow{\bot_{X,Z}} Z \;.$$

- Lifted $\mathcal{K}\ell(F) : \mathcal{K}\ell(T) \to \mathcal{K}\ell(T)$ is **monotonic**:

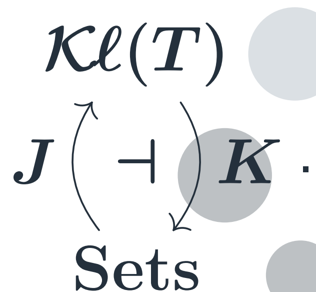
$$X \underset{f}{\overset{g}{\rightrightarrows}} Y \;(f \sqsubseteq g) \quad\Longrightarrow\quad FX \underset{\mathcal{K}\ell(F)(f)}{\overset{\mathcal{K}\ell(F)(g)}{\rightrightarrows}} FY \;(\mathcal{K}\ell(F)(f) \sqsubseteq \mathcal{K}\ell(F)(g))$$

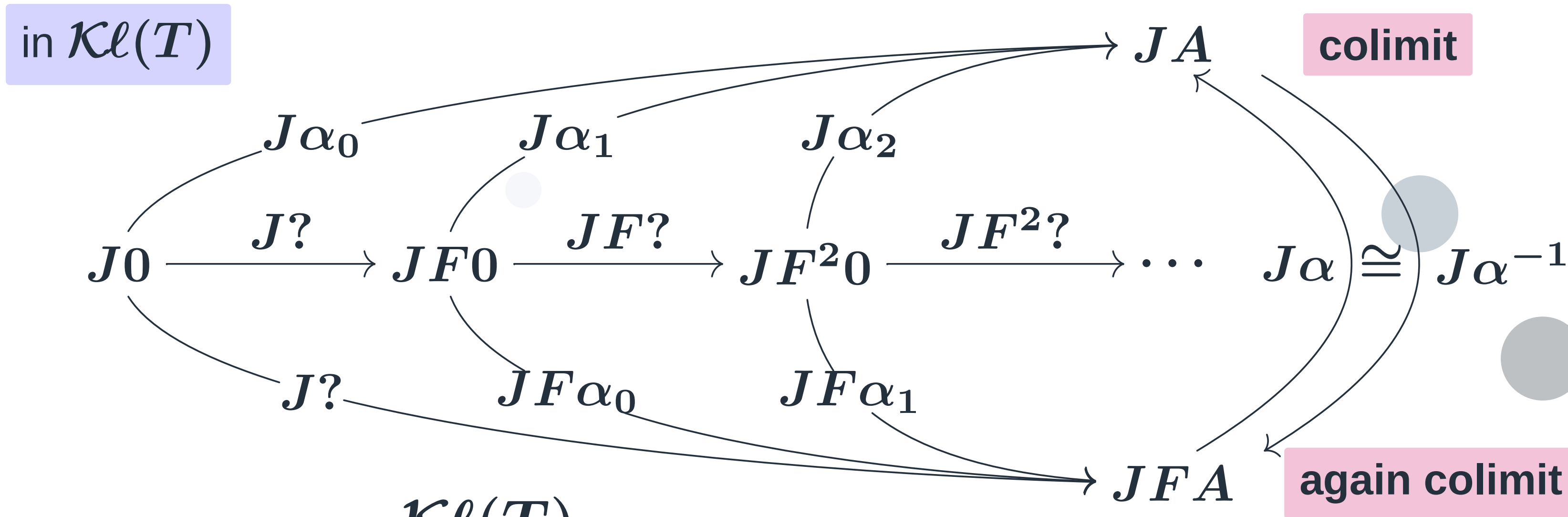
- True for $T = \mathcal{L}, \mathcal{P}, \mathcal{D}$.

Proof: sketch



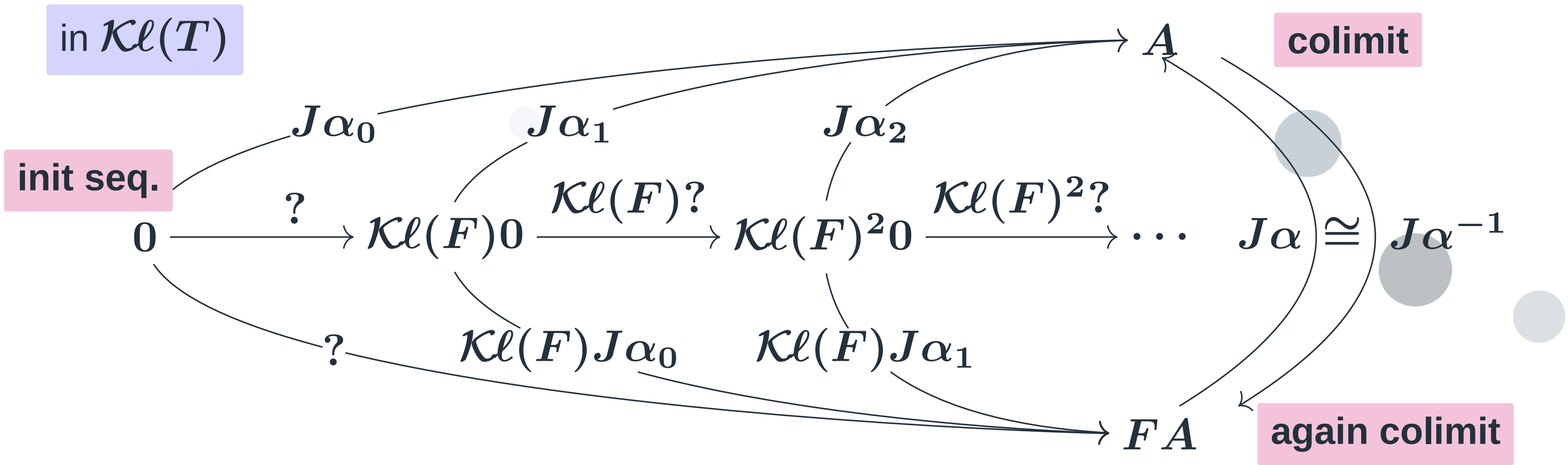
- Initial sequence construction in $\mathbf{Sets}$.
- Let's map by J in



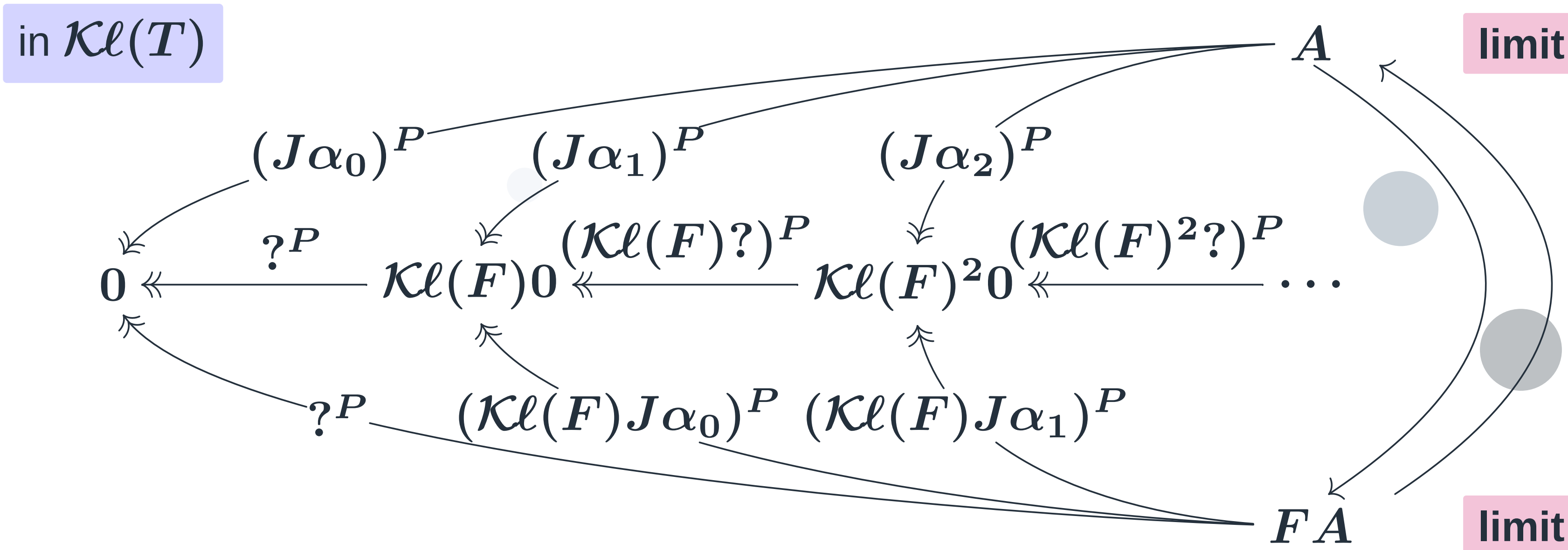


- Mapped by J in $\mathcal{K}\ell(T) \; J \dashv K \; \mathbf{Sets}$.
- Left adjoint preserves colimits.
- We shall show:
  - ☐ The sequence is the initial sequence for $\mathcal{K}\ell(F)$.
  - ☐ The upper cone is mapped by $\mathcal{K}\ell(F)$ to the lower one.

Proof: sketch

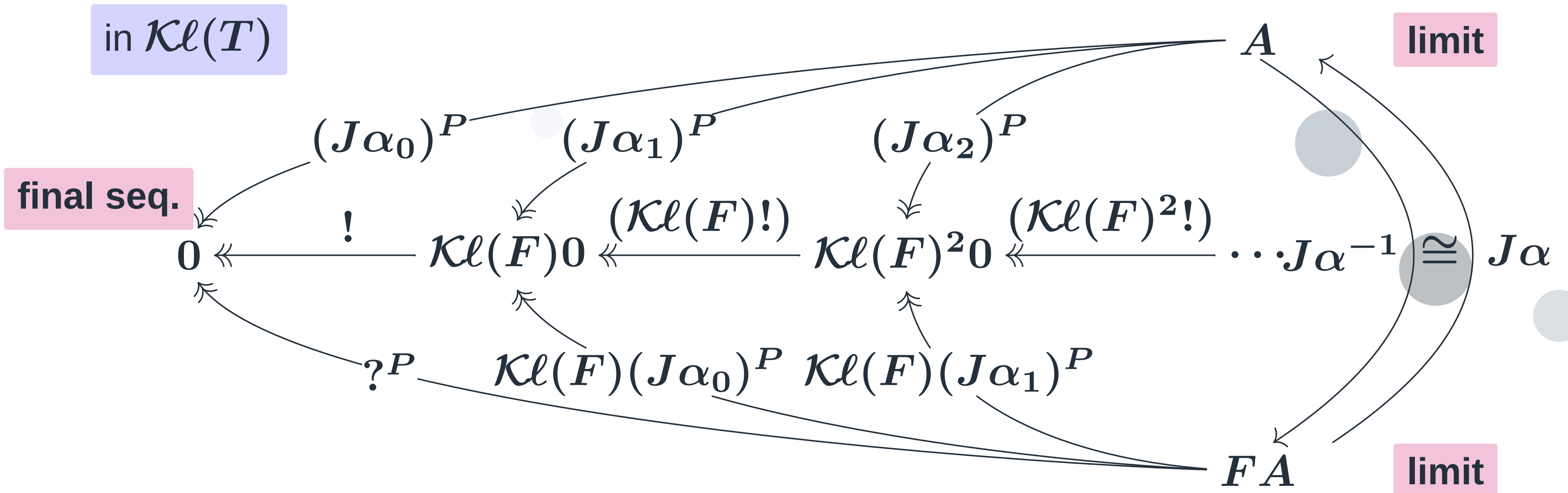


- This proves that $\begin{array}{c} FA \\ J\alpha \downarrow \cong \\ A \end{array}$ is an initial $\mathcal{K}\ell(F)$-algebra.
- All arrows are embeddings. We take the corresponding projections.



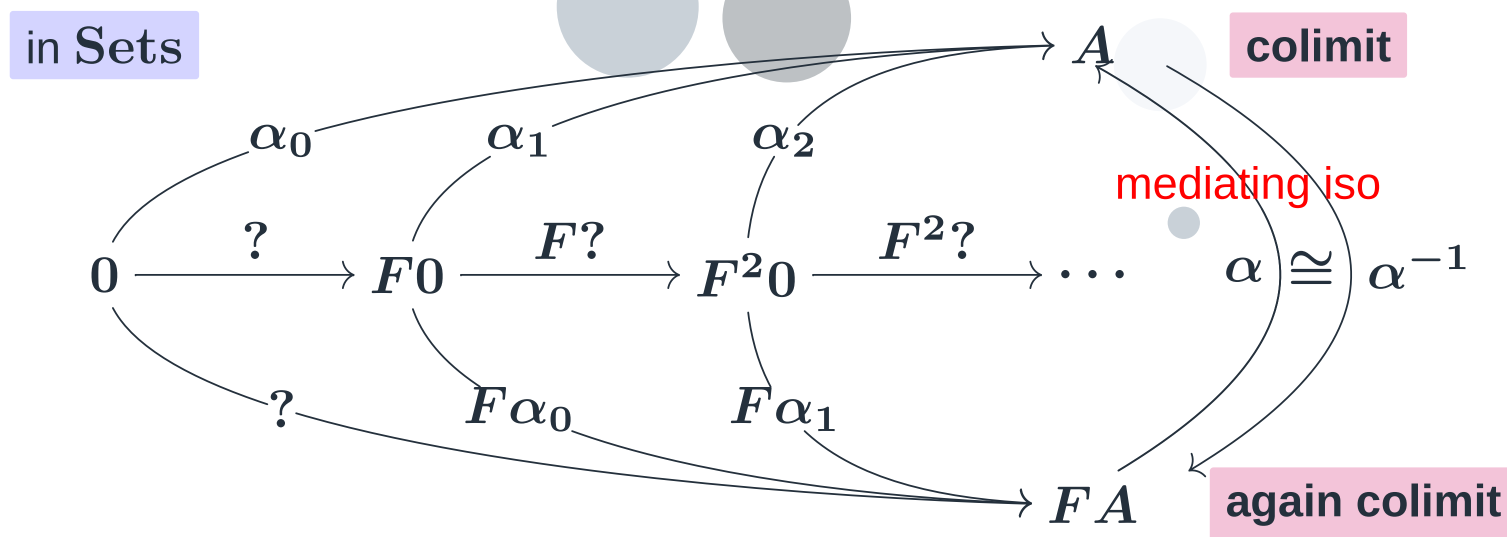
- We used **Limit-colimit coincidence!**
- We show:
  - ☐ The sequence is the final sequence:
  - ☐ The upper cone is mapped by $\mathcal{K}\ell(F)$ to the lower one.

Proof: sketch

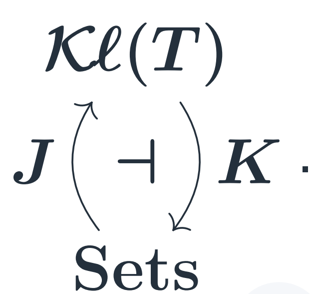


- In a $\mathbf{DCpo}_\perp$-enriched category, an initial object is final as well.
- This proves that $\begin{array}{c} FA \\ J\alpha^{-1}\uparrow\cong \\ A \end{array}$ is a final $\mathcal{K}\ell(F)$-algebra. Q.E.D.

Proof: in detail



- Initial sequence construction in $\mathbf{Sets}$.
- Let's map by J in



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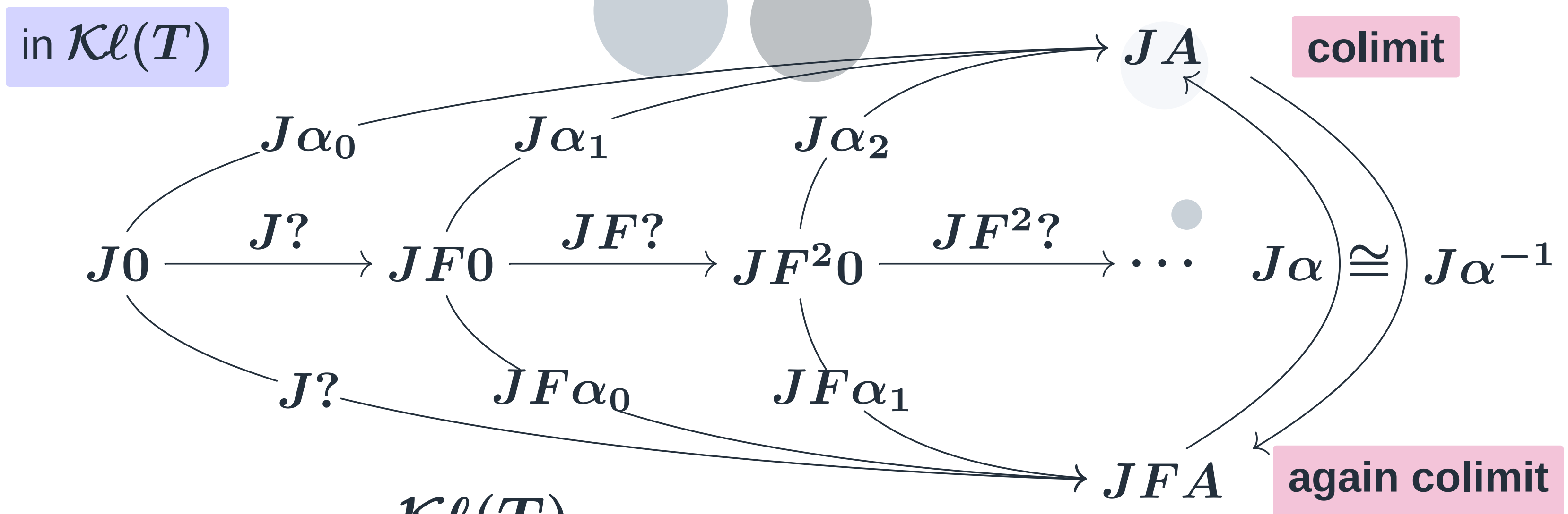
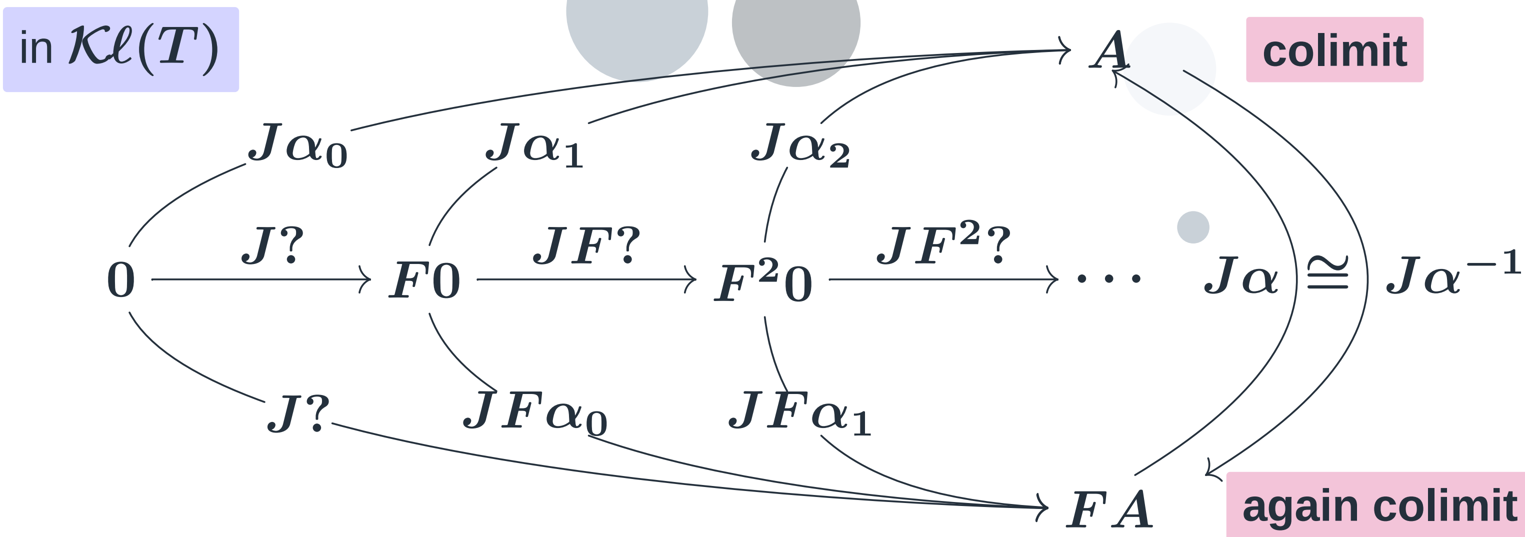
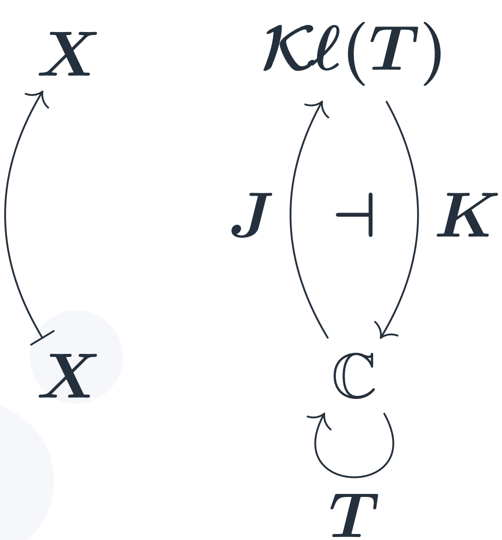


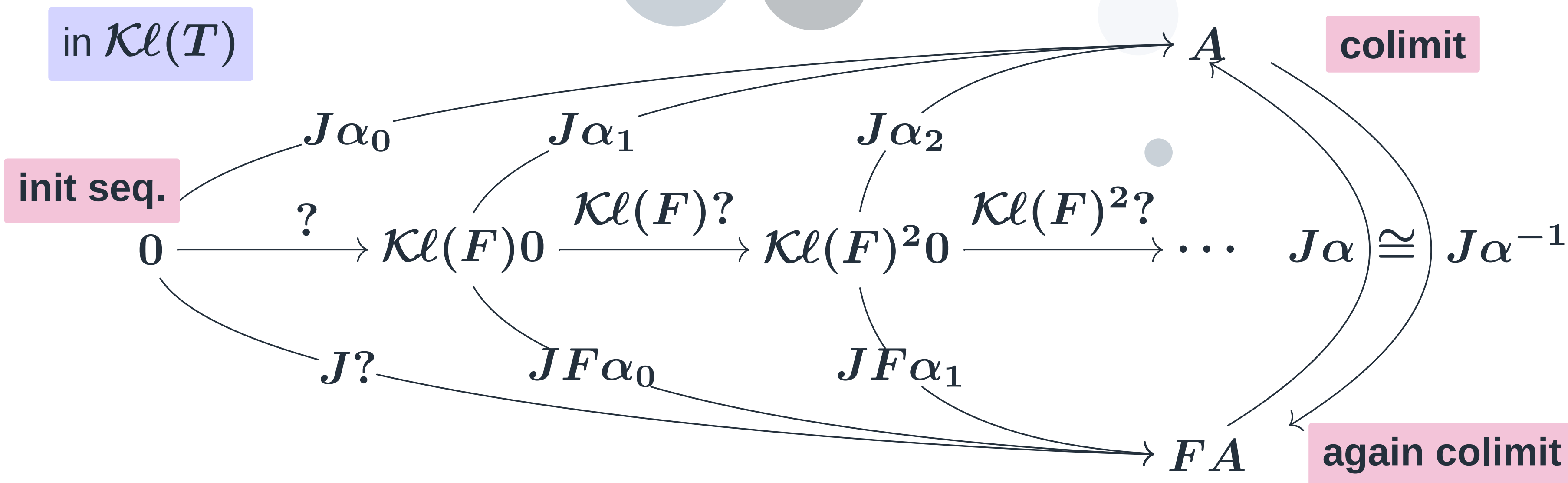
- Mapped by J in $\mathcal{K}\ell(T) \underset{K}{\overset{J}{\rightleftarrows}} \mathbf{Sets}$ with $J \dashv K$.
- Left adjoint preserves colimits.

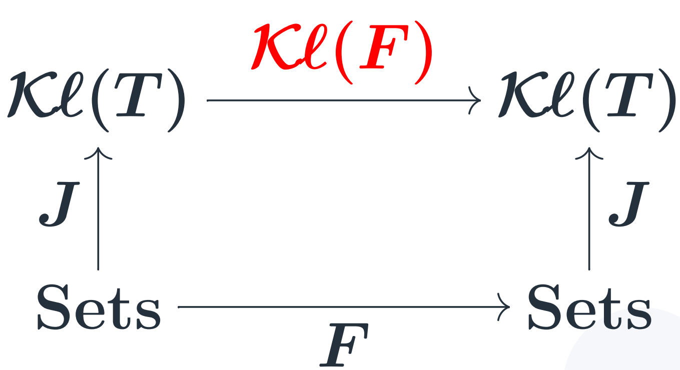
Proof: in detail



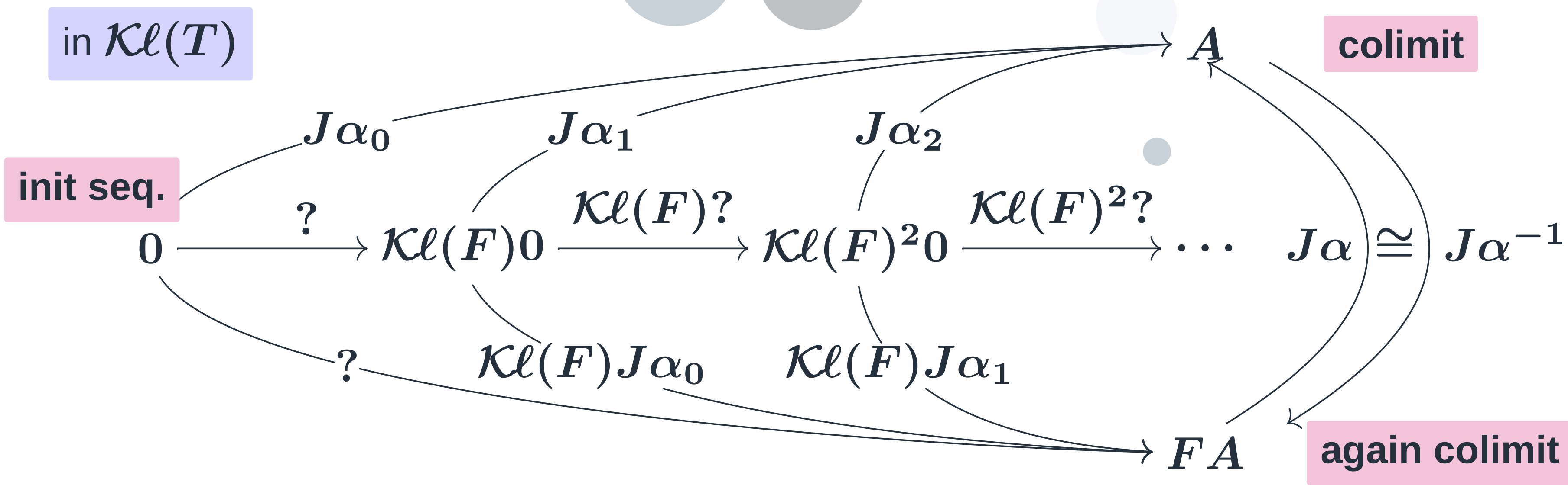
- 

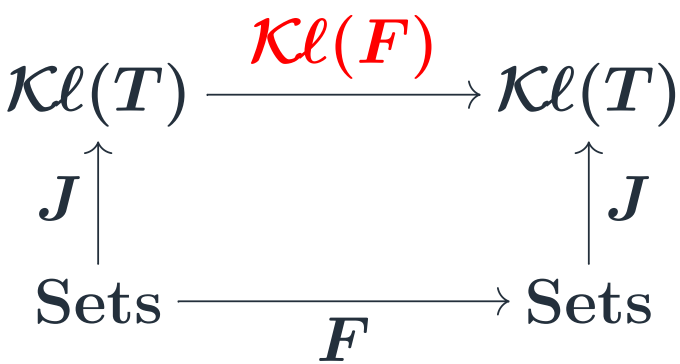
Proof: in detail



- J (left-adjoint) preserves initial object 0.
- 

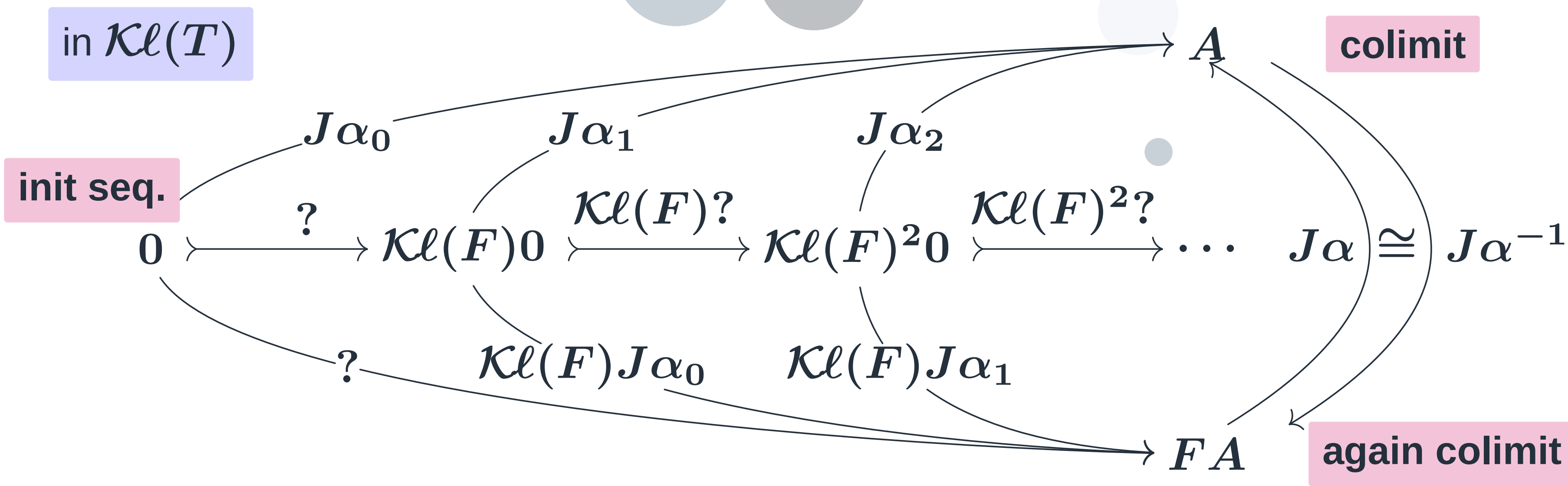
Proof: in detail



- 

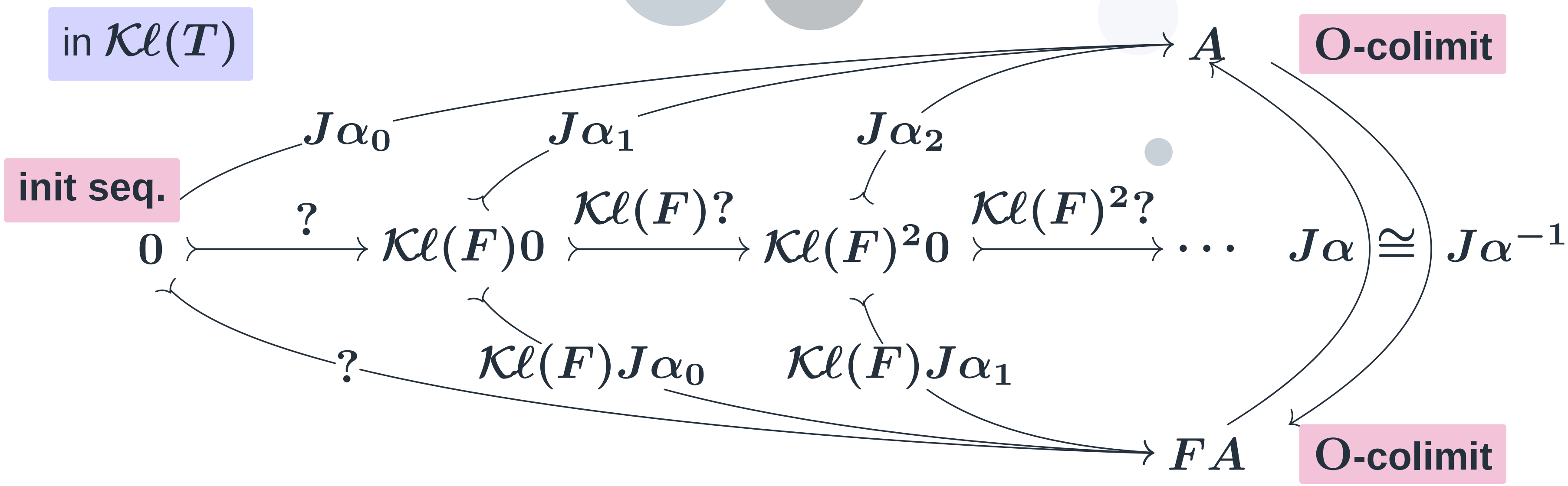
- This proves that $J\alpha \downarrow \cong$ from FA to A is an initial $\mathcal{K}\ell(F)$-algebra.

Proof: in detail

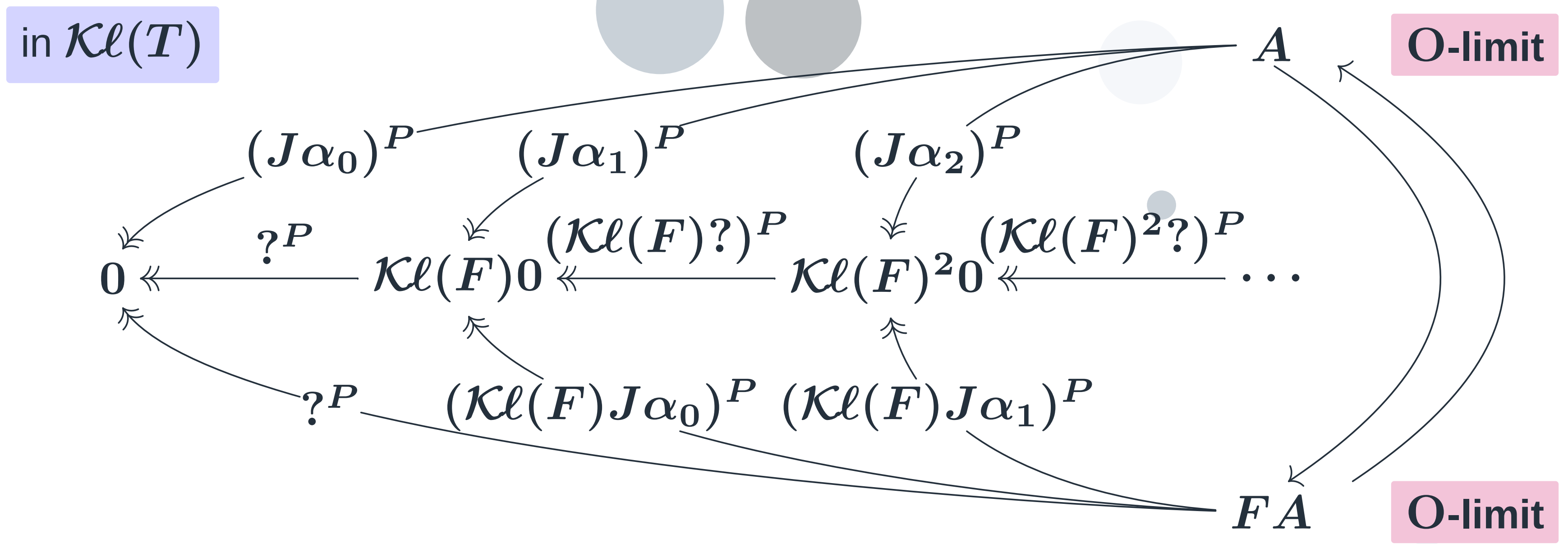


- Arrows in the initial sequence are embeddings.

Proof: in detail

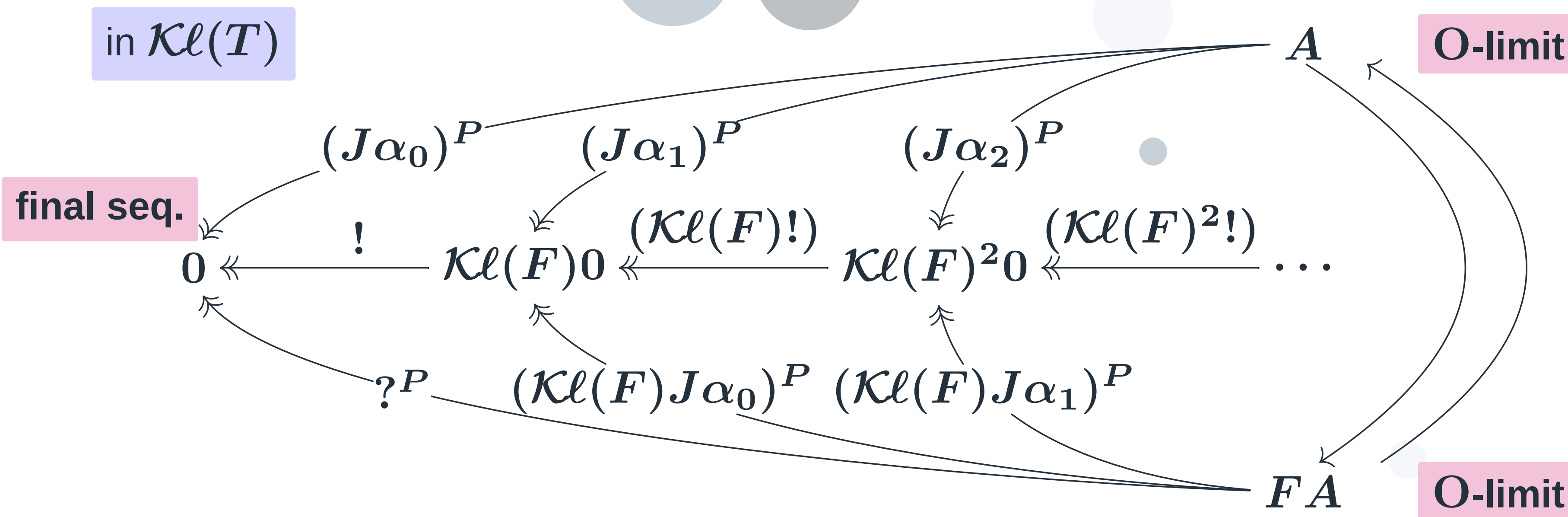


- Hence arrows in colimits are also embeddings.
- Colimits are $\mathbf{O}$-colimits.
- Let's take the corresponding projections...



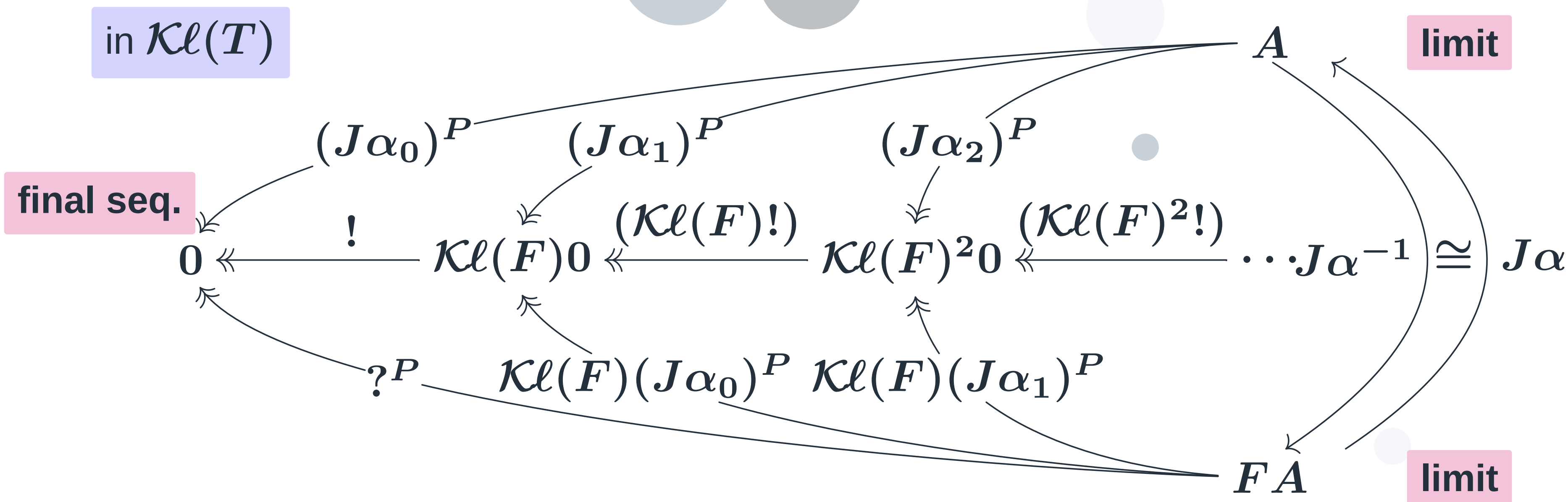
- We need to show:
  - ☐ The sequence is the final sequence:
  - ☐ The upper cone is mapped by $\mathcal{K}\ell(F)$ to the lower one.

Proof: in detail



- 0 is also final in $\mathcal{K}\ell(T)$.
  - Existence $X \xrightarrow{\perp} 0$.
  - Uniqueness $X \xrightarrow{f} 0 \;=\; X \xrightarrow{f} 0 \xrightarrow{\mathrm{id}} 0 \;=\; X \xrightarrow{f} 0 \xrightarrow{\perp} 0$
    $= X \xrightarrow{\perp} 0$.

Proof: in detail



- $\mathbf{O}$-limit $\iff$ limit.
- This proves that $J\alpha^{-1} : A \xrightarrow{\cong} FA$ is a final $\mathcal{K}\ell(F)$-algebra.
- Q.E.D.

Application of the main result

Corollary of the finality result

Let $\begin{array}{c} FA \\ \alpha\downarrow\cong \\ A \end{array}$ in $\mathbf{Sets}$ be an initial F-algebra.

For $\begin{array}{c} TFX \\ c\uparrow \\ X \end{array}$ in $\mathbf{Sets}$, we have unique $X \xrightarrow{\mathsf{tr}_c} TA$ in $\mathbf{Sets}$

s.t.

In $\mathcal{K}\ell(T)$

$$\begin{array}{ccc} \mathcal{K}\ell(F)X & \xrightarrow{\mathcal{K}\ell(F)(\mathsf{tr}_c)} & \mathcal{K}\ell(F)A \\ c\uparrow & & \cong\uparrow J\alpha^{-1} \\ X & \xrightarrow[\mathsf{tr}_c]{} & A \end{array}$$

Example: $T = \mathcal{P}$, $F = 1 + \Sigma \times \_$

- A system $\begin{array}{c} TFX \\ c \uparrow \\ X \end{array}$:
  - **LTS with explicit termination**, or
  - **Nondeterministic automaton**
- $\begin{array}{c} 1 + \Sigma \times \Sigma^* \\ [\mathbf{nil}, \mathbf{cons}] \downarrow \cong \\ \Sigma^* \end{array}$: initial F-algebra
- $X \xrightarrow{\mathbf{tr}_c} \mathcal{P}(\Sigma^*)$ by finality.

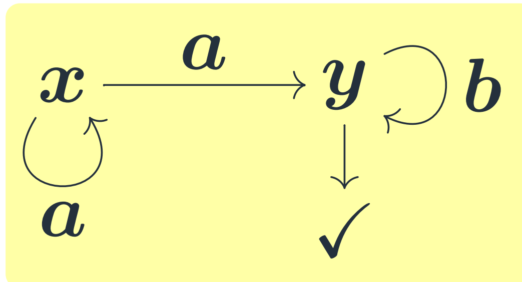
- The diagram of finality

In $\mathcal{K}\ell(\mathcal{P})$

$$\begin{array}{ccc}
\mathcal{K}\ell(F)X & \overset{\mathcal{K}\ell(F)(\mathbf{tr}_c)}{\dashrightarrow} & \mathcal{K}\ell(F)\Sigma^* \\
{\scriptstyle c}\uparrow & & \uparrow{\scriptstyle \cong\; J\alpha^{-1}} \\
X & \underset{\mathbf{tr}_c}{\dashrightarrow} & \Sigma^*
\end{array}$$

amounts to

- $\langle\rangle \in \mathbf{tr}_c(x)$ iff $\checkmark \in c(x)$
- $a \cdot s \in \mathbf{tr}_c(x)$ iff $\exists x' \in X.\ \ (a, x') \in c(x) \ \wedge\ s \in \mathbf{tr}_c(x')$

For 

,

$$\mathbf{tr}(y) \;=\; b^* = \{\langle\rangle, b, bb, bbb, \ldots\}$$

- $\mathbf{tr}(y)$ does **not** include infinite words like b^ω.

Example: $T = \mathcal{D}$, $F = 1 + \Sigma \times \_$

- A system $\begin{array}{c} TFX \\ c\uparrow \\ X \end{array}$: **Generative prob. system** [van Glabbeek, Smolka & Steen]
- $\begin{array}{c} 1 + \Sigma \times \Sigma^* \\ [\mathsf{nil}, \mathsf{cons}] \downarrow \cong \\ \Sigma^* \end{array}$: initial F-algebra
- $X \xrightarrow{\mathsf{tr}_c} \mathcal{D}(\Sigma^*)$ by finality.

Finite traces

- A system $\begin{array}{c} \mathcal{D}(1+\Sigma\times X) \\ c\uparrow \\ X \end{array}$

- The finality diagram

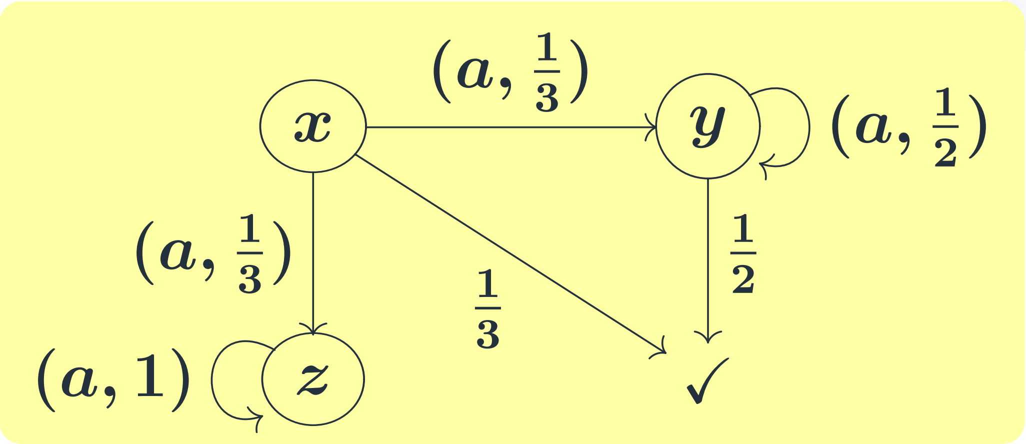
In $\mathcal{K}\ell(\mathcal{D})$

$$\begin{array}{ccc} \mathcal{K}\ell(F)X & \xrightarrow{\mathcal{K}\ell(F)(\mathrm{tr}_c)} & \mathcal{K}\ell(F)\Sigma^* \\ c\uparrow & & \cong\uparrow J\alpha^{-1} \\ X & \xrightarrow[\mathrm{tr}_c]{} & \Sigma^* \end{array}$$

amounts to: $\mathrm{tr}_c(x)$ is a distribution

$$\left[\begin{array}{lcl} \langle\rangle & \mapsto & c(x)(\checkmark) \\ a\cdot\sigma & \mapsto & \sum_{y\in X} c(x)(a,y)\cdot \mathrm{tr}_c(y)(\sigma) \end{array}\right]$$

Finite traces



- $\mathbf{tr}_c(x)$ is a distribution

$$\langle\rangle \mapsto \frac{1}{3} \qquad a \mapsto \frac{1}{3} \cdot \frac{1}{2} \qquad a^2 \mapsto \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} \qquad \cdots$$

- Infinite word a^ω is not in the domain of $\mathbf{tr}_c(x)$
  - Cf. $a^\omega \mapsto 1/3$

Conclusions and future work

Conclusions

- **Generic trace semantics**: coinduction

In $\mathcal{K}\ell(T)$

$$\begin{array}{ccc}
\mathcal{K}\ell(F)X & \xrightarrow{\mathcal{K}\ell(F)(\mathbf{tr}_c)} & \mathcal{K}\ell(F)A \\
{\scriptstyle c}\uparrow & & \uparrow{\scriptstyle \cong\, J\alpha^{-1}} \\
X & \xrightarrow[\mathbf{tr}_c]{} & A
\end{array}$$

- **Initial algebra-final coalgebra coincidence** in a order-enriched settings
- Power of categorical/coalgebraic methods in computer science.

Future work

- Another nondeterminism type:

  combination of $\left\{ \begin{array}{c} \textbf{classical non-det.} \\ \textbf{probability} \end{array} \right\}$

  - Important for system verification: [Vardi, FOCS'85] [Segala, PhD Thesis]
  - Suitable monad/order structure is yet to be found. Cf. [Varacca & Winskel, MSCS to appear]
- Yet another nondeterminism type: monad $\mathcal{PP}$ in [Kupke & Venema, LICS'05].
- **Thank you for your attention!**

  Contact: `www.cs.ru.nl/~ichiro`