

# Compositional Approaches to Games in Model Checking

Kazuki Watanabe<sup>a, b</sup>

a The Graduate University for Advanced Studies (SOKENDAI)

b National Institute of Informatics, Tokyo, Japan

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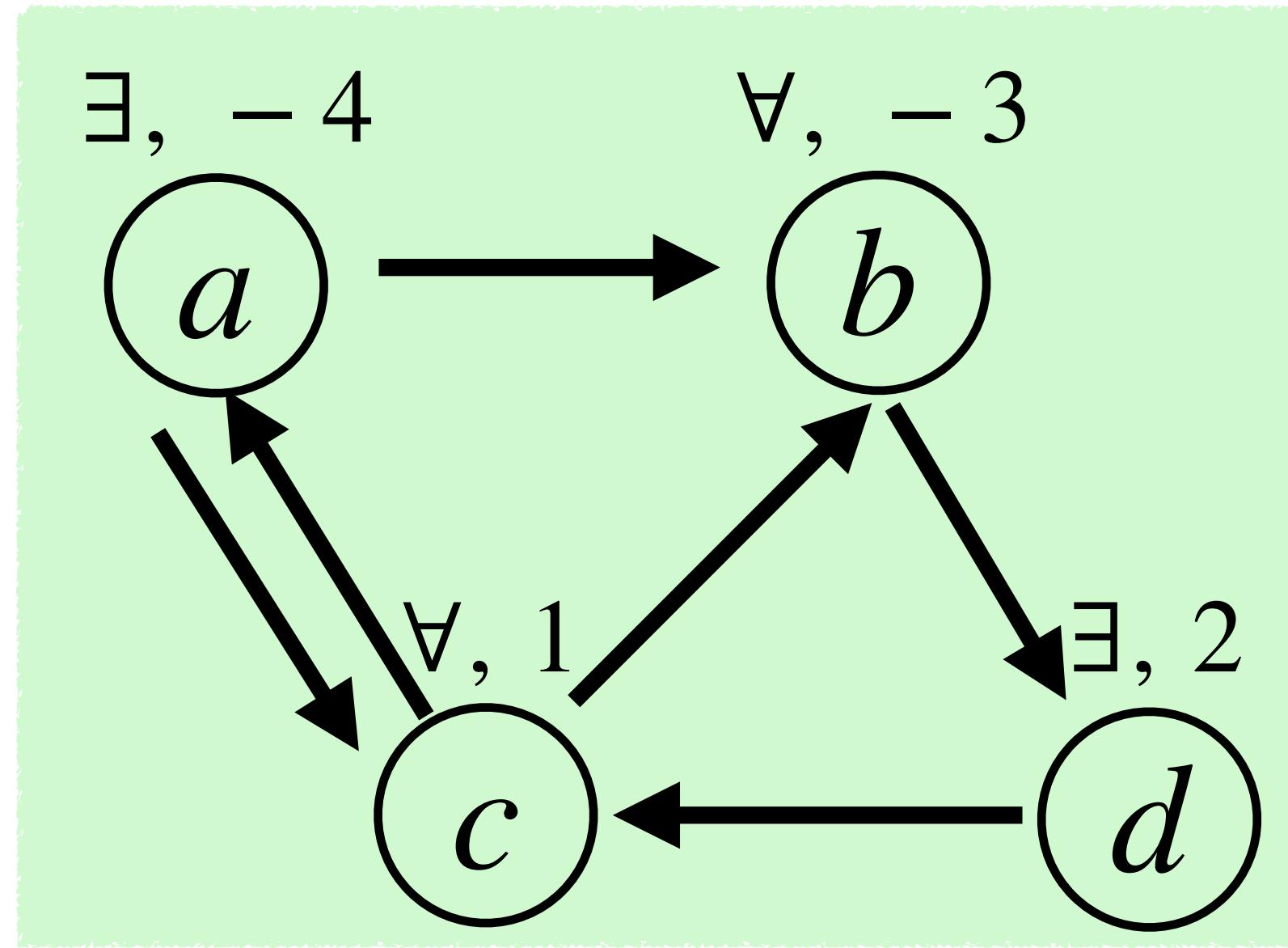
Based on

- Watanabe, Eberhart, Asada & Hasuo, MFPS 2021
- —, ongoing work

# Overview

- Compositional solution of mean payoff games (MPGs)  
(Ongoing work)
  - Compositionality as functoriality
  - Categorical axiomatization
  - Promising experiment results
- Compositional solution of parity games  
[W., Eberhart, Asada, Hasuo, MFPS'21]
  - Using the semantic work of [Grellois & Mellies MFCS'15],  
essentially an instance of the above general categorical framework

# Mean payoff game: quantitative verification method for model checking



Def:

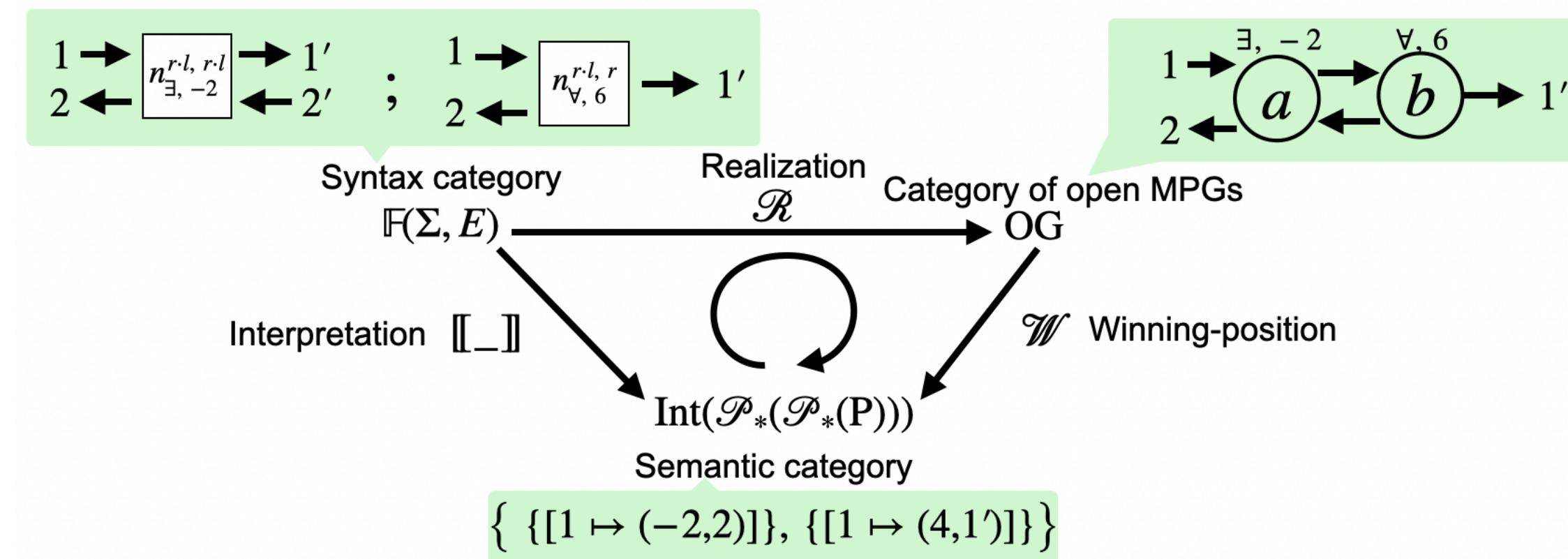
A mean payoff game  $\mathcal{A} = (V, E, r, \omega)$  consists of a directed graph  $(V, E)$ , a role function  $r : V \rightarrow \{\exists, \forall\}$ , and a weight function  $\omega : V \rightarrow \mathbb{Z}$ .

Winning condition:  $\exists$  wins in a infinite play  $v_1 v_2 \dots$  if

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \omega(v_i) \geq 0$$

# Outline

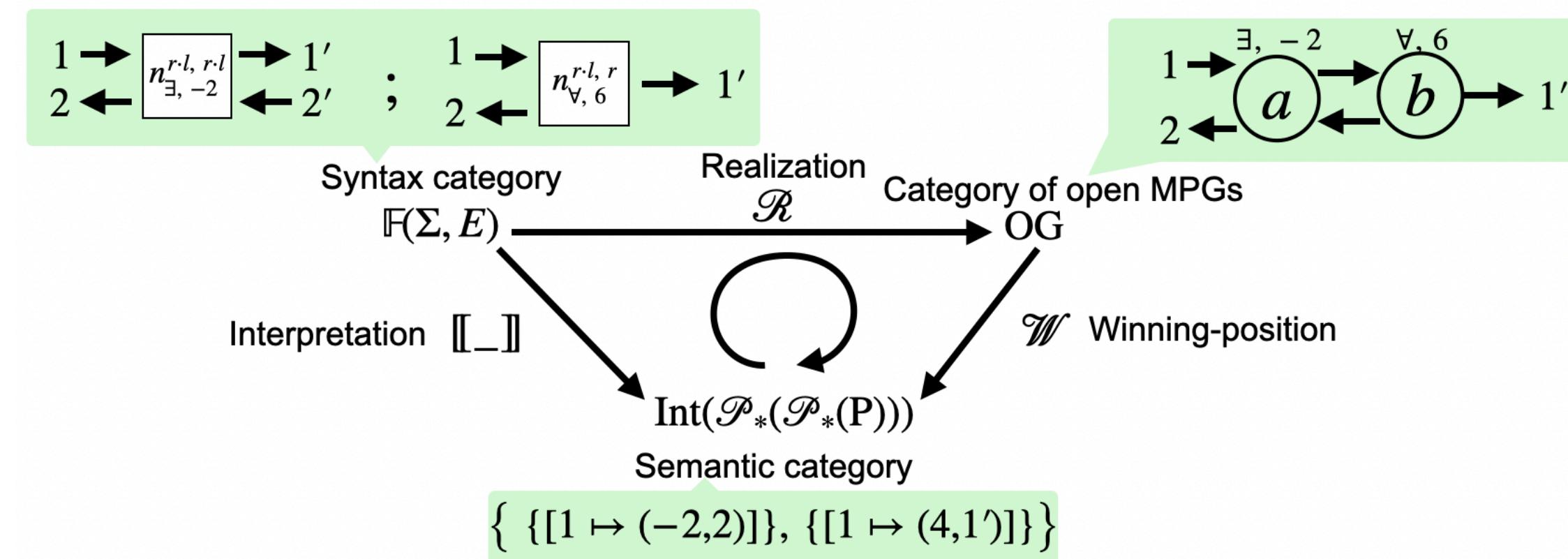
- Categorical framework for compositional solution
  - the triangle



- Compositional algorithm for open MPG: CMGS
- Experiments

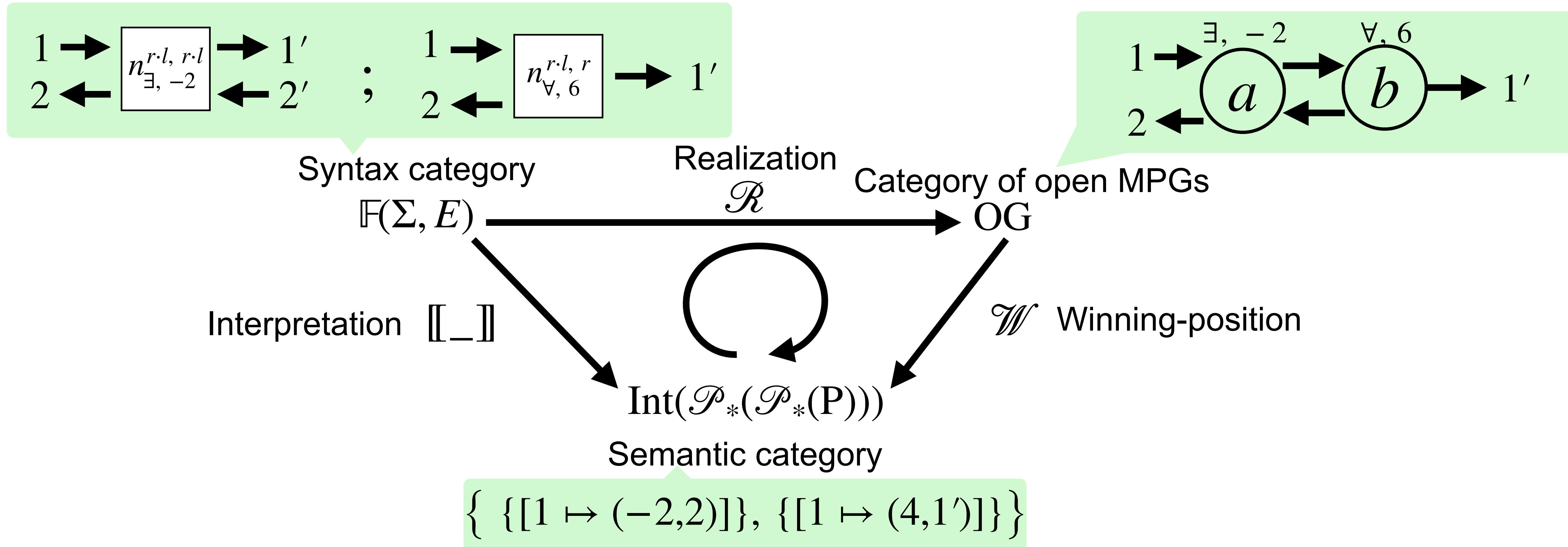
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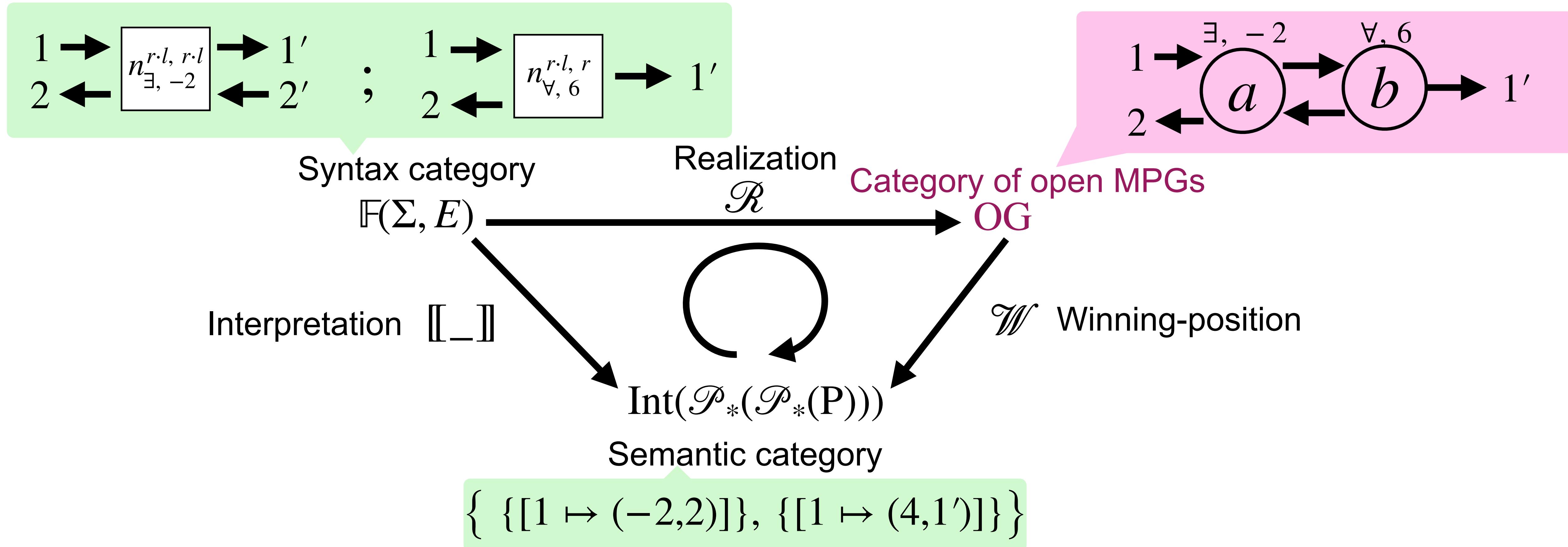
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# Our theoretical contributions: compositional approach to MPG



- winning-position functor  $\mathcal{W}$ : conventional solution of MPG
- realization functor  $\mathcal{R}$  (which is **full**): gives syntactic presentation  $\mathbb{F}(\Sigma, E)$  for MPG
- interpretation functor  $\llbracket \_ \rrbracket$ : a **compact closed functor**, compositionally solving MPG

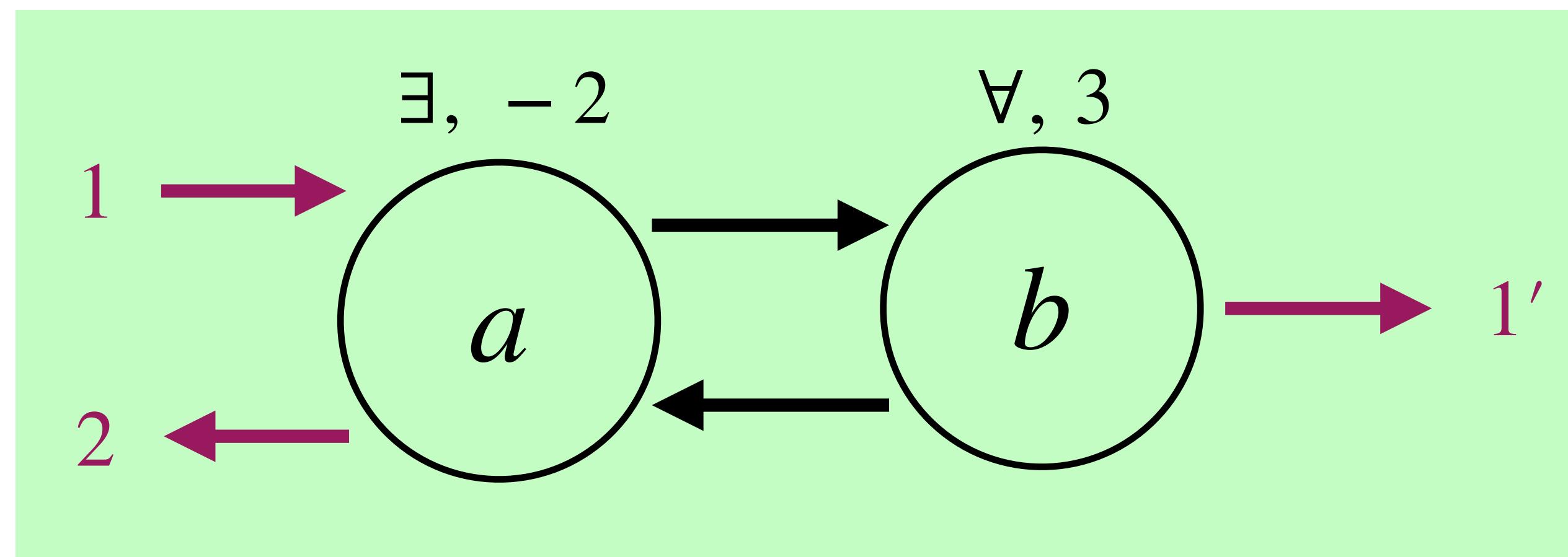
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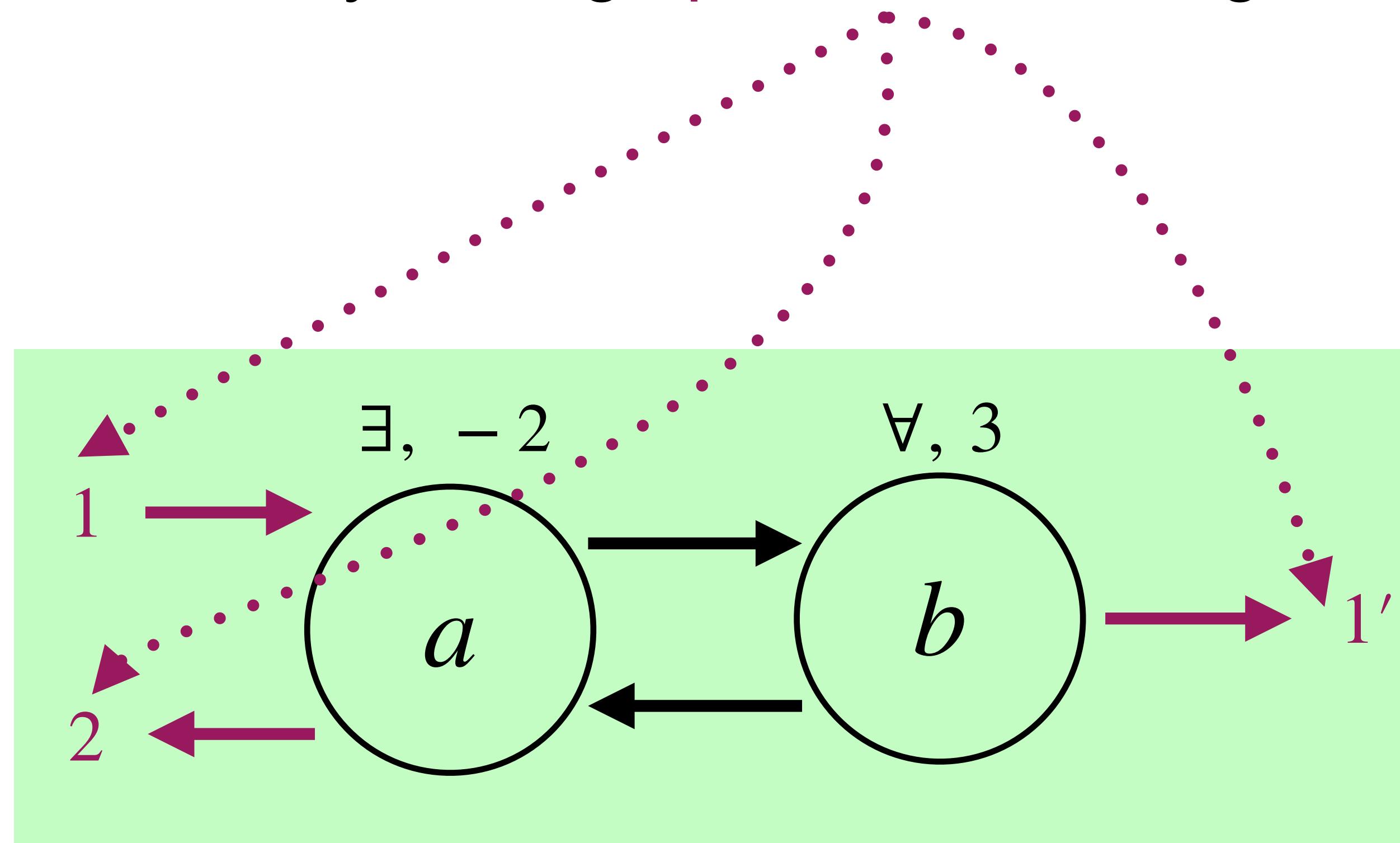
# Open mean payoff game (open MPG)

- Open mean payoff game:  
an extension of MPG by adding open ends to the game.
- Example:



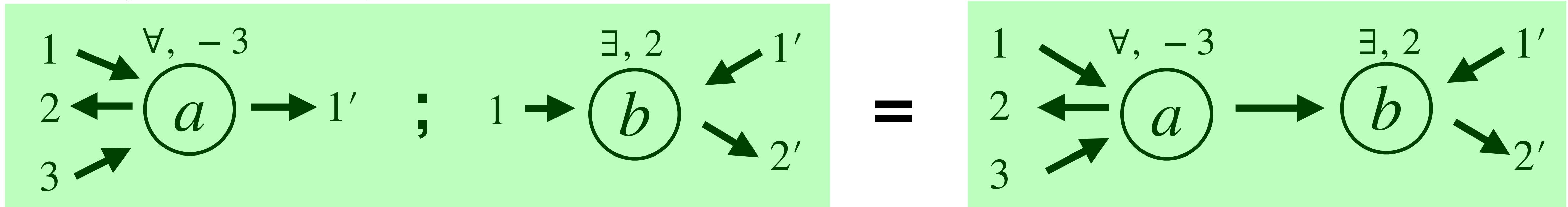
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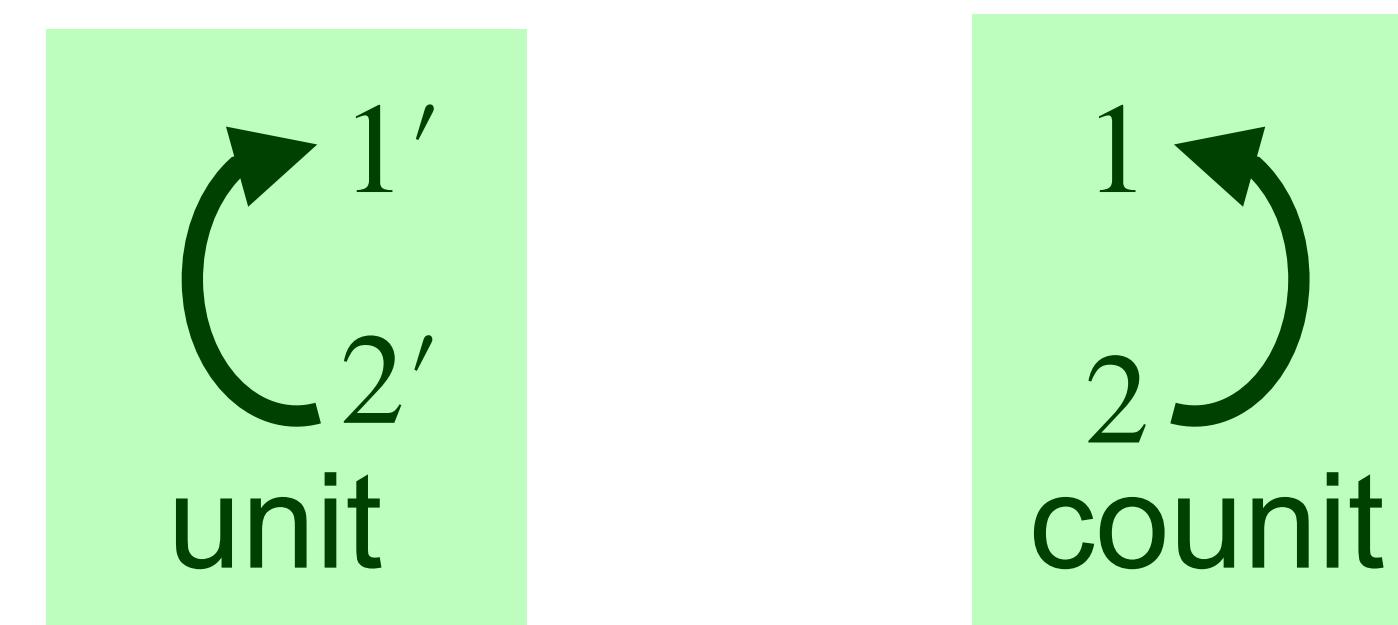


# A compact closed category of open MPGs OG

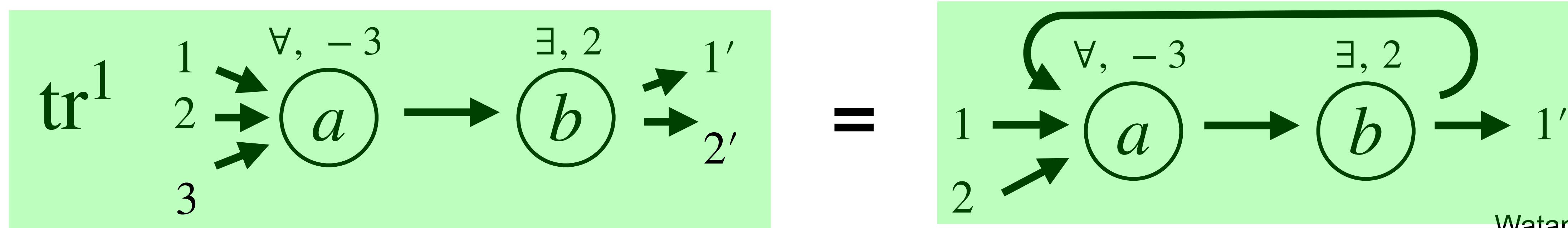
- sequential composition:



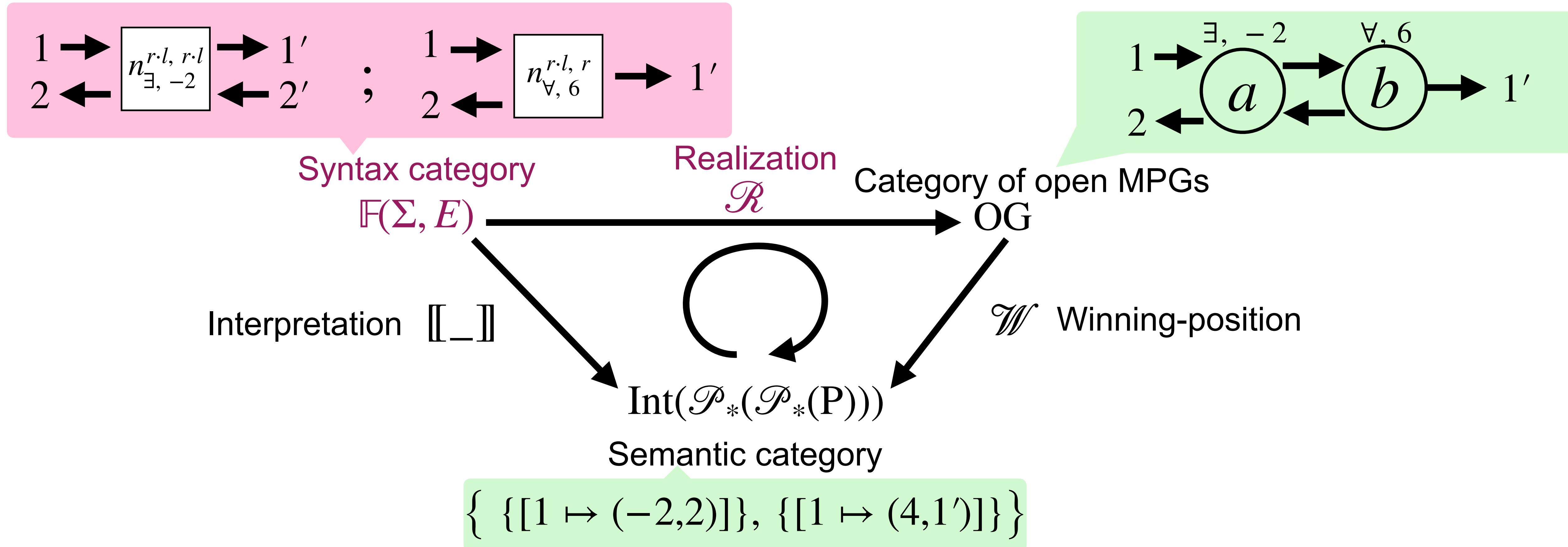
- unit and counit:



- trace operator:



# Our theoretical contributions: compositional approach to MPGs



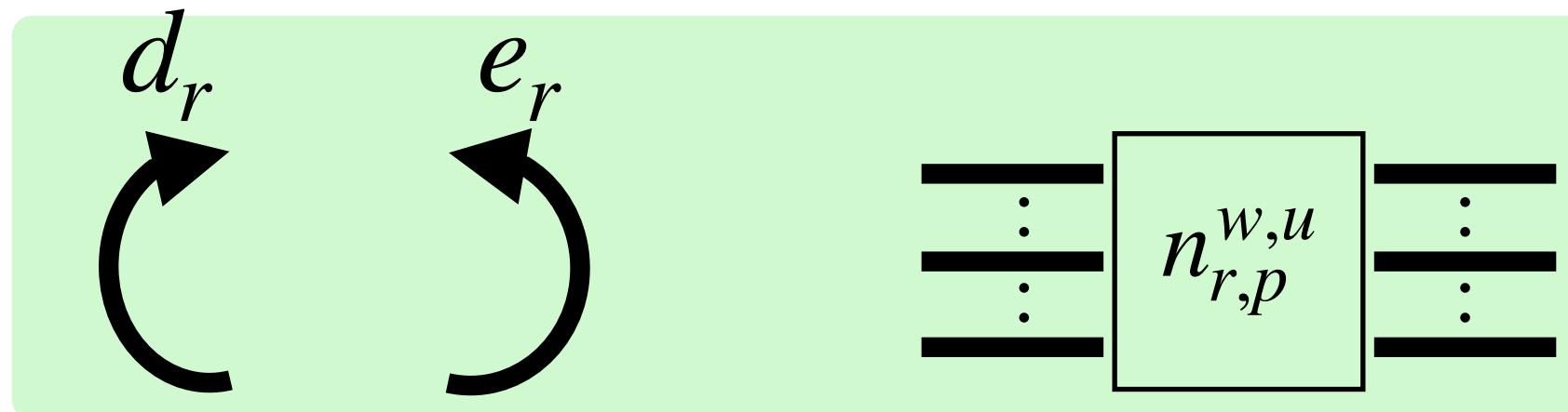
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# The $\{r, l\}$ -prop: $\mathbb{F}(\Sigma, E)$

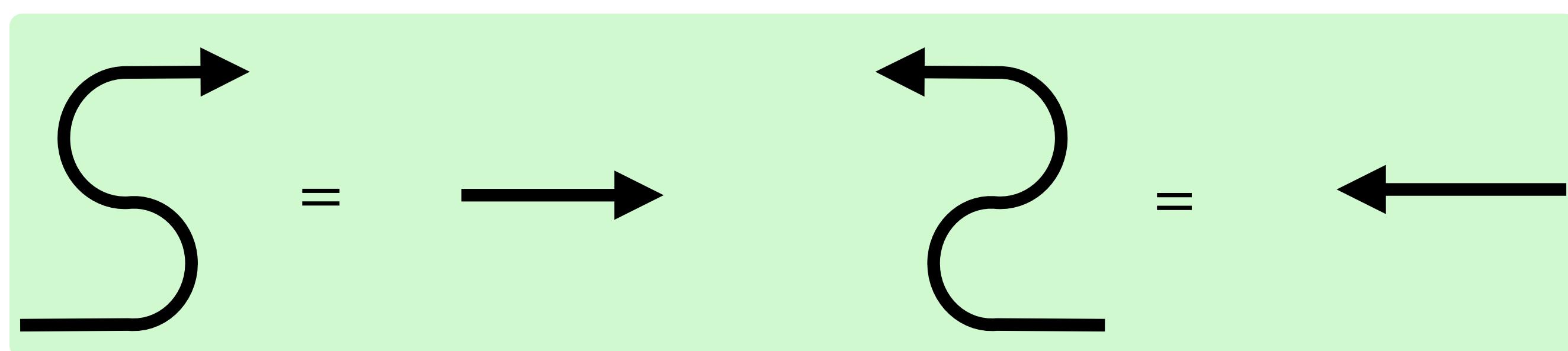
Def ( $C$ -prop) : Let  $C$  be a set (of colors). A  $C$ -prop is a strict symmetric monoidal category whose objects form the free monoid  $C^*$ .

- $\mathbb{F}(\Sigma, E)$ : a free strict symmetric monoidal category generated by a symmetric monoidal theory  $(\Sigma, E)$

- signature  $\Sigma$ :

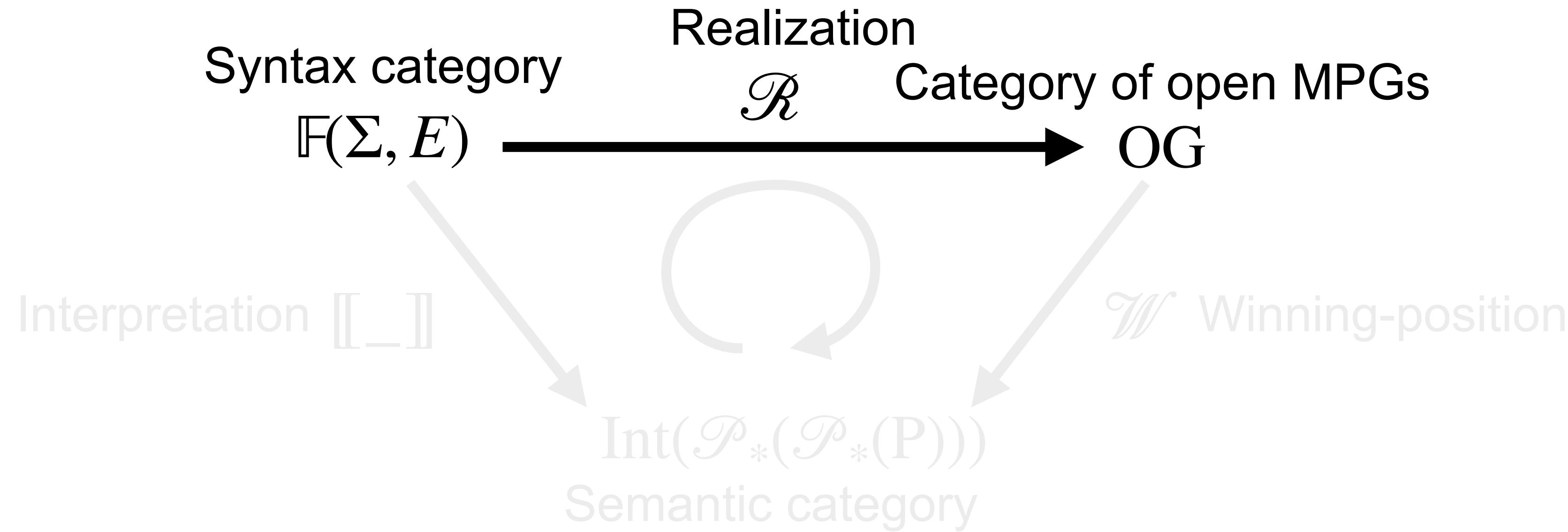


- equations  $E$ :



Thm:  $\mathbb{F}(\Sigma, E)$  is a free compact closed category generated by position generators  $n_{r,p}^{w,u}$ .

# The realization functor $\mathcal{R}$ and fullness



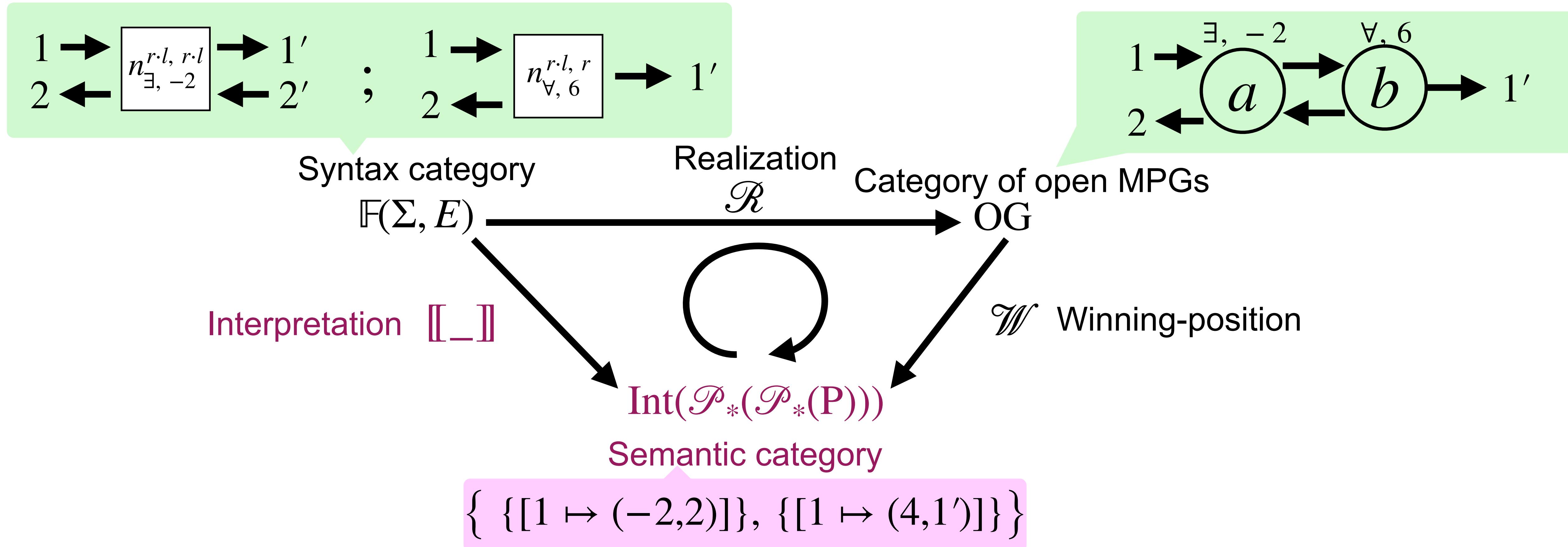
- We define the **realization functor  $\mathcal{R}$**  by the freeness of  $\mathbb{F}(\Sigma, E)$ .

- example

$$\mathcal{R} \left( \begin{array}{ccc} \longrightarrow & \boxed{n_{\forall,2}^{r,l,r}} & \longrightarrow \\ \longleftarrow & & \end{array} \right) = \begin{array}{c} 1 \xrightarrow{\forall,2} n \\ 2 \xleftarrow{} \end{array} \longrightarrow 1'$$

Thm:  $\mathcal{R}$  is full.

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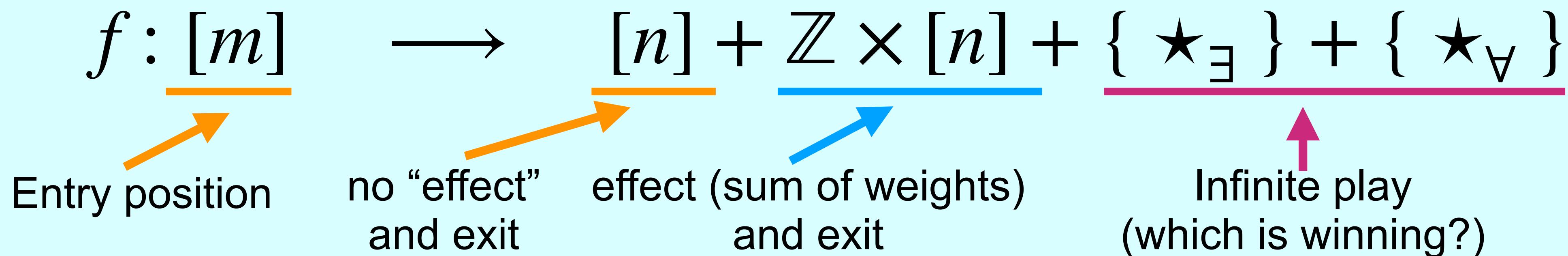
# Category of plays $\mathbf{P}$

Intuition: a morphism in  $\mathbf{P}$  is a result of the play induced by

- a fixed  $\exists$ -strategy and
- a fixed  $\forall$ -strategy

in an open MPG.

Semantic category  
 $\text{Int}(\mathcal{P}_*(\mathcal{P}_*(\mathbf{P})))$



Categorically:  $f: [m] \rightarrow ![n]$  (a Kleisli map), where

$! := \_ + \mathbb{Z} \times \_ + \{\star_{\exists}\} + \{\star_{\forall}\}$  is a monad in Sets.

Category of plays  $\mathbf{P}$ :

objects:  $[m]$  for  $m \in \mathbb{N}$

arrows:  $[m] \longrightarrow [n]$  in  $\mathbf{P}$  is  $[m] \longrightarrow ![n]$  in Sets.

# Category of strategies $\mathcal{P}_*(\mathcal{P}_*(\mathbf{P}))$ by change of base

Semantic category

Int  $\mathcal{P}_*(\mathcal{P}_*(\mathbf{P}))$

The intuition: memoryless strategies are represented by a set of results of plays

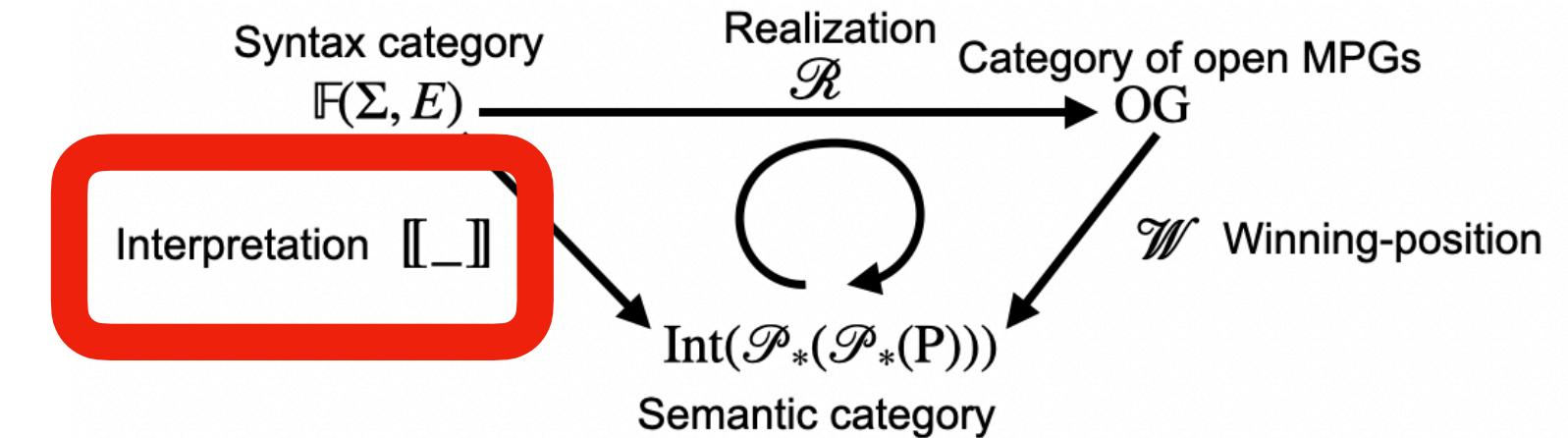
$$f: m \rightarrow n \in \mathcal{P}(\mathcal{P}(\text{hom}_{\mathbf{P}}(m, n)))$$

Memoryless  $\exists$ -strategies      Memoryless  $\forall$ -strategies      Results of plays

Categorically: the powerset functor  $\mathcal{P}$  is a monoidal endo functor on (Sets,  $\mathbf{1}$ ,  $\times$ )

The construction of the category of strategies  $\mathcal{P}_*(\mathcal{P}_*(\mathbf{P}))$  is *change of base*  
[Kelly et al., '82], [Laird, FoSSaCS '17]

# Interpretation functor $\llbracket \_ \rrbracket$



Semantic category  $\text{Int}(\mathcal{P}_*(\mathcal{P}_*(P)))$

Interpretation functor  $\llbracket \_ \rrbracket : \mathbb{F}(\Sigma, E) \rightarrow \text{Int}(\mathcal{P}_*(\mathcal{P}_*(P)))$

- a compact closed functor, freely generated as follows

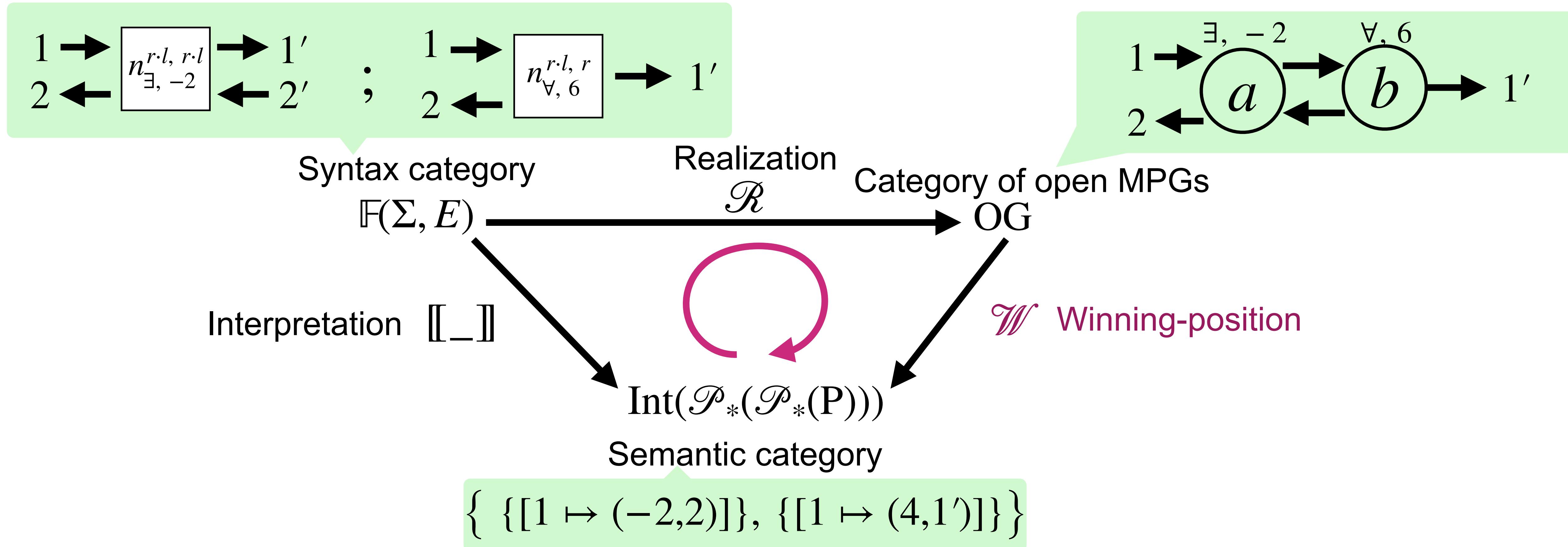
$$\llbracket \begin{bmatrix} 1 \xrightarrow{} n_{\exists, 3}^{r \cdot l, r} \xrightarrow{} 1' \\ 2 \xleftarrow{} \end{bmatrix} \rrbracket = \left\{ \left\{ [1 \mapsto (3, 1')], [1 \mapsto (3, 2)] \right\}, \left\{ [1 \mapsto (3, 1')], [1 \mapsto (3, 2)] \right\} \right\}$$

$$\llbracket \begin{bmatrix} 1 \xrightarrow{} n_{\forall, 3}^{r \cdot l, r} \xrightarrow{} 1' \\ 2 \xleftarrow{} \end{bmatrix} \rrbracket = \left\{ \left\{ [1 \mapsto (3, 1')], [1 \mapsto (3, 2)] \right\} \right\}$$

$$f: \underline{[m]} \rightarrow \underline{[n]} + \underline{\mathbb{Z} \times [n]} + \underline{\{\star_{\exists}\}} + \underline{\{\star_{\forall}\}}$$

Entry position      no “effect” and exit      effect (accum., weight) and exit      Infinite play (which is winning?)

# Our theoretical contributions: compositional approach to MPGs



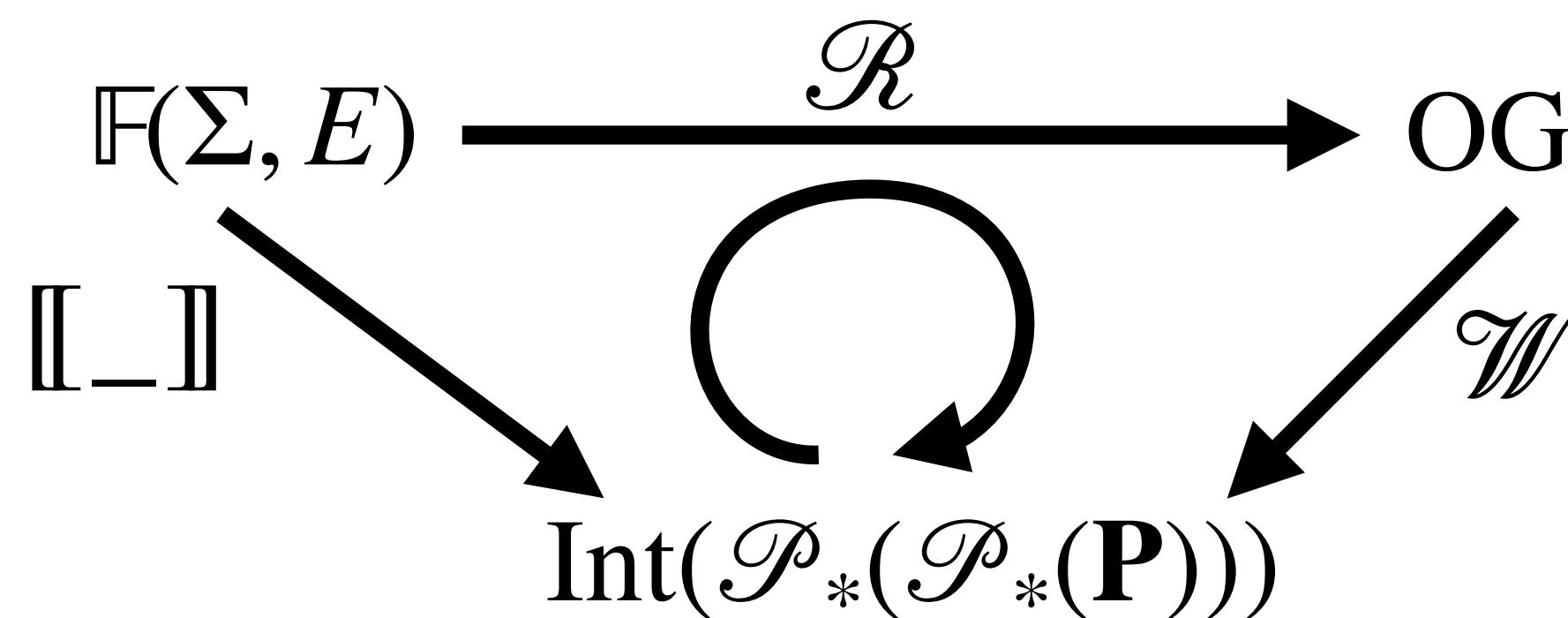
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# Winning position functor $\mathcal{W}$ and the main theorem

- winning-position functor  $\mathcal{W}$  returns whether an entry position is **winning**, **losing**, or **pending**.
  - $\mathcal{W}$  is defined in the traditional way by using plays and memoryless strategies.
  - pending**: intermediate result due to the notion of **open ends**

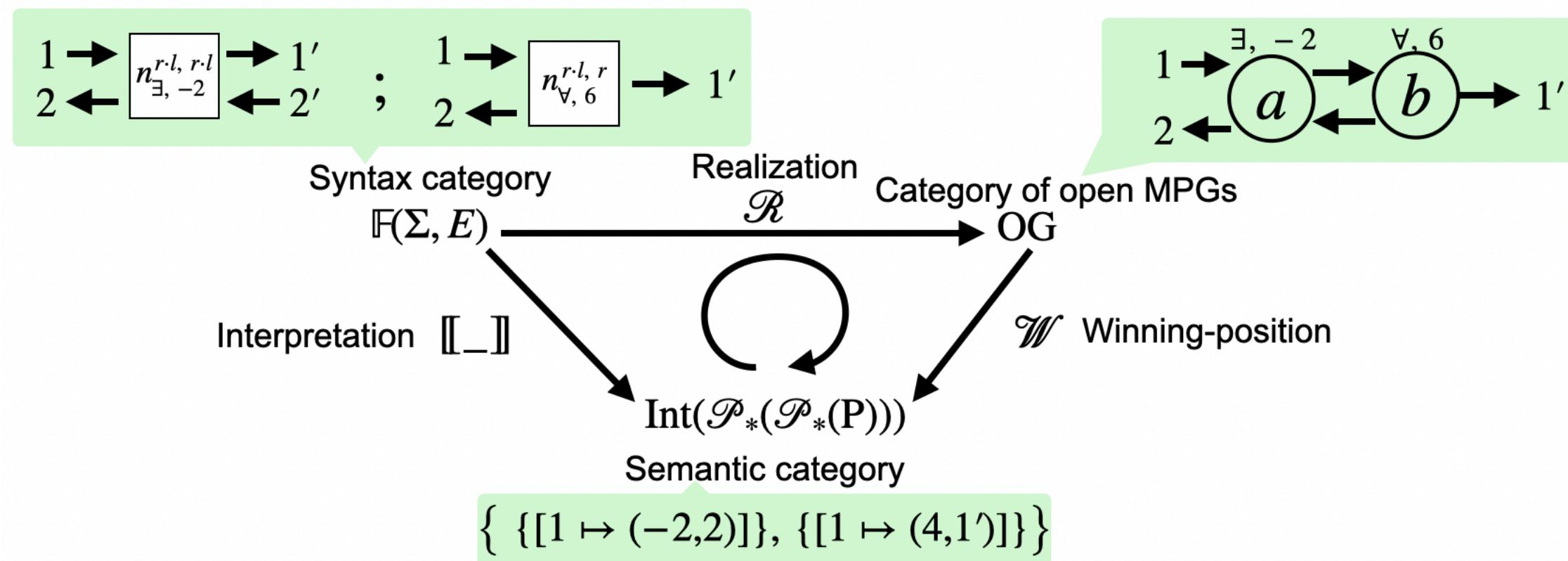
*Thm:  $\mathcal{W}$  is a compact closed functor*

*Thm (main): The triangle commutes:  $\llbracket \_ \rrbracket \simeq \mathcal{W} \circ \mathcal{R}$*



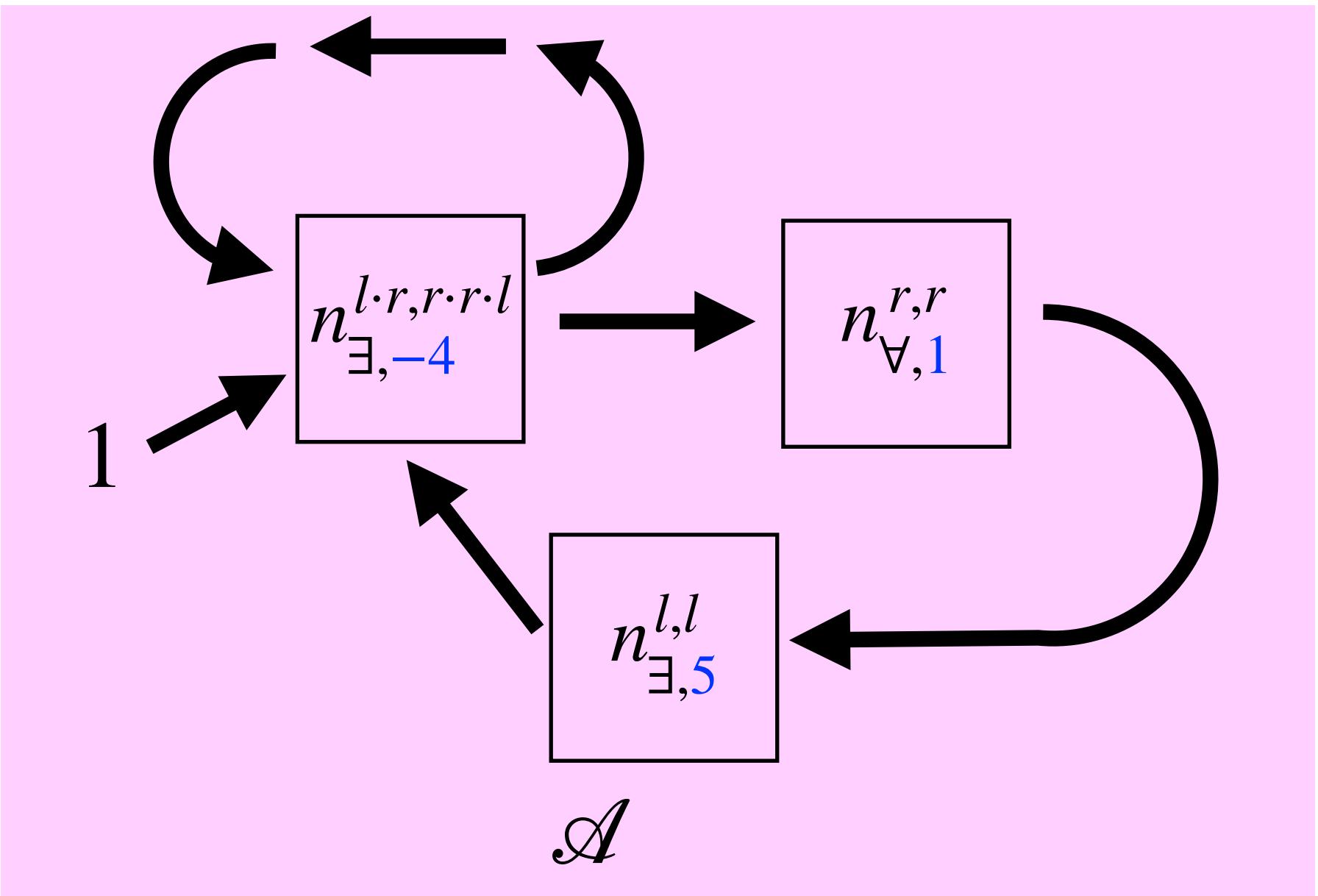
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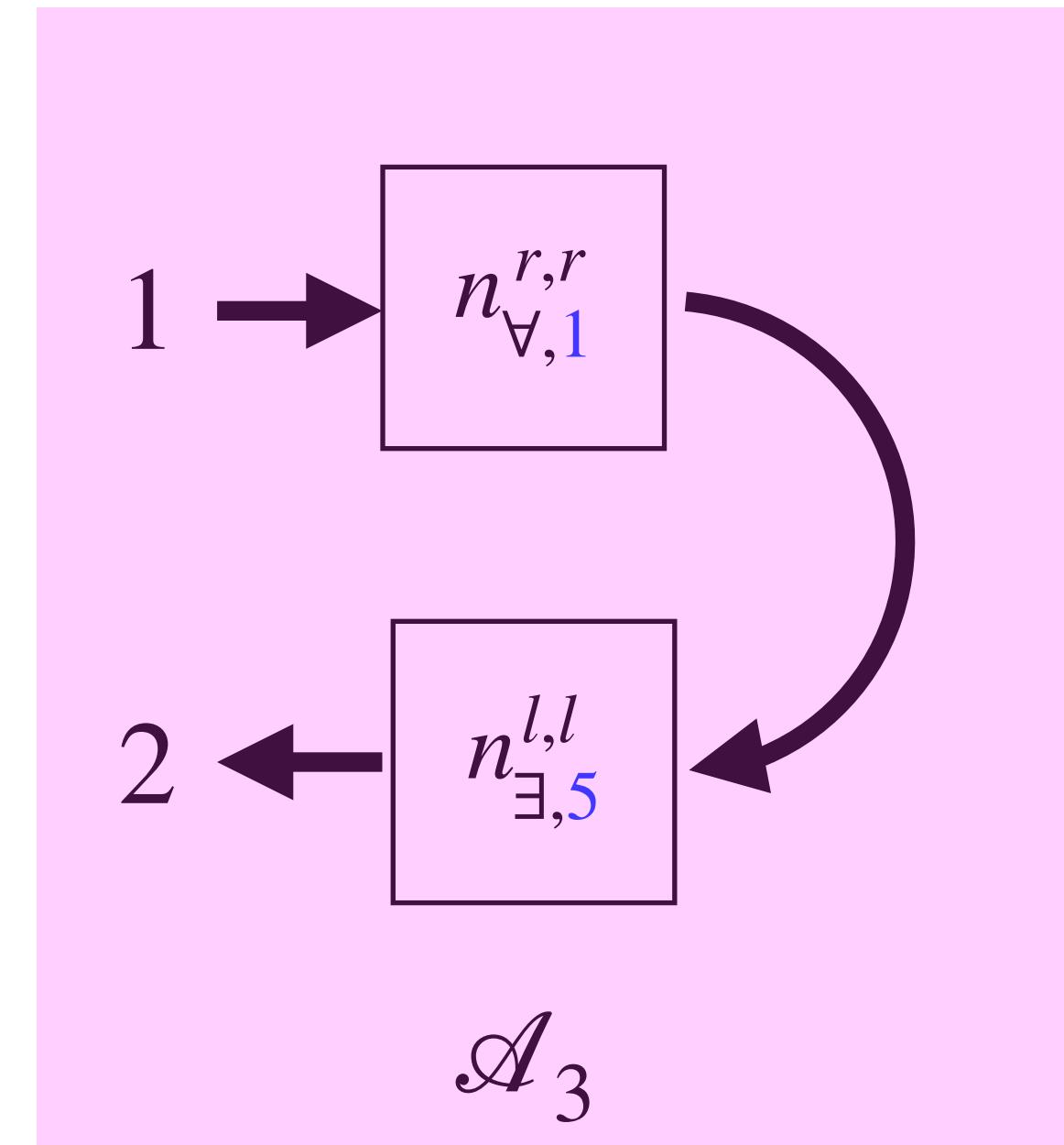
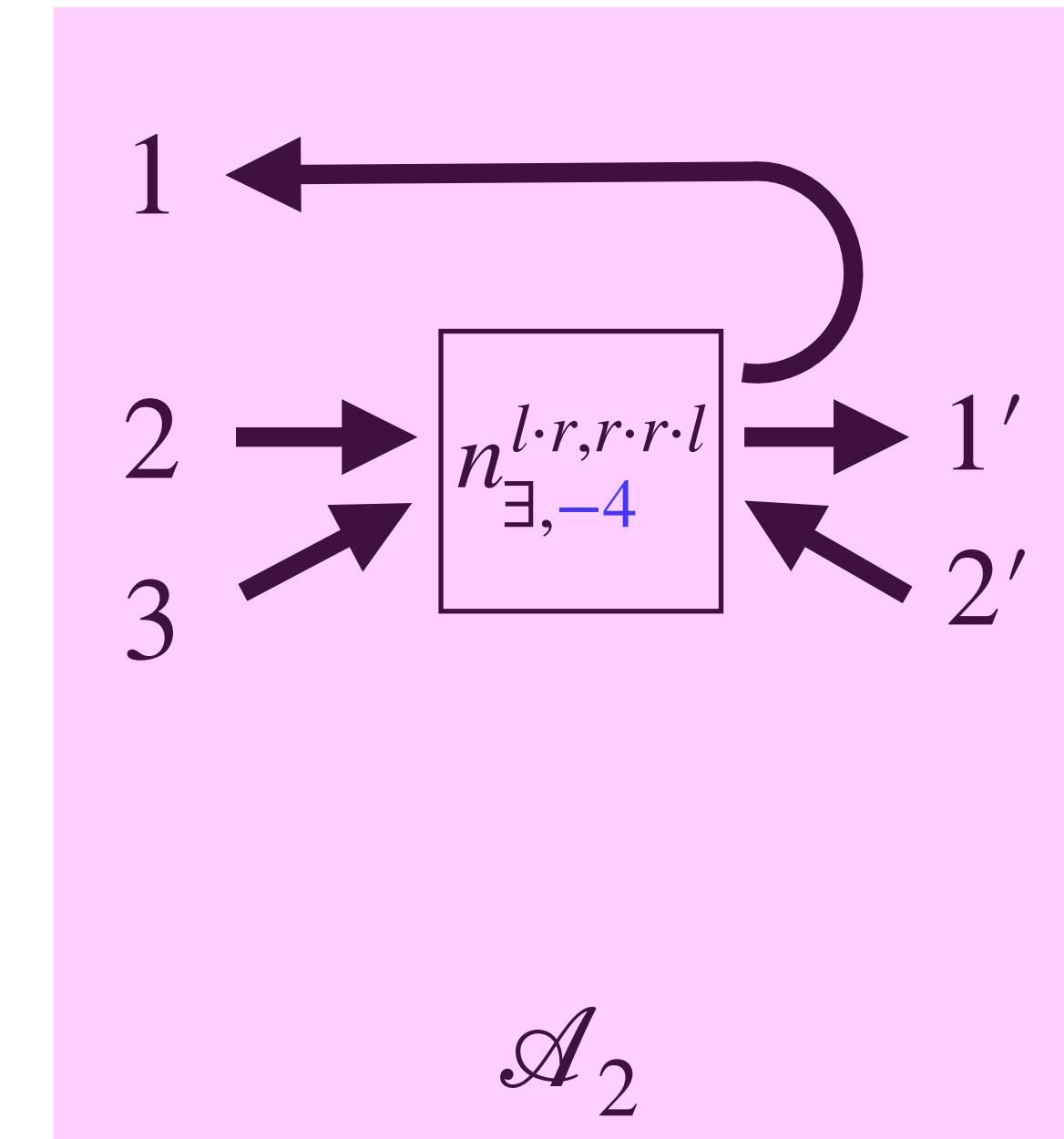
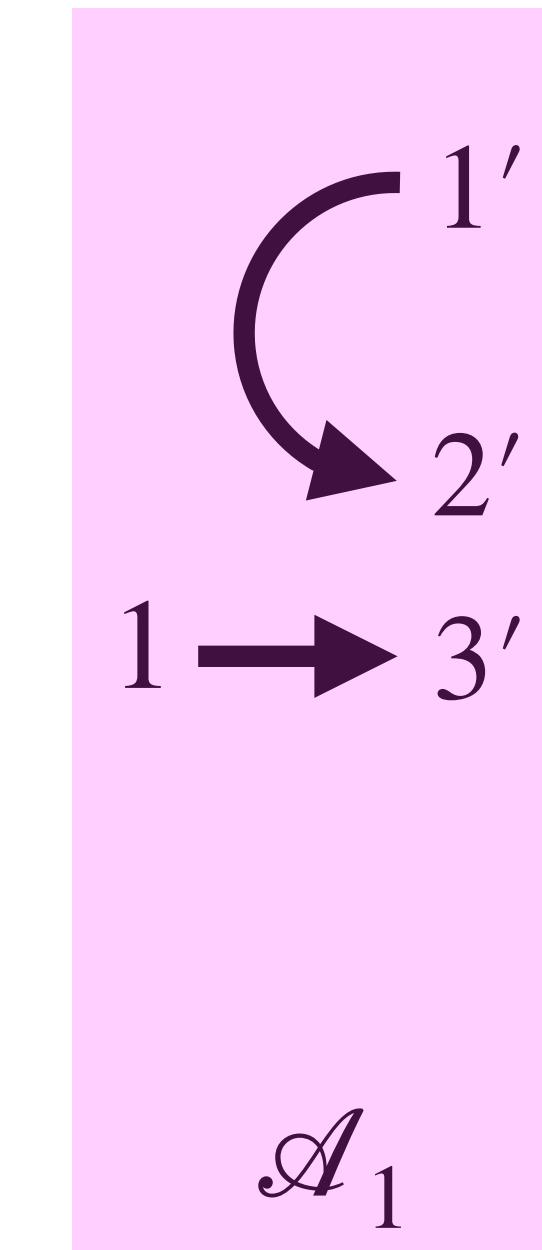
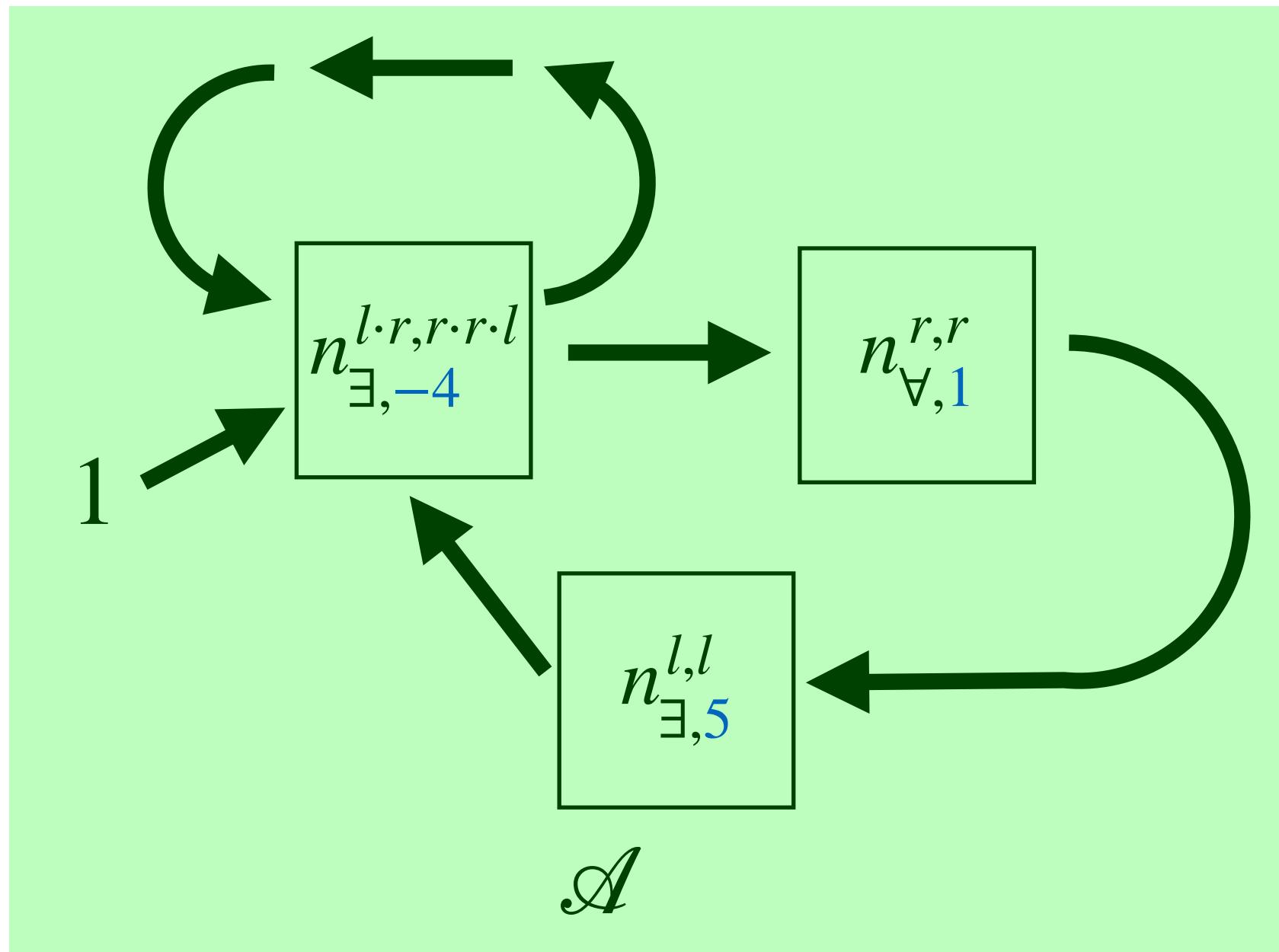


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# Our compositional algorithm: CMGS



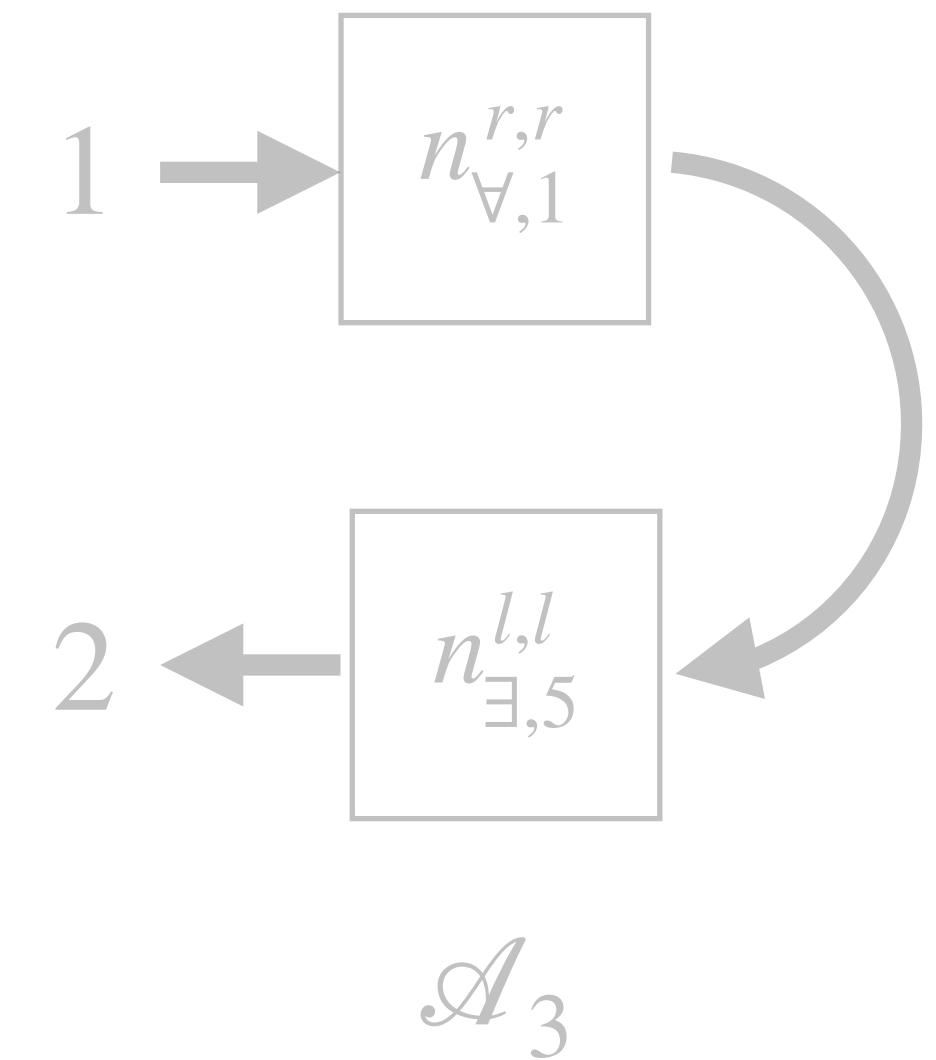
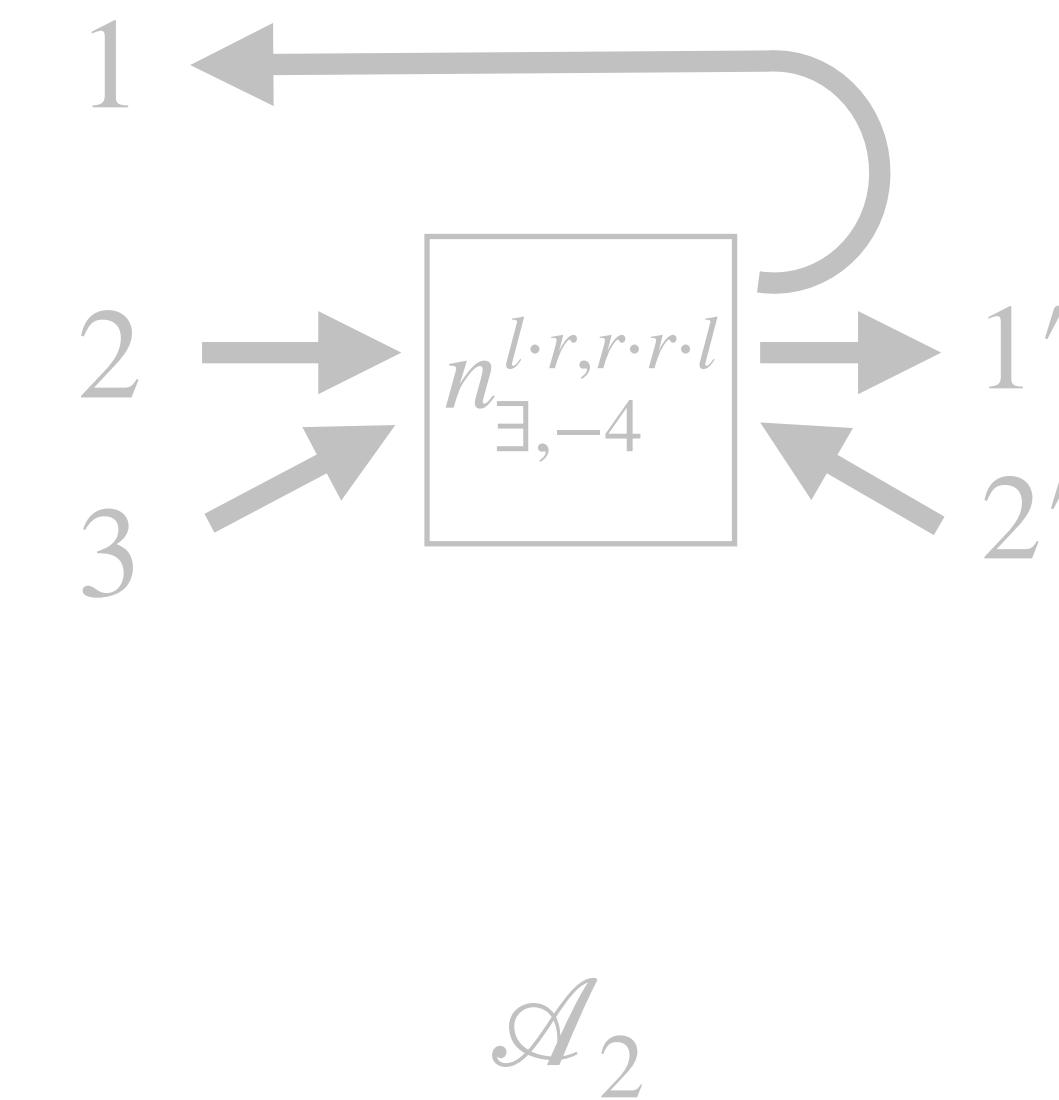
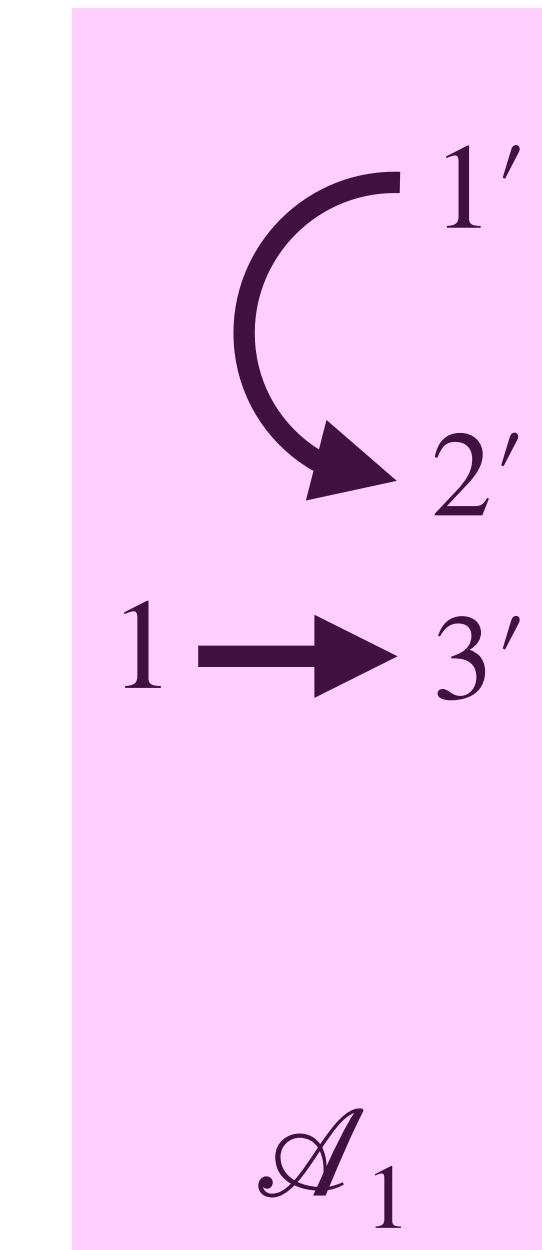
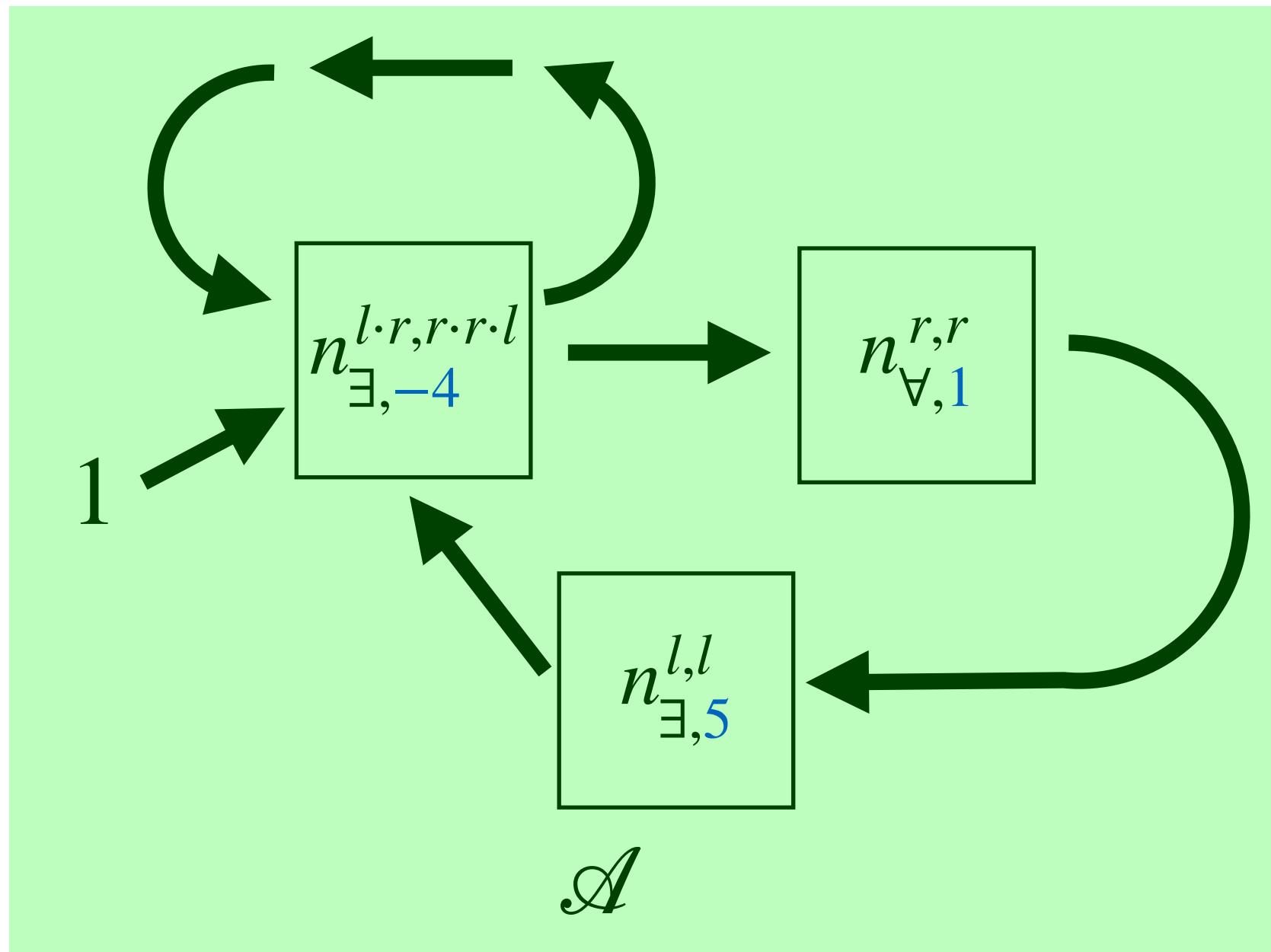
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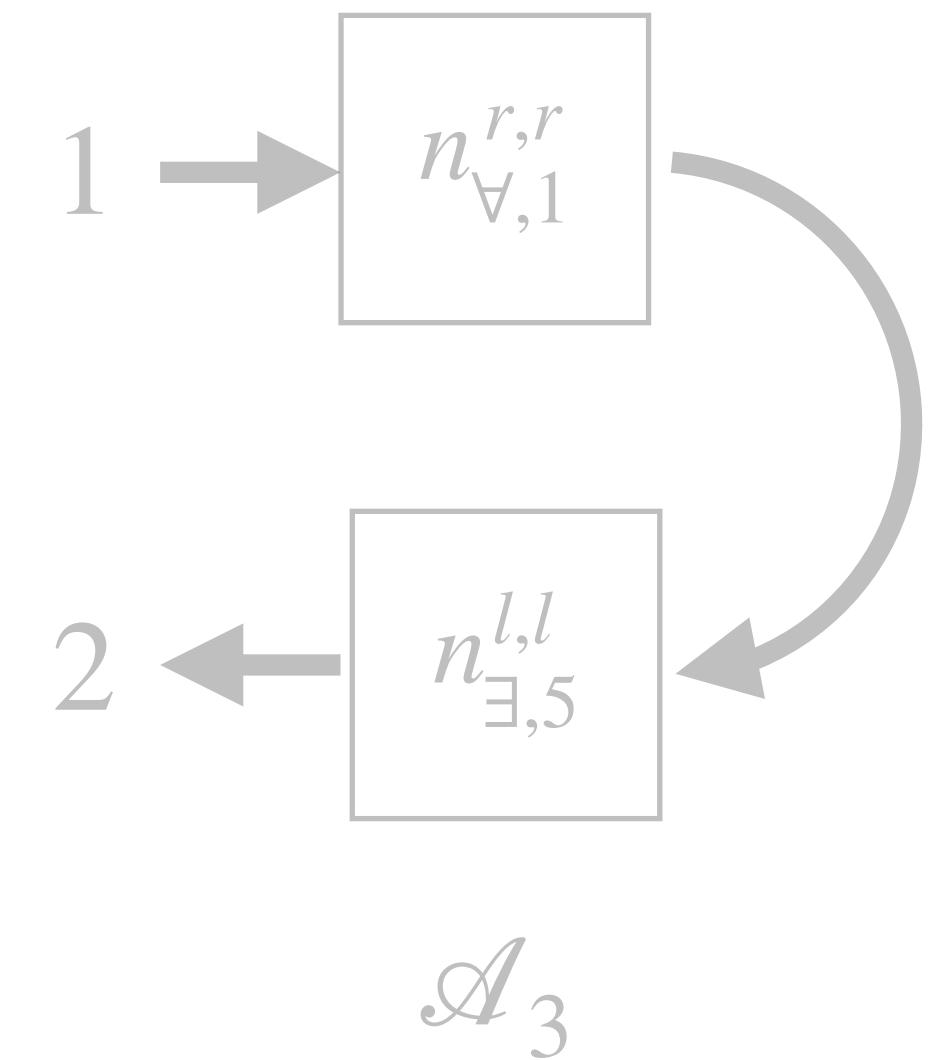
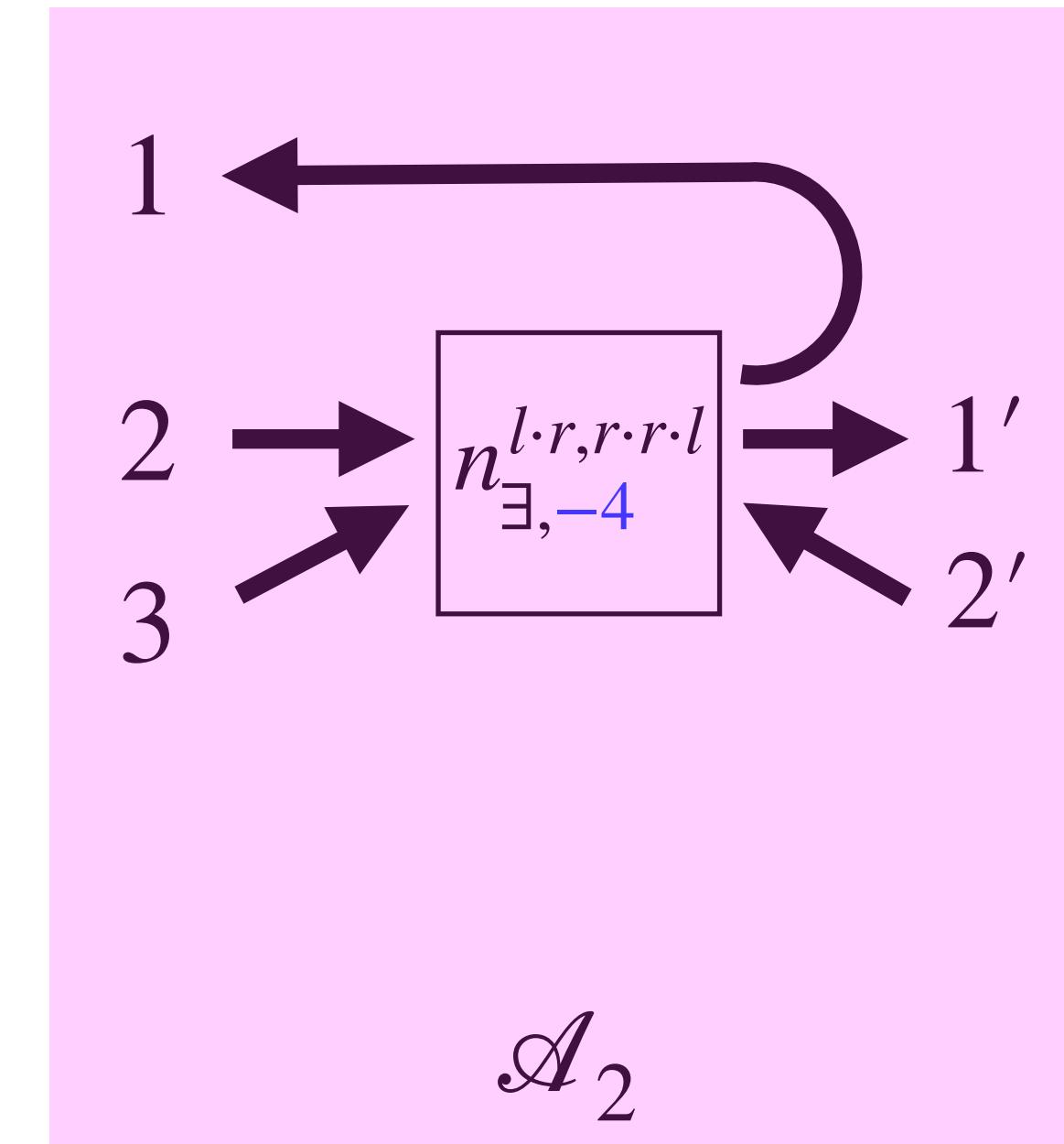
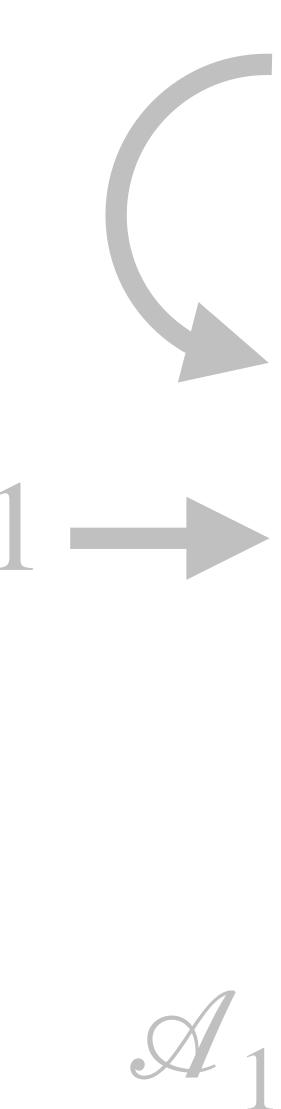
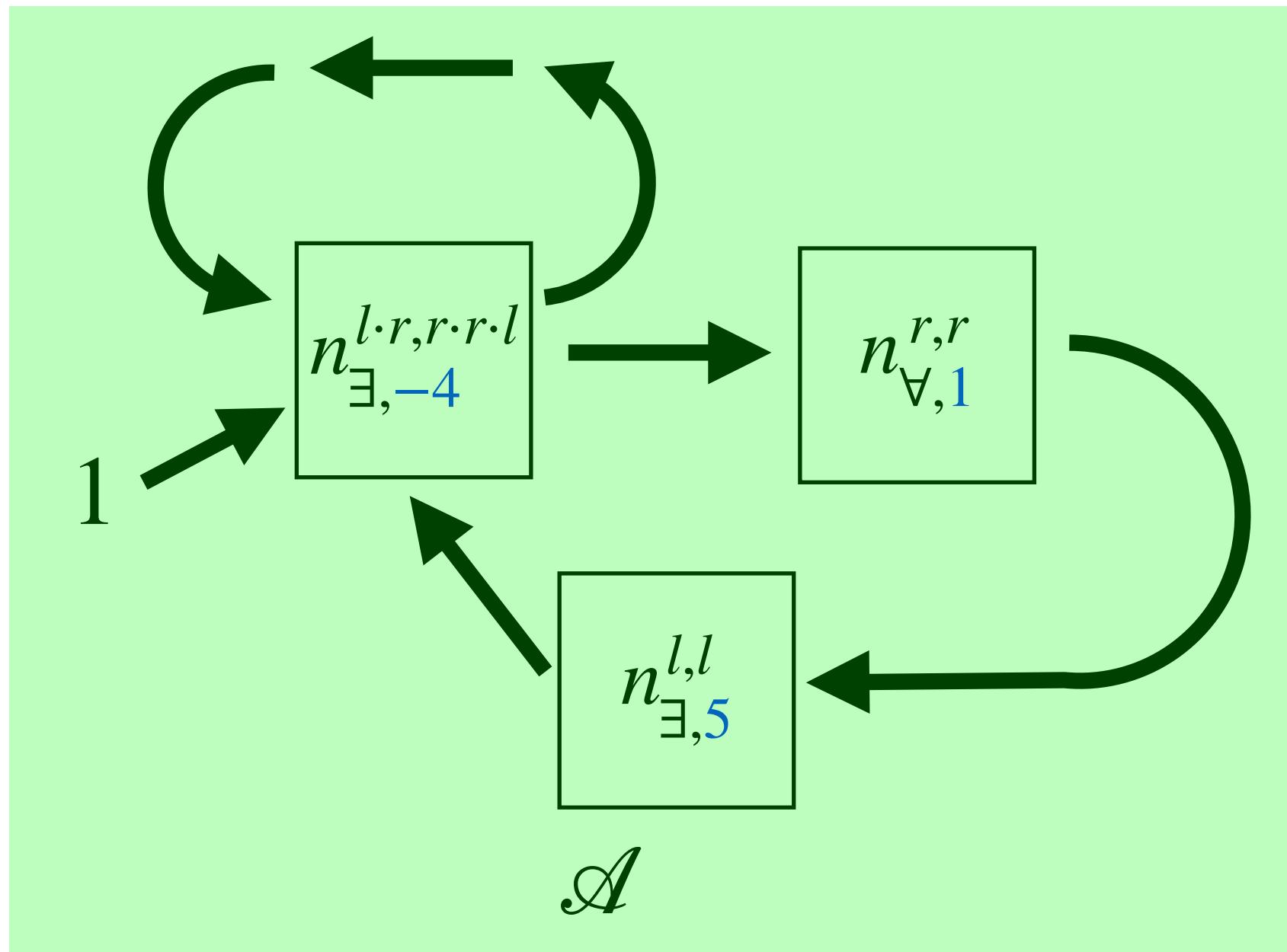
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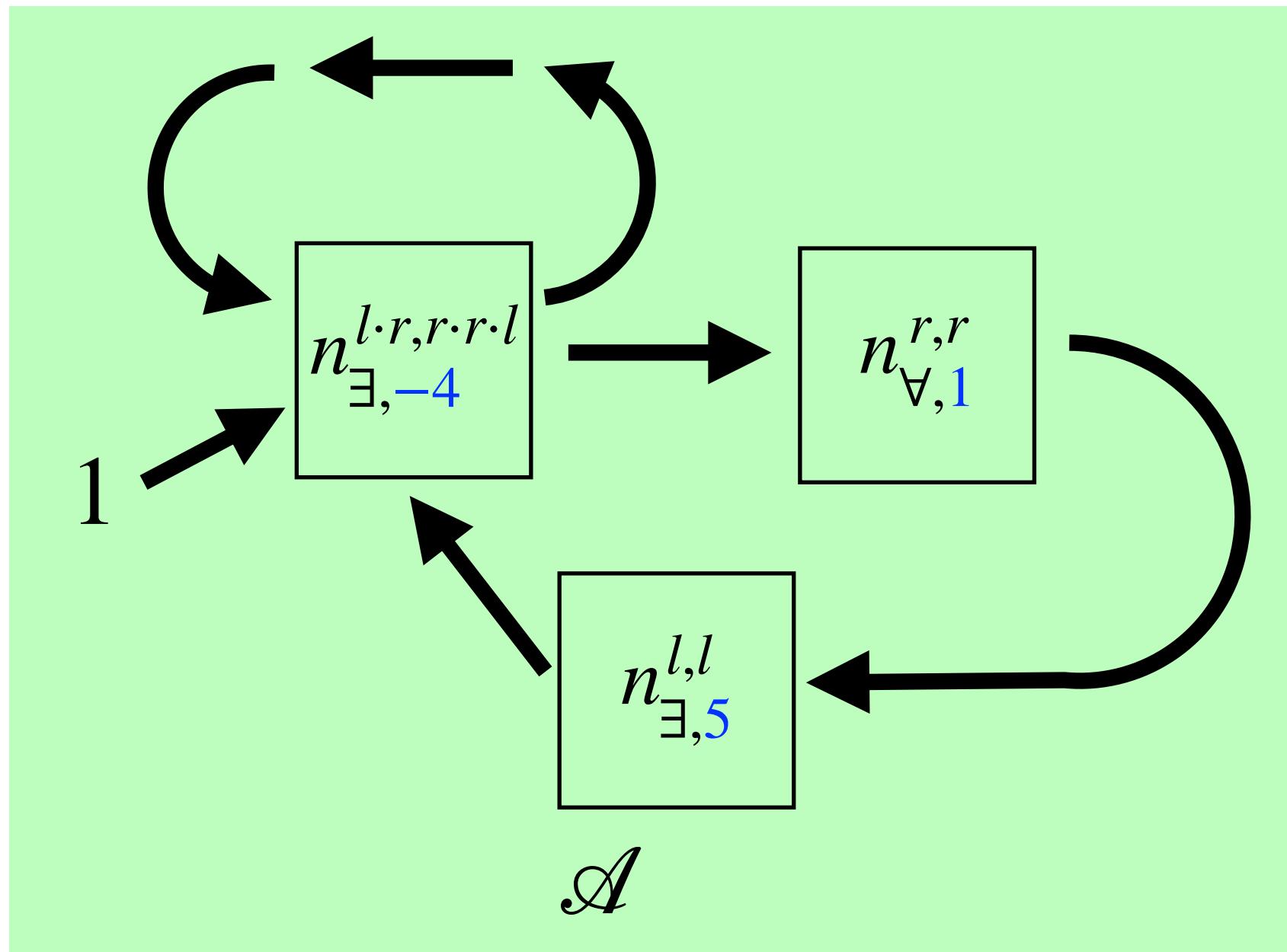
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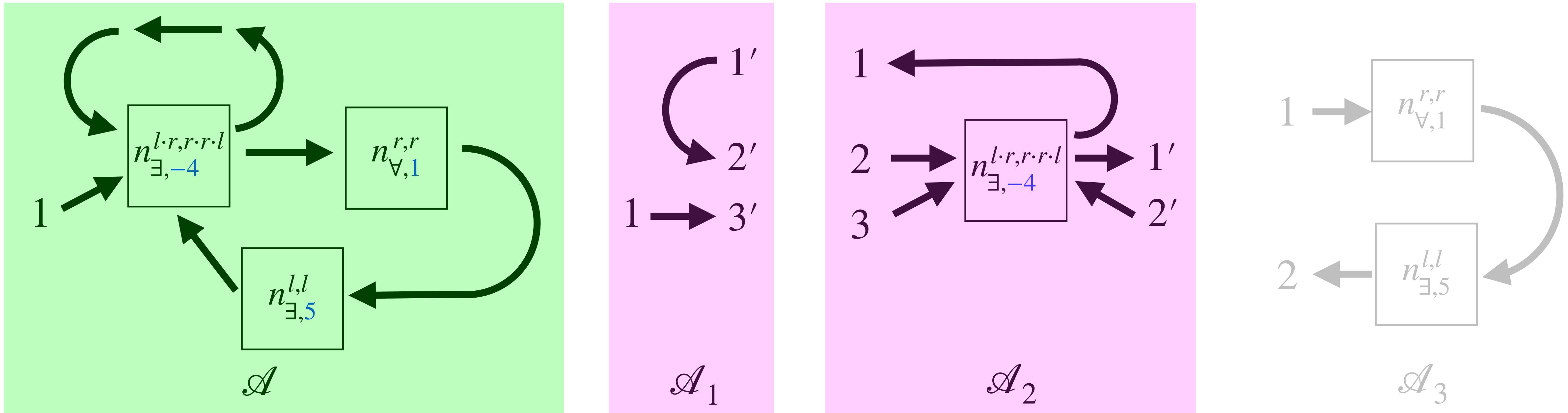
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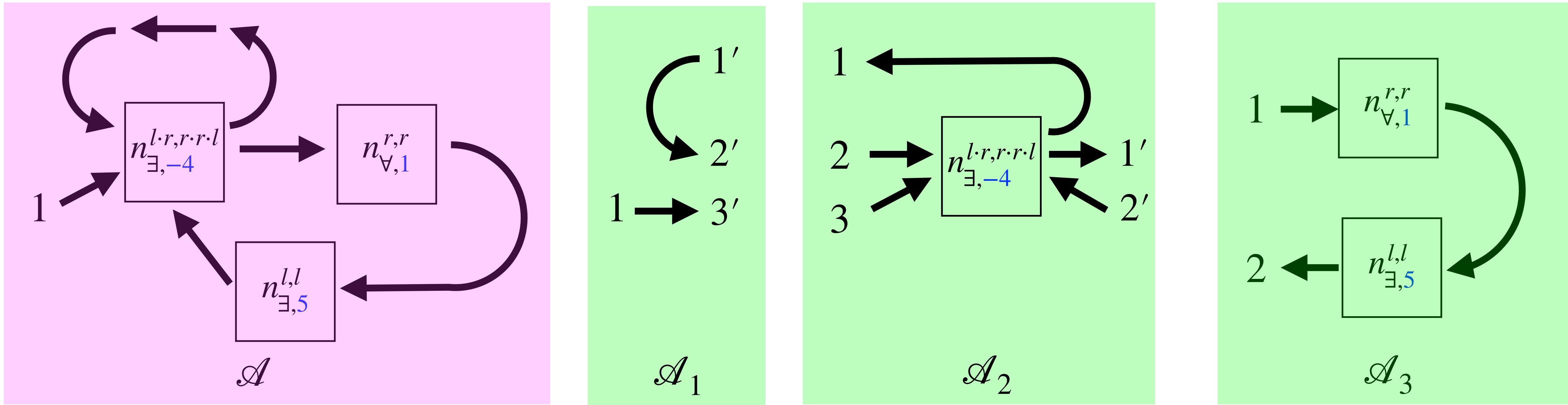
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# Our compositional algorithm: CMGS

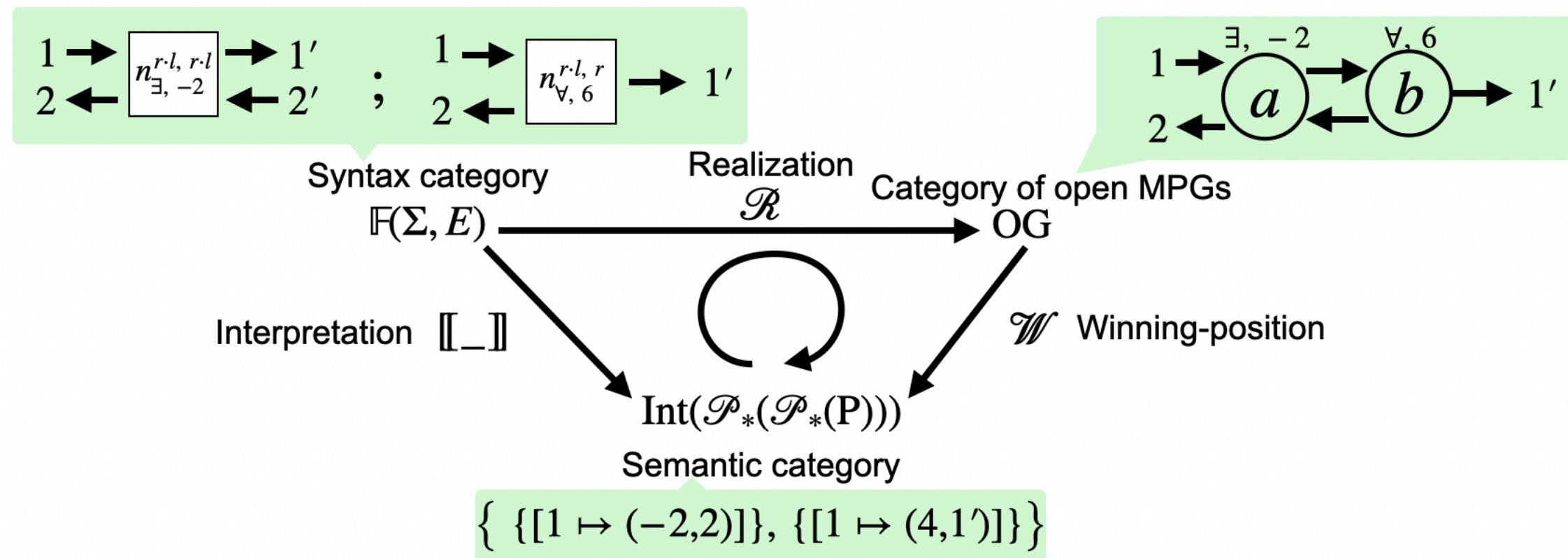


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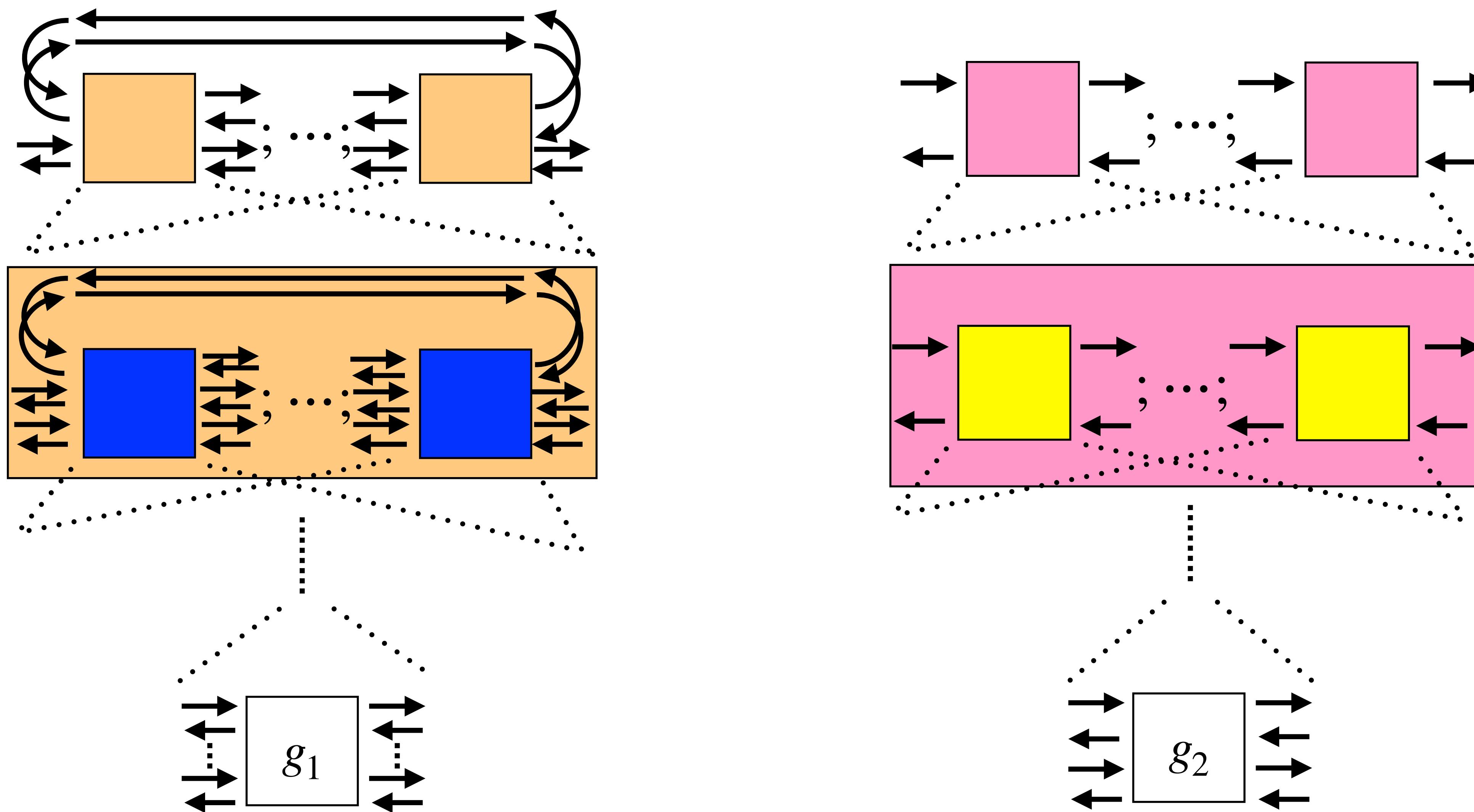
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# Inputs used in experiment (1)

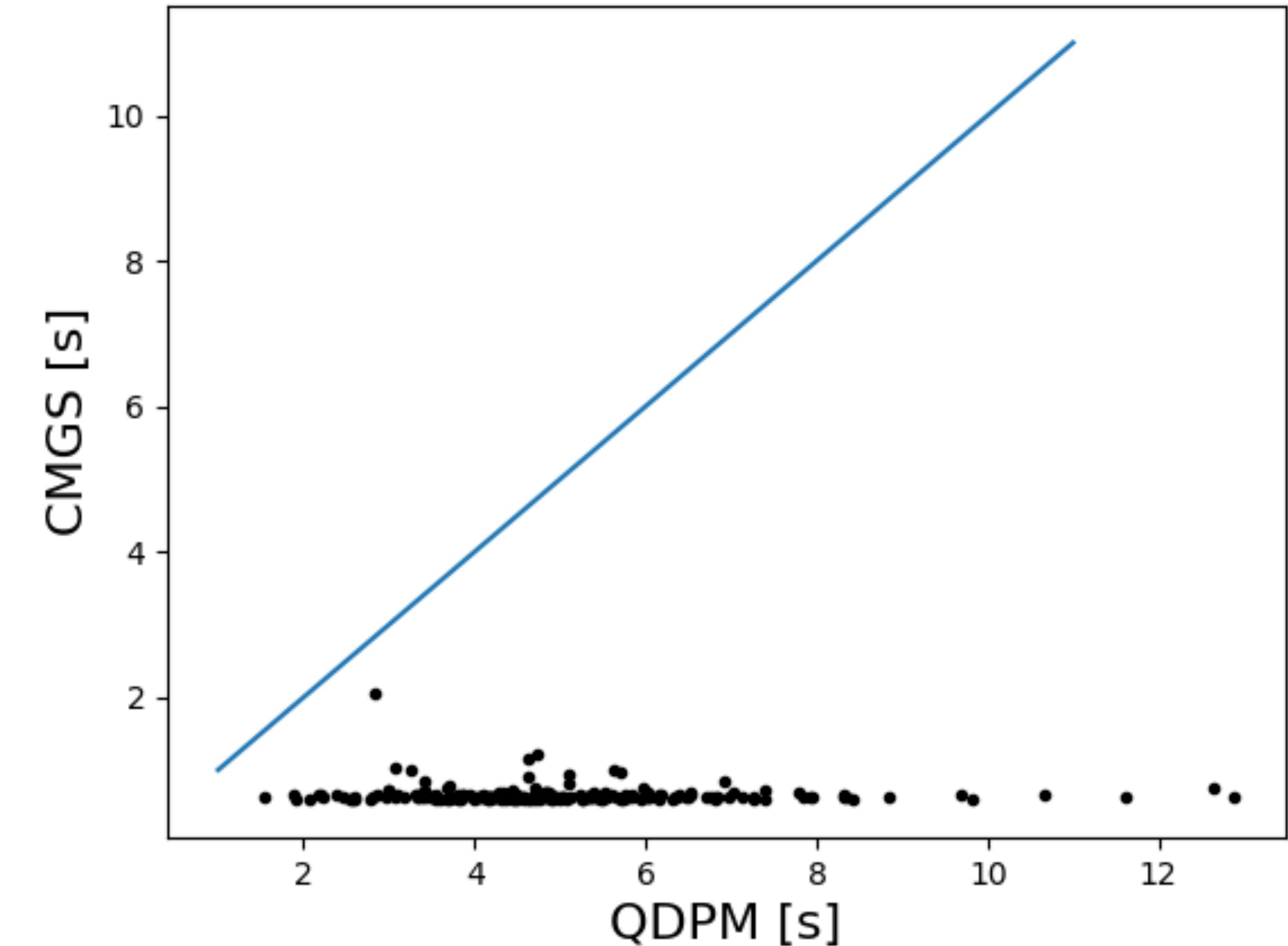
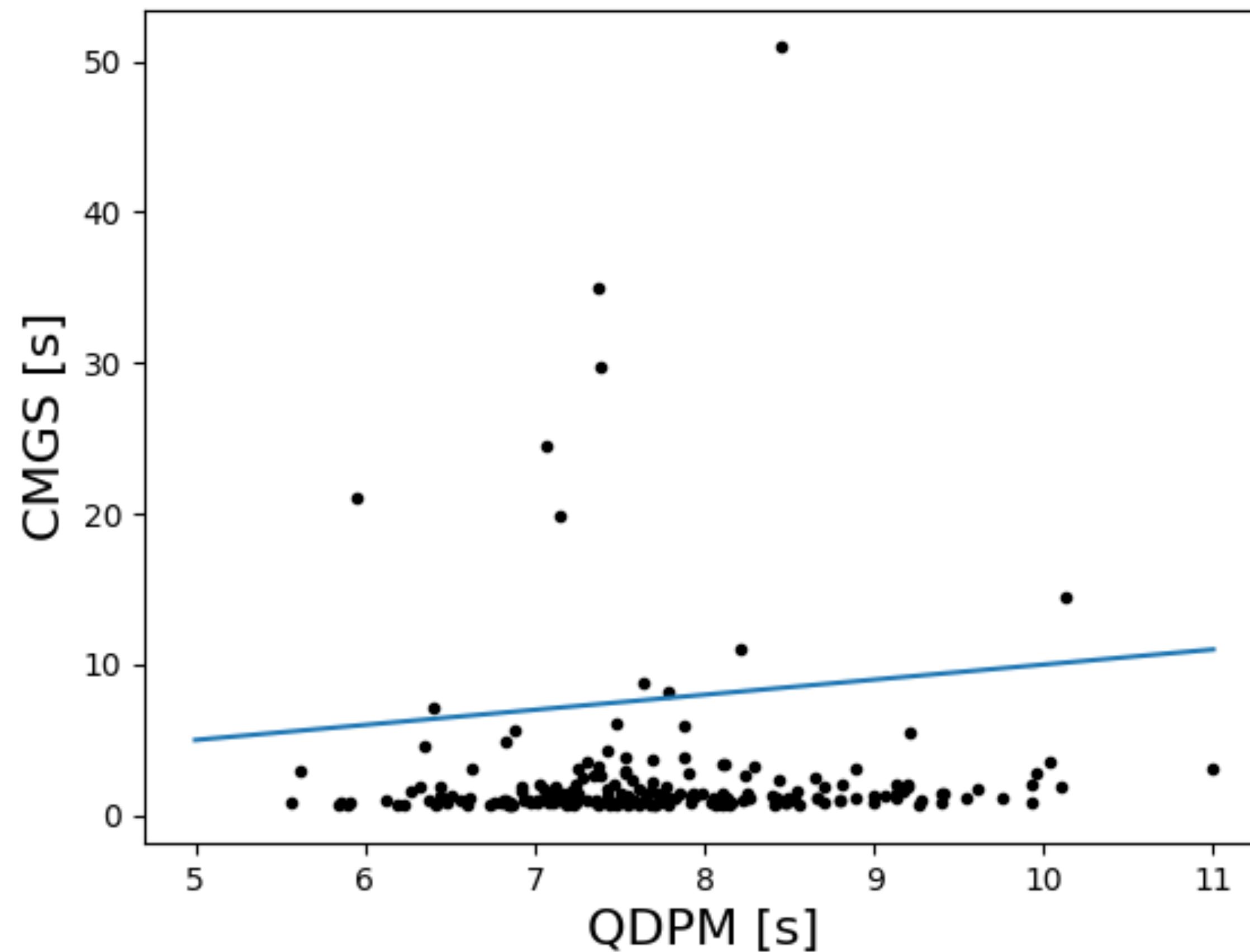


- Open MPGs  $g_1$  and  $g_2$  are randomly generated

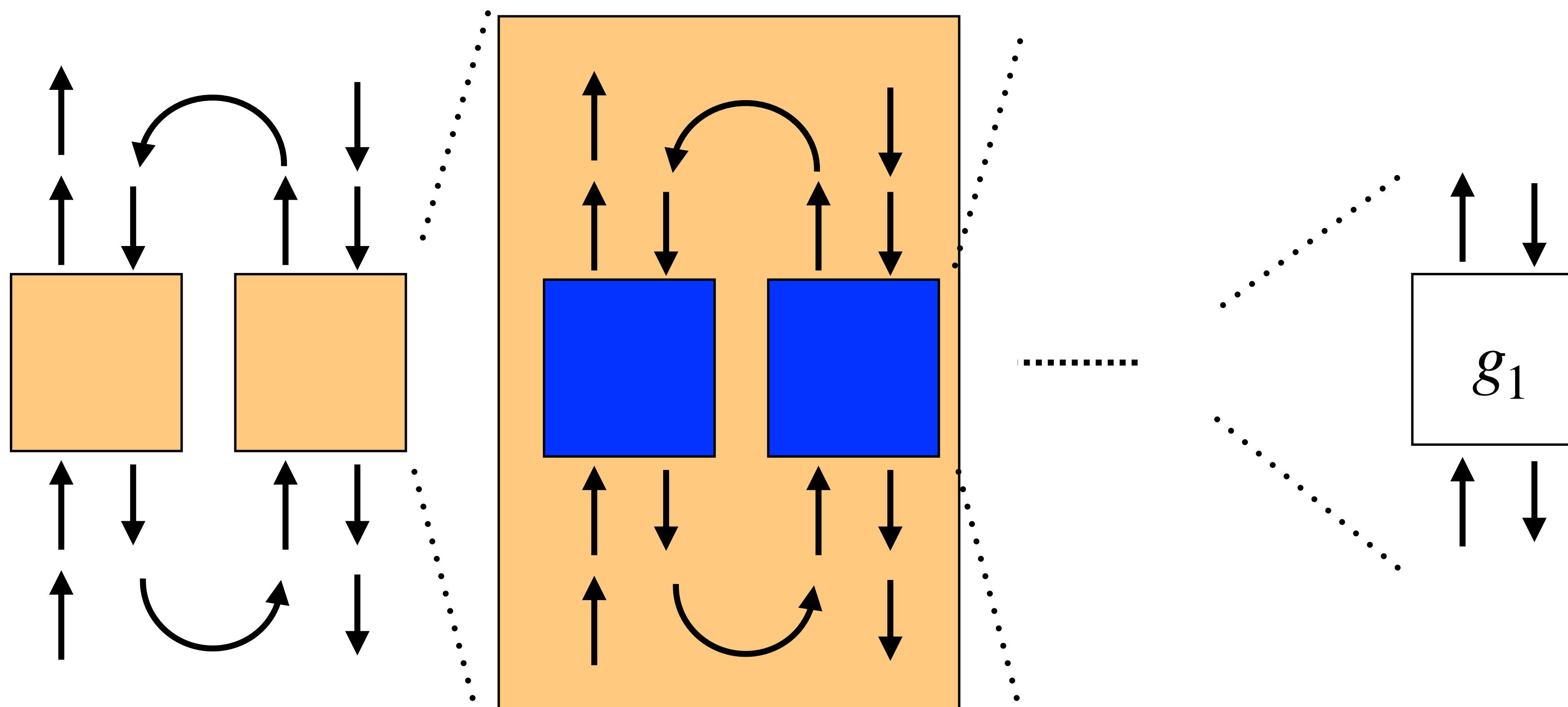
# Experiment result (1)

Env:

Amazon EC2 t2.xlarge instance, 2.30GHz Intel Xeon E5-2686, 4 virtual CPU cores, 16 GB RAM



# Inputs used in experiment (2)

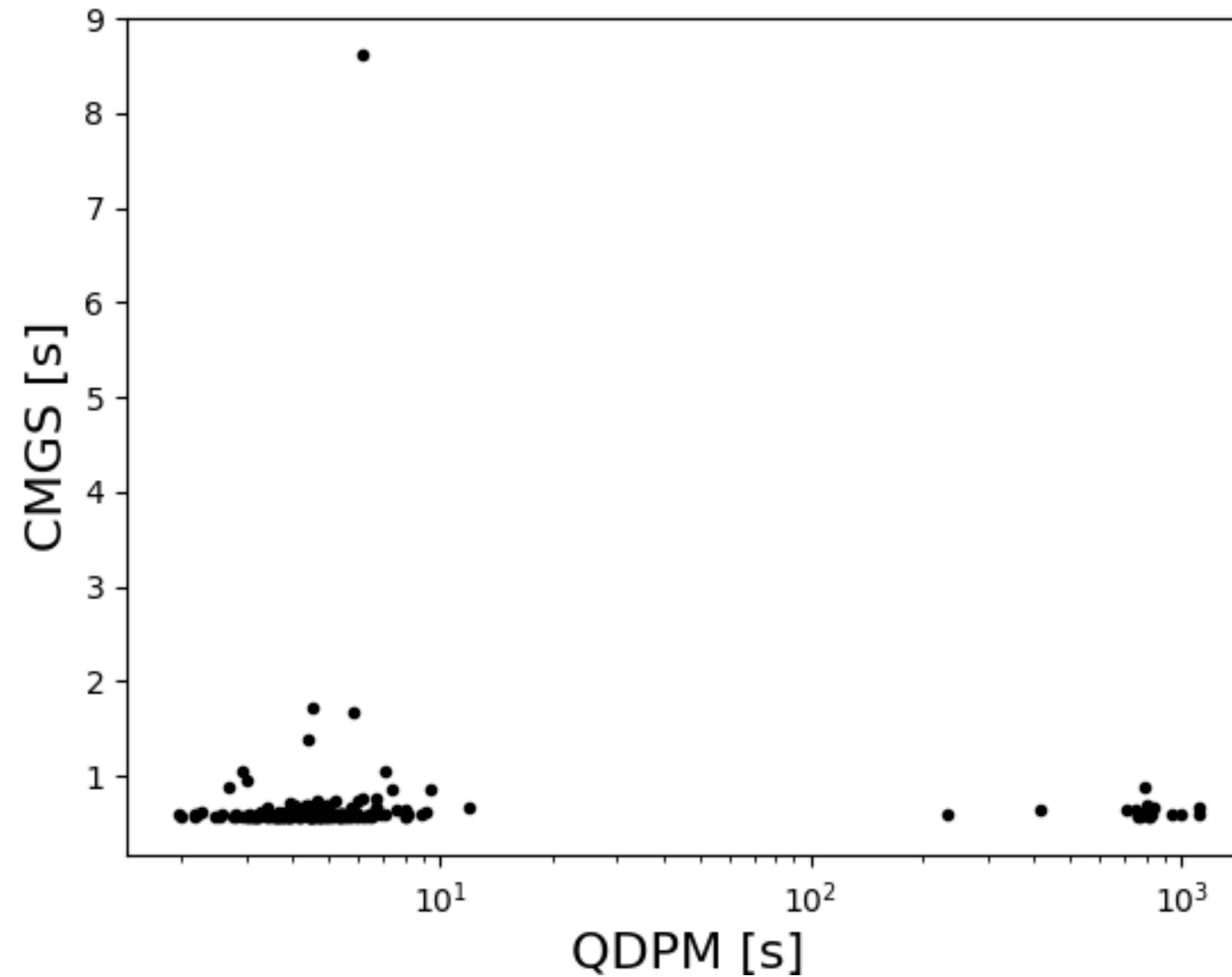


An open MPG  $g_1$  is randomly generated.

# Experiment result (2)

Env:

Amazon EC2 t2.xlarge instance, 2.30GHz Intel Xeon E5-2686, 4 virtual CPU cores, 16 GB RAM



# Observation

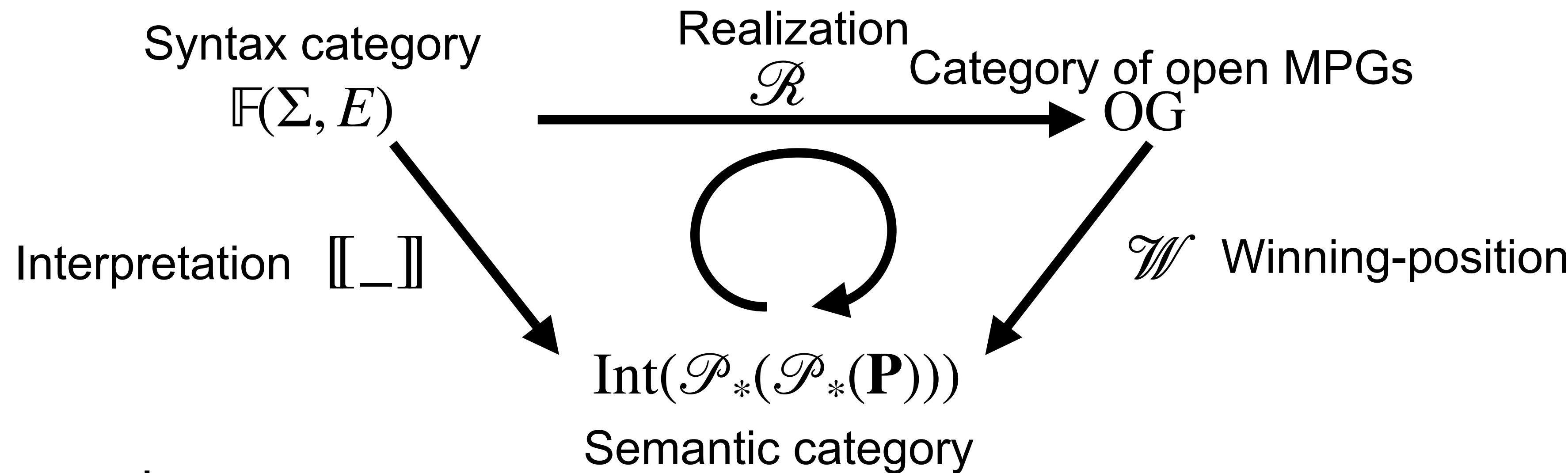
- RQ: What kind of characteristics in the inputs make CMGS faster?
  - Sharing (a lot of) subgames
  - Smaller size of subgame's domains and codomains
- Our framework CMGS outperforms QDPM [Benerecetti, Dell'Erba, Mogavero, TACAS'20] for the data which has the characteristics.

## Related work

- Prop as graphical language
  - Petri net [Bonchi, Holland, Piedeleu, Sobociński, Zanasi, POPL'19 ]
  - Automata [Piedeleu & Zanasi, FoSSaCS'21]
- Open games [Ghani, Hedges, Winschel, Zahn, LICS'18]
  - different kind of games (for economics)
- Open parity games [W., Eberhart, Asada, Hasuo, MFPS'21]
  - different semantic category
  - no algorithms and experiments

# Conclusion and future work

- Conclusion: compositional approach to MPG



- Future work:
  - additional experiments and analysis
    - try with realistic benchmarks
    - find good applications where compositionality are advantageous
  - general compositional framework for positional games
    - discounted game
    - simple stochastic game