Duplicable von Neumann Algebras

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Recently, we have shown that von Neumann algebras form a model for Selinger and Valiron's quantum lambda calculus. In this paper, we explain our choice of interpretation of the duplicability operator "!" by studying those von Neumann algebras that might have served as the interpretation of duplicable types, namely those that carry a (commutative) monoid structure with respect to the spatial tensor product. We show that every such monoid is the (possibly infinite) direct product of the complex numbers, and that our interpretation of the ! operator of the quantum lambda calculus is exactly the free (commutative) monoid.

1 Introduction

The quantum lambda calculus [21], introduced by Selinger and Valiron, is a typed programming language that contains not only a qubit type, but also a function type $A \multimap B$, and a 'duplicable' type $!A$. It has an (affine) linear type system, where each variable may be used at most once unless it has type $!A$, reflecting the no-cloning property of qubits. In [5], we gave a model of the quantum lambda calculus based on the category $\mathbf{vNA}_{\mathrm{MIU}}$ of von Neumann algebras and normal unital $*$-homomorphisms (normal MIU-maps, for short). In this paper, we elaborate on the interpretation of "!" we chose in this model.

Recall that the duplicability operator "!" is what is called the exponential modality in linear logic. It is known that a categorical model of the ! modality is a so-called *linear exponential comonad* L on a symmetric monoidal category (SMC) $(\mathbf{C}, \otimes, I)$, where each object of the form LA is equipped with a commutative comonoid structure in the SMC $\mathbf{C}$:

$$LA \longrightarrow LA \otimes LA \qquad\qquad LA \longrightarrow I$$

The maps corresponds respectively to duplication (contraction rule) and discarding (weakening rule). Benton [3] later gave a reformulation of a linear exponential comonad as a *linear-non-linear adjunction*

$$\begin{tikzcd} (\mathbf{B}, \times, 1) \arrow[r, bend left] \arrow[r, phantom, "\perp"] & (\mathbf{C}, \otimes, I) \arrow[l, bend left] \end{tikzcd} ,$$

which induces a linear exponential comonad on $\mathbf{C}$. We refer to [17, 21] for more details.

To construct a model by von Neumann algebras in [5], we used the linear-non-linear adjunction[1]

$$\begin{tikzcd} (\mathbf{Set}, \times, 1) \arrow[r, bend left, "\ell^\infty"] \arrow[r, phantom, "\perp"] & (\mathbf{vNA}_{\mathrm{MIU}}^{\mathrm{op}}, \otimes, \mathbb{C}). \arrow[l, bend left, "\mathrm{nsp}"] \end{tikzcd}$$

Here $\mathrm{nsp}(\mathscr{A}) = \mathbf{vNA}_{\mathrm{MIU}}(\mathscr{A}, \mathbb{C})$ is the set of normal MIU-maps $\varphi \colon \mathscr{A} \to \mathbb{C}$ for any von Neumann algebra $\mathscr{A}$, and $\ell^\infty(X)$ is the von Neumann algebra of bounded functions $f \colon X \to \mathbb{C}$ for any set X. Then we

[1] The opposite category $\mathbf{vNA}_{\mathrm{MIU}}^{\mathrm{op}}$ is used here, because maps between von Neumann algebras are seen as observable/predicate transformers (Heisenberg's view), which are dual to state transformers (Schrödinger's view).

Preprint
March 2016

interpret $!A$ by $[\![!A]\!] = \ell^\infty(\mathrm{nsp}([\![A]\!]))$, which carries a commutative monoid structure in the SMC $\mathbf{vNA}_{\mathrm{MIU}}$ with the spatial tensor product $\otimes$. The interpretation indeed works well in the sense that our model is adequate with respect to the operational semantics of the quantum lambda calculus.

However, von Neumann algebras of the form $\ell^\infty(X)$ are arguably quite special, excluding general Abelian von Neumann algebras like $L^\infty([0,1])$. The natural question arises whether there is a broader class of von Neumann algebras that might serve as the interpretation of duplicable types. This paper gives an answer to this question, by the definition and theorem below.

Definition 1. A von Neumann algebra $\mathscr{A}$ is **duplicable** if there is a **duplicator** on $\mathscr{A}$, that is, a positive subunital linear map $\delta\colon \mathscr{A}\otimes\mathscr{A} \to \mathscr{A}$ with a **unit** $u \in \mathscr{A}$ such that $0 \le u \le 1$, satisfying

$$\delta(a\otimes u) \;=\; a \;=\; \delta(u\otimes a) \quad \text{for all } a \in \mathscr{A}.$$

The unit u can be identified with a positive subunital map $\tilde{u}\colon \mathbb{C} \to \mathscr{A}$ via $\tilde{u}(\lambda) = \lambda u$. The definition is motivated by the fact that the interpretation of $!A$ must carry a commutative monoid structure in $\mathbf{vNA}_{\mathrm{MIU}}$. The condition is weaker, requiring the maps to be only positive subunital, and dropping associativity and commutativity. Nevertheless this is sufficient to prove:

Theorem 2. *A von Neumann algebra $\mathscr{A}$ is duplicable if and only if $\mathscr{A}$ is MIU-isomorphic to $\ell^\infty(X)$ for some set X. In that case, the duplicator (δ, u) is unique, given by $\delta(a\otimes b) = a\cdot b$ and $u = 1$.*

Thus, to interpret duplicable types, we can really only use von Neumann algebras of the form $\ell^\infty(X)$. It also follows that a von Neumann algebra is duplicable precisely when it is a (commutative) monoid in $\mathbf{vNA}_{\mathrm{MIU}}$, or in the symmetric monoidal category $\mathbf{vNA}_{\mathrm{CPsU}}$ of von Neumann algebras and normal completely positive subunital (CPsU) maps.

We further justify our choice, $[\![!A]\!] = \ell^\infty(\mathrm{nsp}([\![A]\!]))$, by proving that $\ell^\infty(\mathrm{nsp}(\mathscr{A}))$ is the free (commutative) monoid on $\mathscr{A}$ in $\mathbf{vNA}_{\mathrm{MIU}}$. As a corollary, we also obtain that $\ell^\infty(\mathbf{vNA}_{\mathrm{CPsU}}(\mathscr{A},\mathbb{C}))$ is the free (commutative) monoid on $\mathscr{A}$ in $\mathbf{vNA}_{\mathrm{CPsU}}$.

The paper is organised as follows. Preliminaries are given in Section 2, and the proof of Theorem 2 in Section 3. We study monoids in the SMCs $\mathbf{vNA}_{\mathrm{MIU}}$ and $\mathbf{vNA}_{\mathrm{CPsU}}$ in Section 4, proving that $\ell^\infty(\mathrm{nsp}(\mathscr{A}))$ is the free monoid. We end the paper with a discussion of variations (Dauns' categorical tensor product, C^*-algebras), and related work (on broadcasting and cloning of states) in Section 5.

2 Preliminaries

We refer the reader to the literature for details on von Neumann algebras [12, 20, 22] and complete positivity [19]. Given a von Neumann algebra $\mathscr{A}$, we write $[0,1]_{\mathscr{A}} = \{a \in \mathscr{A} \mid 0 \le a \le 1\}$, and we shorten $1-a$ to $a^\perp$ for all $a \in [0,1]_{\mathscr{A}}$.

2.1 Monoids in monoidal categories

Let $(\mathbf{C}, \otimes, I)$ be a symmetric monoidal category (SMC). A **monoid** in $\mathbf{C}$ is an object $A \in \mathbf{C}$ with a 'multiplication' map $m\colon A\otimes A \to A$ and a 'unit' map $u\colon I \to A$ satisfying the associativity and the unit law, i.e. making the following diagrams commute.

$$\begin{tikzcd}
(A\otimes A)\otimes A \arrow[rr, "m\otimes\mathrm{id}"] \arrow[d, "\alpha"'] & & A\otimes A \arrow[d, "m"] \\
A\otimes(A\otimes A) \arrow[r, "\mathrm{id}\otimes m"'] & A\otimes A \arrow[r, "m"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
I\otimes A \arrow[r, "u\otimes\mathrm{id}"] \arrow[dr, "\lambda"'] & A\otimes A \arrow[d, "m"] & A\otimes I \arrow[l, "\mathrm{id}\otimes u"'] \arrow[dl, "\rho"] \\
& A &
\end{tikzcd}$$

Here α, λ, ρ respectively denote the associativity isomorphism, and the left and the right unit isomorphism. A monoid (A,m,u) is **commutative** if $m \circ \gamma = m$, where $\gamma\colon A \otimes A \to A \otimes A$ is the symmetry isomorphism. A **monoid morphism** between monoids (A_1,m_1,u_1) and (A_2,m_2,u_2) is an arrow $f\colon A_1 \to A_2$ that satisfies $m_2 \circ (f \otimes f) = f \circ m_1$ and $u_2 = f \circ u_1$. We denote the category of monoids and monoid morphisms in $\mathbf{C}$ by $\mathrm{Mon}(\mathbf{C}, \otimes, I)$ or simply by $\mathrm{Mon}(\mathbf{C})$. We write $\mathrm{CMon}(\mathbf{C}) \subseteq \mathrm{Mon}(\mathbf{C})$ for the full subcategory of commutative monoids.

2.2 The Model of the Quantum Lambda Calculus by von Neumann algebras

We quickly review some details of the authors' previous work [5] on a model of the quantum lambda calculus by von Neumann algebras, because we will need them in this paper. Let $(\mathbf{Set}, \times, 1)$ be the cartesian monoidal category of sets and functions; $(\mathbf{vNA}_{\mathrm{MIU}}, \otimes, \mathbb{C})$ be the SMC of von Neumann algebras and normal MIU-maps; and $(\mathbf{vNA}_{\mathrm{CPsU}}, \otimes, \mathbb{C})$ be the SMC of von Neumann algebras and normal CPsU-maps ($\otimes$ is the spatial tensor product). The model is constituted by the following adjunctions.

$$\begin{tikzcd} (\mathbf{Set}^{\mathrm{op}}, \times, 1) \arrow[r, bend right, "\ell^\infty"'] \arrow[r, phantom, "\perp"] & (\mathbf{vNA}_{\mathrm{MIU}}, \otimes, \mathbb{C}) \arrow[l, bend right, "\mathrm{nsp}"'] \arrow[r, hook, bend right, "\mathcal{J}"'] \arrow[r, phantom, "\perp"] & (\mathbf{vNA}_{\mathrm{CPsU}}, \otimes, \mathbb{C}) \arrow[l, bend right, "\mathcal{F}"'] \end{tikzcd} \tag{1}$$

Although we do not use it in the present work, it is worth mentioning that $\mathbf{vNA}_{\mathrm{MIU}}$ is a co-closed SMC [14], which we used (together with the adjunction $\mathcal{F} \dashv \mathcal{J}$) to interpret function types $A \multimap B$ in [5].

As is explained in the introduction, the left-hand side of (1) models the ! operator. Recall that $\mathrm{nsp}(\mathscr{A}) = \mathbf{vNA}_{\mathrm{MIU}}(\mathscr{A}, \mathbb{C})$ and $\ell^\infty(X) = \{f\colon X \to \mathbb{C} \mid \sup_x |f(x)| < \infty\}$. They are indeed functors via $\mathrm{nsp}(g)(\varphi) = \varphi \circ g$ and $\ell^\infty(h)(f) = f \circ h$. It is straightforward to check that there is a (dual) adjunction $\mathrm{nsp} \dashv \ell^\infty$ via "swapping arguments" $g(a)(x) = h(x)(a)$ for normal MIU-maps $g\colon \mathscr{A} \to \ell^\infty(X)$ and functions $h\colon X \to \mathrm{nsp}(\mathscr{A})$. In [5] we further observed that:

Lemma 3. *The functors* nsp *and* ℓ^∞ *are strong symmetric monoidal, and there is a symmetric monoidal adjunction* $\mathrm{nsp} \dashv \ell^\infty$. *Moreover the counit of the adjunction is an isomorphism, i.e.* $\mathrm{nsp}(\ell^\infty(X)) \cong X$. □

Corollary 4 ([16, Theorem IV.3.1])**.** *The right adjoint functor* ℓ^∞ *is full and faithful.* □

The following is an important observation for the present work.

Corollary 5. *Von Neumann algebras of the form* $\ell^\infty(X)$ *carry a commutative monoid structure in* $\mathbf{vNA}_{\mathrm{MIU}}$.

Proof. Any set X carries a canonical commutative comonoid structure via the diagonal $X \to X \times X$ and the unique map $X \to 1$, so that it is a commutative monoid in $(\mathbf{Set}^{\mathrm{op}}, \times, 1)$. Because ℓ^∞ is strong symmetric monoidal, it preserves the monoid structure as

$$\ell^\infty(X) \otimes \ell^\infty(X) \cong \ell^\infty(X \times X) \longrightarrow \ell^\infty(X) \qquad\qquad \mathbb{C} \cong \ell^\infty(1) \longrightarrow \ell^\infty(X)\ . \qquad \square$$

Now we turn to the right-hand side of (1). The functor $\mathcal{J}\colon \mathbf{vNA}_{\mathrm{MIU}} \hookrightarrow \mathbf{vNA}_{\mathrm{CPsU}}$ is the inclusion, which is strict symmetric monoidal. It can be shown [23] via Freyd's Adjoint Functor Theorem that:

Lemma 6. *The inclusion* $\mathcal{J}\colon \mathbf{vNA}_{\mathrm{MIU}} \hookrightarrow \mathbf{vNA}_{\mathrm{CPsU}}$ *has a left adjoint* $\mathcal{F}\colon \mathbf{vNA}_{\mathrm{CPsU}} \to \mathbf{vNA}_{\mathrm{MIU}}$. □

We remark that the Kleisli category of the comonad $\mathcal{F}\mathcal{J}$ on $\mathbf{vNA}_{\mathrm{MIU}}$ is isomorphic to $\mathbf{vNA}_{\mathrm{CPsU}}$. Thus the right-hand side of (1) is almost a model of Moggi's computational lambda calculi. That is why our model can accommodate *quantum operations*, which are in general not MIU but CPsU-maps.

3 Characterisation of Duplicable von Neumann Algebras

We will prove Theorem 2 in this section. Let us give a rough sketch of the proof. To show that every duplicable von Neumann algebra is MIU-isomorphic to $\ell^\infty(X)$ for some set X, we first prove that $\mathscr{A}$ is Abelian (in Lemma 9). This reduces the situation to a measure theoretic problem, because $\mathscr{A} \cong L^\infty(X)$ for some appropriate (i.e. localisable) measure space X. For simplicity, we only consider measure spaces with $\mu(X) < \infty$ (which are localisable). This restriction turns out to be harmless (in the proof of Theorem 2). The measure space X splits in a discrete D and a continuous part C, so $X \cong D \oplus C$ (see Lemma 16). Since $L^\infty(D) \cong \ell^\infty(D')$ for some set D', and $L^\infty(X) \cong \ell^\infty(D') \oplus L^\infty(C)$, our task will be to show that $L^\infty(C) = \{0\}$. Since $L^\infty(X)$ is duplicable, we will see that $L^\infty(C)$ is duplicable as well (see Corollary 12). Thus the crux of the proof is that $L^\infty(C)$ cannot be duplicable unless $\mu(C) = 0$ (see Lemma 17).

Lemma 7. *Let δ be a duplicator with unit u on a von Neumann algebra $\mathscr{A}$. Then $u = 1$ and $\delta(1 \otimes 1) = 1$.*

Proof. Since $1 = \delta(u \otimes 1) \leq \delta(1 \otimes 1) \leq 1$, we have $\delta(u^\perp \otimes 1) = 0$. But then $u^\perp = 0$, and thus $u = 1$, because $u^\perp = \delta(u^\perp \otimes u) \leq \delta(u^\perp \otimes 1) = 0$. Hence $u = 1$. Thus $1 = \delta(1 \otimes u) = \delta(1 \otimes 1)$. □

The following consequence of Tomiyama's theorem is based on Lemma 8.3 of [11].

Lemma 8. *Let $\mathscr{A}$ be a unital C^*-algebra, and let $f\colon \mathscr{A} \oplus \mathscr{A} \to \mathscr{A}$ be a positive unital map with $f(a,a) = a$ for all $a \in \mathscr{A}$. Then $p := f(1,0)$ is central, and for all $a,b \in \mathscr{A}$ we have $f(a,b) = ap + bp^\perp$.*

Proof. By Tomiyama's theorem we have, for all $a,b,c,d \in \mathscr{A}$,

$$a\,f(c,d)\,b \;=\; f(\,acb\,,\,adb\,).$$

In particular, for all $a \in \mathscr{A}$, we have $ap = af(1,0) = f(a,0) = f(1,0)a = pa$. Thus p is central. Similarly, $f(0,b) = bp^\perp$ for all $b \in \mathscr{A}$. Then $f(a,b) = f(a,0) + f(0,b) = ap + bp^\perp$ for all $a,b \in \mathscr{A}$. □

Lemma 9. *Let $\delta\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ be a duplicator on a von Neumann algebra $\mathscr{A}$. Then $\mathscr{A}$ is Abelian and $\delta(a \otimes b) = a \cdot b$ for all $a,b \in \mathscr{A}$.*

Proof. We must show that all $a \in \mathscr{A}$ are central. It suffices to show that all $p \in [0,1]_{\mathscr{A}}$ are central (by the usual reasoning). Similarly, we only need to prove that $\delta(a \otimes p) = a \cdot p$ for all $a \in \mathscr{A}$ and $p \in [0,1]_{\mathscr{A}}$.

Let $p \in [0,1]_{\mathscr{A}}$ be given. Define $f\colon \mathscr{A} \oplus \mathscr{A} \to \mathscr{A}$ by $f(a,b) = \delta(a \otimes p + b \otimes p^\perp)$ for all $a,b \in \mathscr{A}$. Then f is positive, unital, $f(1,0) = p$, and $f(a,a) = a$ for all $a \in \mathscr{A}$. Thus, by Lemma 8, we get that p is central, and $f(a,b) = ap + bp^\perp$ for all $a,b \in \mathscr{A}$. Then $a \cdot p = f(a,0) = \delta(a \otimes p)$. □

Remark 10. The special case of Lemma 9 in which δ is *completely* positive can be found in the literature, see for example Theorem 6 of [15] (in which $\mathscr{A}$ is also finite dimensional).

Corollary 11. *Let $\mathscr{A}$ be a von Neumann algebra. Then $\mathscr{A}$ is duplicable iff there is a positive linear map $\delta\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ with $\mu(a \otimes b) = a \cdot b$ for all $a,b \in \mathscr{A}$.*

Corollary 12. *Von Neumann algebras $\mathscr{A}$ and $\mathscr{B}$ are duplicable when $\mathscr{A} \oplus \mathscr{B}$ is duplicable.*

Proof. Let $\delta\colon (\mathscr{A} \oplus \mathscr{B}) \otimes (\mathscr{A} \oplus \mathscr{B}) \longrightarrow \mathscr{A} \oplus \mathscr{B}$ be a duplicator on $\mathscr{A} \oplus \mathscr{B}$. By Lemma 9 we know that $\mathscr{A} \oplus \mathscr{B}$ is Abelian, and that $\delta((a_1,b_1) \otimes (a_2,b_2)) = (a_1a_2, b_1b_2)$ for all $a_1,a_2 \in \mathscr{A}$ and $b_1,b_2 \in \mathscr{B}$.

Let $\kappa_1\colon \mathscr{A} \to \mathscr{A} \oplus \mathscr{B}$ be the normal MIU-map given by $\kappa_1(a) = (a,0)$ for all $a \in \mathscr{A}$. Let $\delta_{\mathscr{A}}$ be the composition of $\mathscr{A} \otimes \mathscr{A} \xrightarrow{\kappa_1 \otimes \kappa_1} (\mathscr{A} \oplus \mathscr{B}) \otimes (\mathscr{A} \oplus \mathscr{B}) \xrightarrow{\delta} \mathscr{A} \oplus \mathscr{B} \xrightarrow{\pi_1} \mathscr{A}$. Then $\delta_{\mathscr{A}}$ is positive, and $\delta_{\mathscr{A}}(a_1 \otimes a_2) = \pi_1(\delta((a_1,0) \otimes (a_2,0))) = \pi_1(a_1a_2, 0) = a_1a_2$ for all $a_1,a_2 \in \mathscr{A}$. Thus, by Corollary 11, $\mathscr{A}$ is duplicable. □

We will now work towards the proof that if C is a finite measure space, then $L^\infty(C)$ cannot be duplicable unless $\mu(C) = 0$, see Lemma 18. Let us first fix some terminology from measure theory (see [10]).

Definition 13. Let X be a measure space.

1. The measurable subsets of X are denoted by Σ_X.
2. A measurable subset A of X is **atomic** if $0 < \mu(A) < \infty$, and $\mu(A') = \mu(A)$ for all $A' \in \Sigma_X$ with $A' \subseteq A$ and $\mu(A') > 0$.
3. X is **discrete** if X is covered by atomic measurable subsets.
4. X is **continuous** if X contains no atomic subsets.

Definition 14. Given a measure space X with $\mu(X) < \infty$, let $L^\infty(X)$ denote the von Neumann algebra of bounded functions $f\colon X \to \mathbb{C}$. Two such functions f and g are identified in $L^\infty(X)$ when $f(x) = g(x)$ for almost all $x \in X$. Multiplication, addition, involution in $L^\infty(X)$ are all computer coordinatewise, and the norm $\|f\|$ of $f \in L^\infty(X)$ is the least number $r > 0$ such that $|f(x)| \leq r$ for almost all $x \in X$. (For more details, see e.g. Example IX/7.2 of [7].)

The following lemma, which will be very useful, is a variation on Zorn's Lemma that does not require the axiom of choice.

Lemma 15. *Let X be a measure space with $\mu(X) < \infty$. Let $\mathcal{S}$ be a collection of measurable subsets of X such that $\bigcup_n A_n \in \mathcal{S}$ for all $A_1 \subseteq A_2 \subseteq \cdots$ in $\mathcal{S}$. Then for all $A \in \mathcal{S}$, there is $B \in \mathcal{S}$ with $A \subseteq B$ which is maximal in the sense that $\mu(B') = \mu(B)$ for all $B' \in \mathcal{S}$ with $B \subseteq B'$.*

Proof. The trick is to consider for every $C \in \Sigma_X$ the quantity $\beta_C \;=\; \sup\{\, \mu(D) \mid C \subseteq D \text{ and } D \in \mathcal{S} \,\}$.

Note that $\mu(C) \leq \beta_C \leq \mu(X)$ for all $C \in \Sigma_X$, and $\beta_{C_2} \leq \beta_{C_1}$ for all $C_1, C_2 \in \Sigma_X$ with $C_1 \subseteq C_2$. To prove this lemma, it suffices to find $B \in \Sigma_X$ with $A \subseteq B$ and $\mu(B) = \beta_B$.

Define $B_1 := B$. Pick $B_2 \in \mathcal{S}$ such that $B_1 \subseteq B_2$ and $\beta_{B_1} - \mu(B_2) \leq 1/2$. Pick $B_3 \in \mathcal{S}$ such that $B_2 \subseteq B_3$ and $\beta_{B_2} - \mu(B_3) \leq 1/3$. Proceeding in this fashion, we get a sequence $B \equiv B_1 \subseteq B_2 \subseteq \cdots$ in $\mathcal{S}$ with $\beta_{B_n} - \mu(B_{n+1}) \leq 1/n$ for all $n \in \mathbb{N}$. Define $B := \bigcup_n B_n$. Then $B \in \mathcal{S}$. Moreover,

$$\mu(B_1) \;\leq\; \mu(B_2) \;\leq\; \cdots \;\leq\; \mu(B) \;\leq\; \beta_B \;\leq\; \cdots \;\leq\; \beta_{B_2} \;\leq\; \beta_{B_1}.$$

Since for every $n \in \mathbb{N}$ we have both $\mu(B_{n+1}) \leq \mu(B) \leq \beta_B \leq \beta_{B_n}$ and $\beta_{B_n} - \mu(B_{n+1}) \leq 1/n$, we get $\beta_B - \mu(B) \leq 1/n$, and so $\beta_B = \mu(B)$. □

Lemma 16. *Let X be a measure space with $\mu(X) < \infty$. Then there is a measurable subset $D \subseteq X$ such that D is discrete and $X \backslash D$ is continuous.*

Proof. Since clearly the countable union of discrete measurable subsets of X is again discrete, there is by Lemma 15 a discrete measurable subset D of X which is maximal in the sense that $\mu(D') = \mu(D)$ for every discrete measurable subset D' of X with $D \subseteq D'$. To show that $X \backslash D$ is continuous, we must prove that $X \backslash D$ contains no atomic measurable subsets. If $A \subseteq X \backslash D$ is an atomic measurable subset of X, then $D \cup A$ is a discrete measurable subset of X which contains D, and $\mu(D \cup A) = \mu(D) \cup \mu(A) > \mu(D)$. This contradicts the maximality of D. Thus $X \backslash D$ is continuous. □

Lemma 17. *Let X be a continuous measure space with $\mu(X) < \infty$. Then for every $r \in [0, \mu(X)]$ there is a measurable subset A of X with $\mu(X) = r$.*

Proof. Let us quickly get rid of the case that $\mu(X)=0$. Indeed, then $r=0$, and so $A=\varnothing$ will do. For the remainder, assume that $\mu(X)>0$.

For starters, we show that for every $\varepsilon>0$ and $B\in\Sigma_X$ with $\mu(B)>0$ there is $A\in\Sigma_X$ with $A\subseteq B$ and $0<\mu(A)<\varepsilon$. Define $A_1:=B$. Since $\mu(B)>0$, and A_1 is not atomic (because X is continuous) there is $A\in\Sigma_X$ with $A\subseteq A_1$ and $\mu(A)\neq\mu(A_1)$. Since $\mu(A)+\mu(A_1\backslash A)=\mu(A_1)$, either $0<\mu(A)\leq\frac{1}{2}\mu(A_1)$ or $0<\mu(X\backslash A)\leq\frac{1}{2}\mu(A_1)$. In any case, there is $A_2\subseteq A_1$ with $A_2\in\Sigma_X$ and $0<\mu(A_2)\leq\frac{1}{2}\mu(A_1)$. Similarly, since A_2 is not atomic (because X is continuous), there is $A_3\subseteq A_2$ with $A_3\in\Sigma_X$ and $0<\mu(A_3)\leq\frac{1}{2}\mu(A_2)$. Proceeding in a similar fashion, we obtain a sequence $B\equiv A_1\supseteq A_2\supseteq\cdots$ of measurable subsets of X with $0<\mu(A_n)\leq 2^{-n}\mu(X)$. Then, for every $\varepsilon>0$ there is $n\in\mathbb{N}$ such that $0<\mu(A_n)\leq\varepsilon$ and $A_n\subseteq B$.

Now, let us prove that there is $A\in\Sigma_X$ with $\mu(A)=r$. By Lemma 15 there is a measurable subset A of X with $\mu(A)\leq r$ and which is maximal in the sense that $\mu(A')=\mu(A)$ for all $A'\in\Sigma_X$ with $\mu(A)\leq r$ and $A\subseteq A'$. In fact, we claim that $\mu(A)=r$. Indeed, suppose that $\varepsilon:=r-\mu(A)>0$ towards a contradiction. By the previous discussion, there is $C\in\Sigma_X$ with $C\subseteq X\backslash A$ such that $\mu(C)\leq\varepsilon$. Then $A\cup C$ is measurable, and $\mu(A\cup C)=\mu(A)+\mu(C)\leq\mu(A)+\varepsilon\leq r$, which contradicts the maximality of A. □

Lemma 18. *Let X be a continuous measure space with $\mu(X)<\infty$. If $L^\infty(X)$ is duplicable, then $\mu(X)=0$.*

Proof. Suppose that $L^\infty(X)$ is duplicable and $\mu(X)>0$ towards a contradiction. Let δ be a duplicator on $L^\infty(X)$. By Lemma 9 we know that $\delta(f\otimes g)=f\cdot g$ for all $f,g\in L^\infty(X)$.

Let $\omega\colon L^\infty(X)\to\mathbb{C}$ be given by $\omega(f)=\frac{1}{\mu(X)}\int f\,d\mu$ for all $f\in L^\infty(X)$. Then ω is normal, positive and unital. Also, ω is faithful, or in other words, for all $f\in L^\infty(X)$ with $f\geq 0$ and $\omega(f)=0$ we have $f=0$. It is known (see e.g. Corollary 5.12 of [22]) that there is a faithful normal positive unital linear map $\omega\otimes\omega\colon L^\infty(X)\otimes L^\infty(X)\to\mathbb{C}$ with $(\omega\otimes\omega)(f\otimes g)=\omega(f)\cdot\omega(g)$ for all $f,g\in L^\infty(X)$. We will use $\omega\otimes\omega$ to tease out a contradiction, but first we will need a second ingredient.

Since X is continuous, we may partition X into two measurable subsets of equal measure with the aid of Lemma 17, that is, there are measurable subsets X_1 and X_2 of X with $X=X_1\cup X_2$, $X_1\cap X_2=\varnothing$, and $\mu(X_1)=\mu(X_2)=\frac{1}{2}\mu(X)$. Similarly, X_1 can be split into two measurable subsets, X_{11} and X_{12}, of equal measure, and so on. In this way, we obtain for every word w over the alphabet $\{1,2\}$ — in symbols, $w\in\{1,2\}^*$ — a measurable subset X_w of X such that $X_w=X_{w1}\cup X_{w2}$, $X_{w1}\cap X_{w2}=\varnothing$, and $\mu(X_{w1})=\mu(X_{w2})=\frac{1}{2}\mu(X_w)$. It follows that $\mu(X_w)=\frac{1}{2^{\#w}}\mu(X)$, where $\#w$ is the length of the word w.

Now, let $p_w=\mathbf{1}_{X_w}$ be the indicator function of X_w for every $w\in\{1,2\}^*$. Let $w\in\{1,2\}^*$ be given. Then p_w is a projection in $L^\infty(X)$, and $\omega(p_w)=2^{-\#w}$. Moreover, $p_w=p_{w1}+p_{w2}$, and so

$$\begin{aligned} p_w\otimes p_w \;&=\; p_{w1}\otimes p_{w1}+p_{w1}\otimes p_{w2}+p_{w2}\otimes p_{w1}+p_{w2}\otimes p_{w2}\\ &\leq\; p_{w1}\otimes p_{w1}+p_{w2}\otimes p_{w2}. \end{aligned}$$

Thus, if we define $q_N:=\sum_{w\in\{1,2\}^N}p_w\otimes p_w$ for every natural number N, where $\{1,2\}^N$ is the set of words over $\{1,2\}$ of length N, then we get a descending sequence $q_1\geq q_2\geq q_3\geq\cdots$ of projections in $L^\infty(X)\otimes L^\infty(X)$. Let q be the infimum of $q_1\geq q_2\geq\cdots$ in the set of self-adjoint elements of $L^\infty(X)\otimes L^\infty(X)$. Do we have $q=0$?

On the one hand, we claim that $\delta(q)=1$, and so $q\neq 0$. Indeed, $\delta(p_w\otimes p_w)=p_w\cdot p_w=p_w$ for all $w\in\{1,2\}^N$. Thus $\delta(q_N)=\sum_{w\in\{1,2\}^N}\delta(p_w\otimes p_w)=\sum_{w\in\{1,2\}^N}p_w=1$ for all $N\in\mathbb{N}$. Hence $\delta(q)=\bigwedge_n\delta(q_N)=1$. On the other hand, we claim that $(\omega\otimes\omega)(q)=0$, and so $q=0$ since $\omega\otimes\omega$ is faithful and $q\geq 0$. Indeed, $(\omega\otimes\omega)(q_N)=\sum_{w\in\{1,2\}^N}\omega(p_w)\cdot\omega(p_w)=\sum_{w\in\{1,2\}^N}2^{-N}\cdot 2^{-N}=2^{-N}$ for all $N\in\mathbb{N}$, and so $(\omega\otimes\omega)(q)=\bigwedge_N(\omega\otimes\omega)(q_N)=\bigwedge_N 2^{-N}=0$. Thus, $q=0$ and $q\neq 0$, which is impossible. □

Lemma 19. *Let A be an atomic measure space. Then $L^\infty(A)\cong\mathbb{C}$.*

Proof. Let $f \in L^\infty(A)$ be given. It suffices to show that there is $z \in \mathbb{C}$ such that $f(x) = z$ for almost all $x \in A$. Moreover, we only need to consider the case that f takes its values in $\mathbb{R}$ (because we may split f in its real and imaginary parts, and in turn split these in positive and negative parts).

Since, writing $I_n = (n, n+1]$, we have $\mu(A) = \sum_{n\in\mathbb{Z}} f^{-1}(I_n)$, and A is atomic, there a (unique) $n \in \mathbb{N}$ with $\mu(A) = \mu(f^{-1}(I_n))$. Similarly, writing $J_1 = (n, \frac{2n+1}{2}]$ and $J_2 = (\frac{2n+1}{2}, n+1]$, we have $\mu(A) = \mu(J_1) + \mu(J_2)$, and so there is a (unique) $m \in \{1,2\}$ with $\mu(A) = \mu(f^{-1}(J_m))$.

Continuing in this way, we can find real numbers $s_1 \leq s_2 \leq \cdots \leq t_2 \leq t_1$ with $t_n - s_n \leq 2^{-n}$ and $\mu(A) = \mu(f^{-1}(\,(s_n, t_n]\,))$ for all $n \in \mathbb{N}$. Of course, we also have $\mu(A) = \mu(f^{-1}(\,[s_n, t_n]\,))$, and so

$$\mu(A) \;=\; \bigwedge_n \mu(f^{-1}(\,[s_n, t_n]\,)) \;=\; \mu(f^{-1}(\,\bigcap_n [s_n, t_n]\,)).$$

Since $t_n - s_n \to 0$ as $n \to \infty$, there is a real number $\lambda \in \mathbb{R}$ with $\{\lambda\} = \bigcap_n [s_n, t_n]$, and so $\mu(A) = f^{-1}(\{\lambda\})$. Hence $f(x) = \lambda$ for almost all $x \in A$. □

Lemma 20. *Let X be a measure space with $\mu(\mathscr{A}) < \infty$. Then for every partition $\mathcal{A}$ of X consisting of measurable subsets, we have $L^\infty(X) \cong \bigoplus_{A\in\mathcal{A}} L^\infty(A)$.*

Proof. Note that since $\sum_{A\in\mathcal{A}} \mu(A) = \mu(X)$, the set $\mathcal{A}' = \{A \in \mathcal{A} \mid \mu(A) > 0\}$ is countable. We will also need the fact that $A_0 = \bigcup\{A \in \mathcal{A} \mid \mu(A) = 0\}$ is negligible. To see this, note that $X \backslash A_0$ is the union of $\mathcal{A}'$ and thus measurable. Further, we have $\mu(X\backslash A_0) = \sum_{A\in\mathcal{A}'} \mu(A) = \sum_{A\in\mathcal{A}} \mu(A) = \mu(X)$, and thus $\mu(A_0) = 0$.

Let $A \in \mathcal{A}$ and $f \in L^\infty(X)$ be given. Then the restriction $f|A\colon A \to \mathbb{C}$ is an element of $L^\infty(A)$. It is not hard to see that $f \mapsto f|_A$ gives a MIU-map $R_A\colon L^\infty(X) \to L^\infty(A)$. Let $R\colon L^\infty(X) \to \bigoplus_{A\in\mathcal{A}} L^\infty(A)$ be given by $R(f) = (R_A(f))_{A\in\mathcal{A}}$. We claim that R is bijective (and thus a normal MIU-isomorphism).

To show that R is injective, let $f \in L^\infty(X)$ with $R(f) = 0$ be given. Then for every $A \in \mathcal{A}$ there is a negligible subset N_A of A with $f(x) = 0$ for all $x \in A\backslash N_A$. Thus $f(x) = 0$ for all $x \in X \backslash \bigcup_{A\in\mathcal{A}} N_A$. Since $N := \bigcup_{A\in\mathcal{A}} N_A \subseteq A_0 \cup \bigcup_{A\in\mathcal{A}'} N_A$, we see that N is negligible, and so $f(x) = 0$ for almost all $x \in X$. Thus $f = 0$ in $L^\infty(X)$, and thus R is injective.

To show that R is surjective, let $f \in \bigoplus_{A\in\mathcal{A}} L^\infty(A)$ be given, and define $g\colon X \to \mathbb{C}$ by $g(x) = f_A(x)$ for all $A \in \mathcal{A}$ and $x \in A$. We claim that $g \in L^\infty(X)$, and then clearly $R(g) = f$. To begin, we show that g is measurable. Let U be a measurable subset of $\mathbb{C}$. We must show that $g^{-1}(U)$ is measurable. We have

$$g^{-1}(U) \;=\; \bigcup_{A\in\mathcal{A}} f_A^{-1}(U) \;=\; \bigcup_{A\in\mathcal{A}'} f_A^{-1}(U) \;\cup\; \bigcup_{A\in\mathcal{A}\backslash\mathcal{A}'} f_A^{-1}(U).$$

Note that $\bigcup_{A\in\mathcal{A}'} f_A^{-1}(A)$ is measurable (being a countable union of measurable sets), and that $\bigcup_{A\in\mathcal{A}\backslash\mathcal{A}'} f_A^{-1}(A)$ is negligible (being a subset of the negligible set A_0). Thus $g^{-1}(U)$ is measurable. Hence g is measurable. It remains to be shown that g is essentially bounded, that is, that there is $r > 0$ such that $|f(x)| \leq r$ for almost all $r \in X$. For every $A \in \mathcal{A}$ there is a negligible subset N_A of A such that $|f_A(x)| \leq \|f_A\|$ for all $x \in A\backslash N_A$. Since $\|f\| = \sup_{A\in\mathcal{A}} \|f_A\|$, we have $|g| \leq \|f\|$ for all $x \in X\backslash(\bigcup_{A\in\mathcal{A}} N_A)$. Since $N := \bigcup_{A\in\mathcal{A}} N_A \subseteq A_0 \cup \bigcup_{A\in\mathcal{A}'} A$, we see that N is negligible, and thus that g is essentially bounded. Of course, $R(g) = f$, and thus R is surjective. □

Corollary 21. *For every discrete measure space X with $\mu(X) < \infty$ there is a set Y with $L^\infty(X) \cong \ell^\infty(Y)$.*

We are now ready to give the proof the main result of this paper.

Proof of Theorem 2. We have already seen that $\ell^\infty(X)$ can be equipped with a commutative monoid structure in $\mathbf{vNA}_{\mathrm{MIU}}$ for any set X, and is thus duplicable. Conversely, let $\delta\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ be a duplicator with unit u on a von Neumann algebra $\mathscr{A}$. By Lemma 7, we know that $u = 1$, and by Lemma 9, we know that $\mathscr{A}$ is Abelian and $\delta(a \otimes b) = a \cdot b$ for all $a, b \in \mathscr{A}$. Thus, the only thing that remains to be shown is that $\mathscr{A}$ is MIU-isomorphic to $\ell^\infty(Y)$ for some set Y.

It is known that any Abelian von Neumann algebra must be of the form $L^\infty(X)$, where X is a measure space. Moreover, X can be taken to be a *localisable* measure space, but we will not need the general theory of localisable measure spaces here. Instead, we can get away with using the fact (obtained by inspecting the proof of Proposition 1.18.1 of [20]), that there is a family of measure spaces $(X_i)_{i \in I}$ with $\mathscr{A} \cong \bigoplus_{i \in I} L^\infty(X_i)$ and $\mu(X_i) < \infty$ for all $i \in I$. To prove that $\mathscr{A} \cong \ell^\infty(Y)$ for some set Y it suffices to show that there is for every $i \in I$ a set Y_i with $L^\infty(X_i) \cong \ell^\infty(Y_i)$, because then

$$\mathscr{A} \;\cong\; \textstyle\bigoplus_{i \in I} \ell^\infty(Y_i) \;\cong\; \ell^\infty\big(\bigcup_{i \in I} Y_i\big).$$

Let $i \in I$ be given. We must find a set Y such that $L^\infty(X_i) \cong \ell^\infty(Y)$. Since $\mathscr{A} \cong L^\infty(X_i) \oplus \bigoplus_{j \neq i} L^\infty(X_j)$ is duplicable, $L^\infty(X_i)$ is duplicable by Corollary 12. By Lemma 16, there is a measurable subset D of X_i such that D is discrete, and $C := X \backslash D$ is continuous. We have $L^\infty(X_i) \cong L^\infty(D) \oplus L^\infty(C)$ by Lemma 20, and so $L^\infty(D)$ and $L^\infty(C)$ are duplicable (again by Corollary 12).

By Lemma 18, $L^\infty(C)$ can only be duplicable if $\mu(C) = 0$, and so $L^\infty(C) \cong \{0\}$. On the other hand, since D is discrete, we have $L^\infty(D) \cong \ell^\infty(Y)$ for some set Y (by Corollary 21). Thus we have

$$L^\infty(X_i) \;\cong\; L^\infty(D) \oplus L^\infty(C) \;\cong\; \ell^\infty(Y) \oplus \{0\} \;\cong\; \ell^\infty(Y).$$

Hence $\mathscr{A} \cong \ell^\infty(Z)$ for some set Z. □

4 Monoids in $\mathbf{vNA}_{\mathrm{MIU}}$ and $\mathbf{vNA}_{\mathrm{CPsU}}$

First we characterise duplicable von Neumann algebras in terms of monoids.

Proposition 22. *Let $\mathscr{A}$ be a von Neumann algebra. The following are equivalent.*

1. *$\mathscr{A}$ is duplicable.*
2. *$\mathscr{A}$ carries a monoid structure in $(\mathbf{vNA}_{\mathrm{MIU}}, \otimes, \mathbb{C})$.*
3. *$\mathscr{A}$ carries a monoid structure in $(\mathbf{vNA}_{\mathrm{CPsU}}, \otimes, \mathbb{C})$.*

In that case, $\mathscr{A}$ is a commutative monoid (both in $\mathbf{vNA}_{\mathrm{MIU}}$ and $\mathbf{vNA}_{\mathrm{CPsU}}$), and the monoid structure $(m\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}, u\colon \mathbb{C} \to \mathscr{A})$ is a duplicator on $\mathscr{A}$ (when u is identified with $u(1) \in \mathscr{A}$). By Theorem 2, a monoid structure on $\mathscr{A}$ is unique.

Proof. ($1 \Rightarrow 2$) By Theorem 2, $\mathscr{A}$ is MIU-isomorphic to $\ell^\infty(X)$ for some set X. By Corollary 5, $\ell^\infty(X)$ carries a commutative monoid structure in $\mathbf{vNA}_{\mathrm{MIU}}$. Thus we can equip $\mathscr{A}$ with a commutative monoid structure in $\mathbf{vNA}_{\mathrm{MIU}}$ via the isomorphism $\mathscr{A} \cong \ell^\infty(X)$.

($2 \Rightarrow 3$) Trivial.

($3 \Rightarrow 1$) Let $m\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ and $u\colon \mathbb{C} \to \mathscr{A}$ be a monoid structure on $\mathscr{A}$ in $\mathbf{vNA}_{\mathrm{CPsU}}$. Then $m(u(1) \otimes a) = (m \circ (u \otimes \mathrm{id}))(1 \otimes a) = \lambda(1 \otimes a) = 1 \cdot a = a$. and similarly $m(a \otimes u(1)) = a$. Thus $\mathscr{A}$ is duplicable via m and $u(1)$. □

It follows that there is no distinction between (commutative) monoids in $\mathbf{vNA}_{\mathrm{MIU}}$ and in $\mathbf{vNA}_{\mathrm{CPsU}}$.

Proposition 23. $\mathrm{CMon}(\mathbf{vNA}_{\mathrm{MIU}}) = \mathrm{Mon}(\mathbf{vNA}_{\mathrm{MIU}}) = \mathrm{CMon}(\mathbf{vNA}_{\mathrm{CPsU}}) = \mathrm{Mon}(\mathbf{vNA}_{\mathrm{CPsU}})$.

Proof. By Proposition 22, $\mathrm{Mon}(\mathbf{vNA}_{\mathrm{MIU}})$ and $\mathrm{Mon}(\mathbf{vNA}_{\mathrm{CPsU}})$ have the same objects, and $\mathrm{CMon}(\mathbf{vNA}_{\mathrm{MIU}}) = \mathrm{Mon}(\mathbf{vNA}_{\mathrm{MIU}})$ and $\mathrm{CMon}(\mathbf{vNA}_{\mathrm{CPsU}}) = \mathrm{Mon}(\mathbf{vNA}_{\mathrm{CPsU}})$. Let $f\colon (\mathscr{A}_1, m_1, u_1) \to (\mathscr{A}_2, m_2, u_2)$ be a morphism in $\mathrm{Mon}(\mathbf{vNA}_{\mathrm{CPsU}})$. Then $f(1) = (f \circ u_1)(1) = u_2(1) = 1$ and $f(ab) = (f \circ m_1)(a \otimes b) = (m_2 \circ (f \otimes f))(a \otimes b) = f(a)f(b)$, using the fact that the monoid structure is a unique duplicator. Thus f is a normal MIU-map, and we are done since $\mathrm{Mon}(\mathbf{vNA}_{\mathrm{MIU}}) \subseteq \mathrm{Mon}(\mathbf{vNA}_{\mathrm{CPsU}})$. □

It turns out that $\ell^\infty(\mathrm{nsp}(\mathscr{A}))$, our interpretation of the ! operator in the quantum lambda calculus, is exactly the free (commutative) monoid on $\mathscr{A}$ in $\mathbf{vNA}_{\mathrm{MIU}}$.

Theorem 24. *Let $\mathscr{A}$ be a von Neumann algebra, and let $\eta\colon \mathscr{A} \to \ell^\infty(\mathrm{nsp}(\mathscr{A}))$ be the normal MIU-map given by $\eta(a)(\varphi) = \varphi(a)$. Then $\ell^\infty(\mathrm{nsp}(\mathscr{A}))$ is the free (commutative) monoid on $\mathscr{A}$ in $\mathbf{vNA}_{\mathrm{MIU}}$ via η.*

Proof. Let $\mathscr{B}$ be a monoid on $\mathbf{vNA}_{\mathrm{MIU}}$, and let $f\colon \mathscr{A} \to \mathscr{B}$ be a normal MIU-map We must show that there is a unique monoid morphism $g\colon \ell^\infty(\mathrm{nsp}(\mathscr{A})) \to \mathscr{B}$ such that $g \circ \eta = f$.

By Theorem 2, we may assume that $\mathscr{B} = \ell^\infty(Y)$ for some set Y. Since $\mathrm{nsp}\colon \mathbf{vNA}_{\mathrm{MIU}}^{\mathrm{op}} \to \mathbf{Set}$ is left adjoint to $\ell^\infty\colon \mathbf{Set} \to \mathbf{vNA}_{\mathrm{MIU}}^{\mathrm{op}}$ with unit η (Lemma 3), there is a unique map $h\colon Y \to \mathrm{nsp}(\mathscr{A})$ with $\ell^\infty(h) \circ \eta = f$. Since ℓ^∞ is full and faithful by Corollary 4, the only thing that remains to be shown is that $\ell^\infty(h)$ is a monoid morphism. Indeed it is, since the monoid multiplication on $\ell^\infty(\mathrm{nsp}(\mathscr{A}))$ and $\ell^\infty(Y)$ is given by ordinary multiplication, which is preserved by $\ell^\infty(h)$, being a MIU-map. □

Corollary 25. *Let $\mathscr{A}$ be a von Neumann algebra. Then $\ell^\infty(\mathbf{vNA}_{\mathrm{CPsU}}(\mathscr{A}, \mathbb{C}))$ is the free (commutative) monoid on $\mathscr{A}$ in $\mathbf{vNA}_{\mathrm{CPsU}}$.*

Proof. Theorem 24 asserts that $\ell^\infty \circ \mathrm{nsp}$ is a left adjoint to the forgetful functor $\mathrm{Mon}(\mathbf{vNA}_{\mathrm{MIU}}) \to \mathbf{vNA}_{\mathrm{MIU}}$. By Prop. 23, the forgetful functor $\mathrm{Mon}(\mathbf{vNA}_{\mathrm{CPsU}}) \to \mathbf{vNA}_{\mathrm{CPsU}}$ factors through $\mathbf{vNA}_{\mathrm{MIU}}$ as:

$$\begin{tikzcd} \mathrm{Mon}(\mathbf{vNA}_{\mathrm{CPsU}}) \arrow[r, equal] & \mathrm{Mon}(\mathbf{vNA}_{\mathrm{MIU}}) \arrow[r, "\perp"] & \mathbf{vNA}_{\mathrm{MIU}} \arrow[l, bend right, "\ell^\infty \circ \mathrm{nsp}"'] \arrow[r, hook, "\perp"] & \mathbf{vNA}_{\mathrm{CPsU}} \arrow[l, bend right, "\mathcal{F}"'] \end{tikzcd}$$

where $\mathcal{F}$ is the adjoint functor of Lemma 6. Thus the free monoid on $\mathscr{A}$ in $\mathbf{vNA}_{\mathrm{CPsU}}$ is given by:

$$(\ell^\infty \circ \mathrm{nsp} \circ \mathcal{F})(\mathscr{A}) = \ell^\infty(\mathbf{vNA}_{\mathrm{MIU}}(\mathcal{F}\mathscr{A}, \mathbb{C})) \cong \ell^\infty(\mathbf{vNA}_{\mathrm{CPsU}}(\mathscr{A}, \mathbb{C})) \ .$$ □

Finally we observe that duplicable von Neumann algebras and monoids in $\mathbf{vNA}_{\mathrm{MIU}}$ (or in $\mathbf{vNA}_{\mathrm{CPsU}}$) are simply identified with sets. Let $\mathbf{DvNA}_{\mathrm{MIU}} \subseteq \mathbf{vNA}_{\mathrm{MIU}}$ denote the full subcategory consisting of duplicable von Neumann algebras.

Proposition 26. $\mathrm{Mon}(\mathbf{vNA}_{\mathrm{MIU}}) \cong \mathbf{DvNA}_{\mathrm{MIU}} \simeq \mathbf{Set}^{\mathrm{op}}$.

Proof. ($\mathrm{Mon}(\mathbf{vNA}_{\mathrm{MIU}}) \cong \mathbf{DvNA}_{\mathrm{MIU}}$) By Proposition 22, we have the forgetful functor $U\colon \mathrm{Mon}(\mathbf{vNA}_{\mathrm{MIU}}) \to \mathbf{DvNA}_{\mathrm{MIU}}$ that is bijective on objects. We need to prove that the functor is full, namely that any normal MIU-map between monoids is a monoid morphism. The monoid structure is preserved by normal MIU-maps since the monoid structure is a duplicator by Proposition 22, and the duplicator is given by the multiplication and unit of the von Neumann algebra by Theorem 2.

($\mathbf{DvNA}_{\mathrm{MIU}} \simeq \mathbf{Set}^{\mathrm{op}}$) We have a functor $\ell^\infty\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{DvNA}_{\mathrm{MIU}}$, which is full and faithful by Corollary 4, and also essentially surjective by Theorem 2. □

Remark 27. Note that $\ell^\infty(\mathrm{nsp}(\mathscr{A}))$ is a free commutative comonoid on $\mathscr{A}$ in $\mathbf{vNA}_{\mathrm{MIU}}^{\mathrm{op}}$. Thus $\mathbf{vNA}_{\mathrm{MIU}}^{\mathrm{op}}$ gives a model of intuitionistic linear logic formulated by Lafont (called a *Lafont category*), see e.g. [17]. We leave it for future work to check if the explicit construction of the free commutative comonoid by Melliès et al. [18] works for $\mathbf{vNA}_{\mathrm{MIU}}^{\mathrm{op}}$.

5 Variations and Related Work

5.1 Categorical tensor product

Dauns [8], and later Kornell [14], have considered an alternative to the spatial tensor product, $\otimes$, called the categorical tensor product, $\tilde{\otimes}$. We bring this up, because, while only the von Neumann algebras of the form $\ell^\infty(X)$ carry a monoid structure in $(\mathbf{vNA}_{\mathrm{MIU}}, \otimes, \mathbb{C})$, *all* Abelian von Neumann algebras carry a commutative monoid structure in $(\mathbf{vNA}_{\mathrm{MIU}}, \tilde{\otimes}, \mathbb{C})$, and in fact, the category of (commutative) monoids on $(\mathbf{vNA}_{\mathrm{MIU}}, \tilde{\otimes}, \mathbb{C})$, is equivalent to $\mathbf{CvNA}_{\mathrm{MIU}}$. We will only sketch a proof of these statements here.

Let us quickly describe $\tilde{\otimes}$. Let $\mathscr{A}$ and $\mathscr{B}$ be von Neumann algebras. Then $\mathscr{A}\tilde{\otimes}\mathscr{B}$ is a von Neumann algebra equipped with a map $\tilde{\otimes}\colon \mathscr{A}\times\mathscr{B}\to\mathscr{A}\tilde{\otimes}\mathscr{B}$, and $\mathscr{A}\tilde{\otimes}\mathscr{B}$ has the following 'universal property'. For every von Neumann algebra $\mathscr{C}$, and for all normal MIU-maps $f\colon \mathscr{C}\to\mathscr{A}$ and $g\colon \mathscr{C}\to\mathscr{B}$ with $f(c)g(d) = g(d)f(c)$ for all $c,d\in\mathscr{C}$, there is a unique normal MIU-map $h\colon \mathscr{A}\tilde{\otimes}\mathscr{B}\to\mathscr{C}$ with $h(a\tilde{\otimes}b) = f(a)g(b)$ for all $a,b\in\mathscr{A}$. (This universal property differs slightly from the one given in Definition 6.2 of [14] but nonetheless correct, as one easily sees by inspecting the proof of Proposition 6.1 of [14].)

With the universal property, one can extend $\tilde{\otimes}$ to a bifunctor on $\mathbf{vNA}_{\mathrm{MIU}}$ (via $(f_1\tilde{\otimes}f_2)(a_1\tilde{\otimes}a_2) = f_1(a_1)\tilde{\otimes}f_2(a_2)$), and easily sees that $(\mathbf{vNA}_{\mathrm{MIU}}, \tilde{\otimes}, \mathbb{C})$ is a monoidal category (cf. Theorem III of [8]).

Let $\mathscr{A}$ be an Abelian von Neumann algebra. Since $ab = ba$ for all $a,b\in\mathscr{A}$, there is a unique normal MIU-map $\delta\colon \mathscr{A}\tilde{\otimes}\mathscr{A}\to\mathscr{A}$ with $\delta(a\tilde{\otimes}b) = ab$. Then $(\mathscr{A},\delta,!)$ is a monoid in $(\mathbf{vNA}_{\mathrm{MIU}}, \tilde{\otimes}, \mathbb{C})$, where $!\colon \mathbb{C}\to\mathscr{A}$ is the unique normal MIU-map of this type. Conversely, let $(\mathscr{A},m,u)$ be a monoid in $(\mathbf{vNA}_{\mathrm{MIU}}, \tilde{\otimes}, \mathbb{C})$. Then clearly $u = !$. Further, by replacing $\otimes$ by $\tilde{\otimes}$ in the proof of Lemma 9, we see that $\mathscr{A}$ is Abelian and $m(a\tilde{\otimes}b) = ab$. In particular, any normal MIU-map between monoids in $(\mathbf{vNA}_{\mathrm{MIU}}, \tilde{\otimes}, \mathbb{C})$ is a monoid morphism. Hence the category of (commutative) monoids on $(\mathbf{vNA}_{\mathrm{MIU}}, \tilde{\otimes}, \mathbb{C})$ is equivalent to $\mathbf{CvNA}_{\mathrm{MIU}}$.

One may wonder if there is a free monoid in $(\mathbf{vNA}_{\mathrm{MIU}}, \tilde{\otimes}, \mathbb{C})$ on a von Neumann algebra $\mathscr{A}$. This boils down to finding a left adjoint to the inclusion $\mathcal{J}\colon \mathbf{CvNA}_{\mathrm{MIU}}\to\mathbf{vNA}_{\mathrm{MIU}}$. Let us just mention that there is a left adjoint to $\mathcal{J}$, which maps a von Neumann algebra $\mathscr{A}$ to $\{a\in\mathscr{A} \mid \forall b,c\in\mathscr{A}\ [\, abc = cab\,]\}$.

Recall that the spatial tensor product $\otimes$ is used to built our model of the quantum lambda calculus. We would like to mention that $\tilde{\otimes}$ cannot serve the same role, because $(\mathbf{vNA}_{\mathrm{MIU}}, \tilde{\otimes}, \mathbb{C})$ is not monoidal closed (see Corollary 6.5 of [14]), and we do not know if $\tilde{\otimes}$ extends to a functor on $\mathbf{vNA}_{\mathrm{CPsU}}$.

5.2 C^*-algebras

Let $\mathbf{C}^*_{\mathrm{MIU}}$ be the category of unital C^*-algebras and MIU-maps. Then $(\mathbf{C}^*_{\mathrm{MIU}}, \otimes, \mathbb{C})$ is an SMC where $\otimes$ is the spatial tensor product (see Proposition 2.7 of [4]). In contrast with $\mathbf{vNA}_{\mathrm{MIU}}$, *all* commutative C^*-algebras carry a commutative monoid structure in $(\mathbf{C}^*_{\mathrm{MIU}}, \otimes, \mathbb{C})$, the category of (commutative) monoids on $(\mathbf{C}^*_{\mathrm{MIU}}, \otimes, \mathbb{C})$ is equivalent to the category $\mathbf{CC}^*_{\mathrm{MIU}}$ of commutative C^*-algebras, and the free (commutative) monoid on $(\mathbf{C}^*_{\mathrm{MIU}}, \otimes, \mathbb{C})$ on a C^*-algebra $\mathscr{A}$ is $C(\mathrm{sp}(\mathscr{A}))$, where $\mathrm{sp}(\mathscr{A})$ is the spectrum of $\mathscr{A}$ (i.e. the compact Hausdorff space of MIU-maps from $\mathscr{A}\to\mathbb{C}$). We only have space for some hints here.

To see that every commutative C^*-algebra $\mathscr{A}$ carries a commutative monoid structure, note that the spatial tensor product $\mathscr{A}\otimes\mathscr{A}$ coincides with the projective tensor product $\mathscr{A}\otimes_{\max}\mathscr{A}$ (by Lemma 4.18 of [22]), which has a universal property similar to that of the categorical tensor product of von Neumann algebras (see Proposition 4.7 of [22]) by which one can define a MIU-map $\delta\colon \mathscr{A}\otimes\mathscr{A}\to\mathscr{A}$ with $\delta(a\otimes b) = ab$ for all $a,b\in\mathscr{A}$. Conversely, if $(\mathscr{A},m,u)$ is a monoid in $(\mathbf{C}^*_{\mathrm{MIU}}, \otimes, \mathbb{C})$, then u is the unique map $\mathbb{C}\to\mathscr{A}$, and by an easy variation on Lemma 9 for unital C^*-algebras, we see that $\mathscr{A}$ is commutative, and that $m(a\otimes b) = ab$ for all $a,b\in\mathscr{A}$. Thus the category of (commutative) monoids on $(\mathbf{C}^*_{\mathrm{MIU}}, \otimes, \mathbb{C})$

is equivalent to $\mathbf{CC}^*_{\mathrm{MIU}}$. Further, using the obvious strong monoidal adjunction

$$\begin{tikzcd}(\mathbf{CH}, \times, 1) \arrow[r, bend left, "C"] \arrow[r, phantom, "\perp"] & ((\mathbf{C}^*_{\mathrm{MIU}})^{\mathrm{op}}, \otimes, \mathbb{C}) \arrow[l, bend left, "\mathrm{sp}"]\end{tikzcd} ,$$

where $\mathbf{CH}$ is the category of compact Hausdorff spaces and continuous maps, one can show (by the same reasoning as in Theorem 24), that $C(\mathrm{sp}(\mathscr{A}))$ is the free (commutative) monoid on $\mathbf{C}^*_{\mathrm{MIU}}$.

5.3 Relation to cloning and broadcasting of states

Our work is motivated by the study of a model of the quantum lambda calculus, which contains a duplicability operator "!" based on linear logic. The need of the linear type system in quantum programming comes from the well-known fact that quantum states cannot be cloned [9,24] nor broadcast [2]. Recently (no-)cloning and (no-)broadcasting have been studied in categorical quantum mechanics [1,6], and also in operator algebras [13].

The notion of cloning and broadcasting can be formulated in von Neumann algebras as follows (the definition is much the same as [13]). Recall that a normal state of a von Neumann algebra $\mathscr{A}$ is a normal (completely) positive unital map $\omega\colon \mathscr{A} \to \mathbb{C}$.

Definition 28. Let $\mathscr{A}$ be a von Neumann algebra.

1. A normal state $\omega\colon \mathscr{A} \to \mathbb{C}$ is **cloned** by a normal CPsU-map $\delta\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ if $\omega \circ \delta = \omega \otimes \omega$, or equivalently, $\omega(\delta(a \otimes b)) = \omega(a)\omega(b)$ for all $a, b \in \mathscr{A}$.
2. A normal state $\omega\colon \mathscr{A} \to \mathbb{C}$ is **broadcast** by a normal CPsU-map $\delta\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ if $\omega(\delta(1 \otimes a)) = \omega(a) = \omega(\delta(a \otimes 1))$ for all $a \in \mathscr{A}$.

There is a clear relationship between duplicability and broadcasting.

Proposition 29. *A von Neumann algebra $\mathscr{A}$ is duplicable if and only if there exists a normal CPsU-map $\delta\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ which broadcasts all normal states of $\mathscr{A}$.*

Proof. Suppose that $\mathscr{A}$ is duplicable via $\delta\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ and $u \in \mathscr{A}$. By Theorem 2 and Proposition 22, δ is a normal CPsU-map and $u = 1$. Then clearly all normal states are broadcast by δ.

Conversely, suppose that there is $\delta\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ such that $\omega(\delta(1 \otimes a)) = \omega(a) = \omega(\delta(a \otimes 1))$ for all $a \in \mathscr{A}$, and all normal states ω. Then, since the set of normal states are separating, we obtain $\delta(1 \otimes a) = a = \delta(a \otimes 1)$ for all $a \in \mathscr{A}$, and so δ is a duplicator. $\square$

Note that cloning is stronger notion than broadcasting: if ω is cloned by δ, then $\omega(\delta(1 \otimes a)) = \omega(1)\omega(a) = \omega(a)$ and similarly $\omega(\delta(a \otimes 1)) = \omega(a)$. Even in a duplicable von Neumann algebra, not every normal state can be cloned. Indeed, it is easy to see that a normal state $\omega\colon \mathscr{A} \to \mathbb{C}$ on a duplicable von Neumann algebra is cloned by its duplicator $\delta\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ if and only if ω preserves the multiplication, i.e. ω is a normal MIU-map. It follows, by $\mathbf{vNA}_{\mathrm{MIU}}(\ell^\infty(X), \mathbb{C}) \cong X$ (Lemma 3), that the set of normal states of $\ell^\infty(X)$ cloned by its duplicator is precisely the set X.

Acknowledgements

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement n$^{\mathrm{o}}$ 320571.

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