A Characterisation of Lambda Definability with Sums via ⊤⊤-Closure Operators

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Abstract. We give a new characterisation of morphisms that are definable by the interpretation of the simply typed lambda calculus with sums in *any* bi-Cartesian closed category. The ⊤⊤-closure operator will be used to *construct* the category in which the collection of definable morphisms at sum types can be characterised as the coproducts of such collections at lower types.

1 Introduction

The *λ-definability problem* is to characterise the semantic elements that are definable by denotational / categorical semantics of the simply typed λ-calculus. A characterisation of the λ-definable elements in full type hierarchies was first given by Plotkin using *Kripke logical relations*. This result was later generalized by Jung and Tiuryn to any Henkin model using *Kripke logical predicates with varying arity* [15]; its categorical formulation was also given by Alimohamed [1].

These precursors considered the definability problem in the simply typed lambda calculus with only arrow types (and possibly product types). The problem becomes more subtle when sum types are added. There is a natural definition of coproducts for Kripke predicates with varying arity, but these coproducts are not sufficient to characterise the definable elements at sum types. In [11] Fiore and Simpson overcome this difficulty by introducing a new concept called *Grothendieck logical predicates*. They used Grothendieck topology to improve the definition of coproducts of Kripke predicates. By constructing a suitable category of worlds and topology on it, Fiore and Simpson succeeded in giving a characterisation of definable morphisms in any bi-CCC with *stable* coproducts.

In this paper we approach the definability problem in the simply typed lambda calculus with sums using a different technique called ⊤⊤-*closure operator*. The main contribution of this paper is the following:

1. We characterise the definability predicate (=the collection of definable morphisms) at sum types by means of the standard coproducts for Kripke predicates and the *semantic* ⊤⊤-*closure operator*:

   $$\mathrm{Def}\ 0 = \dot{0}^{\top\top}, \qquad \mathrm{Def}(\tau + \tau') = (\mathrm{Def}\ \tau \dotplus \mathrm{Def}\ \tau')^{\top\top}.$$

   This characterisation holds with respect to the interpretation of the lambda calculus with sums in *any* bi-CCC. We also give a characterisation of morphisms definable by the simply typed lambda calculus with sums by means of *weak logical predicates*.

2. We analyse the underlying categorical essence of the above arguments, and present it as the *restriction theorem*. The statement of the theorem is the following: let P be a logical predicate in a sufficiently rich fibration $p : \mathbb{P} \to \mathbb{C}$. If P respects product and arrow types, then we can restrict P to a full reflective subcategory $\mathbb{P}^{\top\top}$ of $\mathbb{P}$ so that P respects sum types as well.

The characterisation stated in item 1 implies that in the category $\mathbb{K}^{\top\top}$ of $\top\top$-closed objects the definability predicates at sum types are given by the coproducts of the definability predicates at lower types. By employing $\mathbb{K}^{\top\top}$ as a gluing category, we also show that the inclusion from the free distributive category $L_0(B)$ over the set B of base types to the free bi-CCC $L_1(B)$ over B is full. We note that this result is proved in [10], but there a different gluing category is employed.

Preliminary We define categories and functors by the following table:

A category / functor is ...	when it has / preserves ...
Cartesian	finite products
co-Cartesian	finite coproducts
bi-Cartesian	finite products and finite coproducts
Cartesian closed (CC)	finite products and exponentials
bi-Cartesian closed (bi-CC)	finite products, finite coproducts and exponentials

2 Definability of Calculi with Sums

We deal with the definability problem in the context of functorial semantics, where syntatic theories are treated as freely generated categories, and interpretations are represented by structure-preserving functors. We fix a small Cartesian category L that plays the role of a syntactic theory, a bi-CCC $\mathbb{C}$ that plays the role of a semantic domain, and a strict Cartesian functor F:

$$L \xrightarrow{F} \mathbb{C}$$

that gives an interpretation of the syntactic theory. We first review some basic properties of the definability predicate for F in this setting, then extend L and F with additional structures (coproducts / exponentials) toward the main theorems of this paper. At this moment we do not require that L is a freely generated category, as the freeness does not play any role in the following discussion.

2.1 Kripke Predicates with Varying Arity

We first introduce the poset $\mathbf{Ctx}_L$ that plays the role of Kripke structure for Kripke predicates with varying arity. The carrier of the poset is $(\mathbf{Obj}(L))^*$, the set of finite sequences of L-objects, and these sequences are ordered by the prefix ordering (that is, $\tau \le \sigma$ if τ is a prefix of σ). Below we treat $\mathbf{Ctx}_L$ as a category.

When L is identified as a syntactic (type) theory, the poset $\mathbf{Ctx}_L$ expresses inclusions of typing contexts. We associate these inclusions with projections in L by the following functor $|-| : \mathbf{Ctx}_L \to L^{op}$:

$$|\tau_1 \cdots \tau_n| = (\cdots((1 \times \tau_1) \times \tau_2) \cdots) \times \tau_n, \qquad |\tau_1 \cdots \tau_n \leq \tau_1 \cdots \tau_{n+m}| = \mathrm{id} \circ \overbrace{\pi \circ \cdots \circ \pi}^{m},$$

where π is the first projection for appropriate objects.

We next define the category $\mathbb{K}_F$ of *Kripke predicates with varying arity*.

- An object of $\mathbb{K}_F$ is a pair (C, X) where C is a $\mathbb{C}$-object and X is a subpresheaf of the contravariant presehaf $\mathbb{C}(F|-|, C)$ on $\mathbf{Ctx}_L$.
- A morphism from (C, X) to (D, Y) is a $\mathbb{C}$-morphism $f : C \to D$ such that for any $\mathbf{Ctx}_L$-object Γ and $\mathbb{C}$-morphism $g \in X\Gamma$, we have $f \circ g \in Y\Gamma$.

The category $\mathbb{K}_F$ is constructed as follows. Let $H_F : \mathbb{C} \to [\mathbf{Ctx}_L, \mathbf{Set}]$ be a functor defined by $H_F(C) = \mathbb{C}(F|-|, C)$. Then $\mathbb{K}_F$ is the vertex of the change-of-base (pullback) of the subobject fibration $p : \mathbf{Sub}([\mathbf{Ctx}_L, \mathbf{Set}]) \to [\mathbf{Ctx}_L, \mathbf{Set}]$ along H_F (Figure 1). This construction is an instance of *subscone* [22] or *categorical gluing* [1].

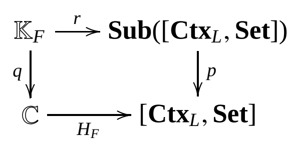


Fig. 1. Derivation of q

Proposition 1. *The leg $q : \mathbb{K}_F \to \mathbb{C}$ of the change-of-base (Figure 1) is a strict bi-CC functor and a partial order fibration with fibred small products.*

Proof. Since p is a partial order fibration with fibred small products, q inherits these structures via the change-of-base along the Cartesian functor H_F. The bi-CC structure of $\mathbb{K}_F$, which is strictly preserved by q, is given as follows (see also Section 1.5, [1]):

$$\begin{aligned}
\dot{1} &= (1, \{!_{F|\Gamma|}\}_\Gamma) \\
(C, X) \dot{\times} (D, Y) &= (C \times D, \{h \mid \pi_1 \circ h \in X\Gamma, \pi_2 \circ h \in Y\Gamma\}_\Gamma) \\
\dot{0} &= (0, \emptyset) \\
(C, X) \dot{+} (D, Y) &= (C + D, (\{inl \circ f \mid f \in X\Gamma\} \cup \{inr \circ f \mid f \in Y\Gamma\})_\Gamma) \\
(C, X) \dot{\Rightarrow} (D, Y) &= (C \Rightarrow D, \{f \mid \forall \Gamma' \geq \Gamma \,.\, \forall x \in X\Gamma' \,.\, ev \circ \langle f \circ F|\Gamma \leq \Gamma'|, x\rangle \in Y\Gamma'\}_\Gamma).
\end{aligned}$$

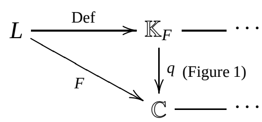
(here $\{\cdots\}_\Gamma$ denotes a presheaf described by an auxiliary parameter $\Gamma \in \mathbf{Ctx}_L$).

We define the target of our study, the *definability functor* $\mathrm{Def} : L \to \mathbb{K}_F$, by

$$\begin{aligned}
\mathrm{Def}\,\tau &= (\tau, \{Fg \in \mathbb{C}(F|\Gamma|, F\tau) \mid g \in L(|\Gamma|, \tau)\}_\Gamma) \\
\mathrm{Def}\,f &= Ff.
\end{aligned}$$

We call $\mathrm{Def}\,\tau$ the *definability predicate* (of F) at τ. We refer to the presheaf part of $\mathrm{Def}\,\tau$ by $D\tau$; in other words, $D = r \circ \mathrm{Def}$ (r is the other leg of the change-of-base).

Proposition 2. $\mathrm{Def} : L \to \mathbb{K}_F$ *is a full strict Cartesian functor, and* $q \circ \mathrm{Def} = F$.



$\mathbb{K}_F$	bi-CCC $(\dot{1}, \dot{\times}, \dot{0}, \dot{+}, \dot{\Rightarrow})$
$\mathbb{C}$	bi-CCC $(1, \times, 0, +, \Rightarrow)$
q	strict bi-CC functor and partial order fibration with fibered small products
L	small Cartesian category
F	strict Cartesian functor
Def	full strict Cartesian functor

Fig. 2. Categories and Functors for the Argument of Definability

Figure 2 summarises categories and functors we have introduced so far.

One important property of the functor Def, which is implicit in the characterisation of λ-definability by Jung and Tiuryn [15], is the following:

Lemma 1. *For any L-objects τ, τ' and $\mathbf{Ctx}_L$-object Γ, we have*

$$f \in r(\mathrm{Def}\,\tau \mathrel{\dot{\Rightarrow}} \mathrm{Def}\,\tau')(\Gamma) \iff \lambda^{-1}(f) \in D\tau'(\Gamma\tau),$$

(here λ^{-1} denotes the uncurrying operator).

2.2 A Characterisation of Definability with Sums

We next assume that the category L in Figure 2 is a distributive category (in the sense of Walters [26, 7]) and F is a strict bi-Cartesian functor. Recall that a distributive category $\mathbb{C}$ is a bi-Cartesian category such that the canonical morphism

$$[A \times inl, A \times inr] : (A \times B) + (A \times C) \to A \times (B + C)$$

has the inverse (called *distributive law*) [1] :

$$m^{\mathbb{C}}_{A,B,C} : A \times (B + C) \to (A \times B) + (A \times C).$$

We note that F strictly preserves distributive laws, that is, $F(m^L_{\tau,\tau',\rho}) = m^{\mathbb{C}}_{F\tau,F\tau',F\rho}$.

The functor Def in Figure 2 is still full strict Cartesian from Proposition 2, but not co-Cartesian. We merely have the following inequations:

$$\mathrm{Def}\,0 \geq \dot{0}, \qquad \mathrm{Def}(\tau + \tau') \geq \mathrm{Def}\,\tau \mathrel{\dot{+}} \mathrm{Def}\,\tau'. \tag{1}$$

Interestingly, these inequations are equated when applied to the contravariant functor $(- \mathrel{\dot{\Rightarrow}} \mathrm{Def}\,\rho)$.

Lemma 2. *For any L-objects τ, τ', ρ, we have*

$$\begin{aligned} \mathrm{Def}\,0 \mathrel{\dot{\Rightarrow}} \mathrm{Def}\,\rho &= \dot{0} \mathrel{\dot{\Rightarrow}} \mathrm{Def}\,\rho \\ \mathrm{Def}(\tau + \tau') \mathrel{\dot{\Rightarrow}} \mathrm{Def}\,\rho &= (\mathrm{Def}\,\tau \mathrel{\dot{+}} \mathrm{Def}\,\tau') \mathrel{\dot{\Rightarrow}} \mathrm{Def}\,\rho. \end{aligned}$$

[1] The distributive law implies that the unique map $0 \to A \times 0$ is the isomorphism; see [7].

Proof. We leave the proof of the first equation to the reader. We show the second equation. Let τ, τ', ρ be L-objects. The inequation (1) implies half of the equation to be proved. We therefore show the other half displayed below:

$$\mathrm{Def}(\tau + \tau') \mathbin{\dot\Rightarrow} \mathrm{Def}\,\rho \geq (\mathrm{Def}\,\tau \mathbin{\dot+} \mathrm{Def}\,\tau') \mathbin{\dot\Rightarrow} \mathrm{Def}\,\rho.$$

Let Γ be a $\mathbf{Ctx}_L$-object and $f \in r((\mathrm{Def}\,\tau \mathbin{\dot+} \mathrm{Def}\,\tau') \mathbin{\dot\Rightarrow} \mathrm{Def}\,\rho)(\Gamma)$. The isomorphism

$$(\mathrm{Def}\,\tau \mathbin{\dot+} \mathrm{Def}\,\tau') \mathbin{\dot\Rightarrow} \mathrm{Def}\,\rho \cong (\mathrm{Def}\,\tau \mathbin{\dot\Rightarrow} \mathrm{Def}\,\rho) \mathbin{\dot\times} (\mathrm{Def}\,\tau' \mathbin{\dot\Rightarrow} \mathrm{Def}\,\rho)$$

implies that $\lambda(ev \circ (f \times inl)) \in r(\mathrm{Def}\,\tau \mathbin{\dot\Rightarrow} \mathrm{Def}\,\rho)(\Gamma)$ and $\lambda(ev \circ f \times inr) \in r(\mathrm{Def}\,\tau' \mathbin{\dot\Rightarrow} \mathrm{Def}\,\rho)(\Gamma)$. From Lemma 1, we obtain

$$(g_1 =)\ ev \circ (f \times inl) \in D\rho(\Gamma\tau), \qquad (g_2 =)\ ev \circ (f \times inr) \in D\rho(\Gamma\tau').$$

We thus take L-morphisms $h_1 : |\Gamma\tau| \to \rho$ and $h_2 : |\Gamma\tau'| \to \rho$ such that $g_1 = Fh_1$ and $g_2 = Fh_2$. Since F is a strict bi-Cartesian functor, we have

$$[g_1, g_2] \circ m^{\mathbb{C}}_{F|\Gamma|, F\tau, F\tau'} = F([h_1, h_2] \circ m^{L}_{|\Gamma|, \tau, \tau'}).$$

The left hand side is equal to $ev \circ (f \times (F\tau + F\tau'))$:

$$\begin{aligned} [g_1, g_2] \circ m &= ev \circ (f \times (F\tau + F\tau')) \circ [F|\Gamma| \times inl, F|\Gamma| \times inr] \circ m \\ &= ev \circ (f \times (F\tau + F\tau')). \end{aligned}$$

Hence $ev \circ (f \times (F\tau + F\tau')) \in D\rho(\Gamma(\tau + \tau'))$. From Lemma 1, we obtain $f \in r(\mathrm{Def}(\tau + \tau') \mathbin{\dot\Rightarrow} \mathrm{Def}\,\rho)(\Gamma)$.

We combine this lemma and the *semantic* $\top\top$*-closure operator* [16] to extract $\mathbb{K}_F$'s full reflective sub bi-CCC whose coproducts can characterise the definability predicates at sum types. The semantic $\top\top$-closure operator (we may drop the word "semantic" thereafter) is a semantic analogue of Pitt's $\top\top$-closure technique in [25], and is an instance of the author's semantic $\top\top$-lifting [16]. The $\top\top$-closure operator in this section is specialised to the argument of definability. In Section 3 it will be re-introduced in more general form, together with the proofs of propositions and theorems in this section.

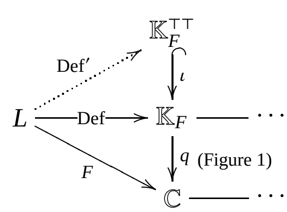
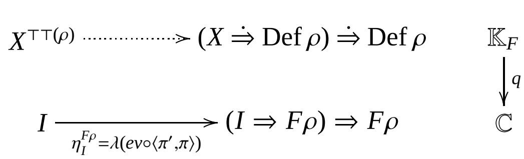


Fig. 3. Restriction of Definability Functor

The $\top\top$-closure operator is defined as follows. Let X be a $\mathbb{K}_F$-object above a $\mathbb{C}$-object I. For each L-object ρ, we define $X^{\top\top(\rho)}$ to be the vertex of the following inverse image in the fibration $q : \mathbb{K}_F \to \mathbb{C}$:



where $\eta_I^{F\rho}$ is the unit of the *continuation monad*. We then define $X^{\top\top}$, the $\top\top$-closure of X by

$$X^{\top\top} = \bigwedge_{\rho \in \mathbf{Obj}(L)} X^{\top\top(\rho)}.$$

Proposition 3. *The assignment $X \mapsto X^{\top\top}$ extends to a monad over $\mathbb{K}_F$ whose unit and multiplication are vertical (c.f. Proposition 7).*

Below we call the assignment *(semantic) $\top\top$-closure operator*. It indeed gives an idempotent closure operator at every fibre, as unit and multiplication are vertical.

Corollary 1. *We have $X \leq X^{\top\top}$ and $(X^{\top\top})^{\top\top} = X^{\top\top}$, and the monad is idempotent (c.f. Corollary 2).*

We then consider $\mathbb{K}_F$'s full reflective subcategory $\mathbb{K}_F^{\top\top}$ consisting of $\top\top$-closed objects (that is, objects X such that $X^{\top\top} = X$); see Figure 3. Some calculation shows that $\top\top$-closed objects form an *exponential ideal*. Therefore we obtain the following:

Proposition 4. *The category $\mathbb{K}_F^{\top\top}$ is a bi-CCC and $q \circ \iota$ is a strict bi-CC functor (c.f. Theorem 4).*

The CC structure in $\mathbb{K}_F^{\top\top}$ is inherited from $\mathbb{K}_F$, while the co-Cartesian structure is given by $\dot{0}^{\top\top}$ and $(X \dot{+} Y)^{\top\top}$. That the $\top\top$-closure operator is defined in terms of the definability predicates themselves implies the following important property:

Proposition 5. *For every L-object ρ,* $\mathrm{Def}\,\rho$ *is $\top\top$-closed (c.f. Proposition 8).*

Thus functor Def can be restricted to the full Cartesian functor (Def′ in Figure 3) to $\mathbb{K}_F^{\top\top}$. Furthermore, from Lemma 2 we obtain a *characterisation of the definability predicates at sum types* (c.f. Theorem 5-2):

$$\mathrm{Def}\,0 = (\mathrm{Def}\,0)^{\top\top} = \dot{0}^{\top\top} \tag{2}$$

$$\mathrm{Def}(\tau + \tau') = (\mathrm{Def}(\tau + \tau'))^{\top\top} = (\mathrm{Def}\,\tau \dot{+} \mathrm{Def}\,\tau')^{\top\top}. \tag{3}$$

This is equivalent to saying that Def′ is a strict co-Cartesian functor. To summarise:

Theorem 1 (Restriction Theorem for Definability Functor). *In Figure 3, assume that L is a small distributive category and F is a strict bi-Cartesian functor. Then* Def′ *is a full strict bi-Cartesian functor.*

We next let L be a small bi-CCC and $F : L \to \mathbb{C}$ be a strict bi-CC functor. Under this situation, the restriction of the definability functor to $\mathbb{K}_F^{\top\top}$ becomes a bi-CC functor. Since any bi-CCC is a distributive category, Def′ in Figure 3 is full bi-Cartesian from the restriction theorem. Moreover, as shown in [1] (c.f. [15]), the functor Def (and Def′) strictly preserves exponentials. Therefore we obtain the following theorem:

Theorem 2. *In figure 3, assume that L is a small bi-CCC and F is a strict bi-CC functor. Then* Def′ *in Figure 3 is a full strict bi-CC functor.*

2.3 Fullness of Free Distributive Categories in Free Bi-CCCs

As an application of the restriction theorem, we show that the canonical inclusion from the free distribute category to the free bi-CCC is full. We note that this result (and faithfulness) is proved in [10] using a different gluing category.

We fix the set B of base types and regard it as a discrete category. In this paper, by the free distributive category $(L_0(B), \eta_0 : B \to L_0(B))$ over B, we mean the distributive category with the following universal property: for any distributive category $\mathbb{C}$ and a functor $F : B \to \mathbb{C}$, there exists a unique strict bi-Cartesian functor $\overline{F} : L_0(B) \to \mathbb{C}$ such that $\overline{F} \circ \eta_0 = F$. We also define the free bi-CCC $(L_1(B), \eta_1 : B \to L_1(B))$ over B as the one having the similar universal property. Such free categories arise as term categories of the simply typed (lambda) calculus with sums. We omit the detail of the construction of free categories due to lack of space; see e.g. [18].

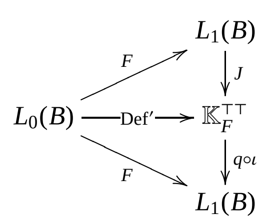
We instantiate Figure 3 with the following data:

1. $L = L_0(B)$, the free distributive category over B.
2. $\mathbb{C} = L_1(B)$, the free bi-CCC over B.
3. $F = \overline{\eta_1} : L_0(B) \to L_1(B)$, the strict bi-Cartesian functor derived from the universal property of $L_0(B)$.

Lafont applied categorical gluing to show that any small Cartesian category $\mathbb{C}$ can be fully embedded into the CCC that is relatively free with respect to $\mathbb{C}$ [17]. We apply his proof technique to the show that $\overline{\eta_1}$ is full. Here we use $\mathbb{K}_F^{\top\top}$ as a substitute for the gluing category.

Theorem 3. *The strict bi-Cartesian functor $\overline{\eta_1} : L_0(B) \to L_1(B)$ is full.*

Proof. Below we write F for $\overline{\eta_1}$. From Theorem 1 we obtain a full strict bi-Cartesian functor $\mathrm{Def}' : L_0(B) \to \mathbb{K}_F^{\top\top}$. From Proposition 4, $\mathbb{K}_F^{\top\top}$ is a bi-CCC; hence we obtain a strict bi-CC functor $J = \overline{\mathrm{Def}' \circ \eta_0} : L_1(B) \to \mathbb{K}_F^{\top\top}$. Furthermore, $q \circ \iota$ is a strict bi-CC functor, so $q \circ \iota \circ J = \mathrm{Id}$ by the universal property of $L_1(B)$. This implies that J is faithful.



In the above diagram, the upper half of the triangle commutes from the universal property of $L_0(B)$. The lower half of the triangle also commutes from Figure 3. We now show that F is full. Let $f : F\tau \to F\sigma$ be a $L_1(B)$-morphism. We seek for a $L_0(B)$-morphism g such that $f = Fg$. We first have $Jf : \mathrm{Def}'\,\tau \to \mathrm{Def}'\,\sigma$. Since Def' is full, there exists a $L_0(B)$-morphism $g : \tau \to \sigma$ such that $Jf = \mathrm{Def}'\,g = J(Fg)$. Since J is faithful, we obtain $f = Fg$.

3 $\top\top$-Closure Operators and the Restriction Theorem

In this section we focus on the general scheme that underlies in the derivation of the restriction theorem (Theorem 1), and re-establish it in more general form.

We first identify the class of fibrations in which we can consider $\top\top$-closure operators. If a functor $U : \mathbb{P} \to \mathbb{C}$ satisfies the following conditions:

$$\begin{array}{c} \mathbb{P} \\ \downarrow U \\ \mathbb{C} \end{array} \qquad \begin{array}{ll} \mathbb{P} & \text{bi-CCC } (\dot{1}, \dot{\times}, \dot{0}, \dot{+}, \dot{\Rightarrow}, \dot{!}, \dot{\pi}, \dot{\pi}', \dot{\lambda}, \dot{ev}, \cdots) \\ \mathbb{C} & \text{bi-CCC } (1, \times, 0, +, \Rightarrow, !, \pi, \pi', \lambda, ev, \cdots) \\ U & \text{strict bi-CC functor and partial order} \\ & \text{fibration with fibered small products} \end{array}$$

we say that U *admits* $\top\top$*-closure operators*. Below we give a sufficient condition for ensuring that a fibration admits $\top\top$-closure operators.

Proposition 6. *Let $p : \mathbb{E} \to \mathbb{B}$ be a partial order bifibration such that $\mathbb{B}$ is a bi-CCC, p has fibred small products, fibred finite coproducts, fibred exponentials and simple products (see e.g. Jacobs [14]). Then p admits $\top\top$-closure operators.*

3.1 $\top\top$-Closure Operators

We fix a fibration $U : \mathbb{P} \to \mathbb{C}$ which admits $\top\top$-closure operators. Each $\top\top$-closure operator takes a $\mathbb{P}$-object as a parameter called *closure parameter*. Let S be a closure parameter. For a $\mathbb{P}$-object X, we define $X^{\top\top(S)}$ to be the vertex of the following inverse image:

$$\begin{array}{ccc} X^{\top\top(S)} & \dashrightarrow & (X \dot{\Rightarrow} S) \dot{\Rightarrow} S \\ & & \\ UX & \xrightarrow[\eta^{US}_{UX} = \lambda(ev \circ \langle \pi', \pi \rangle)]{} & (UX \Rightarrow US) \Rightarrow US \end{array} \qquad \begin{array}{c} \mathbb{P} \\ \downarrow U \\ \mathbb{C} \end{array}$$

We note that the $\mathbb{C}$-morphism η^{US}_{UX} is the unit of the continuation monad $(- \Rightarrow US) \Rightarrow US$. This construction exactly coincides with the *semantic* $\top\top$*-lifting* [16] of the identity monad.

Proposition 7. *[16] Let S be a closure parameter. The assignment $X \mapsto X^{\top\top(S)}$ extends to an endofunctor $(-)^{\top\top(S)} : \mathbb{P} \to \mathbb{P}$ such that $U \circ (-)^{\top\top(S)} = U$. Furthermore, there exists vertical natural transformations $\eta^{\top\top(S)}$ and $\mu^{\top\top(S)}$ that make the triple $((-)^{\top\top(S)}, \eta^{\top\top(S)}, \mu^{\top\top(S)})$ a monad.*

Corollary 2. *Let S be a closure parameter. For any $\mathbb{P}$-object X, we have*

$$X \le X^{\top\top(S)}, \qquad (X^{\top\top(S)})^{\top\top(S)} = X^{\top\top(S)}, \qquad S = S^{\top\top(S)}.$$

Proof. In this proof we simply write $\top\top$ for $\top\top(S)$. The first two (in)equations are immediate consequences of the previous lemma. To show $S^{\top\top} = S$, it is sufficient to show $S^{\top\top} \le S$. We consider the following diagram:

$$\begin{array}{ccccc} S^{\top\top} & \dashrightarrow & (S \dot{\Rightarrow} S) \dot{\Rightarrow} S & \xrightarrow[\dot{ev} \circ \langle id, \dot{\lambda}(\dot{\pi}') \circ \dot{!} \rangle]{} & S \\ & & & & \\ US & \xrightarrow[\eta^{US}_{US}]{} & (US \Rightarrow US) \Rightarrow US & \xrightarrow[ev \circ \langle id, \lambda(\pi') \circ ! \rangle]{} & US \end{array} \qquad \begin{array}{c} \mathbb{P} \\ \downarrow U \\ \mathbb{C} \end{array}$$

The composite of morphisms in $\mathbb{C}$ is the identity. Hence $S^{\top\top} \leq S$ holds.

We next generalise $\top\top$-closure operators to take multiple closure parameters. Let $\mathbf{S} = \{S_i\}_{i \in I}$ be a set-indexed family of closure parameters. We define $(-)^{\top\top(\mathbf{S})}$ by

$$X^{\top\top(\mathbf{S})} = \bigwedge_{i \in I} X^{\top\top(S_i)}$$

where $\bigwedge$ denotes the fibred product. Below we only consider set-indexed family of closure parameters.

Proposition 8. *Let $\mathbf{S} = \{S_i\}_{i \in I}$ be a family of closure parameters. The mapping $X \mapsto X^{\top\top(\mathbf{S})}$ extends to a monad over $\mathbb{P}$ whose unit and multiplication are vertical. Furthermore, for any $\mathbb{P}$-object X, we have*

$$X \leq X^{\top\top(\mathbf{S})}, \qquad (X^{\top\top(\mathbf{S})})^{\top\top(\mathbf{S})} = X^{\top\top(\mathbf{S})}, \qquad S_i = S_i^{\top\top(\mathbf{S})} \quad (i \in I).$$

3.2 Full Reflective Subcategory of $\top\top$-Closed Objects

We investigate the structure of the full reflective subcategory of $\top\top$-closed objects. Let $\mathbf{S}$ be a family of closure parameters. We write $\mathbb{P}^{\top\top(\mathbf{S})}$ for $\mathbb{P}$'s full reflective subcategory consisting of $\top\top(\mathbf{S})$-closed objects (that is, objects X such that $X^{\top\top(\mathbf{S})} = X$). We write ι for the inclusion functor from $\mathbb{P}^{\top\top(\mathbf{S})}$ to $\mathbb{P}$.

Since $\mathbb{P}$ is bi-Cartesian, $\mathbb{P}^{\top\top(\mathbf{S})}$ is also bi-Cartesian (see e.g. Proposition 3.5.3 and 3.5.4, [6]). The Cartesian structure is inherited from $\mathbb{P}$, while the co-Cartesian structure is given by the following diagram:

$$X \xrightarrow{\dot{inl}} X \dot{+} Y \xrightarrow{\leq} (X \dot{+} Y)^{\top\top(\mathbf{S})} \xleftarrow{\geq} X \dot{+} Y \xleftarrow{\dot{inr}} Y \qquad (4)$$

We next show that $\top\top(\mathbf{S})$-closed objects form an *exponential ideal*.

Lemma 3. *Let $\mathbf{S} = \{S_i\}_{i \in I}$ be a family of closure parameters. Then for any $\mathbb{P}$-object X and Y above I and J respectively, $Y^{\top\top(\mathbf{S})} = Y$ implies $(X \dot{\Rightarrow} Y)^{\top\top(\mathbf{S})} = X \dot{\Rightarrow} Y$.*

Proof. Below we only show $(X \dot{\Rightarrow} Y)^{\top\top(\mathbf{S})} \leq X \dot{\Rightarrow} Y$; the other direction is clear as $(-)^{\top\top(\mathbf{S})}$ is a closure operator. Let $i \in I$. We define $\dot{w} : (X \dot{\Rightarrow} Y)^{\top\top(\mathbf{S})} \to (((X \dot{\Rightarrow} Y) \dot{\Rightarrow} S_i) \dot{\Rightarrow} S_i)$ to be the composite of $\mathbb{P}$-morphisms in the following diagram:

$$\begin{array}{ccccc}
(X \dot{\Rightarrow} Y)^{\top\top(\mathbf{S})} & & & & \\
\downarrow{\scriptstyle \leq} & & & & \\
(X \dot{\Rightarrow} Y)^{\top\top(S_i)} & \cdots\cdots\to & ((X \dot{\Rightarrow} Y) \dot{\Rightarrow} S_i) \dot{\Rightarrow} S_i & & \mathbb{P} \\
 & & & & \downarrow{\scriptstyle U} \\
I \Rightarrow J & \xrightarrow[\eta^{US_i}_{I \Rightarrow J}]{} & ((I \Rightarrow J) \Rightarrow US_i) \Rightarrow US_i & & \mathbb{C}
\end{array}$$

From the diagram, $\dot{w}$ is above $\eta^{US_i}_{I \Rightarrow J}$. We also define a $\mathbb{P}$-morphism $\dot{c} : X \dot{\times} (Y \dot{\Rightarrow} S_i) \to (X \dot{\Rightarrow} Y) \dot{\Rightarrow} S_i$ by

$$\dot{c} = \dot{\lambda}(\dot{ev} \circ (id \dot{\times} \dot{ev}) \circ \langle \dot{\pi}' \circ \dot{\pi}, \langle \dot{\pi}', \dot{\pi} \circ \dot{\pi} \rangle \rangle),$$

which is above the following $\mathbb{C}$-morphism $c : I \times (J \Rightarrow US_i) \to (I \Rightarrow J) \Rightarrow US_i$:

$$c = \lambda(ev \circ (id \times ev) \circ \langle \pi' \circ \pi, \langle \pi', \pi \circ \pi \rangle \rangle).$$

By combining these, we obtain a $\mathbb{P}$-morphism

$$\dot{\lambda}(\dot{ev} \circ (\dot{w} \dot{\times} \dot{c}) \circ \dot{a}) : (X \dot{\Rightarrow} Y)^{\top\top(\mathbf{S})} \dot{\times} X \to (Y \dot{\Rightarrow} S_i) \dot{\Rightarrow} S_i$$

above

$$\eta_J^{US_i} \circ ev_{I,J} = \lambda(ev \circ (\eta_{I \Rightarrow J}^{US_i} \times c) \circ a) : (I \Rightarrow J) \times I \to (J \Rightarrow US_i) \Rightarrow US_i$$

(where a and $\dot{a}$ are associativity morphisms in $\mathbb{C}$ and $\mathbb{P}$ respectively). This implies that the following inequation holds for every $i \in I$ in the fibre over $(I \Rightarrow J) \times I$:

$$(X \dot{\Rightarrow} Y)^{\top\top(\mathbf{S})} \dot{\times} X \leqslant ev_{I,J}^*(Y^{\top\top(S_i)}).$$

Therefore we have

$$(X \dot{\Rightarrow} Y)^{\top\top(\mathbf{S})} \dot{\times} X \leqslant \bigwedge_{i \in I} ev_{I,J}^*(Y^{\top\top(S_i)}) = ev_{I,J}^*(Y^{\top\top(\mathbf{S})}).$$

Now the composite, say $\dot{v}$, of $\mathbb{P}$-morphisms in the following diagram is above $ev_{I,J}$:

$$\begin{array}{ccccc}
(X \dot{\Rightarrow} Y)^{\top\top(\mathbf{S})} \dot{\times} X & & & & \\
\downarrow{\scriptstyle \leq} & & & & \\
ev_{I,J}^*(Y^{\top\top(\mathbf{S})}) & \cdots\cdots\cdots\succ & Y^{\top\top(\mathbf{S})} & & \mathbb{P} \\
 & & & & \downarrow{\scriptstyle U} \\
(I \Rightarrow J) \times I & \xrightarrow[ev_{I,J}]{} & J & & \mathbb{C}
\end{array}$$

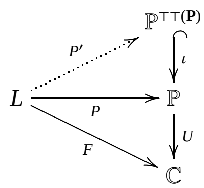
so $\dot{\lambda}(\dot{v}) : (X \dot{\Rightarrow} Y)^{\top\top(\mathbf{S})} \to X \dot{\Rightarrow} Y^{\top\top(\mathbf{S})}$ is above $\lambda(ev_{I,J}) = \mathrm{id}$. Hence $(X \dot{\Rightarrow} Y)^{\top\top(\mathbf{S})} \leq X \dot{\Rightarrow} Y^{\top\top(\mathbf{S})} = X \dot{\Rightarrow} Y$.

Theorem 4. *For any family* $\mathbf{S}$ *of closure parameters,* $\top\top(\mathbf{S})$*-closed objects form a full reflective sub bi-CCC* $\mathbb{P}^{\top\top(\mathbf{S})}$ *of* $\mathbb{P}$*, and* $U \circ \iota : \mathbb{P}^{\top\top(\mathbf{S})} \to \mathbb{C}$ *is a faithful strict bi-CC functor.*

Proof. That $\mathbb{P}^{\top\top(\mathbf{S})}$ is a bi-CCC follows from Lemma 3 and Day's reflection theorem [8]. The CC structure on $\mathbb{P}^{\top\top(\mathbf{S})}$ is inherited from $\mathbb{P}$; so ι is a strict CC functor. In general, ι is not a co-Cartesian functor, but the coproduct diagram in (4) is strictly mapped to the coproduct diagram in $\mathbb{C}$ by $U \circ \iota$. Hence $U \circ \iota$ is a strict bi-CC functor. The faithfulness is obvious.

3.3 Restriction Theorem

We next consider a small category L and functors $F : L \to \mathbb{C}$ and $P : L \to \mathbb{P}$ such that $U \circ P = F$ (see the lower half of the commutative diagram in Figure 4). The functor P specifies a family of closure parameters $\mathbf{P} = \{P\tau\}_{\tau \in \mathbf{Obj}(L)}$.



$\mathbb{P}$	bi-CCC $(\dot{1}, \dot{\times}, \dot{0}, \dot{+}, \dot{\Rightarrow})$
$\mathbb{C}$	bi-CCC $(1, \times, 0, +, \Rightarrow)$
U	strict bi-CC functor and partial order fibration with fibred small products
L	*small* category
P, F	functors
$\mathbb{P}^{\top\top(\mathbf{P})}$	bi-CCC of $\top\top(\mathbf{P})$-closed objects
ι	strict CC inclusion functor

Fig. 4. Restriction of P to $\top\top(\mathbf{P})$-Closed Objects

Proposition 9. *The functor $P : L \to \mathbb{P}$ restricts to $\mathbb{P}^{\top\top(\mathbf{P})}$ (see P' in Figure 4).*

Proof. From Proposition 8, $(P\tau)^{\top\top(\mathbf{P})} = P\tau$ holds for any L-object τ, that is, $P\tau$ is an object in the full subcategory $\mathbb{P}^{\top\top(\mathbf{P})}$ of $\mathbb{P}$. Hence P restricts to $\mathbb{P}^{\top\top(\mathbf{P})}$.

Theorem 5 (Restriction Theorem). *In the commutative diagram in Figure 4,*

1. *If L is a Cartesian (closed) category and F and P are strict Cartesian (closed) functors, then P' is also a strict Cartesian (closed) functor.*
2. *If F is a strict co-Cartesian functor and P satisfies*

$$P0 \dot{\Rightarrow} P\rho = \dot{0} \dot{\Rightarrow} P\rho, \qquad P(\tau + \tau') \dot{\Rightarrow} P\rho = (P\tau \dot{+} P\tau') \dot{\Rightarrow} P\rho$$

   then P' is a strict co-Cartesian functor.
3. *If L is a bi-CCC, F is a strict bi-CC functor and P is a strict CC functor, then P satisfies the above equations (hence P' is a strict bi-CC functor).*

Proof. 1. The Cartesian (closed) structure in $\mathbb{P}^{\top\top(\mathbf{P})}$ is the restriction of that in $\mathbb{P}$ to $\mathbb{P}^{\top\top(\mathbf{P})}$. Since P strictly preserves Cartesian (closed) structure, so does P'.

2. Suppose $P(\tau + \tau') \dot{\Rightarrow} P\rho = (P\tau \dot{+} P\tau') \dot{\Rightarrow} P\rho$. From the definition of $\top\top(\mathbf{P})$, we have

$$\begin{aligned} P(\tau + \tau') &= P(\tau + \tau')^{\top\top(\mathbf{P})} \\ &= \bigwedge_{\rho \in \mathbf{Obj}(L)} (\eta^{F\rho}_{F\tau + F\tau'})^*((P(\tau + \tau') \dot{\Rightarrow} P\rho) \dot{\Rightarrow} P\rho) \\ &= \bigwedge_{\rho \in \mathbf{Obj}(L)} (\eta^{F\rho}_{F\tau + F\tau'})^*(((P\tau \dot{+} P\tau') \dot{\Rightarrow} P\rho) \dot{\Rightarrow} P\rho) \\ &= (P\tau \dot{+} P\tau')^{\top\top(\mathbf{P})}. \end{aligned}$$

   One can similarly show $P0 = (P0)^{\top\top(\mathbf{P})}$.

3. We show that the equations in 2 holds for each strict CC functor P such that $U \circ P = F$. In any bi-CCC $\mathbb{D}$ there is an isomorphism

$$(A + B) \Rightarrow C \underset{\beta^{A,B,C}_{\mathbb{D}}}{\overset{\alpha^{A,B,C}_{\mathbb{D}}}{\rightleftarrows}} (A \Rightarrow C) \times (B \Rightarrow C)$$

which is preserved by strict bi-CC functors. Consider the following diagram:

$$
\begin{array}{ccccc}
P(\tau+\tau') \dot{\Rightarrow} P\rho & & & & \\
\| & & & & \\
P((\tau+\tau')\Rightarrow\rho) & \xrightarrow{P(\alpha_L^{\tau,\tau',\rho})} & P((\tau\Rightarrow\rho)\times(\tau'\Rightarrow\rho)) & & \mathbb{P} \\
 & & \| & & \downarrow U \\
(P\tau \dot{+} P\tau') \dot{\Rightarrow} P\rho & \xleftarrow[\beta_{\mathbb{P}}^{P\tau,P\tau',P\rho}]{} & (P\tau \dot{\Rightarrow} P\rho) \dot{\times} (P\tau' \dot{\Rightarrow} P\rho) & & \\
(F\tau + F\tau') \Rightarrow F\rho & \underset{\beta_{\mathbb{C}}^{F\tau,F\tau',F\rho}}{\overset{\alpha_{\mathbb{C}}^{F\tau,F\tau',F\rho}}{\rightleftarrows}} & (F\tau\Rightarrow F\rho)\times(F\tau'\Rightarrow F\rho) & & \mathbb{C}
\end{array}
$$

From $U \circ P = F$, the morphism $P(\alpha_L^{\tau,\tau',\rho})$ is above $\alpha_{\mathbb{C}}^{F\tau,F\tau',F\rho}$. Therefore the composition of morphisms in $\mathbb{P}$ is above $\beta_{\mathbb{C}}^{F\tau,F\tau',F\rho} \circ \alpha_{\mathbb{C}}^{F\tau,F\tau',F\rho} = \mathrm{id}$. Thus we obtain $P(\tau + \tau') \dot{\Rightarrow} P\rho \le (P\tau \dot{+} P\tau') \dot{\Rightarrow} P\rho$. The other direction, $P(\tau + \tau') \dot{\Rightarrow} P\rho \ge (P\tau \dot{+} P\tau') \dot{\Rightarrow} P\rho$, follows from a similar argument.
We leave the proof of $P0 \dot{\Rightarrow} P\rho = \dot{0} \dot{\Rightarrow} P\rho$ to the reader.

Theorem 1 is an instance of this general restriction theorem. In Figure 4 we instantiate U with $q : \mathbb{K}_F \to \mathbb{C}$, L with a small distributive category, F with a bi-Cartesian functor and P with the definability functor of F. From Proposition 2, Lemma 2 and Theorem 5-2, we obtain Theorem 1.

3.4 A Characterisation of Definable Morphisms by Weak Logical Predicates

We finally give a characterisation of morphisms definable by the simply typed lambda calculus with sums by means of *weak logical predicates*. Let B be the set of base types, $F : L_1(B) \to \mathbb{C}$ be a bi-CC functor and $U : \mathbb{P} \to \mathbb{C}$ be a fibration admitting $\top\top$-closure operators. An $\mathbf{Obj}(L_1(B))$-indexed family P of $\mathbb{P}$-objects is called *weak logical predicate* (with respect to F and U) if the following holds for any $L_1(B)$-objects τ, τ', ρ:

– $P\tau$ is above $F\tau$,
– $P(\tau \times \tau') = P\tau \dot{\times} P\tau'$, $P1 = \dot{1}$, $P(\tau \Rightarrow \tau') = P\tau \dot{\Rightarrow} P\tau'$, and
– $(P\tau \dot{+} P\tau) \dot{\Rightarrow} P\rho = P((\tau + \tau') \Rightarrow \rho)$, $\dot{0} \dot{\Rightarrow} P\rho = P(0 \Rightarrow \rho)$; (c.f. Theorem 5-2).

We say that a $\mathbb{C}$-morphism $f : F\tau \to F\tau'$ is *invariant under P* if there exists a (necessarily unique) $\mathbb{P}$-morphism $g : P\tau \to P\tau'$ above f.

Lemma 4 (Basic Lemma for Weak Logical Predicates). *Let P be a weak logical predicate with respect to a bi-CC functor $F : L_1(B) \to \mathbb{C}$ and a fibration $U : \mathbb{P} \to \mathbb{C}$ admitting $\top\top$-closure operators. Then for any $L_1(B)$-morphism $f : \tau \to \sigma$, Ff is invariant under P.*

Theorem 6. *Let $\mathbb{C}$ be a bi-CCC and $F : L_1(B) \to \mathbb{C}$ be a bi-CC functor. Then a $\mathbb{C}$-morphism f is definable by F (i.e. f is in the image of F) if and only if f is invariant under any weak logical predicate with respect to any fibration $U : \mathbb{P} \to \mathbb{C}$ admitting $\top\top$-closure operators.*

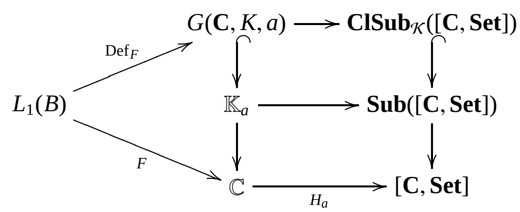


Fig. 5. Construction of the Category of Grothendieck Predicates

Proof. If f is invariant under any weak logical predicate, then it should be so under Def with respect to F and q in Section 2. Since Def is full, f is definable by F. The converse is immediate from Lemma 4.

4 Related Work

4.1 Grothendieck Logical Predicates

We briefly review Fiore and Simpson's *Grothendieck logical predicates* [11]. They are a further refinement of Jung and Tiuryn's Kripke predicates with varying arity using *Grothendieck topology*. Let $\mathbf{C}$ be a small category, K be a Grothendieck topology on $\mathbf{C}$ and $a : \mathbf{C} \to \mathbb{C}$ be a functor called *arity functor*. The topology K induces an *idempotent monad* $\mathcal{K}$ over the category $\mathbf{Sub}([\mathbf{C}^{op}, \mathbf{Set}])$ of subpresheaves [4], and one obtains the full reflective subcategory $\mathbf{ClSub}_{\mathcal{K}}([\mathbf{C}, \mathbf{Set}]) \to \mathbf{Sub}([\mathbf{C}, \mathbf{Set}])$ of $\mathcal{K}$-closed subobjects. One can verify that $\mathbf{ClSub}_{\mathcal{K}}([\mathbf{C}, \mathbf{Set}])$ is a bi-CCC, and the composite of functors $\mathbf{ClSub}_{\mathcal{K}}([\mathbf{C}, \mathbf{Set}]) \to \mathbf{Sub}([\mathbf{C}, \mathbf{Set}]) \to [\mathbf{C}, \mathbf{Set}]$ strictly preserves the bi-CC structure. We then take the pullback of the composite along $H_a : \mathbb{C} \to [\mathbf{C}^{op}, \mathbf{Set}]$ defined by $H_a(C) = \mathbb{C}(a-, C)$. This yields the category $G(\mathbf{C}, K, a)$ of Grothendieck predicates, which is also a bi-CCC (see Figure 5). Every Grothendieck logical predicate is then formulated as a bi-CC functor from $L_1(B)$ (the free bi-CCC over the set B of base types) to $G(\mathbf{C}, K, a)$.

Let $\mathbb{C}$ be a bi-CCC whose coproducts are stable and $F : L \to \mathbb{C}$ be a strict bi-CC functor. For the characterisation of the morphisms definable by F, Fiore and Simpson instantiated $\mathbf{C}$ with a syntactically constructed category of constrained contexts, K with a suitable topology on $\mathbf{C}$ and a with the interpretation of contexts by F. They showed that the functor $\mathrm{Def} : L_1(B) \to G(\mathbf{C}, K, a)$ that captures the morphisms definable by F is a bi-CC functor, that is, a Grothendieck logical predicate.

We give an informal comparison of their approach and our approach.

1. In our approach the parameter category for Kripke predicates is the partial order $\mathbf{Ctx}_L$ of context inclusions, while in [11] a non-partial order category of constrained contexts and renamings is used (although it can be switched to the partial order called *Diaconescu cover* without affecting the result; see Section 5, [11]).

2. The closure operator $\mathcal{K}$ can be restricted to the one $\mathcal{K}|_{\mathbb{K}_a}$ over $\mathbb{K}_a$, and $G(\mathbf{C}, K, a)$ can be seen as the full reflective subcategory of the $\mathcal{K}|_{\mathbb{K}_a}$-closed subobjects. In our approach we derived the ($\top\top$-)closure operator over $\mathbb{K}_F$ from the definability predicates itself, and considered the full reflective subcategory $\mathbb{K}_F^{\top\top}$ of $\top\top$-closed subobjects. Both approaches perform a similar categorical construction to obtain the category for characterising the definability predicates, but with different closure operators.
3. One drawback of our characterisation is that the definability predicates at sum types are *not* inductively characterised, although they are coproducts of the definability predicates at lower types. This is because the $\top\top$-closure operator used in equations (2) and (3) refers to the definability predicates at every type. On the other hand, in Fiore and Simpson's work the definability predicates at sum types are completely determined by those at lower types.
4. One advantage of our characterisation is that it holds for any interpretation of the simply typed lambda calculus with sums in *any* bi-CCC.

4.2 Other Related Work

Pitts introduced $\top\top$-closure operator for capturing the concept of admissible relations in the syntactic study of a polymorphic functional language [24]. Operators that are similar to the $\top\top$-closure had already appeared in various forms: the duality operator in the phase-space semantics of linear logic [13] and Parigot's technique of the strong normalisation of the second order classical natural deduction [23] are such instances. The notion of $\top\top$-closure operators also appears in other studies [21, 5].

Hinted from Pitts' $\top\top$-closure operator, Lindley and Stark introduced a new technique called $\top\top$-lifting for extending the strong normalisation proof using computability predicate technique to Moggi's computational metalanguage [20, 19]. Their $\top\top$-lifting was later categorically formulated as a method to lift strong monads on the base category of a fibration to the one on its total category [16] by the author. There, $\top\top$-closure operators are formulated as the $\top\top$-lifting of the identity monad.

It is widely recognised that re-establishing properties that hold in the lambda calculus with only arrow types is difficult under the presence of sum types. For instance, the design of a confluent and strongly normalising rewriting system (with β-reduction and η-expansion) for the simply typed lambda calculus with sums [12] and the proof of the completeness of the equational theory of the lambda calculus with sums in **Set** [9] exhibits the intrinsic difficulty in handling sums. In this stream of research Grothendieck logical predicates are shown to be an effective tool in reasoning about the lambda calculus with sums. They are applied to the correctness of the normalisation-by-evaluation algorithm [2] and the proof of the extensional normalisation [3] for the lambda calculus with sums.

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