Freyd is Kleisli, for Arrows

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Introduction

- Interfaces for ***structured computations*** (as opposed to ***pure functions***):
  - ☐ ***monads*** [Moggi'91] for structured output
  - ☐ ***comonads*** for structured input
  - ☐ ***(monad + comonad + distr. law)*** for structured input/output

- ***Arrows*** [Hughes'00] generalize them

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  All come with notions of

  - ☐ Kleisli categories
  - ☐ Eilenberg-Moore algebras
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  All come with notions of
    - ☐ Kleisli categories
    - ☐ Eilenberg-Moore algebras
- ***A**rrows* [Hughes'00] generalize them

Question

$$\text{What are } \left\{ \begin{array}{c} \text{Kleisli categories} \\ \text{(Eilenberg-Moore) algebras} \end{array} \right\} \text{ for Arrows?}$$

Introduction

- ***Kleisli*** for Arrows:

  Freyd categories [Power, Robinson, Thielecke]

☐

$$\begin{tikzcd}
\{\text{Arrows on } \mathbb{C}\} \arrow[d, "\cong"] \\
\{\text{Freyd categories on } \mathbb{C}\}
\end{tikzcd}$$

[Heunen&Jacobs, MFPS'06]

Introduction

- ***Kleisli*** for Arrows:

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  ☐

$$\begin{tikzcd} \{\text{Arrows on } \mathbb{C}\} \arrow[d, "\textbf{\textit{Kleisli!}}\ \text{[This paper]}"', "\cong\ \text{[Heunen\&Jacobs, MFPS'06]}"] \\ \{\text{Freyd categories on } \mathbb{C}\} \end{tikzcd}$$

  ☐ 2-categorical characterization (Cf. [Street'72])

Introduction

- ***Kleisli*** for Arrows:

  Freyd categories [Power, Robinson, Thielecke]

  ☐ **Kleisli!** [This paper]

$$\begin{tikzcd} \{\text{Arrows on } \mathbb{C}\} \arrow[d, "\cong"] \\ \{\text{Freyd categories on } \mathbb{C}\} \end{tikzcd}$$

  [Heunen&Jacobs, MFPS'06]

  ☐ 2-categorical characterization (Cf. [Street'72])

- ***(Eilenberg-Moore) algebras*** for Arrows:

  our notion is comparable to monad-algebras $T \Downarrow \mathrm{id}$

  ☐ Not carried by specific object(s)

Introducing Arrows

Use of monads [Moggi'91]

$\mathbb{C}$: category of types and pure functions. $\mathbb{C} = \mathbf{Sets}, \mathbf{Cpo}$, etc.

- ***Monad*** $\boldsymbol{T} : \mathbb{C} \longrightarrow \mathbb{C}$ is a functor equipped with:
  - ☐ ***unit*** $X \xrightarrow{\eta_X} TX$ for each X
  - ☐ ***extension*** $(X \xrightarrow{f} TY) \xmapsto{(-)^*} (TX \xrightarrow{f^*} TY)$

Use of monads [Moggi'91]

$\mathbb{C}$: category of types and pure functions. $\mathbb{C} = \mathbf{Sets}, \mathbf{Cpo}$, etc.

- ***Monad*** $T : \mathbb{C} \longrightarrow \mathbb{C}$ is a functor equipped with:
  - ☐ ***unit*** $X \xrightarrow{\eta_X} TX$ for each X
  - ☐ ***extension*** $(X \xrightarrow{f} TY) \overset{(-)^*}{\longmapsto} (TX \xrightarrow{f^*} TY)$
- Useful as an interface for computations ***with structured output***, or computations ***with effect*** $X \longrightarrow TY$. We can...
  - ☐ ***embed pure functions*** due to unit

$$\begin{tikzcd} & TY \\ X \arrow[r, "p"'] & Y \arrow[u, "\eta_Y"'] \end{tikzcd}$$

  - ☐ ***compose computations*** due to extension

$$\begin{tikzcd} & TY & & & TZ \\ X \arrow[ur, "f"] & & \text{and} & Y \arrow[ur, "g"] & \end{tikzcd}$$

compose $\Longrightarrow$

$$\begin{tikzcd} & TY \arrow[r, "g^*"] & TZ \\ X \arrow[ur, "f"] & & \end{tikzcd}$$

Kleisli category for monads

A monad T gives rise to a ***Kleisli category***.

$$\begin{tikzcd}\mathbb{C} \arrow[rrr, "\text{embedding}", "\text{id. on objects}"'] & & & \mathbb{C}_T\end{tikzcd}$$

base category
cat. of ***pure functions***

Kleisli category
cat. of ***computations with str. output***

$$\frac{X \longrightarrow Y \text{ in } \mathbb{C}_T}{X \longrightarrow TY \text{ in } \mathbb{C}}$$

- Embedding is due to unit.
- $\mathbb{C}_T$ is a category.
  - In particular, composition of arrows is due to extension $(-)^*$.

Strong monad

Assume $\mathbb{C}$ has $\times$.

Strong monad = monad + compatibility with $\times$

- **Defn.** A strong monad T is a monad with ***strength***

$$(TX) \times Y \xrightarrow{\mathrm{st}_{X,Y}} T(X \times Y)$$

- Allows us to add ***environments*** $(-) \times Z$:

$$\begin{tikzcd} X \arrow[r, "f"] & TY \end{tikzcd}$$

Computation from X to Y

add env. $\Longrightarrow$

$$\begin{tikzcd} X \times Z \arrow[r, "f \times Z"] & (TY) \times Z \arrow[r, "\mathrm{st}_{Y,Z}"] & T(Y \times Z) \end{tikzcd}$$

Computation from $X \times Z$ to $Y \times Z$

Use of comonads, (monad + comonad + distr. law)

Similarly,

- A **comonad** M is an interface for computation with ***structured input*** $MX \longrightarrow Y$
    - ☐ Brookes & Geva '92, Kieburtz '99, Uustalu & Vene '05
    - ☐ $$\mathbb{C} \xrightarrow{\text{embedding}} \mathbb{C}_M$$

base category
cat. of ***pure functions***

co-Kleisli category
cat. of ***computations with str. input***

$$\begin{array}{c} X \longrightarrow Y \text{ in } \mathbb{C}_M \\ \hline\hline MX \longrightarrow Y \text{ in } \mathbb{C} \end{array}$$

Use of comonads, (monad + comonad + distr. law)

- Monad T + comonad M + distributive law $MT \overset{\lambda}{\Rightarrow} TM$ is an interface for computation with ***structured input & output*** $MX \longrightarrow TY$
  - ☐ Uustalu & Vene, '05
  - ☐ $$\begin{tikzcd}\mathbb{C} \arrow[r, "\text{embedding}"] & \mathbb{C}_{T,M,\lambda}\end{tikzcd}$$

base category
cat. of ***pure functions***

bi-Kleisli category
cat. of ***computations with str. I/O***

$$\begin{array}{c} X \longrightarrow Y \text{ in } \mathbb{C}_{T,M,\lambda} \\ \hline\hline MX \longrightarrow TY \text{ in } \mathbb{C} \end{array}$$

Arrows [Hughes'00]

An ***Arrow*** on $\mathbb{C}$ is...

- a bifunctor $A(-,+) : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{Sets}$,
- equipped with

$$\begin{aligned} \mathrm{arr}_{X,Y} &: \mathrm{Hom}_{\mathbb{C}}(X,Y) \to A(X,Y) \\ \ggg_{X,Y,Z} &: A(X,Y) \times A(Y,Z) \to A(X,Z) \\ \mathrm{first}_{X,Y,Z} &: A(X,Y) \to A(X \times Z, Y \times Z) \end{aligned}$$

- with coherence conditions

$$\begin{aligned} (a \ggg b) \ggg c &= a \ggg (b \ggg c), \\ \mathrm{arr}(g \circ f) &= \mathrm{arr}(f) \ggg \mathrm{arr}(g), \\ \mathrm{arr}\,\mathrm{id} \ggg a &= a = a \ggg \mathrm{arr}\,\mathrm{id}, \\ \mathrm{first}(a) \ggg \mathrm{arr}(\pi_1) &= \mathrm{arr}(\pi_1) \ggg a, \\ \mathrm{first}(a) \ggg \mathrm{arr}(\mathrm{id} \times f) &= \mathrm{arr}\,(\mathrm{id} \times f) \ggg \mathrm{first}(a), \\ \mathrm{first}(a) \ggg \mathrm{arr}(\alpha) &= \mathrm{arr}(\alpha) \ggg \mathrm{first}(\mathrm{first}(a)), \\ \mathrm{first}(\mathrm{arr}(f)) &= \mathrm{arr}(f \times \mathrm{id}) \\ \mathrm{first}(a \ggg b) &= \mathrm{first}(a) \ggg \mathrm{first}(b) \end{aligned}$$

Arrows [Hughes'00]

An ***Arrow*** on $\mathbb{C}$ is...

- a bifunctor $A(-,+) : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{Sets}$,
  $A(X,Y)$: set of structured computations from X to Y
- equipped with

$$\begin{aligned} \mathrm{arr}_{X,Y} &: \mathrm{Hom}_{\mathbb{C}}(X,Y) \to A(X,Y) \end{aligned}$$

embeds pure functions

$$\ggg_{X,Y,Z} : A(X,Y) \times A(Y,Z) \to A(X,Z)$$

composes structured computations

$$\mathrm{first}_{X,Y,Z} : A(X,Y) \to A(X \times Z, Y \times Z)$$

allows for handling environment

- Straightforward to ***enrich*** the whole setting,
  by replacing $\mathbf{Sets}$ by symmetric monoidal $\mathbb{V}$

Arrows generalize (co)monads

- Strong monad T gives rise to an Arrow A_T by

$$A_T(X, Y) = \mathrm{Hom}_{\mathbb{C}}(X, TY)$$

- Comonad M gives rise to an Arrow A_M by

$$A_M(X, Y) = \mathrm{Hom}_{\mathbb{C}}(MX, Y)$$

- Monad T + comonad M + distributive law $MT \overset{\lambda}{\Rightarrow} TM$ gives rise to an Arrow $A_{T,M,\lambda}$ by

$$A_{T,M,\lambda}(X, Y) = \mathrm{Hom}_{\mathbb{C}}(MX, TY)$$

Comparing strong monad vs. Arrow

	Strong monad T	Arrow $A(-,+)$
structured computation from X to Y	$X \longrightarrow TY$ in $\mathbb{C}$	$a \in A(X,Y)$
pure function is embedded due to	unit $\begin{tikzcd} TY \\ Y \arrow[u, "\eta_Y"'] \end{tikzcd}$	$\mathrm{Hom}_{\mathbb{C}}(X,Y) \xrightarrow{\mathrm{arr}} A(X,Y)$
composition of computation due to	extension $\begin{tikzcd} (X \xrightarrow{f} TY) \arrow[d, mapsto, "(-)^*"] \\ (TX \xrightarrow{f^*} TY) \end{tikzcd}$	$\begin{tikzcd} A(X,Y) \times A(Y,Z) \arrow[d, ">\!\!>\!\!>"] \\ A(X,Z) \end{tikzcd}$
compatible with × (i.e. environment) due to	strength $\begin{tikzcd} (TX) \times Y \arrow[d, "\mathrm{st}"] \\ T(X \times Y) \end{tikzcd}$	$\begin{tikzcd} A(X,Y) \arrow[d, "\mathrm{first}"] \\ A(X \times Z, Y \times Z) \end{tikzcd}$

Arrows, like Monads, are monoids [Heunen&Jacobs,MFPS'06]

This similarity can be put more formally:

- Monad with η, $(-)^*$

  = ***monoid*** in functor category $[\mathbb{C}, \mathbb{C}]$

- Strong monad with η, $(-)^*$, $\mathbf{st}$

  = ***monoid*** in $[\mathbb{C}, \mathbb{C}]$

  + (compatibility with $\times$)

- Arrow with $\mathbf{arr}$, $\ggg$, $\mathbf{first}$

  = ***monoid*** in $[\mathbb{C}^{\mathrm{op}} \times \mathbb{C}, \mathbf{Sets}]$

  + (compatibility with $\times$)

$[\mathbb{C}, \mathbb{C}]$: monoidal

tensor: $F \otimes G = F \circ G$

unit: $\mathbf{id}$

$[\mathbb{C}^{\mathrm{op}} \times \mathbb{C}, \mathbf{Sets}]$: monoidal

tensor: like composition

unit: $\mathbf{Hom}_{\mathbb{C}}$

Kleisli categories for Arrows

Kleisli categories for Arrows

Obvious definition:

Kleisli category $\mathbb{C}_A$ for A is

arrow
$$\frac{X \xrightarrow{a} Y \quad \text{in } \mathbb{C}_A}{a \in A(X, Y)}$$

obj.
$$\frac{X \in \mathbb{C}_A}{X \in \mathbb{C}}$$

Kleisli categories for Arrows

Obvious definition:

Kleisli category $\mathbb{C}_A$ for A is

arrow
$$\frac{X \xrightarrow{a} Y \qquad \text{in } \mathbb{C}_A}{a \in A(X,Y)}$$

obj.
$$\frac{X \in \mathbb{C}_A}{X \in \mathbb{C}}$$

- Inclusion functor

$$\mathbb{C} \longrightarrow \mathbb{C}_A$$

$$X \xrightarrow{f} Y \longmapsto \frac{\mathbf{arr}(f) \in A(X,Y)}{X \xrightarrow{\mathbf{arr}(f)} Y}$$

- $\mathbb{C}_A$ is a category.
  In particular, composition is due to $\ggg$.

Cf.

$$\begin{tikzcd} \mathbf{Hom}_{\mathbb{C}}(X,Y) \arrow[d, "\mathbf{arr}"] \\ A(X,Y) \end{tikzcd}$$

$$\begin{tikzcd} A(X,Y) \times A(Y,Z) \arrow[d, "\ggg"] \\ A(X,Z) \end{tikzcd}$$

Kleisli is Freyd

Let's look at $\mathbb{C}_A$...

- $\mathbb{C}_A$ is ***pre-monoidal*** (which is, almost monoidal)
  - ☐ Instead of a bifunctor $\otimes : \mathbb{C}_A \times \mathbb{C}_A \longrightarrow \mathbb{C}_A$, we have two functors for each Z

$$(-) \ltimes Z : \quad \mathbb{C}_A \longrightarrow \mathbb{C}_A$$
$$Z \rtimes (-) : \quad \mathbb{C}_A \longrightarrow \mathbb{C}_A$$

given by

$$\frac{X \xrightarrow{a} Y \quad \text{in } \mathbb{C}_A}{X \times Z \xrightarrow[\substack{\mathrm{first}(a) \\ \| \\ a \ltimes Z}]{} Y \times Z \quad \text{in } \mathbb{C}_A}$$

Cf.

$$\begin{tikzcd} A(X, Y) \arrow[d, "\mathrm{first}"] \\ A(X \times Z, Y \times Z) \end{tikzcd}$$

Kleisli is Freyd

Let's look at $\mathbb{C}_A$...

- $\mathbb{C}_A$ is ***pre-monoidal*** (which is, almost monoidal)
  - ☐ But not quite monoidal. If it were,

$$\begin{tikzcd}
X \times Z \arrow[r, "X \rtimes g"] \arrow[d, "f \ltimes Z"'] \arrow[dr, "f \otimes g"] & X \times W \arrow[d, "f \ltimes W"] \\
Y \times Z \arrow[r, "Y \rtimes g"'] & Y \times W
\end{tikzcd}$$

should commute,

which is unlikely.
E.g. $A(X, Y)$: "state-based computations from X to Y"

Kleisli is Freyd

In fact, $\mathbb{C} \longrightarrow \mathbb{C}_A$ is a ***Freyd category***.

- A Freyd category on $\mathbb{C}$ is: [Power, Robinson, Thielecke]

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\text{id. on objects}"] & \mathbb{K} \end{tikzcd} ,$$

with finite products

pre-monoidal

preserving appropriate structures.

- 1-1 correspondence between Arrows and Freyd categories: [Heunen&Jacobs, MFPS'06]

$$\begin{tikzcd} \{\text{Arrows on } \mathbb{C}\} \arrow[r, "\cong"] & \{\text{Freyd categories on } \mathbb{C}\} \\ A(-,+) \arrow[r, mapsto] & (\mathbb{C} \rightarrow \mathbb{C}_A) \\ \mathbf{Hom}_{\mathbb{K}}(J-, J+) & (\mathbb{C} \xrightarrow{J} \mathbb{K}) \arrow[l, mapsto] \end{tikzcd}$$

2-categorical characterization of Kleisli categories

Why “Kleisli”?

Why is this “Kleisli”?

$$\begin{tikzcd}
\{\text{Arrows on } \mathbb{C}\} \arrow[r, "\cong", "\text{“Kleisli”}"'] & \{\text{Freyd categories on } \mathbb{C}\} \\
A \arrow[r, mapsto] & (\mathbb{C} \longrightarrow \mathbb{C}_A)
\end{tikzcd}$$

- For Arrows induced by (co)monads, “Kleisli” (for Arrows) coincides with usual Kleisli (for monads).
- 2-categorical characterization. Details now

$$\begin{tikzcd}
\mathbf{Freyd} \arrow[r, bend left, "\mathit{Ins}"] \arrow[r, phantom, "\top"] & \mathbf{Arr(Freyd)} \arrow[l, bend left, "\text{“Kleisli”}"]
\end{tikzcd}$$

, just like

$$\begin{tikzcd}
\mathbf{Cat} \arrow[r, bend left, "\mathit{Ins}"] \arrow[r, phantom, "\top"] & \mathbf{Mnd(Cat_*)} \arrow[l, bend left, "\text{Kleisli}"]
\end{tikzcd}$$

for monads [Street’72]

Eilenberg-Moore construction as right 2-adjoint

2-categorical characterization [Street'72] of

- Eilenberg-Moore construction of algebras
- for monads

is presented as a showcase.

$$\begin{tikzcd}
\mathbf{Cat} \arrow[r, bend left, "\boldsymbol{Ins}"] \arrow[r, phantom, "\perp"] & \mathbf{Mnd}(\mathbf{Cat}) \arrow[l, bend left, "\boldsymbol{EMAlg}"]
\end{tikzcd}$$

2-category $\mathbf{Mnd}(\mathbf{Cat})$

object: $(\mathbb{C}, \boldsymbol{T})$

1-cell:

$$(\mathbb{C}, \boldsymbol{T}) \xrightarrow{(\boldsymbol{H},\boldsymbol{\sigma})} (\mathbb{D}, \boldsymbol{S})$$

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\boldsymbol{H}"] \arrow[d, "\boldsymbol{T}"'] \arrow[dr, phantom, "\Swarrow \boldsymbol{\sigma}"] & \mathbb{D} \arrow[d, "\boldsymbol{S}"] \\
\mathbb{C} \arrow[r, "\boldsymbol{H}"'] & \mathbb{D}
\end{tikzcd}$$

2-cell: ...

Eilenberg-Moore construction as right 2-adjoint

2-categorical characterization [Street'72] of

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is presented as a showcase.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, mapsto] & (\mathbb{C}, \mathbf{id}) \\
\mathbf{Cat} \arrow[r, bend left, "Ins"] \arrow[r, phantom, "\perp"] & \mathbf{Mnd(Cat)} \arrow[l, bend left, "EMAlg"] \\
\mathbb{C}^{T} & (\mathbb{C}, T) \arrow[l, mapsto]
\end{tikzcd}$$

2-category $\mathbf{Mnd(Cat)}$

object: $(\mathbb{C}, T)$

1-cell:

$$(\mathbb{C}, T) \xrightarrow{(H,\sigma)} (\mathbb{D}, S)$$

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "H"] \arrow[d, "T"'] \arrow[dr, phantom, "\Swarrow \sigma"] & \mathbb{D} \arrow[d, "S"] \\
\mathbb{C} \arrow[r, "H"'] & \mathbb{D}
\end{tikzcd}$$

2-cell: ...

Eilenberg-Moore construction as right 2-adjoint

2-categorical characterization [Street'72] of

- Eilenberg-Moore construction of algebras
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is presented as a showcase.

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, mapsto] & & (\mathbb{C}, \mathbf{id}) \\
\mathbf{Cat} \arrow[rr, bend left, "\mathit{Ins}"] \arrow[rr, phantom, "\perp"] & & \mathbf{Mnd}(\mathbf{Cat}) \arrow[ll, bend left, "\mathit{EMAlg}"] \\
\mathbb{C}^{T} & & (\mathbb{C}, T) \arrow[ll, mapsto]
\end{tikzcd}$$

2-category $\mathbf{Mnd}(\mathbf{Cat})$

object: $(\mathbb{C}, T)$

1-cell:

$$\underline{\underline{(\mathbb{C}, T) \xrightarrow{(H,\sigma)} (\mathbb{D}, S)}}$$

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, "H"] \arrow[d, "T"'] & & \mathbb{D} \arrow[d, "S"] \\
\mathbb{C} \arrow[rr, "H"'] & & \mathbb{D}
\arrow[from=1-1, to=2-3, phantom, "\Swarrow \sigma"]
\end{tikzcd}$$

2-cell: ...

obj. in $\mathbb{C}^{T}$ $\Longleftrightarrow$ $1 \longrightarrow \mathbb{C}^{T}$ in $\mathbf{Cat}$ $\overset{\text{adj.}}{\Longleftrightarrow}$ $(1, \mathbf{id}) \longrightarrow (\mathbb{C}, T)$ in $\mathbf{Mnd}(\mathbf{Cat})$

$$\begin{tikzcd}
& 1 \arrow[r, "X"] \arrow[d, "\mathrm{id}"'] & \mathbb{C} \arrow[d, "T"] & & TX \arrow[d, "\sigma_*"] \\
\Longleftrightarrow & 1 \arrow[r, "X"'] & \mathbb{C} & \Longleftrightarrow & X
\arrow[from=1-2, to=2-3, phantom, "\Swarrow \sigma"]
\end{tikzcd}$$

Kleisli as left 2-adjoint

Similarly, [Street'72]

$$\begin{tikzcd}
\mathbf{Cat} \arrow[r, bend left, "\mathcal{I}ns"] \arrow[r, phantom, "\top"] & \mathbf{Mnd}(\mathbf{Cat}_*) \arrow[l, bend left, "\mathcal{K}\ell"]
\end{tikzcd}$$

Can we do the same for Arrows?

$$\begin{tikzcd}
\mathbf{FPCat} \arrow[r, bend left, "\mathcal{I}ns"] \arrow[r, phantom, "\top"] & \mathbf{Arr}(\mathbf{FPCat}) \arrow[l, bend left, "\mathcal{K}\ell"]
\end{tikzcd}$$

2-category $\mathbf{Arr}(\mathbf{FPCat})$

object: $(\mathbb{C}, A)$

1-cell: ...

2-cell: ...

Kleisli as left 2-adjoint

Similarly, [Street’72]

$$\begin{tikzcd}
\mathbf{Cat} \arrow[r, bend left, "\mathcal{I}ns"] \arrow[r, phantom, "\top"] & \mathbf{Mnd}(\mathbf{Cat}_*) \arrow[l, bend left, "\mathcal{K}\ell"]
\end{tikzcd}$$

Can we do the same for Arrows? **No.**

$$\begin{tikzcd}
\mathbf{FPCat} \arrow[r, bend left, "\mathcal{I}ns"] \arrow[r, phantom, "\top"] & \mathbf{Arr}(\mathbf{FPCat}) \arrow[l, bend left, "\times\ \mathcal{K}\ell"] \\
\mathbb{C}_{A} & (\mathbb{C}, A) \arrow[l, mapsto]
\end{tikzcd}$$

2-category $\mathbf{Arr}(\mathbf{FPCat})$

object: $(\mathbb{C}, A)$

1-cell: ...

2-cell: ...

pre-monoidal,
not with finite products!

Arrows on Freyd categories

We extend notion of Arrows: on $\mathbf{FPCat} \Longrightarrow$ on $\mathbf{Freyd}$

- Arrow $\boldsymbol{A}$ on $\mathbb{C} \in \mathbf{FPCat}$:
  - ☐ $\boldsymbol{A} : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \longrightarrow \mathbf{Sets}$, with $\mathbf{arr}, \ggg, \mathbf{first}$
  - ☐ with coherence conditions
- Arrow $\boldsymbol{A}$ on $(\mathbb{C} \xrightarrow{J} \mathbb{K}) \in \mathbf{Freyd}$:
  - ☐ $\boldsymbol{A} : \mathbb{K}^{\mathrm{op}} \times \mathbb{K} \longrightarrow \mathbf{Sets}$, with $\mathbf{arr}, \ggg, \mathbf{first}$
  - ☐ with similar coherence conditions

    Especially, $\mathbf{first}$ is compatible with Cartesian structure carried from $\mathbb{C}$ to $\mathbb{K}$

Kleisli for Arrows, as left 2-adjoint

Then indeed $\mathcal{K}\ell \dashv \mathcal{I}ns$ for Arrows.

$$\begin{tikzcd}
\mathbf{Freyd} \arrow[rr, bend left, "\mathcal{I}ns"] \arrow[rr, phantom, "\top"] & & \mathbf{Arr(Freyd)} \arrow[ll, bend left, "\mathcal{K}\ell"]
\end{tikzcd}$$

Kleisli for Arrows, as left 2-adjoint

Then indeed $\mathcal{K}\ell \dashv \boldsymbol{\mathit{Ins}}$ for Arrows.

$$\begin{tikzcd}
(\mathbb{C} \xrightarrow{\boldsymbol{J}} \mathbb{K}) \arrow[r, mapsto] & (\mathbb{C} \xrightarrow{\boldsymbol{J}} \mathbb{K}, \mathbf{Hom}_{\mathbb{K}}) \\
\mathbf{Freyd} \arrow[r, bend left, "\boldsymbol{\mathit{Ins}}"] \arrow[r, phantom, "\top"] & \mathbf{Arr(Freyd)} \arrow[l, bend left, "\mathcal{K}\ell"] \\
(\mathbb{C} \xrightarrow{\boldsymbol{J}} \mathbb{K} \xrightarrow{\boldsymbol{J_A}} \mathbb{K}_{\boldsymbol{A}}) & (\mathbb{C} \xrightarrow{\boldsymbol{J}} \mathbb{K}, \boldsymbol{A}) \arrow[l, mapsto]
\end{tikzcd}$$

Kleisli for Arrows, as left 2-adjoint

Then indeed $\mathcal{K}\ell \dashv \mathcal{I}ns$ for Arrows.

$$\begin{tikzcd}
\mathbf{Freyd} \arrow[r, bend left, "\mathcal{I}ns"] \arrow[r, phantom, "\top"] & \mathbf{Arr}(\mathbf{Freyd}) \arrow[l, bend left, "\mathcal{K}\ell"] \\
\mathbf{FPCat} \arrow[u] \arrow[r, bend left, "\mathcal{I}ns"] & \mathbf{Arr}(\mathbf{FPCat}) \arrow[u] \arrow[l, bend left, "\times" description]
\end{tikzcd}$$

Four 2-functors, three 2-adjunctions

$$\begin{tikzcd}
\mathbf{Freyd} & \mathbf{Arr(Freyd)} \\
(\mathbb{C} \xrightarrow{J} \mathbb{K}) & (\mathbb{C} \xrightarrow{J} \mathbb{K},\ A) \arrow[l, mapsto, "U"', "\top"] \\
(\mathbb{C} \xrightarrow{J} \mathbb{K}) \arrow[r, mapsto, "Ins", "\top"'] & (\mathbb{C} \xrightarrow{J} \mathbb{K},\ \mathbf{Hom}_{\mathbb{K}}) \\
(\mathbb{C} \xrightarrow{J} \mathbb{K} \xrightarrow{J_A} \mathbb{K}_A) & (\mathbb{C} \xrightarrow{J} \mathbb{K},\ A) \arrow[l, mapsto, "\mathcal{K}\ell"', "\top"] \\
(\mathbb{C} \xrightarrow{J} \mathbb{K}) \arrow[r, mapsto, "Arr"] & (\mathbb{C} \xrightarrow{\mathrm{id}} \mathbb{C},\ \mathbf{Hom}_{\mathbb{K}}(J-, J+))
\end{tikzcd}$$

- Ins is a full embedding.
  - ☐ $\mathbf{Freyd}$ is a full reflective 2-subcategory of $\mathbf{Arr(Freyd)}$.

Four 2-functors, three 2-adjunctions

$$\begin{tikzcd}
\mathbf{Freyd} & & \mathbf{Arr(Freyd)} \\
(\mathbb{C} \xrightarrow{J} \mathbb{K}) \arrow[rr, phantom, "\top"'] & & (\mathbb{C} \xrightarrow{J} \mathbb{K},\ A) \arrow[ll, mapsto, "U"'] \\
(\mathbb{C} \xrightarrow{J} \mathbb{K}) \arrow[rr, mapsto, "Ins"] \arrow[rr, phantom, "\top"', shift right=2] & & (\mathbb{C} \xrightarrow{J} \mathbb{K},\ {\color{red}\mathbf{Hom}_{\mathbb{K}}}) \\
(\mathbb{C} \xrightarrow{J} \mathbb{K} \xrightarrow{J_A} \mathbb{K}_A) \arrow[rr, phantom, "\top"'] & & (\mathbb{C} \xrightarrow{J} \mathbb{K},\ {\color{red}A}) \arrow[ll, mapsto, "\mathcal{K}\ell"'] \\
(\mathbb{C} \xrightarrow{J} \mathbb{K}) \arrow[rr, mapsto, "Arr"] & & (\mathbb{C} \xrightarrow{\mathrm{id}} \mathbb{C},\ {\color{red}\mathbf{Hom}_{\mathbb{K}}(J-, J+)})
\end{tikzcd}$$

- $$\begin{tikzcd}
\mathbf{Freyd} \arrow[rr, "Arr"] \arrow[rrdd, red, "\cong"] & & \mathbf{Arr(Freyd)} \\
(\mathbb{C} \xrightarrow{J} \mathbb{K}) \arrow[rrdd, mapsto, "Arr"] & & \\
& & \mathbf{Arr(FPCat)} \arrow[uu, "Ins"'] \\
& & (\mathbb{C},\ \mathbf{Hom}_{\mathbb{K}}(J-, J+))
\end{tikzcd}$$

(Eilenberg-Moore) algebras for Arrows

Algebras for monads

For a monad T, consider the following notion of “algebras”...

- **Defn.** An T***-algebra*** is a natural transformation

$$\begin{array}{c} T \\ \Downarrow \\ \mathrm{id} \end{array} \quad \text{in} \quad [\mathbb{C}, \mathbb{C}], \quad \text{compatible with } \eta,\ (-)^*$$

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in $[\mathbb{C}, \mathbb{C}]$, compatible with η, $(-)^*$

- We aim at generalizing this to Arrows

Recall:

☐ Monad

= ***monoid*** in $[\mathbb{C}, \mathbb{C}]$

☐ Arrow

= ***monoid*** in $[\mathbb{C}^{\mathrm{op}} \times \mathbb{C}, \mathbf{Sets}]$

+ (compatibility with $\times$)

$[\mathbb{C}, \mathbb{C}]$: monoidal

tensor: $F \otimes G = F \circ G$

unit: $\mathbf{id}$

$[\mathbb{C}^{\mathrm{op}} \times \mathbb{C}, \mathbf{Sets}]$: monoidal

tensor: like composition

unit: $\mathbf{Hom}_{\mathbb{C}}$

Algebras for Arrows

- **Defn.** An ***algebra*** for Arrow $\boldsymbol{A}$ is a natural transformation

$$\begin{array}{c} \boldsymbol{A}(-,+) \\ \Downarrow \\ \mathbf{Hom}_{\mathbb{C}}(-,+) \end{array} \quad \text{in} \quad [\mathbb{C}^{\mathbf{op}} \times \mathbb{C}, \mathbf{Sets}],$$

  compatible with $\mathbf{arr}$, $\ggg$, $\mathbf{first}$.

- Justification?
  - ☐ For $\boldsymbol{A}_{\boldsymbol{T}}$ induced by a monad $\boldsymbol{T}$,

    this is the same as algebras $\begin{array}{c} \boldsymbol{T} \\ \Downarrow \\ \mathbf{id} \end{array}$.
  - ☐ Also for comonads, (monad + comonad + distr. law).
  - ☐ Proof: non-trivial computation!

Algebras for Arrows

- Justification? (ctn'd)
  - ☐ Characterized as a left-inverse of Kleisli inclusion.
    [Envisaged by John Power]

$$\left\{ \boldsymbol{A}\text{-algebras} \qquad \begin{array}{c} \boldsymbol{A}(-,+) \\ \Downarrow\boldsymbol{\chi} \\ \mathbf{Hom}_{\mathbb{C}}(-,+) \end{array} \right\} \cong \left\{ \begin{array}{l} \text{left inverse of} \\ \mathbb{C} \longrightarrow \mathbb{C}_{\boldsymbol{A}} \end{array} \right\}$$

  - ☐ Also the case for (co)monads. For a monad $\boldsymbol{T}$,

$$\left\{ \boldsymbol{T}\text{-algebras} \quad \begin{array}{c} \boldsymbol{T} \\ \Downarrow\boldsymbol{\sigma} \\ \mathbf{id} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{left inverse of} \\ \mathbb{C} \longrightarrow \mathbb{C}_{\boldsymbol{T}} \end{array} \right\}$$

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$$\begin{tikzcd}
\mathbb{C} & & \mathbb{C}_T \arrow[ll] \\
& TY \arrow[d, "\sigma_Y"] & \\
X \arrow[ur, "f"] & Y & X \xrightarrow{f} Y
\end{tikzcd}$$

Towards more satisfactory notion of algebras

- Specific object(s) as a carrier? E.g.

$$\begin{tikzcd} A(X,Y) \arrow[d] \\ \mathbf{Hom}_{\mathbb{C}}(X,Y) \end{tikzcd}$$

  - ☐ Doesn't work: meaning of "compatibility with $\ggg$" is not clear

- As a 2-categorical dual of Kleisli?

  - ☐ For monads,

$$\left\{ \begin{tikzcd} \mathbf{Cat} \arrow[r, bend left, "\mathcal{I}ns"] \arrow[r, phantom, "\perp"] & \mathbf{Mnd}(\mathbf{Cat}) \arrow[l, bend left, "\mathcal{EMA}lg"] \arrow[d, dashed, "\text{2-cells opposed}"] \\ \mathbf{Cat} \arrow[r, bend left, "\mathcal{I}ns"] \arrow[r, phantom, "\top"] & \mathbf{Mnd}(\mathbf{Cat}_*) \arrow[l, bend left, "\mathcal{K}\ell"] \end{tikzcd} \right.$$

  - ☐ Does not work for

$$\begin{tikzcd} \mathbf{Freyd} \arrow[r, bend left, "\mathcal{I}ns"] \arrow[r, phantom, "\top"] & \mathbf{Arr}(\mathbf{Freyd}) \arrow[l, bend left, "\mathcal{K}\ell"] \end{tikzcd}$$

Towards more satisfactory notion of algebras

- Is the question “What is an algebra for Arrows?” reasonable?
  - If not, why?
  - Examples of Arrow-algebras?

Conclusions & future work

- Arrows generalize (co)monads: they are ***monoids***.
- Kleisli for Arrows:

$$\begin{tikzcd} \{\text{Arrows on } \mathbb{C}\} \arrow[r, "\cong", "\textit{Kleisli}"'] & \{\text{Freyd cat. on } \mathbb{C}\} \\ \boldsymbol{A} \arrow[r, mapsto] & (\mathbb{C} \longrightarrow \mathbb{C}_{\boldsymbol{A}}) \end{tikzcd}$$

  - ☐ 2-categorical characterization
- Eilenberg-Moore algebras for Arrows:

$$\begin{array}{c} \boldsymbol{A}(-,+) \\ \Downarrow \\ \mathbf{Hom}_{\mathbb{C}}(-,+) \end{array} \quad , \quad \text{just like} \quad \begin{array}{c} \boldsymbol{T} \\ \Downarrow \\ \mathbf{id} \end{array} \quad \text{for monads} \quad .$$

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Conclusions & future work

- Arrows generalize (co)monads: they are ***monoids***.
- Kleisli for Arrows:

$$\begin{tikzcd}
\{\text{Arrows on } \mathbb{C}\} \arrow[r, "\cong", "\textit{Kleisli}"'] & \{\text{Freyd cat. on } \mathbb{C}\} \\
\boldsymbol{A} \arrow[r, mapsto] & (\mathbb{C} \to \mathbb{C}_{\boldsymbol{A}})
\end{tikzcd}$$

  ☐ 2-categorical characterization

- Eilenberg-Moore algebras for Arrows:

$$\begin{matrix} \boldsymbol{A}(-,+) \\ \Downarrow \\ \mathbf{Hom}_{\mathbb{C}}(-,+) \end{matrix} \quad , \quad \text{just like} \quad \begin{matrix} \boldsymbol{T} \\ \Downarrow \\ \mathbf{id} \end{matrix} \quad \text{for monads} \quad .$$

  ☐ Examples?

Thank you for your attention!

Contact: **http://www.cs.ru.nl/∼ichiro**