

On Extension of Parameterized Coinduction

Masaki Hara

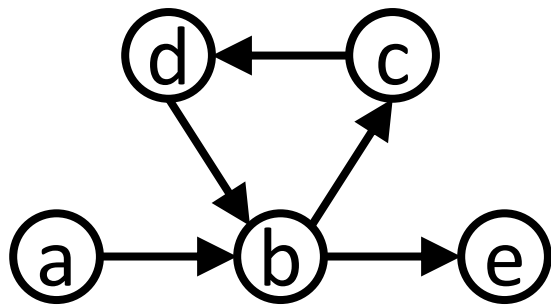
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Tokyo

Background

Formal Verification of (Co)Inductive Properties using Proof Assistants

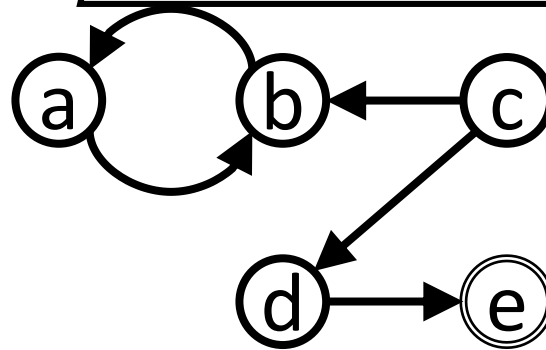
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Formal Verification of (Co)Inductive Properties using Proof Assistants



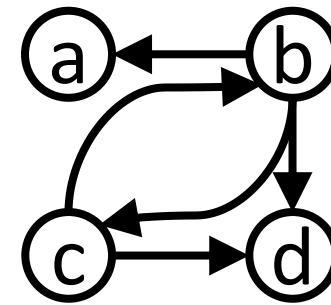
Infinite Path

$$\{a\} \subseteq \nu Z. \diamond Z$$



Reachability

$$\{c\} \subseteq \mu Z. \{e\} \cup \diamond Z$$



Bisimulation

$$\{(b, c)\} \subseteq \nu R. \text{sim}(R)$$

[Hur *et al.*, POPL '13]

Parameterized coinduction
for coinductive properties

Parameterized induction
for inductive properties

No-discovery characterization

How convenient is it?
Mathematical characterization

Our
contribution

[Hur *et al.*, POPL '13]

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Complete lattice and monotone function

Def.

(C, \sqsubseteq) : complete lattice *i.e.*

- $\forall A \subseteq C. \forall x \in C. x \sqsubseteq \bigcap A \Leftrightarrow (\forall y \in A. x \sqsubseteq y)$
- $\forall A \subseteq C. \forall x \in C. \bigcup A \sqsubseteq x \Leftrightarrow (\forall y \in A. y \sqsubseteq x)$
- $x \sqcup y = \bigcup \{x, y\}, \perp = \bigcup \emptyset$
- $x \sqcap y = \bigcap \{x, y\}, \top = \bigcap \emptyset$

$f : C \rightarrow C$: monotone *i.e.*

- $\forall x, y \in C. x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$

Tarski coinduction

Def.

$\nu f = \max\{x \in C \mid x = f(x)\}$ greatest fixpoint

Lem.

$$f(\nu f) = \nu f$$

$$\forall x. x \sqsubseteq f(x) \Rightarrow x \sqsubseteq \nu f \quad (\text{Tarski})$$

Coq's built-in coinduction

Parameterized coinduction

[Hur *et al.*, POPL '13]

Def.

$G_f(x) = \nu y. f(x \sqcup y)$ parameterized greatest fixpoint

Lem.

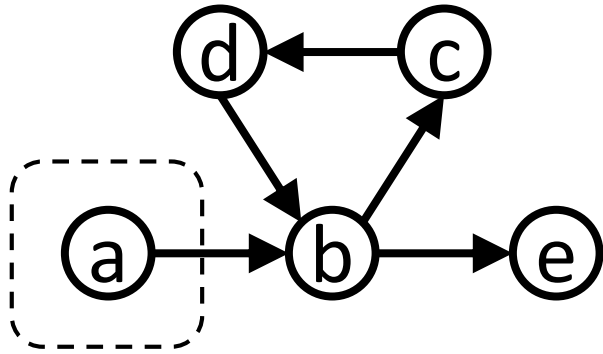
$\forall x, y \in C. y \sqsubseteq f(x \sqcup G_f(x)) \Rightarrow y \sqsubseteq G_f(x)$ (Unfolding)

$\forall x, y \in C. y \sqsubseteq G_f(x \sqcup y) \Rightarrow y \sqsubseteq G_f(x)$ (Accumulation)

Implemented on Coq in [Hur *et al.*, POPL '13]

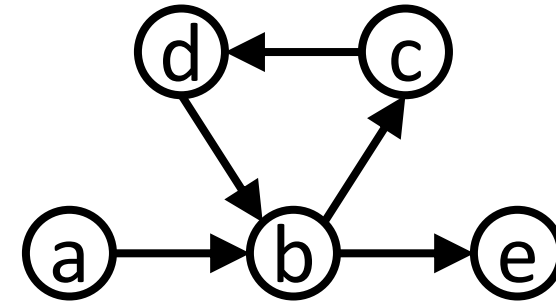
Example

Tarski



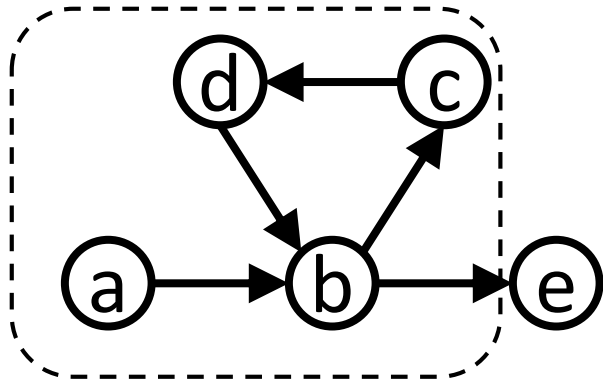
$$\{a\} \subseteq \nu Z. \diamond Z$$

PaCo



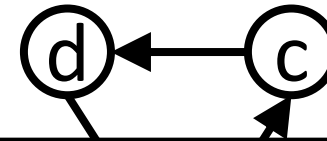
Example

Tarski



$$\{a\} \subseteq vZ. \diamond Z$$
$$\Leftarrow \{a, b, c, d\} \subseteq vZ. \diamond Z$$

PaCo

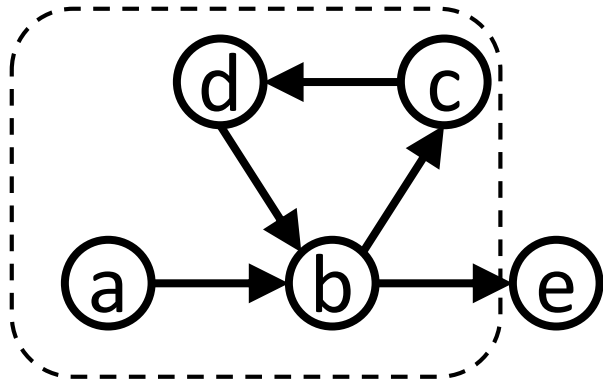


Transitivity

$$x \sqsubseteq vf$$
$$\Leftarrow x \sqsubseteq x', x' \sqsubseteq vf$$

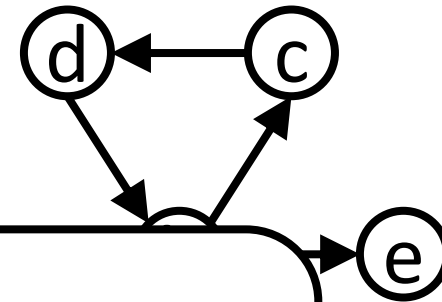
Example

Tarski



$\{a\} \subseteq \nu Z. \diamond Z$
 $\Leftarrow \{a, b, c, d\} \subseteq \nu Z. \diamond Z$
 $\Leftarrow \{a, b, c, d\} \subseteq \diamond \{a, b, c, d\}$

PaCo

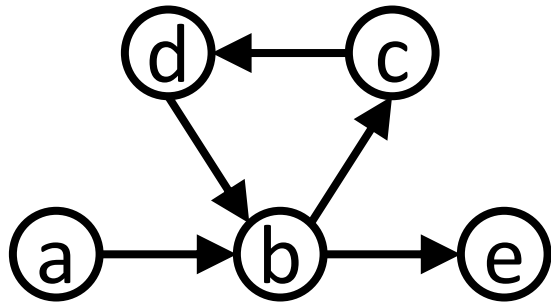


Tarski

$x \sqsubseteq \nu f$
 $\Leftarrow x \sqsubseteq f(x)$

Example

Tarski



$$\{a\} \subseteq \nu Z. \diamond Z$$

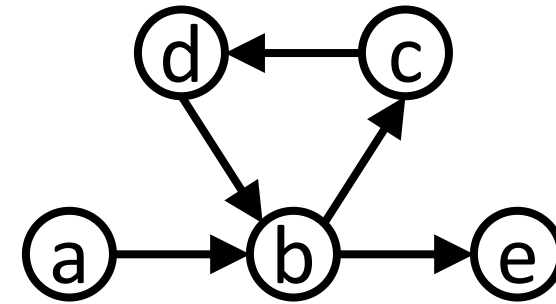
$$\Leftarrow \{a, b, c, d\} \subseteq \nu Z. \diamond Z$$

$$\Leftarrow \{a, b, c, d\} \subseteq \diamond \{a, b, c, d\}$$

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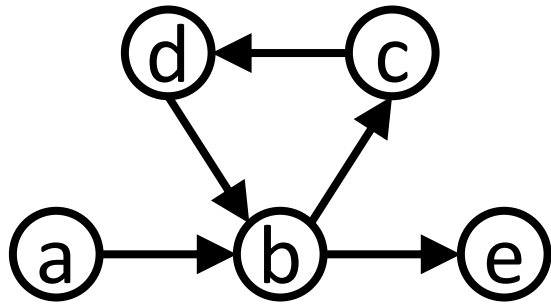
□

PaCo



Example

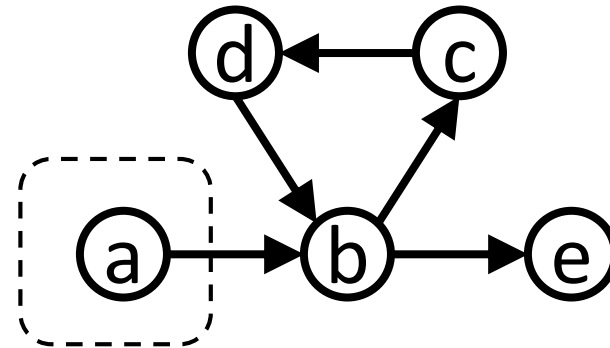
Tarski



$$\begin{aligned} & \{a\} \subseteq \nu Z. \diamond Z \\ \Leftarrow & \{a, b, c, d\} \subseteq \nu Z. \diamond Z \\ \Leftarrow & \{a, b, c, d\} \subseteq \diamond \{a, b, c, d\} \\ \Leftarrow & \{a, b, c, d\} \subseteq \{a, b, c, d\} \end{aligned}$$

□

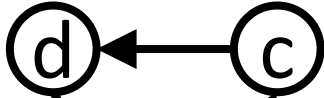
PaCo



$$\{a\} \subseteq G_{\diamond}(\emptyset)$$

Example

Tarski



Unfolding

$$y \sqsubseteq G_f(x)$$

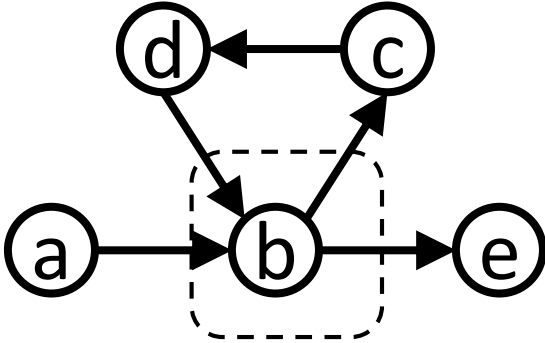
$$\{ \Leftarrow y \sqsubseteq f(x \sqcup G_f(x))$$

$$\Leftarrow \{a, b, c, d\} \sqsubseteq \diamond \{a, b, c, d\}$$

$$\Leftarrow \{a, b, c, d\} \sqsubseteq \{a, b, c, d\}$$

□

PaCo

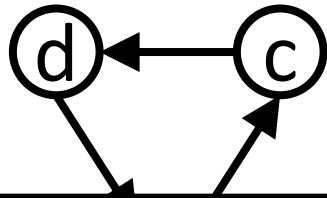


$$\{a\} \sqsubseteq G_\diamond(\emptyset)$$

$$\Leftarrow \{b\} \sqsubseteq G_\diamond(\emptyset)$$

Example

Tarski



Accumulation

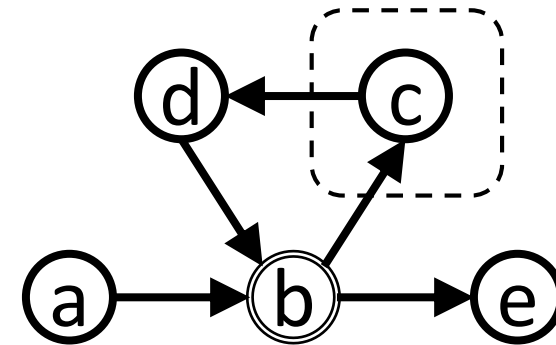
$$y \sqsubseteq G_f(x)$$

$$\Leftarrow y \sqsubseteq G_f(x \sqcup y)$$

$$\Leftarrow \{a, b, c, d\} \sqsubseteq \{a, b, c, d\}$$

□

PaCo



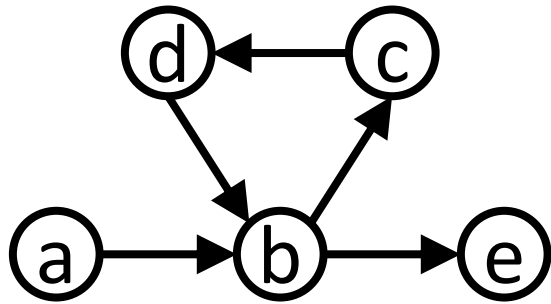
$$\{a\} \sqsubseteq G_\diamond(\emptyset)$$

$$\Leftarrow \{b\} \sqsubseteq G_\diamond(\emptyset)$$

$$\Leftarrow \{c\} \sqsubseteq G_\diamond(\{b\})$$

Example

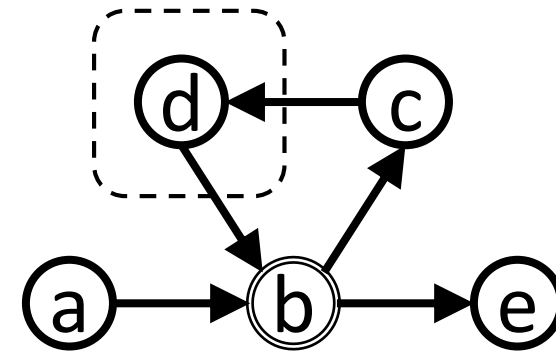
Tarski



$\{a\} \subseteq \nu Z. \diamond Z$
 $\Leftarrow \{a, b, c, d\} \subseteq \nu Z. \diamond Z$
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□

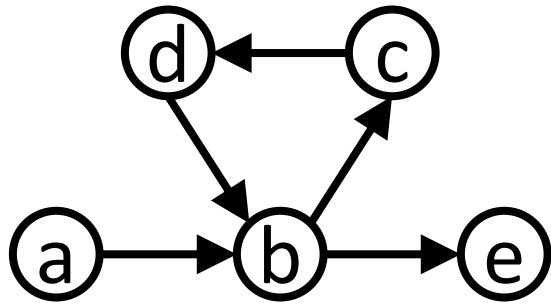
PaCo



$\{a\} \subseteq G_{\diamond}(\emptyset)$
 $\Leftarrow \{b\} \subseteq G_{\diamond}(\emptyset)$
 $\Leftarrow \{c\} \subseteq G_{\diamond}(\{b\})$
 $\Leftarrow \{d\} \subseteq G_{\diamond}(\{b\})$

Example

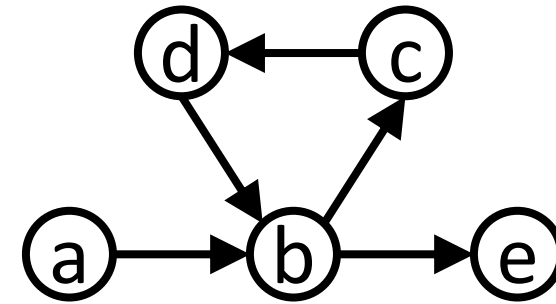
Tarski



$\{a\} \subseteq \nu Z. \diamond Z$
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□

PaCo

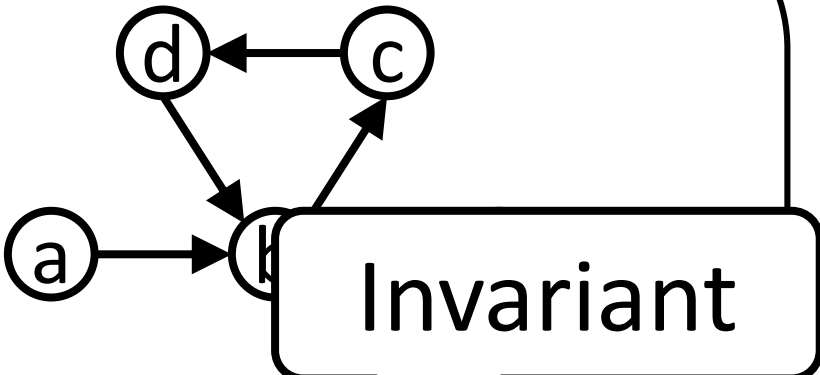


$\{a\} \subseteq G_\diamond(\emptyset)$
 $\Leftarrow \{b\} \subseteq G_\diamond(\emptyset)$
 $\Leftarrow \{c\} \subseteq G_\diamond(\{b\})$
 $\Leftarrow \{d\} \subseteq G_\diamond(\{b\})$
 $\Leftarrow \{b\} \subseteq \{b\}$

□

Example

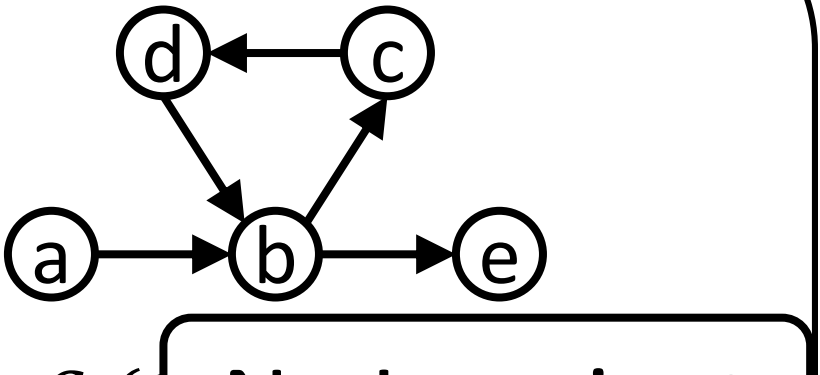
Tarski



$\{a\} \subseteq \nu Z. \diamond Z$
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 $\Leftarrow \{a, b, c, d\} \subseteq \{a, b, c, d\}$

□

PaCo



$\{a\} \subseteq G_\diamond(\{b\})$
 $\Leftarrow \{b\} \subseteq G_\diamond(\emptyset)$
 $\Leftarrow \{c\} \subseteq G_\diamond(\{b\})$
 $\Leftarrow \{d\} \subseteq G_\diamond(\{b\})$
 $\Leftarrow \{b\} \subseteq \{b\}$

□

Functional characterization

Def.

For $F : C \rightarrow C$ a function,

- F satisfies *Unfolding*, if $\forall x, y \in C. y \sqsubseteq f(x \sqcup F(x)) \Rightarrow y \sqsubseteq F(x)$
- F satisfies *Accumulation*, if $\forall x, y \in C. y \sqsubseteq F(x \sqcup y) \Rightarrow y \sqsubseteq F(x)$

Thm.

For F satisfying Unfolding and Accumulation,

$$G_f(x) \sqsubseteq F(x).$$

[Hur *et al.*, POPL '13]

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Parameterized induction
for inductive properties

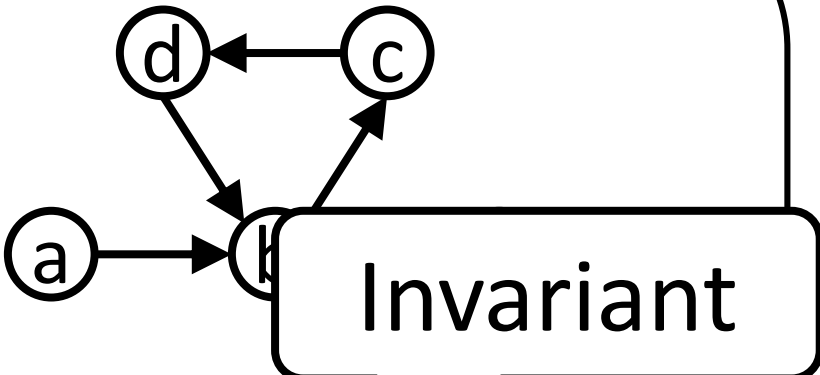
No-discovery characterization

How convenient is it?
Mathematical characterization

Our
contribution

Review : comparison of two proofs

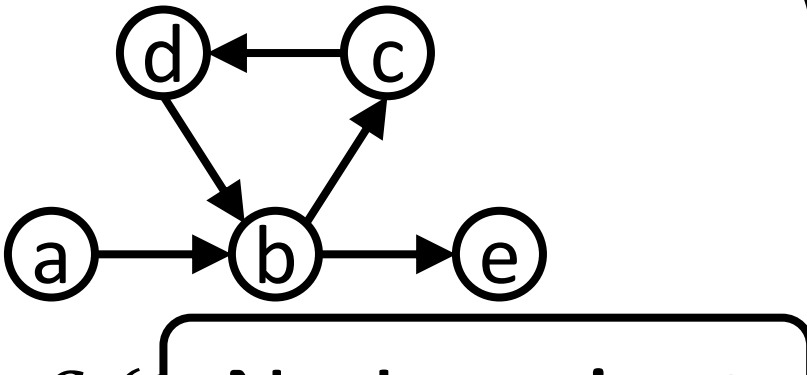
Tarski



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□

PaCo



$\{a\} \subseteq G_\diamond(\emptyset)$
 $\Leftarrow \{b\} \subseteq G_\diamond(\emptyset)$
 $\Leftarrow \{c\} \subseteq G_\diamond(\{b\})$
 $\Leftarrow \{d\} \subseteq G_\diamond(\{b\})$
 $\Leftarrow \{b\} \subseteq \{b\}$

□

Review : comparison of two proofs

Tarski

PaCo

Question:

1. What is invariant discovery?
2. Doesn't PaCo need invariant discovery?

$\{a\} \subseteq$

$\Leftarrow \{a, a\}$

$\Leftarrow \{a, b\}$

$\Leftarrow \{a, b, c, d\} \subseteq \{a, b, c, d\}$

□

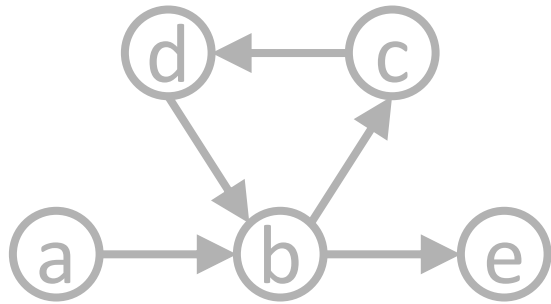
$\Leftarrow \{b\} \subseteq \{b\}$

□

ant

Invariant discovery : informal definition

Tarski



$$\{a\} \subseteq \forall Z. \diamond Z$$

$$\Leftarrow \{a, b, c, d\} \subseteq \forall Z. \diamond Z$$

$$\Leftarrow \{a, b, c, d\} \subseteq \diamond \{a, b, c, d\}$$

$$\Leftarrow \{a, b, c, d\} \subseteq \{a, b, c, d\}$$

□

Def. (Informal)

Invariant Discovery is a proof like this:

$$x \sqsubseteq \forall f$$

$$\Leftarrow x \sqsubseteq x', x' \sqsubseteq \forall f$$

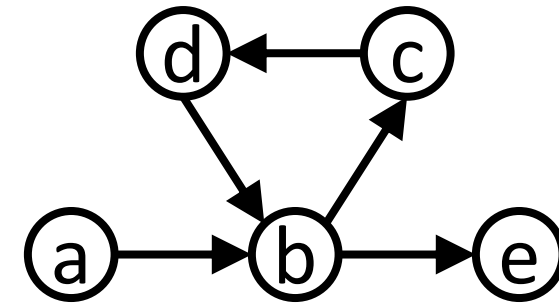
$$\Leftarrow x \sqsubseteq x', x' \sqsubseteq f(x')$$

Invariant discovery : example

	With invariant	Without invariant	
		Finite	Infinite
Tarski	✓	✗	✗
PaCo	✓	✓	✗

Invariant discovery : example

- Proof with invariant discovery
- $\{a\} \subseteq \forall Z. \diamond Z$
 $\Leftarrow \{a, b, c, d\} \subseteq \forall Z. \diamond Z$
 $\Leftarrow \{a, b, c, d\} \subseteq \diamond \{a, b, c, d\}$
 $\Leftarrow \{a, b, c, d\} \subseteq \{a, b, c, d\} \quad \square$



	With invariant	Without invariant	
		Finite	Infinite
Tarski	✓	✗	✗
PaCo	✓	✓	✗

Invariant discovery : example

- Proof without invariant discovery : Tarski case

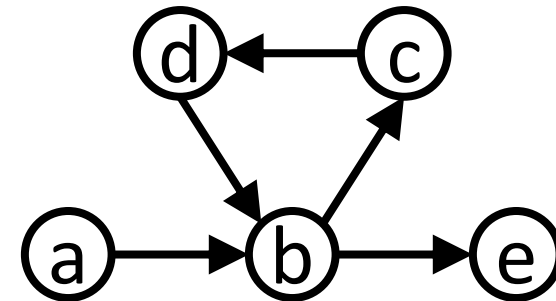
- $\{a\} \subseteq \nu Z. \diamond Z$

- $\Leftarrow \{b\} \subseteq \nu Z. \diamond Z$

- $\Leftarrow \{b\} \subseteq \diamond \{b\}$

- $\Leftrightarrow \{b\} \subseteq \{a, d\}$

Unsuccessful proof



	With invariant	Without invariant	
		Finite	Infinite
Tarski	✓	✗	✗
PaCo	✓	✓	✗

Invariant discovery : example

- Proof without invariant discovery : Tarski case

- $\{a\} \subseteq \nu Z. \diamond Z$

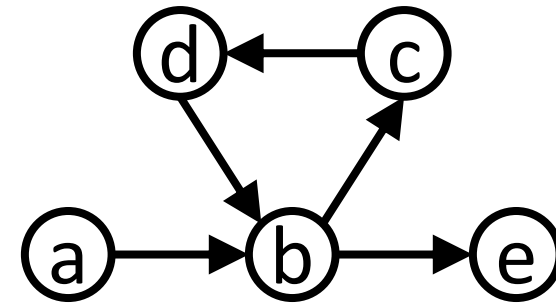
- $\Leftarrow \{b\} \subseteq \nu Z. \diamond Z$

- $\Leftarrow \{c\} \subseteq \nu Z. \diamond Z$

- $\Leftarrow \{d\} \subseteq \nu Z. \diamond Z$

- $\Leftarrow \vdots$

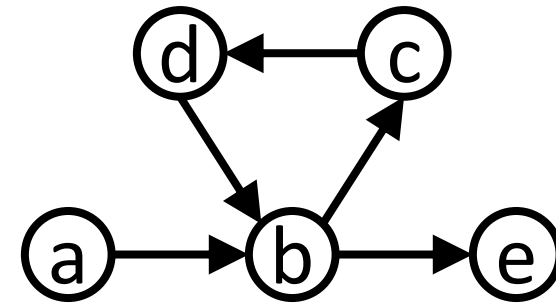
Unsuccessful proof



	With invariant	Without invariant	
		Finite	Infinite
Tarski	✓	✗	✗
PaCo	✓	✓	✗

Invariant discovery : example

- Proof without invariant discovery : PaCo case
- $\{a\} \subseteq G_{\diamond}(\emptyset)$
 - $\Leftarrow \{b\} \subseteq G_{\diamond}(\emptyset)$
 - $\Leftarrow \{c\} \subseteq G_{\diamond}(\{b\})$
 - $\Leftarrow \{d\} \subseteq G_{\diamond}(\{b\})$
 - $\Leftarrow \{b\} \subseteq \{b\} \quad \square$

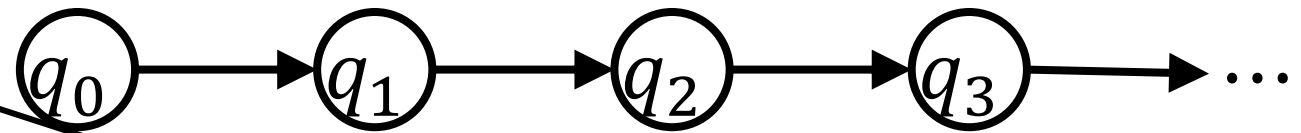


	With invariant	Without invariant	
		Finite	Infinite
Tarski	✓	✗	✗
PaCo	✓	✓	✗

Invariant discovery : example

- Proof without invariant discovery : PaCo case

- $\{q_0\} \subseteq G_\diamond(\emptyset)$
- $\Leftarrow \{q_1\} \subseteq G_\diamond(\{q_0\})$
- $\Leftarrow \{q_2\} \subseteq G_\diamond(\{q_0, q_1\})$
- $\Leftarrow \{q_3\} \subseteq G_\diamond(\{q_0, q_1, q_2\})$
- $\Leftarrow \vdots$



Unsuccessful proof

	With invariant	Without invariant	
		Finite	Infinite
Tarski	✓	✗	✗
PaCo	✓	✓	✗

Invariant discovery : example

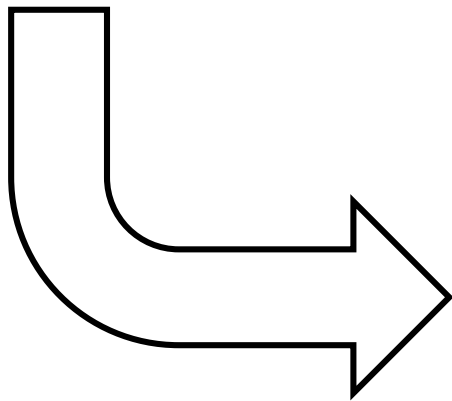
	With invariant	Without invariant	
		Finite	Infinite
Tarski	✓	✗	✗
PaCo	✓	✓	✗

Aim: to make these rigorous

Invariant discovery : analysis

$$\begin{aligned} &\{a\} \subseteq vZ. \diamond Z \\ \Leftarrow &\{a, b, c, d\} \subseteq vZ. \diamond Z \end{aligned}$$

Invariant discovery
uses transitivity



Forget transitivity

$$y \sqsubseteq G_f(x) \longrightarrow P(x, y)$$

Invariant discovery : assumption

Asm.

From now on, we assume that for all x ,
 $\lambda y. f(x \sqcup y)$ preserves directed infima.

Invariant discovery : definition

Def.

For $P(x, y)$: predicate,

- Downward closure :

$$\forall y, y' \in C. y \sqsubseteq y', P(x, y') \Rightarrow P(x, y)$$

$$\text{cf. transitivity } y \sqsubseteq y', y' \sqsubseteq G_f(x) \Rightarrow y \sqsubseteq G_f(x)$$

- Join closure :

$$\forall x \in C. \forall Y \subseteq C. (\forall y \in Y. P(x, y)) \Rightarrow P(x, \sqcup Y)$$

Invariant discovery : definition

Def.

For $P(x, y)$: predicate,

- Unfolding : for all x, y, z , $P(x, y)$ holds if
 - z is minimal among $\{z' \mid y \sqsubseteq f(z')\}$
 - $P(x, z)$.
- Accumulation : $\forall x, y. P(x \sqcup y, y) \Rightarrow P(x, y)$
- Tarski : $\forall x, y. y \sqsubseteq f(x \sqcup y) \Rightarrow P(x, y)$

Invariant discovery : definition

Thm.

- $y \sqsubseteq \nu f$
 $\Leftrightarrow \forall P$. Downward closure, Join closure, Unfolding, Tarski
 $\Rightarrow P(\perp, y)$
- $y \sqsubseteq G_f(x)$
 $\Leftrightarrow \forall P$. Downward closure, Join closure, Unfolding, Accumulation
 $\Rightarrow P(x, y)$

Invariant discovery : definition

Def.

- $y \sqsubseteq \nu f$ is *no-discovery Tarski provable*
 $\Leftrightarrow \forall P. \text{Join closure, Unfolding, Tarski}$
 $\Rightarrow P(\perp, y)$
- $y \sqsubseteq G_f(x)$ is *no-discovery PaCo provable*
 $\Leftrightarrow \forall P. \text{Join closure, Unfolding, Accumulation}$
 $\Rightarrow P(x, y)$

Invariant discovery : main theorem

Thm.

- For finite lattices, $y \sqsubseteq G_f(x)$ is no-discovery PaCo provable, if it holds.
- For infinite lattices, $y \sqsubseteq G_f(x)$ is not necessarily no-discovery PaCo provable.
- $y \sqsubseteq \nu f$ is not necessarily no-discovery Tarski provable.

Invariant discovery : main theorem

Thm.

- For finite lattices, $y \sqsubseteq G_f(x)$ is no-discovery PaCo provable, if it holds.
- For infinite lattices, $y \sqsubseteq G_f(x)$ is not necessarily no-discovery PaCo provable.
- $y \sqsubseteq \nu f$ is not necessarily no-discovery Tarski provable.

	With invariant	Without invariant	
		Finite	Infinite
Tarski	✓	✗	✗
PaCo	✓	✓	✗

[Hur *et al.*, POPL '13]

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for coinductive properties

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No-discovery characterization

How convenient is it?
Mathematical characterization

Our
contribution

Parameterized induction

Def.

$L_f(x) = \mu y. f(x \sqcup y)$ parameterized least fixpoint

Lem.

$\forall x, y. y \sqsubseteq f(x \sqcup L_f(x)) \Rightarrow y \sqsubseteq L_f(x)$ (Unfolding)

$\forall x, y : \alpha' \rightarrow C$ (x increasing).

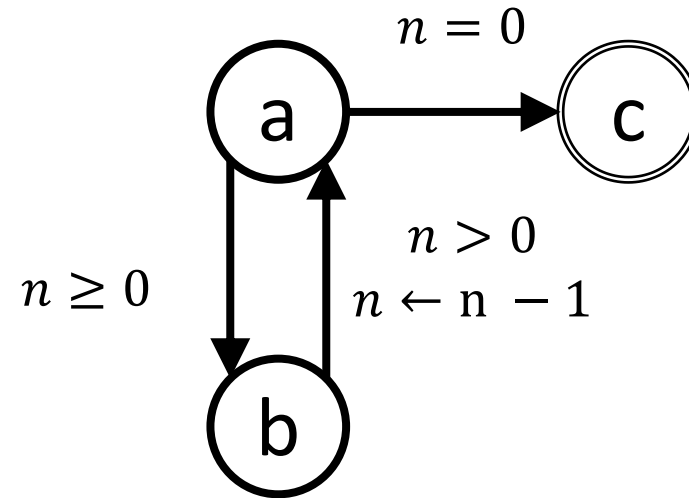
$$\left(\forall \alpha. y(\alpha) \sqsubseteq L_f \left(x(\alpha) \sqcup \bigsqcup_{\beta < \alpha} y(\beta) \right) \right)$$

$$\Rightarrow \left(\forall \alpha. y(\alpha) \sqsubseteq L_f(x(\alpha)) \right)$$

(Indexed Accumulation)

Parameterized induction : pushdown example

- $\forall n. \{a(n)\} \subseteq L_f(\emptyset)$
 $\Leftarrow \forall n. \{a(n)\} \subseteq L_f(\{a(m) \mid m < n\})$
 $\Leftarrow \forall n. \{b(n)\} \subseteq L_f(\{a(m) \mid m < n + 1\})$
 $\Leftarrow \forall n. \{a(n)\} \subseteq \{a(m) \mid m < n + 1\} \quad \square$



$$f(Z) = \{c(n)\} \cup \diamond Z$$

Conclusion & Future Work

- **No-discovery characterization**
 - Gave mathematical framework to characterize power of parameterized coinduction
 - No-go theorem : perhaps useful for future proof-technique development.
- Future work
 - Syntactic Approach
 - Trying non-Boolean examples

Conclusion & Future Work

- **Parameterized Induction**
 - Applied PaCo-like strategy to inductive properties.
 - Perhaps useful for pushdown systems
- **Future Work**
 - Implementation to a proof assistant
 - Generalization to mixed μ/ν formulas

[Hur *et al.*, POPL '13]

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