Trace Semantics for Coalgebras: a Generic Theory

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- **Trace semantics** is defined for various non-det. systems:
  - different input/output types,
  - different “nondeterminism”: e.g. classical non-det. vs. probability.

- They are instances of one categorical construction:
  - **coinduction in a Kleisli category**

- Demonstrates the abstraction power of **category theory, coalgebras** in particular in computer science!
  - Same mathematical principle hidden behind
  - apparently different constructions
Coalgebraic modelling of a non-det. system, which is suitable for trace semantics:

\[
\begin{array}{ccc}
T & F & X \\
\uparrow & c & \uparrow \\
X & & \\
\end{array}
\]

in \text{Sets},

\begin{itemize}
  \item A monad \( T \) specifies the type of non-det.;
  \item An endofunctor \( F \) specifies the input/output type.
\end{itemize}

Here

\begin{itemize}
  \item the \textbf{monad structure} of \( T \) and
  \item a \textbf{distributive law} \( \pi : FT \Rightarrow TF \)
\end{itemize}

play central roles.
Coalgebraic modelling of a non-det. system, which is suitable for trace semantics:

- A monad $T$ specifies the type of non-det.;
- An endofunctor $F$ specifies the input/output type.

Here

- the monad structure of $T$ and
- a distributive law $\pi : FT \Rightarrow TF$

play central roles.
Main theorem

An initial algebra in \( \mathbf{Sets} \) gives rise to

\[ \square \] an initial algebra, and also

\[ \square \] a final coalgebra,

in a Kleisli category \( \mathcal{K}_\ell(T) \).

[Under some order-theoretic assumptions]

Finality yields the finite trace map: in \( \mathcal{K}_\ell(T) \),

\[
\begin{array}{c}
F X \xrightarrow{\mathcal{K}_\ell(F)(\text{tr}_c)} F A \\
\cong \uparrow J \alpha^{-1} \\
X \xrightarrow{\text{tr}_c} A
\end{array}
\]
The proof of main result
(initial algebra-final coalgebra coincidence)
uses:

- a classic result of limit-colimit coincidence in a
suitably order-enriched setting
  [Smyth & Plotkin, Siam J. Comput., ’82]

IH, Bart Jacobs and Ana Sokolova.
Generic Trace Theory.
To appear in CMCS’06.
The result covers:

- **I/O types** almost all polynomial $F$
- **type of “nondeterminism”:**
  - **lift monad** $\mathcal{L} = 1 + _-$
    systems with non-termination, exception
  - **powerset monad** $\mathcal{P}$
    (classical) non-deterministic systems
  - **subdistribution monad** $\mathcal{D}$
    probabilistic systems
The result is **generic**: generalizing our previous papers

- [IH & Jacobs, CALCO’05] $T = \mathcal{P}$
- [IH & Jacobs, CALCO-jnr] $T = \mathcal{D}$

Order-enriched structure is explicitly used for the first time.
We’d rather spend all time for preliminaries...

- various examples of trace semantics
- monads, distributive laws, Kleisli categories
- construction of
  - initial algebra via initial sequence
  - final coalgebra via final sequence
- Smyth & Plotkin’s limit-colimit coincidence

We go slowly, very slowly, ...
Preliminaries I:
trace semantics
Various semantics for non-det. systems...

Compare two non-deterministic systems.

\[ x \] and \[ y \] are
- different wrt. \textit{bisimilarity}, but
Various semantics for non-det. systems...

Compare two non-deterministic systems.

\[ \text{tr}(x) = \text{tr}(y) = \{ab, ac\}. \]

\( x \) and \( y \) are
- different wrt. bisimilarity, but
- equivalent wrt. trace semantics!
For (classical) non-deterministic systems,

\[
\text{trace} = \text{the set of all possible linear-time behavior}
\]

For (classical) non-deterministic systems,

\[
\text{tr}(y) = b^* = \{\langle\rangle, b, bb, bbb, \ldots \}
\]

\[
\text{tr}(x) = (a + a^2 + a^3 + \cdots) \cdot \text{tr}(y) = \{a^{n+1}b^m \mid n, m \in \mathbb{N}\}
\]
Another “nondeterminism”

Another type of nondeterminism: probabilistic systems

Question: What is the “trace” of $x$?
Another “nondeterminism”

Another type of nondeterminism: probabilistic systems

Question: What is the “trace” of $x$?

Answer: the probability distribution over possible linear-time behavior

$$
\langle \rangle \mapsto \frac{1}{3} \quad a \mapsto \frac{1}{3} \cdot \frac{1}{2} \quad a^2 \mapsto \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} \quad \ldots
$$
Consider a **context-free grammar**.

- **Terminal symbols**: 0, s
- **Non-terminal symbol**: T
- **Generation rules**:
  
  - \( T \Rightarrow 0 \)
  - \( T \Rightarrow sT \)

From \( T \), the following parse trees can be generated:

![Parse trees example](image)

This is the “**trace**” of \( T \).
A trace of (a state of) a non-det. system is:

- For **(classical) non-deterministic** systems, the set of possible linear-time behavior
- For **probabilistic** systems, the probability distribution over possible linear-time behavior

- The input/output type specifies what is a “linear-time behavior”.
Preliminaries II: monads, distributive laws, Kleisli categories
A non-det. system is modelled as a coalgebra

\[ \begin{array}{c}
TFX \\
\uparrow c \\
X
\end{array} \quad \text{in Sets} \]

- A monad \( T \) specifies the type of nondeterminism;
- An endofunctor \( F \) specifies the input/output type.
Non-Det. systems as coalgebras

Examples (Details on blackboard...)

- I/O type: $F = 1 + \Sigma \times _$
- Type of nondeterminism: $T = \mathcal{P}$ (classical non-det.)

- I/O type: $F = 1 + \sum \times _$
- Type of nondeterminism: $T = \mathcal{D}$ (probability)
A non-det. system is modelled as a coalgebra

\[
\begin{array}{c}
T F X \\
\downarrow c \\
X
\end{array}
\]

in \( \text{Sets} \)

- A monad \( T \) specifies the type of nondeterminism;
- An endofunctor \( F \) specifies the input/output type.
A non-det. system is modelled as a coalgebra

\[ \begin{array}{c}
T \\
c \\
F \\
\hline
X \\
\end{array} \]

in \( \text{Sets} \)

- A monad \( T \) specifies the type of nondeterminism;
- An endofunctor \( F \) specifies the input/output type.

“Nondeterminism” is modelled due to

- the monad structure of \( T \), and
- a distributive law \( \pi : FT \Rightarrow TF \).
A non-det. system is modelled as a coalgebra

\[ \mathcal{T} \xrightarrow{c} \mathcal{F} \xrightarrow{X} \text{ in } \mathbf{Sets} \]

- A monad $\mathcal{T}$ specifies the type of nondeterminism;
- An endofunctor $\mathcal{F}$ specifies the input/output type.

“Nondeterminism” is modelled due to
- the monad structure of $\mathcal{T}$, and
- a distributive law $\pi : \mathcal{F}\mathcal{T} \Rightarrow \mathcal{T}\mathcal{F}$.

The **Kleisli category** $\mathcal{Kl}(\mathcal{T})$ of $\mathcal{T}$ turns out to be an appropriate base category.
A non-det. system is modelled as a coalgebra

- A monad $T$ specifies the type of nondeterminism;
- An endofunctor $F$ specifies the input/output type.

“Nondeterminism” is modelled due to

- the monad structure of $T$, and
- a distributive law $\pi : FT \Rightarrow TF$.

The **Kleisli category** $\mathcal{Kl}(T)$ of $T$ turns out to be an appropriate base category.
A monad

\[ T : C \to C \]

is an endofunctor with additional structures: for each object \( X \),

\[ X \xrightarrow{\eta_X} TX \]  
unit

\[ T^2 X \xrightarrow{\mu_X} TX \]  
multiplication

such that:

- \( \eta : \text{id} \Rightarrow T \) and \( \mu : T^2 \Rightarrow T \) are natural transformations;
- they are compatible in the sense:

\[ T X \xrightarrow{T \eta_X} T^2 X \xleftarrow{\eta_T X} T X \]

unit law

\[ T^3 X \xrightarrow{T \mu_X} T^2 X \xrightarrow{\mu_X} TX \]

assoc. law

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Monads

- Generalization of notion of **monoids**.

- Examples of our interest: (details on blackboard)
  - **Lift monad** \( \mathcal{L}X = 1 + X \)
  - **Powerset monad** \( \mathcal{P}X = \{ X' \mid X' \subseteq X \} \)
  - **Subdistribution monad** \( \mathcal{D}X = \{ d : X \rightarrow [0, 1] \mid \sum_{x \in X} d(x) \leq 1 \} \)
More generally, an adjunction $\mathcal{L} \dashv \mathcal{R}$ yields a monad $RL : \mathcal{C} \to \mathcal{C}$.

Hence a functor

$$X \mapsto \left[ \text{Free ("term") algebra with variables from } X \right]$$

comes with a monad structure.
More generally, an adjunction $L \xrightarrow{\delta} R$ yields a monad $RL : \mathcal{C} \rightarrow \mathcal{C}$.

Hence a functor $X \mapsto \left[ \text{Free ("term") algebra with variables from } X \right]$ comes with a monad structure.

The converse is also true: every monad arises from an adjunction

- **Eilenberg-Moore** construction (biggest, final)
- **Kleisli** construction (smallest, initial)
**Kleisli categories**

**Kleisli category** $\mathcal{K}\ell(T)$ for $T : \mathcal{C} \to \mathcal{C}$, a monad.

- **Object**
  
  \[
  X \in \mathcal{K}\ell(T) \quad X \in \mathcal{C}
  \]

- **Arrow**
  
  \[
  X \xrightarrow{f} Y \quad X \xrightarrow{f} TY \quad X \xrightarrow{f} TY \xrightarrow{g} Z \quad X \xrightarrow{f} TY \xrightarrow{Tg} T^2Z \xrightarrow{\mu_Z} Z
  \]

- **Composition**
  
  \[
  \begin{align*}
  X & \xrightarrow{f} TY & & \xrightarrow{g \circ f} TZ & & \xrightarrow{Tg} T^2Z \\
  \end{align*}
  \]

- **Id. arrow**
  
  \[
  X \xrightarrow{\eta_X} TX \quad X \xrightarrow{id} X \quad X \xrightarrow{id} X \xrightarrow{\eta_X} TX \quad X \xrightarrow{\eta_X} TX \quad X \xrightarrow{\eta_X} TX
  \]
Examples: $T = \mathcal{L}, \mathcal{P}, \mathcal{D}$. On the blackboard.

There is an **adjunction**:

```
X \xrightarrow{\eta_Y \circ f} Y
```

\[X \xrightarrow{f} Y\]

\[\mathcal{KL}(T)\]

\[\begin{array}{c}
\mathcal{J} \\
\downarrow \bigcirc \\
\mathcal{K}
\end{array}\]

\[\begin{array}{c}
X \xrightarrow{g} Y \\
\downarrow \bigcirc \\
TX \xrightarrow{\mu_Y \circ Tg} TY
\end{array}\]

which yields the monad $T$.

Moreover, this Kleisli adjunction is the initial one among those which yield $T$. 

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Motivation

A system of the form \( TX \) can be iterated:

\[
\begin{align*}
X &\xrightarrow{c^n} TX \\
&\xrightarrow{Tc} T^2X
\end{align*}
\]

In \( \text{Sets} \)

\[
\begin{align*}
T X &\xrightarrow{c} X \\
X &\xrightarrow{\mu X} TX
\end{align*}
\]

In \( \mathcal{K}(T) \)

How about \( TFX \) which is of our interest?
A **distributive law** is a natural transformation

$$\pi : FT \Rightarrow TF$$

which is compatible with the monad structure of $T$.

- It **swaps** $T$ over $F$.
- The direction is opposite in [Bartels, PhD thesis], since:
  - Here the base category is Kleisli,
  - In [Bartels, PhD thesis] the base category is Eilenberg-Moore.
  - Duality in a suitable 2-categorical sense.
If a system comes with a distributive law $\pi : FT \Rightarrow TF$, we can define \textbf{n-th iteration} of $c$:

\[
\begin{array}{c}
TF^n X \\
\downarrow \quad c^n \\
X
\end{array}
\]

\begin{itemize}
\item Construction on the blackboard.
\item Example: $T = \mathcal{P}$ and $F = 1 + \Sigma \times \_$. 
\end{itemize}
Another view, in $\mathcal{K}\ell(T)$

- A distr. law $FT \Rightarrow TF$ lifts $F$

\[
\begin{array}{ccc}
\mathcal{K}\ell(T) & \xrightarrow{\mathcal{K}\ell(F)} & \mathcal{K}\ell(T) \\
J & \xrightarrow{J} & \downarrow J \\
\text{Sets} & \xrightarrow{\text{Sets}} & \text{Sets} \\
K & \xleftarrow{\xrightarrow{\downarrow}} & J
\end{array}
\]

Construction on the blackboard.

- A system is now in the Kleisli category $\mathcal{K}\ell(T)$

\[
\begin{array}{ccc}
TFX & \xrightarrow{c} & X \\
\text{in Sets} & \text{is:} & \mathcal{K}\ell(F)X & \xrightarrow{c} & X \\
\text{in } \mathcal{K}\ell(T)
\end{array}
\]

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A non-det. system is as a coalgebra

Its “nondeterminism” (suitable for trace semantics) is due to

- unit \( \eta \) of \( T \) (“singleton”)
- multiplication \( \mu \) of \( T \) (“union”)
- distr. law \( \mathcal{F}T \Rightarrow TF \)

allowing for iteration of the system

We move to Kleisli category where the system is
Preliminaries III: Initial/final sequences

Overview
Initial sequence
Initial sequence, in $\textbf{Sets}$
Final sequence
Summary

Preliminaries IV: Limit-colimit coincidence
Main technical result
Application of the main result

Conclusions and future work
We sketch: generic construction of
- initial $F$-algebra via **initial sequence**
- final $F$-coalgebra via **final sequence**

for $F : C \to C$.

Assumptions are categorical.
For initial sequence construction,
- existence of initial object $0 \in C$;
- existence of certain colimits in $C$;
- $F$ preserves such colimits.

For illustration the example is $C = \text{Sets}$. Later applied to $C = \mathcal{K}_\ell(T)$. 

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Initial sequence

\[
\begin{align*}
0 & \rightarrow F^0 \rightarrow F^2 \rightarrow F^4 \rightarrow F^6 \rightarrow \ldots
\end{align*}
\]
Initial sequence

\[
\begin{align*}
0 & \xrightarrow{\alpha_0} F^0 & \xrightarrow{\alpha_1} F^2 & \xrightarrow{\alpha_2} F^4 & \xrightarrow{\alpha_3} F^6 & \cdots
\end{align*}
\]
Assume: $F$ preserves the upper colimit.
Assume: $F$ preserves the upper colimit.
\[ \alpha : FA \cong A \] is an initial algebra.
Construction of $f$ in

\[ FA \xrightarrow{\alpha} AX \xrightarrow{f} X \]

induces a cocone over initial sequence:

\[ F^{n+1}0 \xrightarrow{F\beta_n} FX \]

\[ F^{n+1}0 \xrightarrow{\beta_{n+1}} FX \]

\[ \beta_0 \quad \beta_1 \quad \beta_2 \quad \beta_3 \]

\[ 0 \xrightarrow{?} F0 \xrightarrow{F?} F^20 \xrightarrow{F^2?} F^30 \xrightarrow{F^3?} \cdots \xrightarrow{X} \]
Initial sequence

Construction of $f$ in

$$\begin{align*}
FA & \xrightarrow{\cdot} FX \\
\alpha & \downarrow \cong \\
A & \xrightarrow{\cdot} X
\end{align*}$$
Initial sequence, in \textit{Sets}

\[ F = 1 + \Sigma \times _, \quad \text{where } 1 = \{\checkmark\} \text{ and } \Sigma = \{a\}. \]

\textbf{Question} What is an initial algebra?

\[
\begin{array}{c}
0 \xrightarrow{?} F0 \xrightarrow{?} F^20 \xrightarrow{?} F^30 \xrightarrow{?} \ldots \\
\text{initial obj.}
\end{array}
\]
Initial sequence, in \textit{Sets}

\[ F = 1 + \Sigma \times \_ , \quad \text{where} \ 1 = \{ \checkmark \} \ \text{and} \ \Sigma = \{ a \} . \]
Initial sequence, in \textit{Sets}.

\[ F = 1 + \Sigma \times _, \text{ where } 1 = \{ \checkmark \} \text{ and } \Sigma = \{ a \}. \]

\[ \begin{array}{c}
0 \xrightarrow{?} F_0 \xrightarrow{F?} F^2_0 \xrightarrow{F^2?} F^3_0 \xrightarrow{F^3?} \ldots \\
\downarrow 1 \quad \downarrow 1 + \Sigma \quad \downarrow 1 + \Sigma + \Sigma^2 \\
\checkmark \quad \checkmark \quad \checkmark \quad \ldots \\
\downarrow a \quad \downarrow a \quad \ldots \\
\downarrow aa \quad \ldots 
\end{array} \]
Initial sequence, in \textit{Sets}

\[ F = 1 + \Sigma \times \_ , \text{ where } 1 = \{ \checkmark \} \text{ and } \Sigma = \{ a \} . \]

\[
\begin{array}{cccccc}
& & & & & A \\
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \text{colimit} \\\n0 & \rightarrow & F_0 & \rightarrow & F^2_0 & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & 1 + \Sigma & 1 + \Sigma + \Sigma^2 & & & & \cdots \\
\checkmark & \checkmark & \checkmark & & & & \cdots \\
\rightarrow & a & \rightarrow & \rightarrow & a & \rightarrow & \cdots \\
\rightarrow & a & \rightarrow & aa & \rightarrow & \cdots \\
\end{array}
\]
Initial sequence, in \( \text{Sets} \)

\[
F = 1 + \Sigma \times _, \quad \text{where } 1 = \{\checkmark\} \text{ and } \Sigma = \{a\}.
\]

\[
1 + \Sigma + \Sigma^2 + \cdots \rightarrow A
\]

\[
\begin{align*}
F_0 & \xrightarrow{\alpha_0} F_0 \\
F_1 & \xrightarrow{\alpha_1} F^2_0 \\
F_2 & \xrightarrow{\alpha_2} F^3_0 \\
F_3 & \xrightarrow{\alpha_3} \cdots
\end{align*}
\]

\[
\begin{align*}
\checkmark & \xrightarrow{\ \ } \checkmark \\
a & \xrightarrow{\ \ } a \\
aa & \xrightarrow{\ \ } \cdots
\end{align*}
\]

\[
\text{colimit} = \left\{ \begin{array}{c}
\text{coproduct, then} \\
\text{coequalizer}
\end{array} \right\} \quad \text{in } \text{Sets} \quad \text{union}
\]
Initial sequence, in \( \text{Sets} \)

\[
F = 1 + \Sigma \times _, 
\text{where} \ 1 = \{\checkmark\} \text{ and } \Sigma = \{a\}.
\]

\[
\mathbb{N} \xrightarrow{\sim} 1 + \Sigma + \Sigma^2 + \cdots \rightarrow A
\]

\[
0 \xrightarrow{\text{?}} F_0 \xrightarrow{\text{?}} F^2_0 \xrightarrow{\text{?}} F^3_0 \cdots
\]

\[
\begin{align*}
F^n_0 &= \{\text{terms with depth } \leq n\}
\end{align*}
\]
Final sequence

Dual of initial sequence...

Final obj.

\[ 1 \leftarrow F_1 \leftarrow F \leftarrow F^2_1 \leftarrow F^2 \leftarrow F^3_1 \leftarrow F^3 \ldots \]
Final sequence

Dual of initial sequence...

1 \overset{!}{\rightarrow} F_1 \overset{!}{\rightarrow} F_2 \overset{!}{\rightarrow} F_3 \overset{!}{\rightarrow} \cdots

\zeta_0 \quad \zeta_1 \quad \zeta_2 \quad \zeta_3

limit

Z
Final sequence

Dual of initial sequence...
Assume: $F$ preserves the upper limit.
Dual of initial sequence...
Assume: $F$ preserves the upper limit.

Final sequence
Dual of initial sequence...

\[\zeta : Z \cong FZ\] is a final coalgebra.
Summary

- Initial sequence and final sequence.
- In \textbf{Sets} the constructions coincide with familiar structural (co)induction.
- However, the constructions are purely categorical.
- They work also in other categories!
- Later applied in \( \mathcal{K}_\ell(T) \).
- Too much time left? Final sequence in \textbf{Sets}.
Preliminaries IV:
Limit-colimit coincidence
Taking colimit of initial sequence seems taking union of an increasing chain
Taking colimit of initial sequence seems taking union of an increasing chain.

In a certain setting it is!

- $O$-limits (order-theoretic notion) coincide with limits;
- $O$-colimits coincide with colimits.

Obvious duality, coincidence

$O$-limit $\iff$ limit $\iff$ $O$-colimit $\iff$ colimit

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Taking colimit of initial sequence seems taking \textbf{union} of an \textbf{increasing chain}

In a certain setting it is!

\begin{itemize}
  \item \textbf{O}-limits (order-theoretic notion) coincide with limits;
  \item \textbf{O}-colimits coincide with colimits.
\end{itemize}

\begin{itemize}
  \item \textbf{O}-limit \hspace{2cm} \textbf{O}-colimit
  \item \hspace{1cm} obvious duality, coincidence
  \item \hspace{1cm} limit-colimit coincidence
\end{itemize}

[Smyth & Plotkin, SIAM J. Comp., 1982]
Each homset is a dcpo:

- order between arrows $X \xrightarrow{\sqcup} Y$ and
- supremum of increasing $\omega$-chain:

$$\sqcup_{n<\omega} f_n$$

Composition preserves suprema:

$$X \xrightarrow{\sqcup f_n} Y \xrightarrow{\sqcup g_n} Z = X \xrightarrow{\sqcup (g_n \circ f_n)} Z$$

Examples: $\mathcal{K}\ell(T)$ for $T = \mathcal{L}, \mathcal{P}, \mathcal{D}$ (on blackboard)
Embedding-projection pairs

In a DCpo-enriched category,

\[
\begin{array}{ccc}
X & \xrightarrow{e} & Y \\
\downarrow{p} & & \Rightarrow \\
X & \xleftarrow{p} & Y
\end{array}
\]

s.t. \[ p \circ e = \text{id} \quad \text{and} \quad e \circ p \sqsubseteq \text{id} \]

- Diagrammatically,

\[
\begin{array}{ccc}
X & \xrightarrow{e} & Y \\
\downarrow{p} & & \Rightarrow \\
\downarrow{\text{id}} & & \Rightarrow \\
X & \xleftarrow{p} & Y
\end{array}
\]

- \( e \) is mono and \( p \) is epi. Both are split.

- \( p \) is the smallest left-inverse of \( e \)
- \( e \) is the smallest right-inverse of \( p \)

Hence corresponding emb./proj. is unique:

\((e, e^P)\) and \((p^E, p)\).

- Intuition?
DCpo-enriched. Each $e_n$ is an embedding.
DCpo-enriched. Each $e_n$ is an embedding.

Each $\alpha_n$ is also an embedding.
DCpo-enriched. Each $e_n$ is an embedding.

- Each $\alpha_n$ is also an embedding.
- $\{ A \xrightarrow{\alpha_n^P} X_n \xrightarrow{\alpha_n} A \}_{n<\omega}$ is increasing.
$\mathbf{DCpo}$-enriched. Each $e_n$ is an embedding.

Each $\alpha_n$ is also an embedding.

$\{ A \xrightarrow{\alpha_n^P} X_n \xrightarrow{\alpha_n} A \}_{n<\omega}$ is increasing.

Its supremum is $A \xrightarrow{id} A$. 

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DCpo-enriched. Each $e_n$ is an embedding.

Each $\alpha_n$ is also an embedding.

\[ \{ A \xrightarrow{\alpha_n^P} X_n \xrightarrow{\alpha_n} A \}_{n<\omega} \] is increasing.

Its supremum is $A \xrightarrow{id} A$. 

Notion of $O$-colimit!
\textbf{Definition [O-colimit]}

- Each $\alpha_n$ is an embedding.
- The sequence $\{ A \xrightarrow{\alpha_n^P} X_n \xrightarrow{\alpha_n} A \}_{n<\omega}$ is increasing.
- Its supremum is $\xrightarrow{\text{id}} A$.

\textbf{Theorem [Smyth \& Plotkin]}

- An O-colimit is a colimit.
- Conversely, a colimit of $X_0 \overset{e_0}{\rightarrow} X_1 \overset{e_1}{\rightarrow} \cdots$ is an O-colimit.
**Definition [O-limit]**

Each $\beta_n$ is a projection.

$\{ A \xrightarrow{\beta_n} X_n \xleftarrow{\beta_n^{E}} A \}_{n<\omega}$ is increasing.

Its supremum is $A \xrightarrow{id} A$.

**Theorem [Smyth & Plotkin]**

- An O-limit is a limit.
- Conversely, a limit of $X_0 \leftrightarrow X_1 \leftrightarrow \cdots$ is an O-limit.
Limit-colimit coincidence

**Theorem [Smyth & Plotkin]**

DCpo-enriched. Each $e_n$ is an embedding.

If and only if...
Theorem [Smyth & Plotkin]

$D CP o$-enriched. Each $e_n$ is an embedding.
Proof: Limit-colimit coincidence

\textbf{DCpo-enriched. Each }e_n\textbf{ is an embedding.}

\begin{tikzpicture}
  \node (X0) at (0,0) {$X_0$};
  \node (X1) at (2,0) {$X_1$};
  \node (X2) at (4,0) {$X_2$};
  \node (X3) at (6,0) {$X_3$};
  \node (A) at (8,0) {$A$};
  \node (colimit) at (8,-1) {colimit};

  \draw[->] (X0) to node [above] {$e_0$} (X1);
  \draw[->] (X1) to node [above] {$e_1$} (X2);
  \draw[->] (X2) to node [above] {$e_2$} (X3);
  \draw[->] (X3) to node [above] {$e_3$} (A);

  \draw[->] (X0) to node [left] {$\alpha_0$} (A);
  \draw[->] (X1) to node [left] {$\alpha_1$} (A);
  \draw[->] (X2) to node [left] {$\alpha_2$} (A);
  \draw[->] (X3) to node [left] {$\alpha_3$} (A);

  \draw[->] (A) to node [above] {$\cdots$} (A);
\end{tikzpicture}
Proof: Limit-colimit coincidence

$DCpo$-enriched. Each $e_n$ is an embedding.

Colimit of $\omega$-chain of embeddings consists of embeddings.
Proof: Limit-colimit coincidence

DCpo-enriched. Each $e_n$ is an embedding.

\begin{itemize}
  \item \[
    \{ \alpha^P_n : A \to X_n \xrightarrow{\alpha_n} A \}_{n < \omega}
  \]
  is increasing and its supremum is $A \xrightarrow{id} A$.
  
  \item Colimit $\iff$ O-colimit.
\end{itemize}
Proof: Limit-colimit coincidence

DCpo-enriched. Each $e_n$ is an embedding.

\[ \{ A \xrightarrow{\alpha_n^P} X_n \xrightarrow{\alpha_n} A \}_{n<\omega} \] is increasing and its supremum is $A \xrightarrow{id} A$. 
Proof: Limit-colimit coincidence

DCpo-enriched. Each $e_n$ is an embedding.

$\{ A \xrightarrow{\alpha_n^P} X_n \xrightarrow{\left(\alpha_n^P\right)^E} A \}_{n<\omega}$ is increasing and its supremum is $A \xrightarrow{id} A$.

$\alpha_n = \left(\alpha_n^P\right)^E$. 
Proof: Limit-colimit coincidence

DCpo-enriched. Each $e_n$ is an embedding.

\[ \{ A \xrightarrow{\alpha_n^P} X_n \xrightarrow{(\alpha_n^P)^E} A \}_{n<\omega} \text{ is increasing and its supremum is } A \xrightarrow{id} A. \]

Obvious duality between $O$-colimits and $O$-limits!
Proof: Limit-colimit coincidence

\( \mathbf{DCpo} \)-enriched. Each \( e_n \) is an embedding.

\[
\begin{align*}
X_0 & \xleftarrow{e_0^P} X_1 & \xleftarrow{e_1^P} X_2 & \xleftarrow{e_2^P} X_3 & \cdots \\
\alpha_0^P & & \alpha_1^P & & \alpha_2^P & & \alpha_3^P
\end{align*}
\]

- Limit \( \iff \) O-limit.
- Q.E.D.
Summary

- Base category will be $\mathcal{K}\ell(T)$.
- The chain will be initial/final sequences.
- Implies **initial alg.-final coalg. coincidence!**
Main technical result
Initial algebra-final coalgebra coincidence

A system is $T F X \xrightarrow{c} X$ in $\text{Sets}$, i.e. $F X \xrightarrow{c} X$ in $\mathcal{K}(T)$.

**Main theorem**

If $\alpha : FA \cong A$ in $\text{Sets}$: initial algebra. Then

- $J \alpha : \text{initial } \mathcal{K}(F)$-algebra;
- $J \alpha^{-1} : \text{final } \mathcal{K}(F')$-coalgebra.
Initial algebra-final coalgebra coincidence

A system is

\[ TFX \xrightarrow{c} X \text{ in Sets} \]

\[ FX \xrightarrow{c} X \text{ in } \mathcal{KL}(T) \]

[i.e.

\[ FA \xrightarrow{\alpha} A \text{ in Sets} \]

\[ FA \xleftarrow{J \alpha} A \text{ in } \mathcal{KL}(T) \]

\[ FA \xrightarrow{J \alpha^{-1}} A \text{ in } \mathcal{KL}(T) \]

Main theorem

\[ FA \xrightarrow{\alpha} A \text{ in Sets} \]

: initial algebra. Then

\[ FA \xleftarrow{J \alpha} A \text{ in } \mathcal{KL}(T) \]

: initial \( \mathcal{KL}(F) \)-algebra;

\[ FA \xrightarrow{J \alpha^{-1}} A \text{ in } \mathcal{KL}(T) \]

: final \( \mathcal{KL}(F') \)-coalgebra.

\[ \square \]

[monads, distributive laws, Kleisli categories]
Distributive law $FT \Rightarrow TF$.  

Available for

- “shapely” functors $F$,

$$F, G, F_i ::= \text{id} \mid \Sigma \mid F \times G \mid \bigsqcup_{i \in I} F_i,$$

and

- commutative monads $T$. 

Assumptions

- Kleisli category $\mathcal{K}_\ell(T)$ is $\mathbf{DCpo}_{\bot}$-enriched.

- Each homset has the minimum:

$$X \xrightarrow{f \bowtie} Y \xleftarrow{\bot_{x,y}} Z$$

- Composition in $\mathcal{K}_\ell(T)$ is left-strict:

$$X \xrightarrow{f} Y \xrightarrow{\bot_{y,z}} Z = X \xrightarrow{\bot_{x,z}} Z$$

- Lifted $\mathcal{K}_\ell(F) : \mathcal{K}_\ell(T) \to \mathcal{K}_\ell(T)$ is monotonic:

$$X \xrightarrow{f \bowtie} Y \Rightarrow FX \xrightarrow{\bot_{\mathcal{K}_\ell(F)(f)}} FY$$

- True for $T = \mathcal{L}, \mathcal{P}, \mathcal{D}$. 

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Proof: sketch

Initial sequence construction in \( \text{Sets} \).

\[
K \ell(T)
\]

Let's map by \( J \) in \( J \xrightarrow{\sim} K \).

\[
\begin{array}{c}
\text{sets} \\
\end{array}
\]
Proof: sketch

\[ K_\ell(T) \]

Mapped by \( J \) in \( J \leftarrow K \leftarrow \text{Sets} \).

Left adjoint preserves colimits.

We shall show:

- The sequence is the initial sequence for \( K_\ell(F) \).
- The upper cone is mapped by \( K_\ell(F) \) to the lower one.
Proof: sketch

This proves that $\frac{FA}{J\alpha \cong J\alpha^{-1}}$ is an initial $\mathcal{K}\ell(F)$-algebra.

All arrows are embeddings. We take the corresponding projections.
Proof: sketch

- We used **Limit-colimit coincidence**!
- We show:
  - The sequence is the final sequence:
  - The upper cone is mapped by $\mathcal{KL}(F)$ to the lower one.
Proof: sketch

In a $\mathbf{DCpo}_\bot$-enriched category, an initial object is final as well.

This proves that $J_\alpha^{-1} \cong FA$ is a final $\mathcal{Kl}(F)$-algebra. Q.E.D.
Proof: in detail

Initial sequence construction in \( \text{Sets} \).

\[ \mathcal{K} \ell(T) \]

Let’s map by \( J \) in \( J \xrightarrow{-} K \).

\( \xrightarrow{\text{Sets}} \)
Proof: in detail

\[ \text{in } \mathcal{K}(T) \]

\[ J_0 \xrightarrow{J?} JF0 \xrightarrow{JF?} JF^20 \xrightarrow{JF^2?} \cdots \]

\[ J\alpha \cong J\alpha^{-1} \]

- Mapped by \( J \) in \( J \xleftarrow{\rightarrow} K \).
- Left adjoint preserves colimits.
Proof: in detail

in \( \mathcal{K}(T) \)

\[
\begin{array}{ccc}
0 & \xrightarrow{J?} & F0 \\
& J? & \downarrow \quad JF? \\
& & F^20 \\
& & JF^2? \\
& & \vdots \\
& & J\alpha^{-1}
\end{array}
\]

\(J\alpha_0 \xrightarrow{J?} \cdots \xrightarrow{J?} \xrightarrow{J?} F^20 \xrightarrow{JF^2?} A\) again colimit

\(X \xrightarrow{J} X\)

\(\mathcal{K}(T)\)
Proof: in detail

- $J$ (left-adjoint) preserves initial object $0$.
- $\mathcal{K}(T) \xrightarrow{\mathcal{K}(F)} \mathcal{K}(T)$

<table>
<thead>
<tr>
<th>$J$ (left-adjoint) preserves initial object 0.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{K}(T) \xrightarrow{\mathcal{K}(F)} \mathcal{K}(T)$</td>
</tr>
</tbody>
</table>

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Proof: in detail

\[ \begin{array}{c}
\text{in } \mathcal{Kl}(T) \\
0 \rightarrow \mathcal{Kl}(F) \rightarrow \mathcal{Kl}(F)^2 \rightarrow \cdots \rightarrow \mathcal{Kl}(F)^n \rightarrow A \\
\end{array} \]

\[ \begin{array}{c}
\text{init seq.} \\
J_0 \quad J_1 \quad J_2 \\
? \rightarrow \mathcal{Kl}(F) \quad \mathcal{Kl}(F)^2 \quad \cdots \\
\text{again colimit} \\
\end{array} \]

\[ \begin{array}{c}
\text{colimit} \\
A \rightarrow \text{FA} \\
\end{array} \]

\[ \begin{array}{c}
\mathcal{Kl}(T) \xrightarrow{\mathcal{Kl}(F)} \mathcal{Kl}(T) \\
J \uparrow \quad \uparrow J \\
\text{Sets} \xrightarrow{F} \text{Sets} \\
\end{array} \]

This proves that \( \frac{FA}{J_0 \sim A} \) is an initial \( \mathcal{Kl}(F) \)-algebra.
Proof: in detail

- Arrows in the initial sequence are embeddings.
Hence arrows in colimits are also embeddings.

Colimits are $O$-colimits.

Let’s take the corresponding projections...
We need to show:

- The sequence is the final sequence:
- The upper cone is mapped by $\mathcal{K}(F)$ to the lower one.
Proof: in detail

- 0 is also final in $\mathcal{K}\ell(T)$.
  - **Existence** $X \xrightarrow{\perp} 0$.
  - **Uniqueness** $X \xrightarrow{f} 0 = X \xrightarrow{f} 0 \xrightarrow{\text{id}} 0 = X \xrightarrow{f} 0 \xrightarrow{\perp} 0 = X \xrightarrow{\perp} 0$. 
Proof: in detail

in $\mathcal{K}\ell(T)$

0 $\overset{!}{\longrightarrow}$ \(\mathcal{K}\ell(F)\) 0 $\overset{!}{\longrightarrow}$ \(\mathcal{K}\ell(F)\) 0 $\overset{!}{\longrightarrow}$ \(\mathcal{K}\ell(F)^2\) 0 $\overset{!}{\longrightarrow}$ $\cdots$ \(J_\alpha^{-1}\) $\cong$ \(J_\alpha\)

- O-limit $\iff$ limit.
- This proves that $J_\alpha^{-1} \cong A$ is a final $\mathcal{K}\ell(F')$-algebra.
- Q.E.D.
Application of the main result
Corollary of the finality result

Let $F A \xrightarrow{\alpha \cong} A$ in $\text{Sets}$ be an initial $F$-algebra.

For $c : T F X \rightarrow X$ in $\text{Sets}$, we have unique $X \xrightarrow{\text{tr}_c} TA$ in $\text{Sets}$ such that $\text{In } \mathcal{K}_\ell(T) \xrightarrow{\mathcal{K}_\ell(F)(\text{tr}_c)} \mathcal{K}_\ell(F) A$ s.t. $c \cong J \alpha^{-1}$.
Finite traces

Example: \( T = \mathcal{P} \)
\( F = 1 + \Sigma \times \_
\)

- A system:
  \[
  \begin{array}{c}
  T \downarrow^c \\
  X
  \end{array}
  \]

  - LTS with explicit termination, or
  - Nondeterministic automaton

\( 1 + \Sigma \times \Sigma^* \)

- \([\text{nil, cons}] \cong \) initial \( F \)-algebra

\( \Sigma^* \)

\( X \xrightarrow{\text{tr}_c} \mathcal{P}(\Sigma^*) \) by finality.
The diagram of finality

\[
\begin{array}{ccc}
\text{In } \mathcal{K}_\ell(\mathcal{P}) & \rightarrow & \mathcal{K}_\ell(F)(\text{tr}_c) \\
\mathcal{K}_\ell(F)X & \rightarrow & \mathcal{K}_\ell(F)\Sigma^* \\
\mathcal{K}_\ell(F) & \cong & J_\alpha^{-1} \\
\text{tr}_c & \rightarrow & \Sigma^*
\end{array}
\]

amounts to

\[
\begin{align*}
\langle \rangle & \in \text{tr}_c(x) \quad \text{iff} \quad \checkmark \in c(x) \\
\alpha \cdot s & \in \text{tr}_c(x) \quad \text{iff} \quad \exists x' \in X. \quad (a, x') \in c(x) \land s \in \text{tr}_c(x')
\end{align*}
\]
For $x \xrightarrow{a} y \xrightarrow{} b$, we have:

$$\text{tr}(y) = b^* = \{\langle\rangle, b, bb, bbb, \ldots\}$$

$\text{tr}(y)$ does not include infinite words like $b^\omega$. 
Finite traces

Example: \( T = D \)
\[
F = 1 + \Sigma \times _
\]

- A system \( TFX \):

\[
\begin{array}{c}
TFX \\
\uparrow c \\
X
\end{array}
\]

Generative prob. system

[van Glabbeek, Smolka & Steen]

- \( 1 + \Sigma \times \Sigma^* \)

\[
[\text{nil, cons}] \cong \Sigma^*
\]

- initial \( F \)-algebra

- \( X \xrightarrow{\text{tr}_c} D(\Sigma^*) \) by finality.
Finite traces

A system

\[
\mathcal{D}(1 + \Sigma \times X) \\
\begin{array}{c}
\mathcal{D}(1 + \Sigma \times X) \\
\uparrow c \\
\downarrow X
\end{array}
\]

The finality diagram

\[
\begin{array}{cccccccccccccc}
\text{In } \mathcal{KL}(\mathcal{D}) & \mathcal{KL}(F) X & \mathcal{KL}(F)(\text{tr}_c) & \mathcal{KL}(F) \Sigma^* \\
\uparrow c & \downarrow & \uparrow & \Rightarrow \\
X & \text{tr}_c & \downarrow & \Sigma^*
\end{array}
\]

amounts to:

\[
\begin{align*}
\text{tr}_c(x) \text{ is a distribution} \\
\left[
\begin{array}{c}
\langle \rangle \\
a \cdot \sigma
\end{array}
\right] & \mapsto \\
\left[
\begin{array}{c}
\mathcal{D}(x)(\checkmark) \\
\sum_{y \in X} \mathcal{D}(x)(a, y) \cdot \text{tr}_c(y)(\sigma)
\end{array}
\right]
\]

Ichiro Hasuo, RU Nijmegen – 58 / 62
Finite traces

- $\textbf{tr}_c(x)$ is a distribution

\[
\langle \rangle \mapsto \frac{1}{3} \quad a \mapsto \frac{1}{3} \cdot \frac{1}{2} \quad a^2 \mapsto \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} \quad \ldots
\]

- Infinite word $a^\omega$ is not in the domain of $\textbf{tr}_c(x)$

□ Cf. $a^\omega \mapsto 1/3$
Conclusions and future work
Conclusions

- **Generic trace semantics**: coinduction

  \[
  \begin{align*}
  \text{In } \mathcal{K}\ell(T) & \quad \mathcal{K}\ell(F)(tr_c) \\
  \mathcal{K}\ell(F)(X) & \to \mathcal{K}\ell(F)(A) \\
  \overset{c}{\uparrow} & \quad \overset{\cong}{\uparrow} J_{\alpha^{-1}} \\
  X & \to A
  \end{align*}
  \]

- **Initial algebra-final coalgebra coincidence** in an order-enriched setting

- **Power of categorical/coalgebraic methods in computer science.**
Another nondeterminism type:

- combination of classical non-det. probability

- Important for system verification: [Vardi, FOCS’85] [Segala, PhD Thesis]

- Suitable monad/order structure is yet to be found. Cf. [Varacca & Winskel, MSCS to appear]

Yet another nondeterminism type:
monad \( PP \) in [Kupke & Venema, LICS’05].

Thank you for your attention!
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