Semantics for a Quantum Programming Language by Operator Algebras

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QPL 2014
5 June 2014

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Master thesis
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Overview

- Semantics for a first-order functional quantum programming language **QPL** [Selinger 2004]
- Use the category $\mathbf{Wstar}_{\text{CP-PU}}$ of W*-algebras and normal completely positive pre-unital maps
- $\mathbf{Wstar}_{\text{CP-PU}}$ is a $\mathbf{Dcppo}_{\perp}$-enriched SMC with $\mathbf{Dcppo}_{\perp}$-enriched finite products
  - “nice” enough to give a semantics for QPL

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- Use the category $\mathbf{Wstar}_{\text{CP-PU}}$ **?** of W*-algebras and normal completely positive pre-unital maps
  quantum operations in the Heisenberg picture

- $\mathbf{Wstar}_{\text{CP-PU}}$ is a $\mathbf{Dcppo}_{\perp}$-enriched SMC with $\mathbf{Dcppo}_{\perp}$-enriched finite products

  - “nice” enough to give a semantics for QPL

Outline

- Quantum Operation
- Selinger's QPL
- Operator Algebras and Quantum Operation
- Semantics for QPL by W*-algebras
- Future work and Conclusions

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Quantum Operation [Kraus]

(aka. superoperator)

$\mathcal{T}(\mathbb{C}^n) \cong \mathcal{M}_n$ the set of n×n matrices

$\mathcal{H}_1, \mathcal{H}_2$: Hilbert spaces

$\mathcal{T}(\mathcal{H}_i)$: the set of trace class operators on $\mathcal{H}_i$

Def. A linear map $\mathcal{E}\colon \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{H}_2)$ is a ***quantum operation (QO)***
$\stackrel{\text{def}}{\Longleftrightarrow}$ it is **completely positive** and **trace-nonincreasing**.

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ρ : **state** (density operator) on $\mathcal{H}_1$
i.e. positive operator on $\mathcal{H}_1$ with $\mathrm{tr}(\rho) = 1$
(hence $\rho \in \mathcal{T}(\mathcal{H}_1)$ by def.)

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$\longmapsto$ $\mathcal{E}(\rho)$: positive operator on $\mathcal{H}_2$ with $0 \leq \mathrm{tr}(\mathcal{E}(\rho)) \leq 1$
i.e. subnormalised **state** on $\mathcal{H}_2$

Complete positivity

Def. A linear map $\mathcal{E}: \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{H}_2)$ is a ***quantum operation (QO)***

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$\mathcal{E}\colon \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{H}_2)$ is *completely positive (CP)*
$\overset{\text{def}}{\iff}$ $\forall n \in \mathbb{N}$
$\mathrm{id} \otimes \mathcal{E}\colon \mathcal{M}_n \otimes \mathcal{T}(\mathcal{H}_1) \to \mathcal{M}_n \otimes \mathcal{T}(\mathcal{H}_2)$ is positive
$\cong \mathcal{T}(\mathbb{C}^n \otimes \mathcal{H}_1)$ $\cong \mathcal{T}(\mathbb{C}^n \otimes \mathcal{H}_2)$

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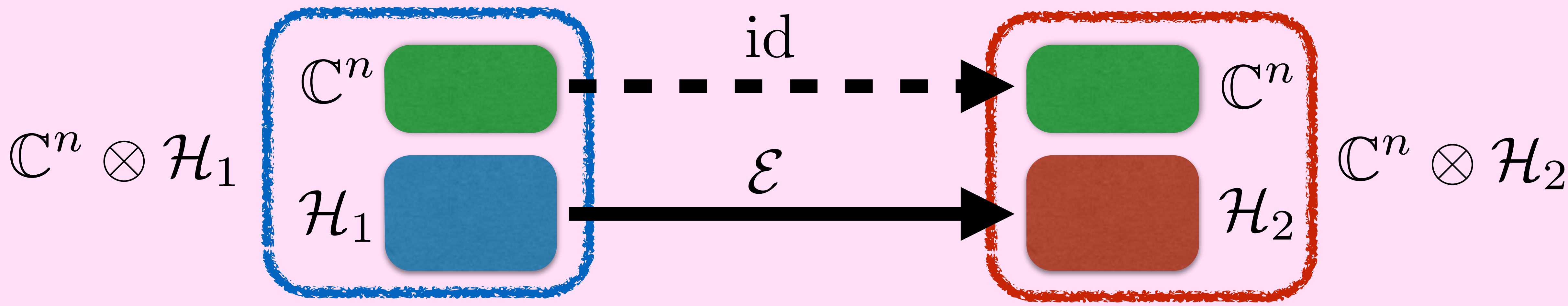
Compatibility with composition (i.e. tensor product) of systems



Dualising Quantum Operations

$\mathcal{B}(\mathcal{H}_i)$: the set of bounded operators on $\mathcal{H}_i$

Fact. There is a 1-1 correspondence:

$$\frac{\mathcal{E}: \mathcal{T}(\mathcal{H}_1) \longrightarrow \mathcal{T}(\mathcal{H}_2) \text{ bounded}}{\mathcal{E}^*: \mathcal{B}(\mathcal{H}_2) \longrightarrow \mathcal{B}(\mathcal{H}_1) \text{ weak*-continuous (called normal)}}$$

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$\mathcal{E} : \mathcal{T}(\mathcal{H}_1) \longrightarrow \mathcal{T}(\mathcal{H}_2)$ bounded

$\mathcal{T}(\mathcal{H}_i)$ is a Banach space wrt. trace norm, and $\mathcal{T}(\mathcal{H}_i)^* \cong \mathcal{B}(\mathcal{H}_i)$

$\mathcal{E}^* : \mathcal{B}(\mathcal{H}_2) \longrightarrow \mathcal{B}(\mathcal{H}_1)$ weak*-continuous (called ***normal***)

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This correspondence restricts to:

$\mathcal{E}: \mathcal{T}(\mathcal{H}_1) \longrightarrow \mathcal{T}(\mathcal{H}_2)$ **QO**, i.e. CP trace-nonincreasing

$\mathcal{E}^*: \mathcal{B}(\mathcal{H}_2) \longrightarrow \mathcal{B}(\mathcal{H}_1)$ normal CP pre-unital (sub-unital)

$\mathcal{E}^*(\mathcal{I}) \leq \mathcal{I}$

Schrödinger vs Heisenberg picture

QOs arise in two equivalent (dual) forms:

$$\frac{\mathcal{E}\colon \mathcal{T}(\mathcal{H}_1) \longrightarrow \mathcal{T}(\mathcal{H}_2) \text{ CP trace-nonincreasing}}{\mathcal{E}^*\colon \mathcal{B}(\mathcal{H}_2) \longrightarrow \mathcal{B}(\mathcal{H}_1) \text{ normal CP pre-unital}}$$

Schrödinger vs Heisenberg picture

QOs arise in two equivalent (dual) forms:

span of density operators

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$\mathcal{E}$ is a QO in the **Schrödinger** picture (**states** evolve)

$\mathcal{E}^*$ is a QO in the **Heisenberg** picture (**observables** evolve)

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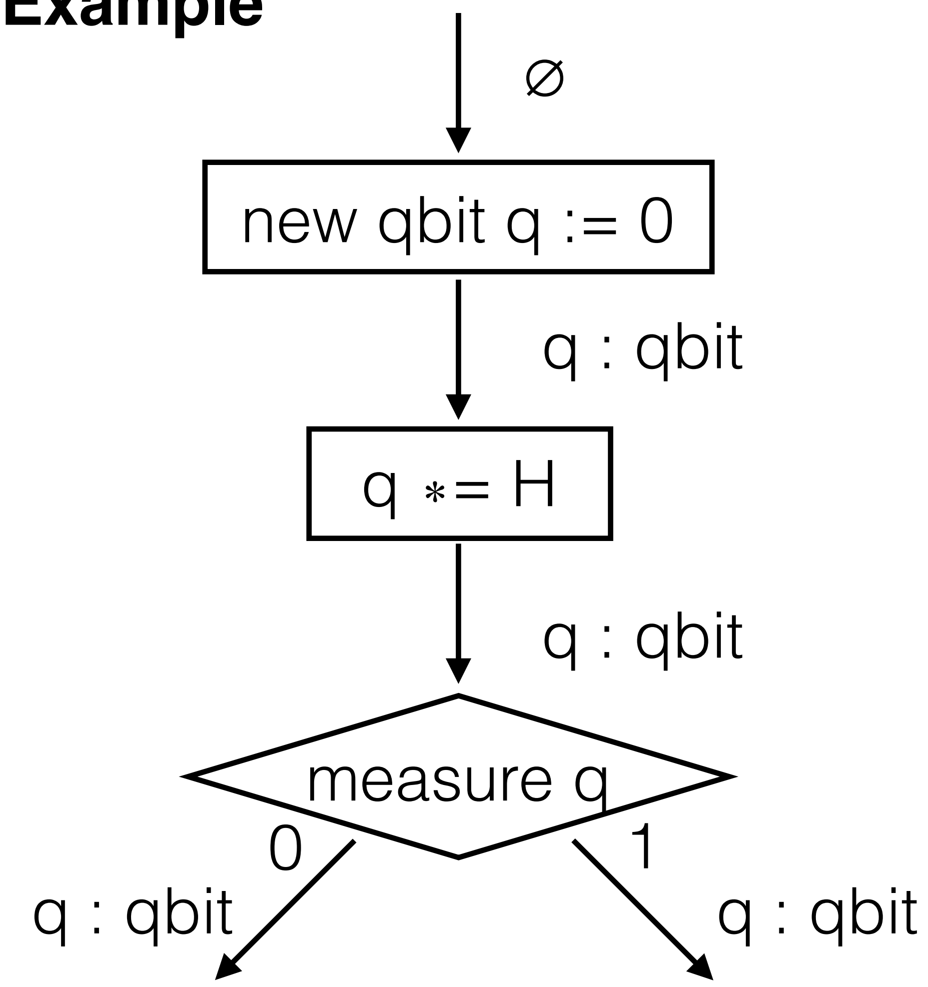
Selinger's QPL

- QPL (QFC) [Selinger 2004]
- First-order functional quantum programming language
- Loop and recursion
- "Quantum data, Classical control"
- Data types: qbit, bit
- Written as a flow chart

Example



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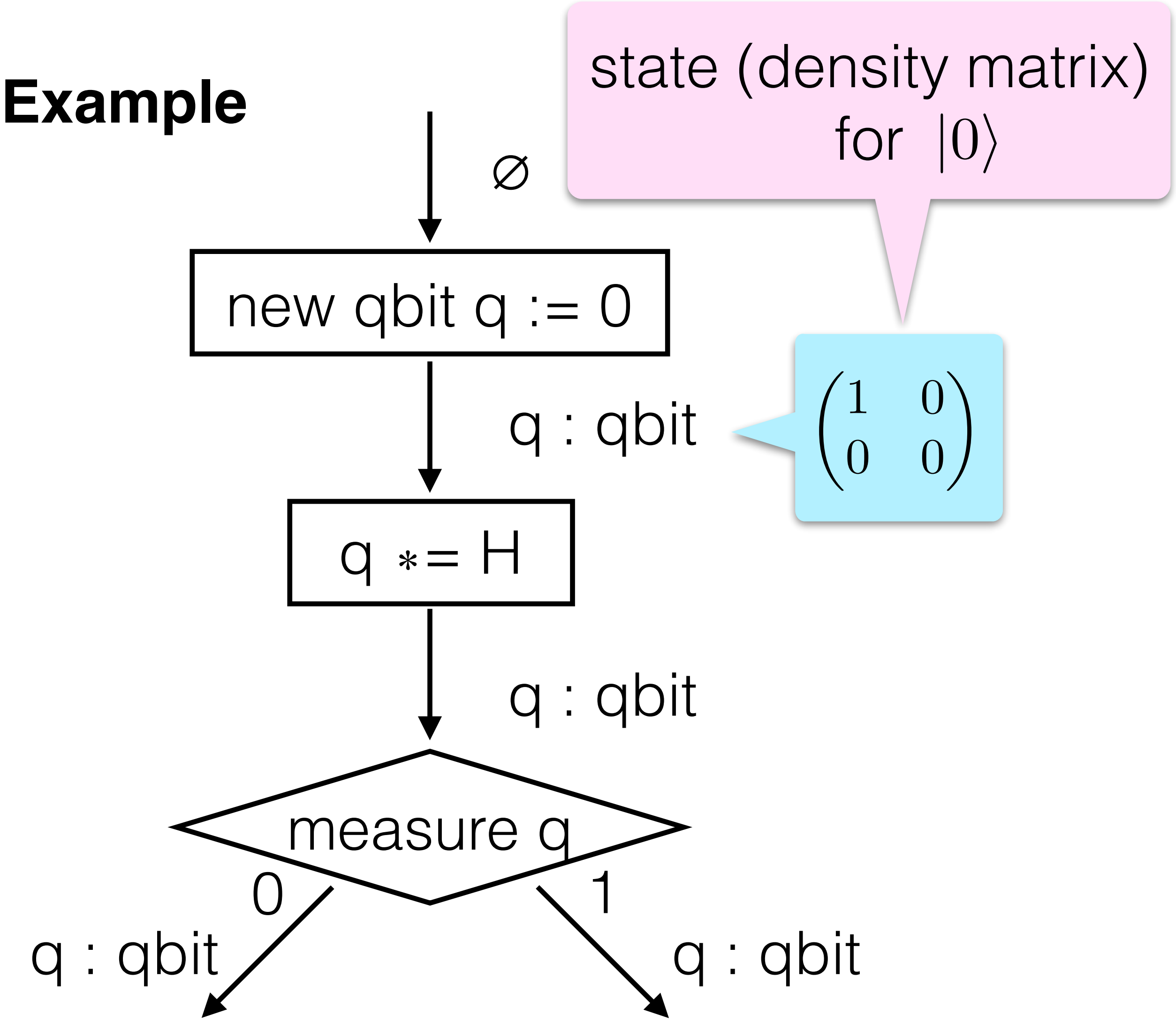

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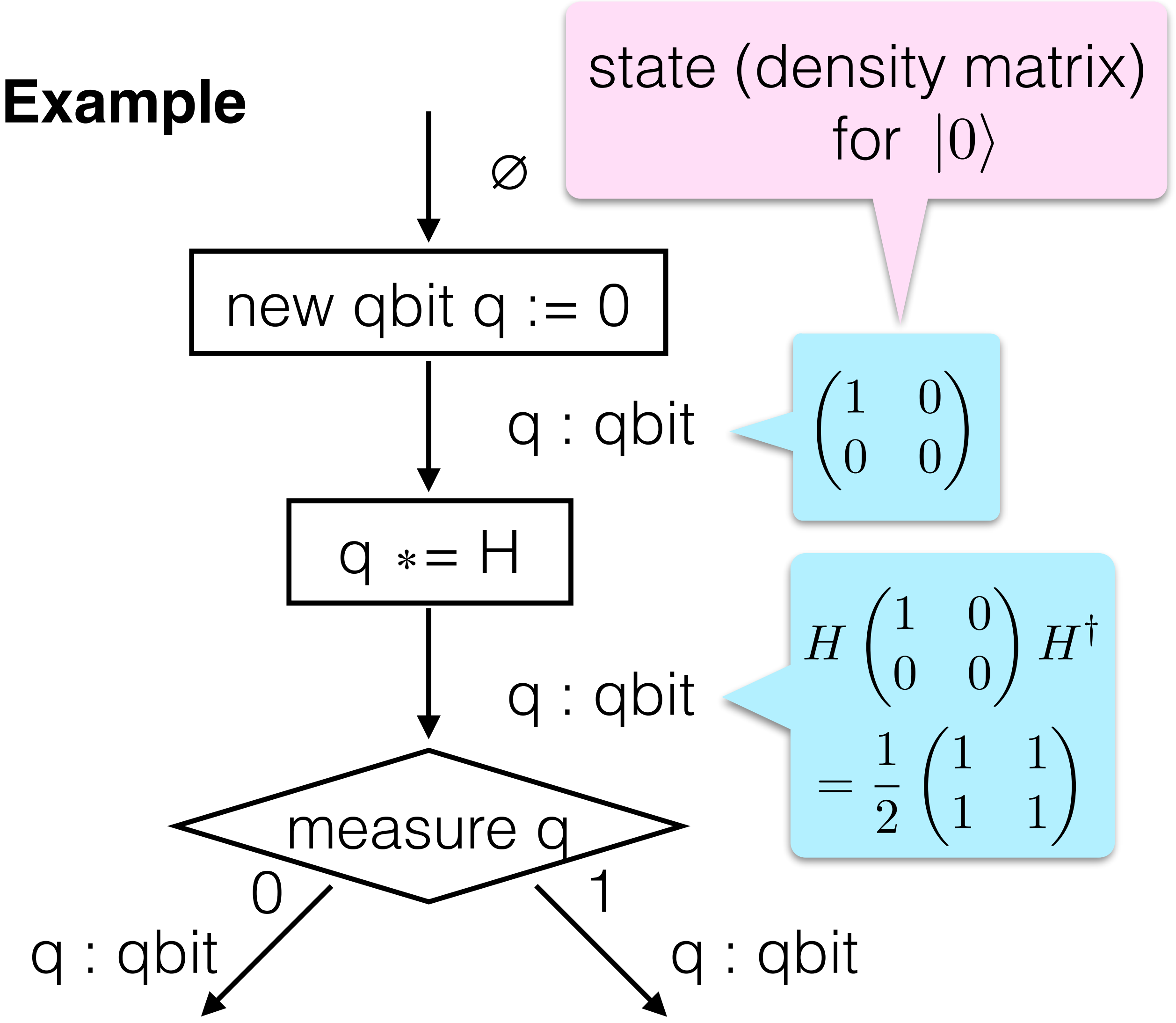

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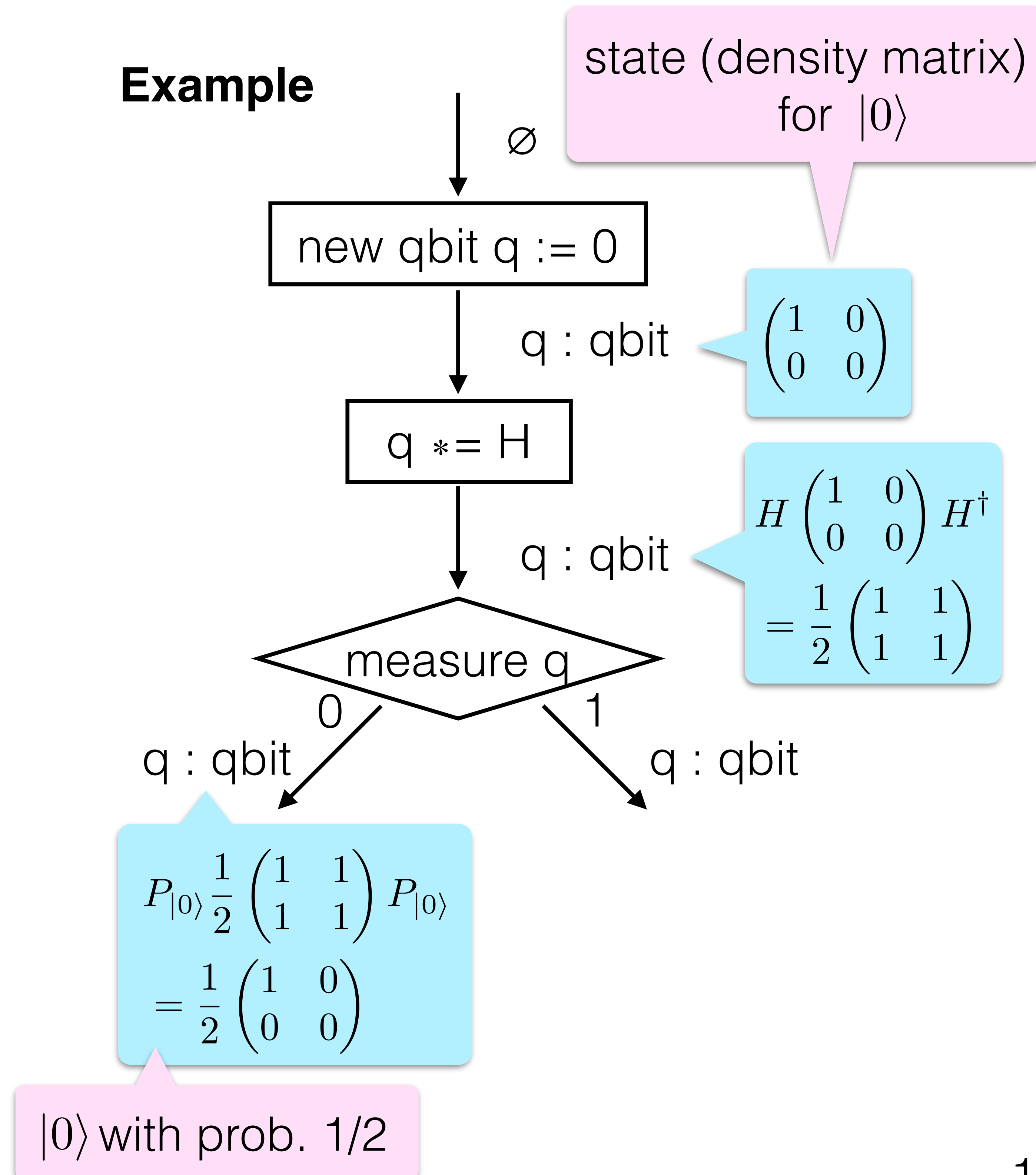

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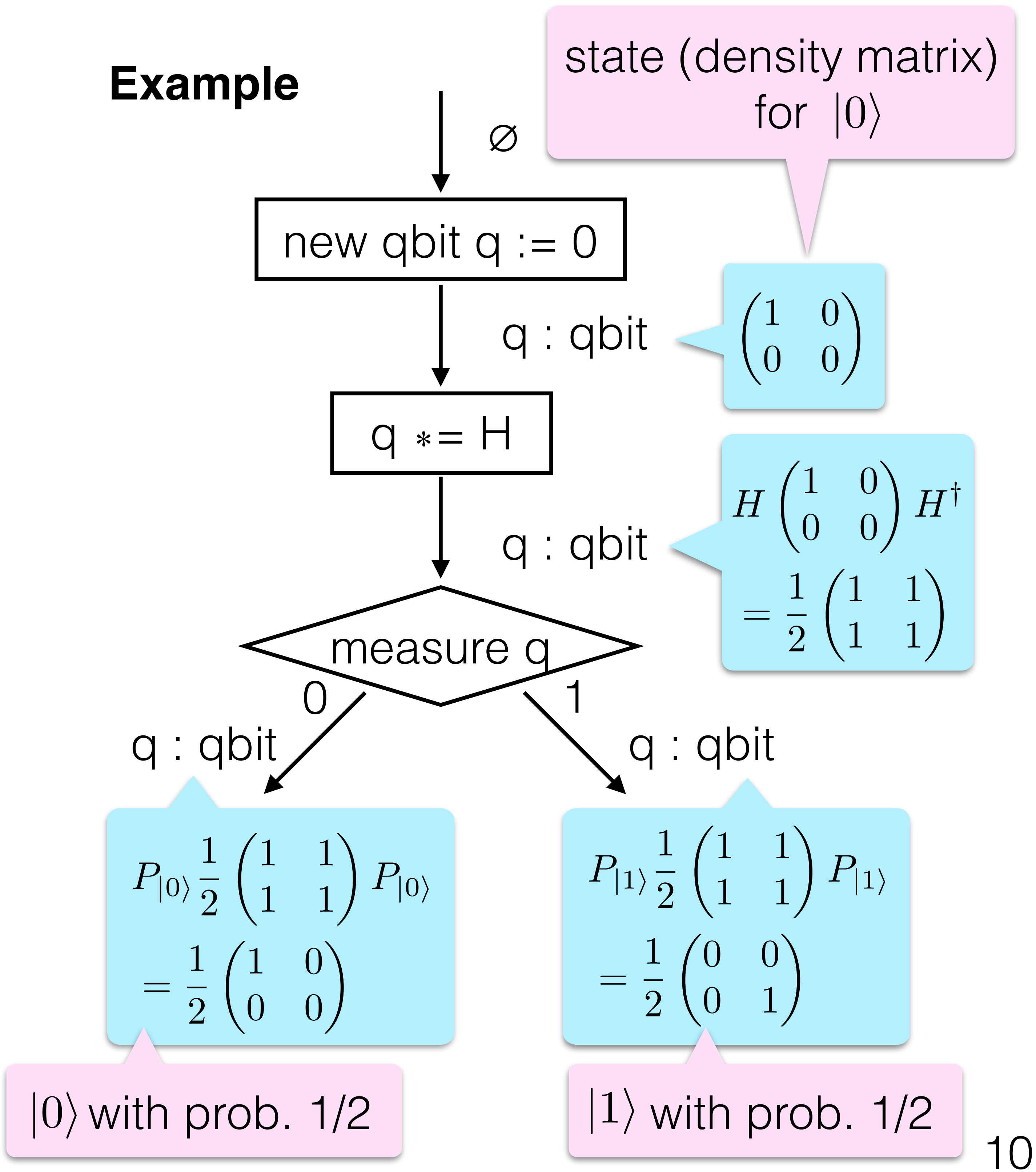
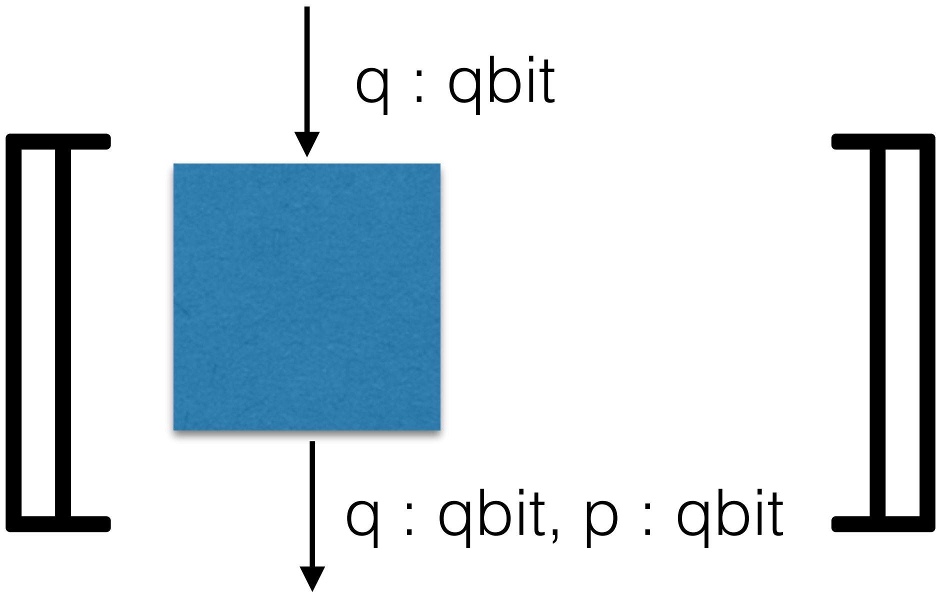


Semantics for QPL



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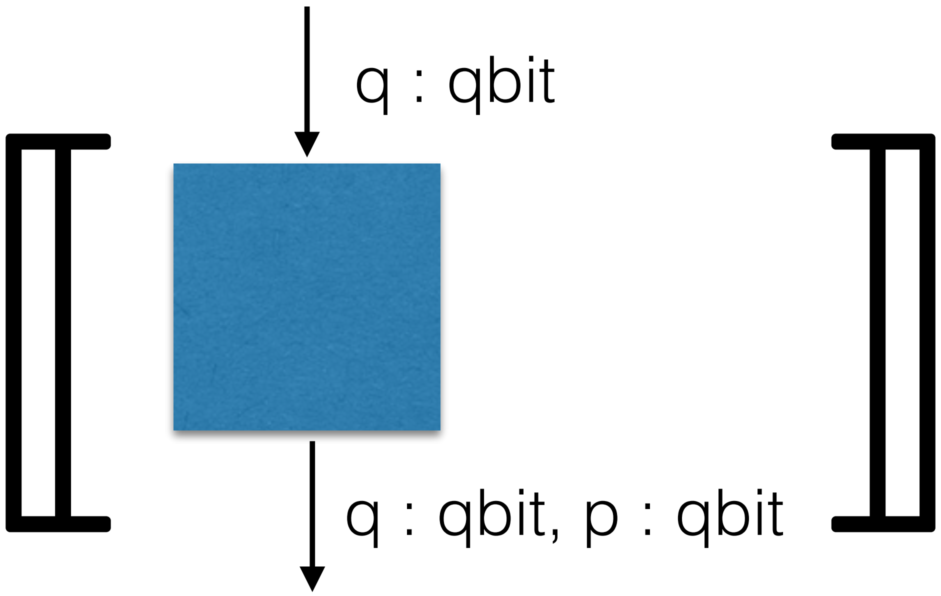

$$: \mathcal{M}_2 \longrightarrow \mathcal{M}_2 \otimes \mathcal{M}_2 \cong \mathcal{M}_4$$

QO (in Schrödinger picture)

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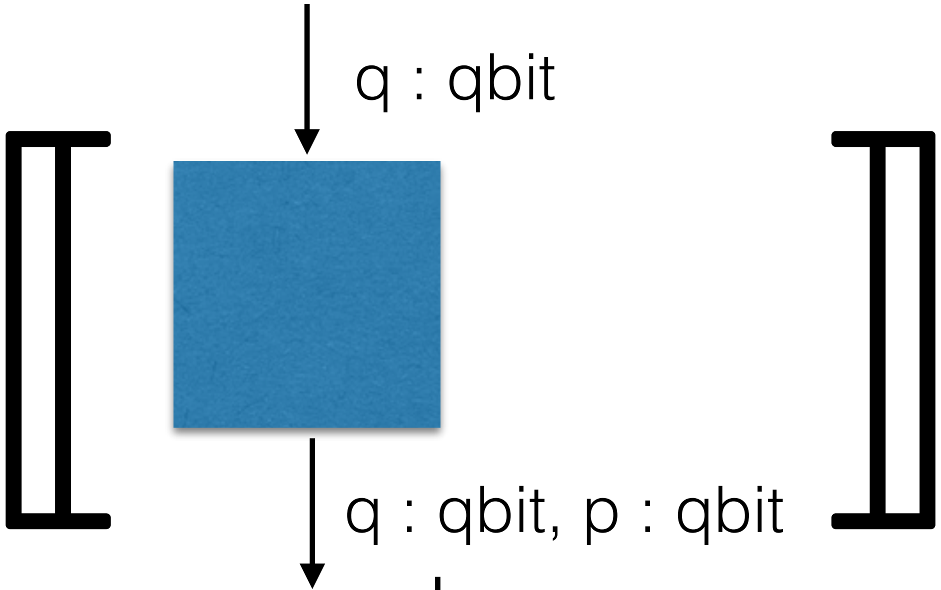

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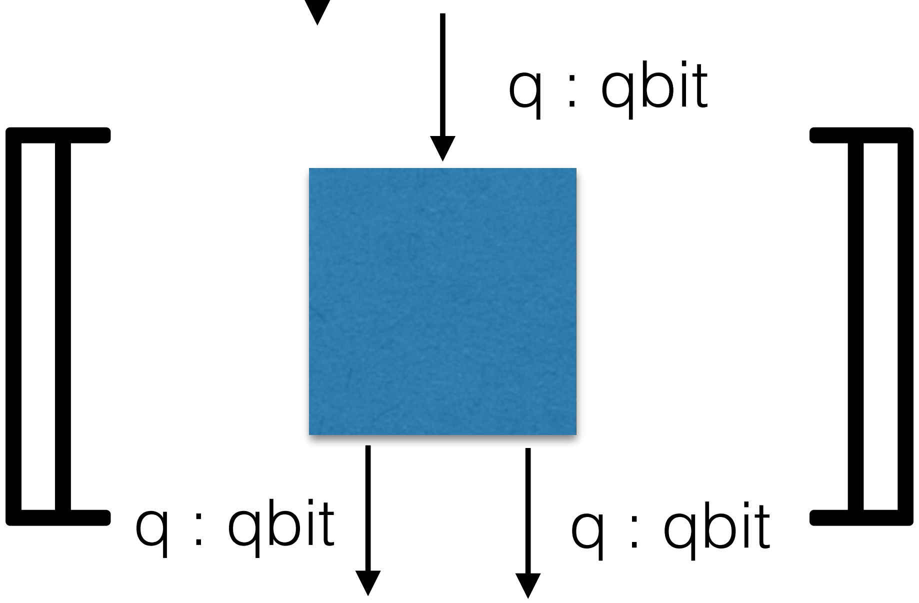
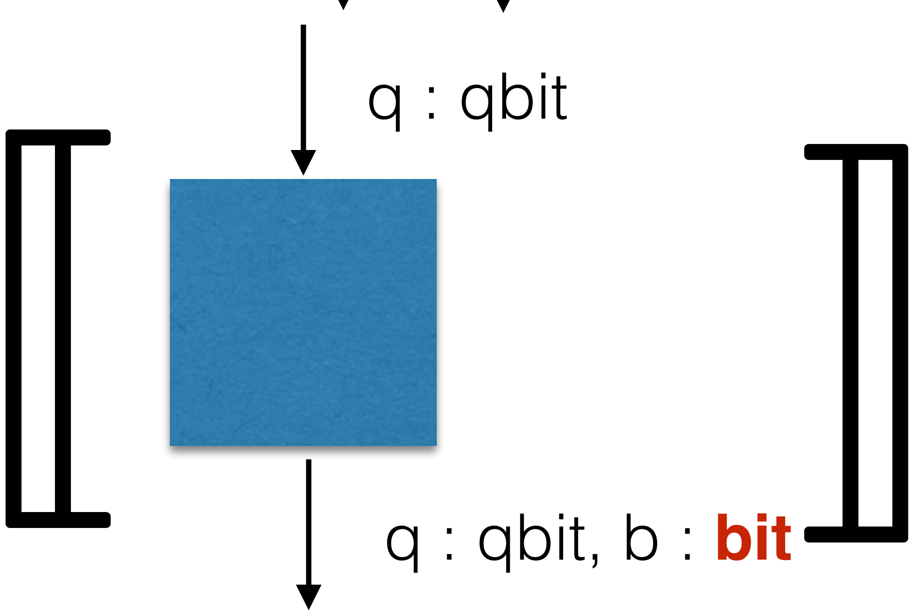


$= ??$



$= ??$

Kraus' "simple" QO is **not** suitable for **classical** control/data

Selinger's QO

Selinger's solution: generalise QOs into maps of type

$$\mathcal{E} : \bigoplus_{j=1}^{k} \mathcal{M}_{n_j} \longrightarrow \bigoplus_{i=1}^{l} \mathcal{M}_{m_i}$$

$\bigoplus$ direct sum (of vector spaces)

Def.
A linear map $\mathcal{E} : \bigoplus_{j=1}^{k} \mathcal{M}_{n_j} \longrightarrow \bigoplus_{i=1}^{l} \mathcal{M}_{m_i}$ is a ***QO*** (in the Schrödinger pic.)

$\stackrel{\text{def}}{\Longleftrightarrow}$ it is CP and trace-nonincreasing.

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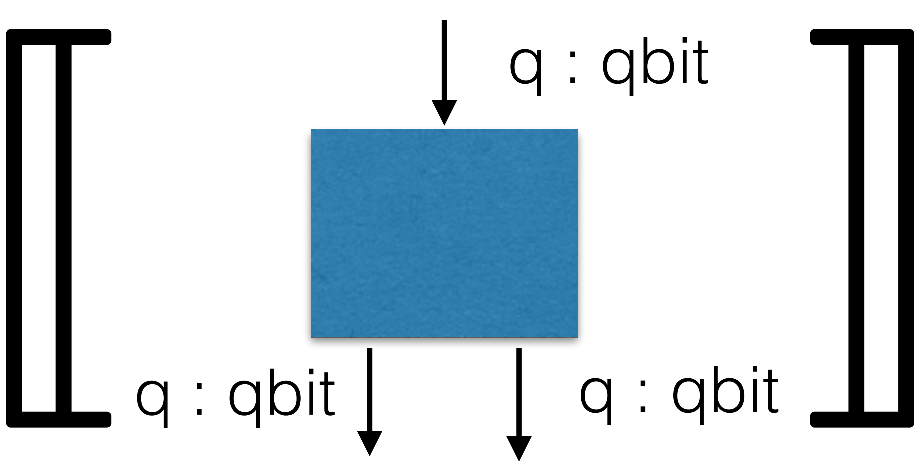

$: \ \mathcal{M}_2 \longrightarrow \mathcal{M}_2 \oplus \mathcal{M}_2$

The category $\mathbf{Q}$

Def. The category $\mathbf{Q}$ is defined as follows.

Objects: $\bigoplus_{j=1}^{k} \mathcal{M}_{n_j}$ for each sequence of natural numbers $(n_1, \ldots, n_k)$

Arrows: $\mathcal{E} \colon \bigoplus_{j=1}^{k} \mathcal{M}_{n_j} \longrightarrow \bigoplus_{i=1}^{l} \mathcal{M}_{m_i}$ Selinger's QO

Categorical Property of $\mathbf{Q}$

$\mathbf{Q}$ is an SMC $(\mathbf{Q}, \otimes, \mathbb{C})$ with finite coproducts $(\oplus, 0)$ such that $\otimes$ distributes over $(\oplus, 0)$:

$$A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C), \quad A \otimes 0 \cong 0 \ .$$

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tensor product

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Thm. With the interpretation of types

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$$[\![\mathrm{bit}]\!] = \mathbb{C} \oplus \mathbb{C}$$

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for loop and recursion

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Sufficient condition to give a semantics for QPL

- $\mathbf{C}$ is an $\boldsymbol{\omega}\mathbf{Cppo}$-enriched SMC $(\mathbf{C}, \otimes, I)$ with $\boldsymbol{\omega}\mathbf{Cppo}$-enriched finite coproducts $(\oplus, 0)$ such that $\otimes$ distributes over $(\oplus, 0)$:

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- An object $[\![\mathrm{qbit}]\!] \in \mathbf{C}$

$([\![\mathrm{bit}]\!] = I \oplus I)$

- Some additional conditions…

$$f \circ \bot = \bot$$
$$f \otimes \bot = \bot$$
$$\iota \colon I \oplus I \to [\![\mathrm{qbit}]\!]$$
$$p \colon [\![\mathrm{qbit}]\!] \to I \oplus I$$
$$p \circ \iota = \mathrm{id}$$

Thm. Such $\mathbf{C}$ gives a semantics for QPL.

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Operator Algebras

Concrete (*-subalgebra of $\mathcal{B}(\mathcal{H})$)	Abstract (Hilbert space-free)
norm-closed	C*-algebra
weakly closed, unital = von Neumann algebra	W*-algebra

- First, von Neumann algebras are introduced by von Neumann, motivated by quantum theory
- In the context of quantum theory, operator algebras are seen as algebras of **observables**
- ***Algebraic*** quantum theory
  - Emphasis on operator algebras, rather than Hilbert spaces
  - Successful in quantum field theory, quantum statistical mechanics

W*-algebra

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Eg. $\mathcal{E}^* : \mathcal{B}(\mathcal{H}_2) \longrightarrow \mathcal{B}(\mathcal{H}_1)$ QO in the Heisenberg picture
is (by def.) a normal CP pre-unital map betw. W*-alg.

The category $\mathbf{Wstar}_{\text{CP-PU}}$

Def. The category $\mathbf{Wstar}_{\text{CP-PU}}$ is defined as follows.
Objects: W*-algebras
Arrows: normal CP pre-unital maps

QO $\mathcal{E}^* : \mathcal{B}(\mathcal{H}_2) \longrightarrow \mathcal{B}(\mathcal{H}_1)$ is an arrow in $\mathbf{Wstar}_{\text{CP-PU}}$

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Moreover, there is one-to-one correspondence:

$$\mathcal{E}\colon \bigoplus_{j=1}^{k} \mathcal{M}_{n_j} \longrightarrow \bigoplus_{i=1}^{l} \mathcal{M}_{m_i}$$ Selinger's QO, i.e. arrow in $\mathbf{Q}$

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$$\mathcal{E}^*\colon \bigoplus_{i=1}^{l} \mathcal{M}_{m_i} \longrightarrow \bigoplus_{j=1}^{k} \mathcal{M}_{n_j} \quad \text{arrow in } \mathbf{Wstar}_{\text{CP-PU}}$$

This gives a full embedding $\mathbf{Q} \longrightarrow (\mathbf{Wstar}_{\text{CP-PU}})^{\text{op}}$

Various Quantum Operations

Kraus' (simple) QO

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$f\colon M \longrightarrow N$ in $\mathbf{Wstar}_{\mathrm{CP\text{-}PU}}$

$\mathbf{Wstar}_{\mathrm{CP\text{-}PU}}$ naturally arises as the category whose arrows are **quantum operations** (in the Heisenberg picture)

The present work shows:

$\mathbf{Wstar}_{\mathrm{CP\text{-}PU}}$ is "nice" enough to give a sem. for QPL

Outline

- Quantum Operation
- Selinger's QPL
- Operator Algebras and Quantum Operation
- **Semantics for QPL by W*-algebras**
- Future work and Conclusions

Sufficient condition to give a semantics for QPL

- $\mathbf{C}$ is an $\boldsymbol{\omega}\mathbf{Cppo}$-enriched SMC $(\mathbf{C}, \otimes, I)$ with $\boldsymbol{\omega}\mathbf{Cppo}$-enriched finite coproducts $(\oplus, 0)$ such that $\otimes$ distributes over $(\oplus, 0)$:

$$A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C), \quad A \otimes 0 \cong 0 \ .$$

- An object $[\![\mathrm{qbit}]\!] \in \mathbf{C}$

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Goal: $(\mathbf{Wstar}_{\text{CP-PU}})^{\text{op}}$ satisfies these conditions

Categorical Property of $\mathbf{Wstar}_{\text{CP-PU}}$

$\mathbf{Wstar}_{\text{CP-PU}}$ is an SMC $(\mathbf{Wstar}_{\text{CP-PU}}, \overline{\otimes}, \mathbb{C})$
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$\mathbf{Wstar}_{\text{CP-PU}}$ is $\boldsymbol{\omega}\mathbf{Cppo}$-enriched?

Yes! In fact, $\mathbf{Wstar}_{\text{CP-PU}}$ is $\mathbf{Dcppo}_{\perp}$-enriched

the category of pointed dcpos and strict Scott-continuous maps

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Thm. Every W*-algebra is **monotone closed**, *i.e.* every norm-bounded directed set of self-adjoint elements has a supremum (which is self-adjoint).

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⇒ For every W*-algebra M, $[0,1]_M$ is a (pointed) dcpo.

W*-algebras and Domain theory

Prop. $f\colon M\to N$ positive pre-unital map betw. W*-alg.
f is normal (i.e. weak*-continuous)
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➡ W*-algebras behave well domain-theoretically

Dcpo structure of W*-algebras "lifts" to hom-set
with an ordering: $f \sqsubseteq g \overset{\text{def}}{\iff} g - f$ is CP

Thm. M, N : W*-algebras.
$\mathbf{Wstar}_{\text{CP-PU}}(M, N)$ is a pointed dcpo.

Moreover, the composition of arrows

$$\mathbf{Wstar}_{\text{CP-PU}}(N,L) \times \mathbf{Wstar}_{\text{CP-PU}}(M,N) \to \mathbf{Wstar}_{\text{CP-PU}}(M,L)$$
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We can also show:

$$\mathbf{Wstar}_{\text{CP-PU}}(M, N) \times \mathbf{Wstar}_{\text{CP-PU}}(M', N') \to \mathbf{Wstar}_{\text{CP-PU}}(M \mathbin{\overline{\otimes}} M', N \mathbin{\overline{\otimes}} N')$$
$$(f, g) \mapsto f \mathbin{\overline{\otimes}} g$$

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$(\mathbf{Wstar}_{\text{CP-PU}})^{\text{op}}$ satisfies all conditions of the following.

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Comparison with Selinger's original semantics

- Recall that there is a full embedding:

$$\mathbf{Q} \longrightarrow (\mathbf{Wstar}_{\text{CP-PU}})^{\text{op}}$$

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$(\mathbf{Wstar}_{\text{CP-PU}})^{\text{op}}$ can be seen as an **infinite** dimensional extension of $\mathbf{Q}$

C* vs W*

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(Note: $\mathbf{FdCstar}_{\text{CP-PU}} = \mathbf{FdWstar}_{\text{CP-PU}} \simeq \mathbf{Q}^{\text{op}}$)

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$$\ell^\infty(X) := \left\{ \varphi \colon X \to \mathbb{C} \;\middle|\; \sup_{x \in X} |\varphi(x)| < \infty \right\}$$

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$$\vdots$$

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Classical computation in commutative W*-algebras

Thm.

There is an embedding

the category of **commutative** W*-algebras

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the category of **commutative** W*-algebras

Classical (deterministic) computation in $\mathbf{Set}$ arises as a map between **commutative** W*-algebras

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- Quantum Operation
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  Q. Is $(\mathbf{Wstar}_{\text{CP-PU}})^{\text{op}}$ monoidal closed?

- Exploit the duality between commutative W*-algebras and measurable space

  cf. Gelfand duality $(\mathbf{CCstar}_{\text{M-I-U}})^{\text{op}} \simeq \mathbf{CompHaus}$

Conclusions

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  - "nice" enough to give a semantics for Selinger's QPL
- W*-algebras give a flexible model for quantum computation
  - accommodate infinite dim. structures and classical (= commutative) computation
- The present work is the first step. A lot of things to do!
  - *cf.* Mathys Rennela's work (MFPS XXX, 2014)