#### **Graded Algebraic Theories**

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#### Outline

Algebraic theory for graded monads

- Graded algebraic theory
- Graded Lawvere theory

Application: combining effects

- Sum
- Tensor

#### Monads

- Semantics for computational effects [Moggi, LICS'89]
- Monads correspond to
  - algebraic theories (= operations + equations)
  - · Lawvere theories [Lawvere, PhD thesis]

Computational effects can be algebraically presented.

• Arity = number of continuations

 $lookup_l(\lambda u.x_u)$  "choose  $x_u$  by value u in l"

• Executed from outermost operation

 $\mathsf{update}_{l,v}(\mathsf{lookup}_l(\lambda u.x_u))$ 

"update l with v and then lookup l"

#### An example of algebraic theories

$$\begin{array}{lll} \mbox{State monad} & S \rightarrow (-) \times S \\ & \mbox{where} & S = V^{\rm Loc} \end{array}$$

Algebraic theory  $\mathcal{T}^{st}$ :

- update<sub>l,v</sub>: unary
- lookup<sub>l</sub>: V-ary
- with several equational axioms
  - e.g.  $\mathsf{update}_{l,v}(\mathsf{lookup}_l(\lambda u.x_u)) = x_v$

#### terms in $\mathcal{T}^{st}$ = stateful computations

#### Effect systems

- · estimate the scope of computational effects
  - · accessed memory locations
  - · raised exceptions
- · via type systems

 $egin{array}{ccc} \Gammadash M:T(e, au) & \Gamma,x: audash N:T(e',\sigma) \ \hline \Gammadash ext{ let }x ext{ be }M ext{ in }N:T(e\cdot e',\sigma) \end{array}$ 

• semantics by graded monads [Katsumata, POPL'14]

 $\mathbf{M} = (\mathbf{M}, \mathbf{I}, \otimes)$ : monoidal category.

M-graded monad on category C consists of

- a functor  $T: \mathbf{M} \to [C, C]$
- two natural transformations

$$\eta: \mathrm{Id}_C o T_I \ \mu_{m_1,m_2}: (T_{m_1}) \circ (T_{m_2}) o T_{m_1 \otimes m_2}$$

• satisfying monad laws.

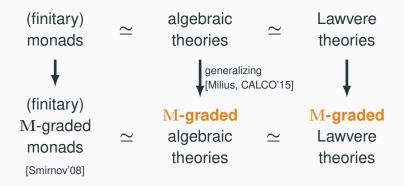
Only a few pieces of work.

- When  $\mathbf{M}=(\mathbb{N},0,+)\text{:}$  [Milius, CALCO'15]

Why not consider

- more general M?
  - "subeffecting"  $m \leq m'$ ?
- "graded" Lawvere theories?

#### Extending algebraic/Lawvere theories



M: small strict monoidal category(M is not necessarily symmetric)

#### **Graded algebraic theories**

#### Graded algebraic theories

#### $\mathcal{T} = (\Sigma, E)$

- $\Sigma$ : operations (equipped with grade)
- E: equational axioms

Each operation has its grade  $m \in M$ .

$$\Sigma = (\Sigma_{n,m})_{n \in \mathbb{N}, m \in \mathbb{M}}$$

## Example (graded state monad).grade: accessed memory locations $M = ((2^{Loc}, \subseteq), \emptyset, \cup)$ $\begin{array}{c} \begin{array}{c} \text{operation} & \text{arity} & \text{grade} \\ \hline \text{lookup}_l & V & \{l\} \\ \text{update}_{l,v} & 1 & \{l\} \end{array}$



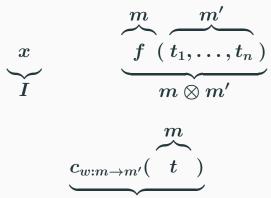
### We define the set of terms with grade $m \in \mathrm{M}$ $T^\Sigma_m(X)$

where X is a set of variables.

Three rules of term formation:

- Variables: x
- Operations:  $f(t_1,\ldots,t_n)$
- Coercions (or subeffecting):  $c_{w:m o m'}(t)$

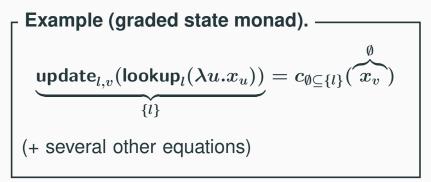
#### **Grade of Terms**



m'

Grades of both sides must be equal.

$$E = \left( E_m \ \subseteq \ T_m^\Sigma(X) imes T_m^\Sigma(X) 
ight)_{m \in \mathrm{M}}$$



#### Equivalence

terms	=	computations
with grade $m{m}$		with effect $m$

Theorem.category ofcategory ofM-graded $\simeq$ algebraic theoriesmonads on Set

<sup>1</sup>graded monad T where  $T_m:\operatorname{Set} o\operatorname{Set}$  is finitary for each  $m\in\operatorname{M}$ 

#### **Graded Lawvere theories**

Independent from specific choice of operations

 because we consider all terms generated by algebraic theories

Independent from base category

 compared to monads, which are defined on one base category

#### Intuition of Lawvere theories

A Lawvere theory is a category L where

• objects: arities

$$\mathrm{ob}L = \mathbb{N}$$

arrows: terms (modulo equational axioms)

$$n \xrightarrow{(t_1,...,t_k)} k$$

composition: substitution

$$n \xrightarrow{(t_1,...,t_k)} k \xrightarrow{s} 1 \quad = \quad n \xrightarrow{s[t_1/x_1,...,t_k/x_k]} 1$$

#### Lawvere theories

#### Definition.

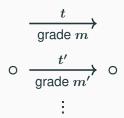
#### A Lawvere theory is a pair of

- a category L with finite products and
- a strict finite-product preserving identity-on-objects functor  $J: \aleph_0^{\text{op}} \to L$ .

Here,  $\aleph_0$  is the full subcategory of Set on natural numbers.

#### **Graded Lawvere theories?**

Different terms have different grades.



#### Take hom-objects not from Set but from [M, Set].

How to express the mathematical structure of grades?

$$n \xrightarrow[m]{(t_1,...,t_k)}{m} k \xrightarrow[m']{} 1 \quad = \quad n \xrightarrow[m' \otimes m]{} rac{s[t_1/x_1,...,t_k/x_k]}{m' \otimes m} 1$$

Consider [M, Set]-enriched categories using the Day tensor monoidal structure.

#### [M, Set]-categories

If C is an [M, Set]-category,

• hom-object:  $C(X,Y): \mathbf{M} \to \mathbf{Set}$ 



• composition:

$$X \xrightarrow{f}_{m} Y \xrightarrow{g}_{m'} Z = X \xrightarrow{g \circ f}_{m' \otimes m} Z$$

#### **Graded Lawvere Theories**

# Definition.A graded Lawvere theory is a pair of• an [M, Set]-category L with<br/> $N_{\rm M}^{\rm op}$ -cotensors and• a $N_{\rm M}^{\rm op}$ -cotensor preserving<br/>identity-on-objects functor $J: N_{\rm M}^{\rm op} \to L.$

 $N_{\mathrm{M}}$  is the full sub-[M, Set]-category of [M, Set] on  $\{n \cdot y(I) \mid n \in \mathbb{N}\}$  (if M is symmetric).

#### Equivalence

#### Theorem.

category of		category of
$\mathbf{M}$ -graded	$\simeq$	finitary <sup>2</sup> M-graded
Lawvere theories		monads on $\operatorname{Set}$

 $^2$ graded monad T where  $T_m:\operatorname{Set} o\operatorname{Set}$  is finitary for each  $m\in\operatorname{M}$ Satoshi Kura (NII, Tokyo)

#### An application: combining effects

#### **Combining effects**

For ordinary algebraic theories [Hyland & Power, TCS'06]

Sum: add operations with no additional equation

$$\mathcal{T}_1 \qquad \mathcal{T}_2 \qquad \mapsto \qquad \mathcal{T}_1 + \mathcal{T}_2$$

Tensor: allow commutation f(g(x)) = g(f(x))between two theories

$$\mathcal{T}_1 \qquad \mathcal{T}_2 \qquad \mapsto \qquad \mathcal{T}_1 \otimes \mathcal{T}_2$$

#### **Combining effects**

For graded algebraic theories: Sum M-graded M-graded  $\mathcal{T}_1 \quad \mathcal{T}_2 \quad \mapsto \quad \mathcal{T}_1 + \mathcal{T}_2$ Tensor  $M_1$ -graded  $M_2$ -graded  $M_1 \times M_2$ -graded  $\mathcal{T}_2 \qquad \mapsto \qquad \mathcal{T}_1 \otimes \mathcal{T}_2$  $\mathcal{T}_1$ 



#### Given two $\mathbf{M}$ -graded algebraic theories

$$\mathcal{T} = (\Sigma, E), \qquad \mathcal{T}' = (\Sigma', E')$$

#### where

- $\Sigma, \Sigma'$ : sets of operations
- E, E': sets of equations

The sum is

$$\mathcal{T} + \mathcal{T}' \coloneqq (\Sigma \cup \Sigma', E \cup E')$$

+ graded exception monad finitary monad  $T: Set \to Set$  $T^{\mathrm{ex}}: \mathrm{M} \to [\mathrm{Set}, \mathrm{Set}]$  $TT^{\mathrm{ex}}: \mathrm{M} \to [\mathrm{Set}, \mathrm{Set}]$ =Cf. exception monad (-) + Exfinitary monad T= T((-) + Ex)

#### Tensors in [Hyland & Power, TCS'06]

Tensors are defined by

$$(\Sigma,E)\otimes (\Sigma',E')\coloneqq (\Sigma\cup\Sigma',E\cup E'\cup E_{\otimes})$$

where  $E_{\otimes}$  consists of

$$f(\lambda i.g(\lambda j.x_{ij})) = g(\lambda j.f(\lambda i.x_{ij}))$$

for any  $f \in \Sigma$  and  $g \in \Sigma'$ .

#### Problem of commutativity

 $\begin{array}{ccc} \text{M-graded} & \text{M-graded} \\ \mathcal{T} & \mathcal{T}' & \mapsto & \mathcal{T} \otimes \mathcal{T}' \end{array} (?)$ 

We cannot define

$$\underbrace{ f \ (\lambda i. \ g \ (\lambda j. x_{ij})) = }_{m} \underbrace{ g \ (\lambda j. x_{ij}) ) }_{m'} \underbrace{ g \ (\lambda j. \ f \ (\lambda i. x_{ij})) }_{m'} \underbrace{ f \ (\lambda i. x_{ij}) ) }_{m'}$$

because  $\mathbf{M}$  is not necessarily symmetric.

$$m\otimes m'
eq m'\otimes m$$

#### Tensors

Example: theories for graded state monad Suppose

- $\mathcal{T}_1$  : 2-graded theory with  $\#\mathsf{Loc}=1$
- $\mathcal{T}_n$  :  $2^n$ -graded theory with  $\#\mathsf{Loc} = n$
- $2 = \{ \perp \leq \top \}$  and  $2^n$  are join-semilattices

We have

$$\mathcal{T}_n = \underbrace{\mathcal{T}_1 \otimes \cdots \otimes \mathcal{T}_1}_{n ext{-fold tensor}}$$

#### Conclusions

- Graded algebraic theories: by assigning grades to each term.
- Graded Lawvere theories: by considering enrichment in [M, Set].
- They correspond to graded monads.
- Sums and tensors can be extended to graded algebraic theories.

#### Term formation

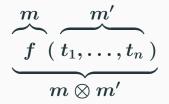
#### Variables: grade = unit I

$$\frac{x\in X}{x\in T^\Sigma_I(X)}$$

$$\overset{x}{\overbrace{I}}$$

Operations: grade = tensor product  $m \otimes m'$ 

$$egin{array}{ccc} f\in \Sigma_{n,m} & t_1,\ldots,t_n\in T^\Sigma_{m'}(X) \ \hline f(t_1,\ldots,t_n)\in T^\Sigma_{m\otimes m'}(X) \end{array}$$



#### **Term formation**

Coercions: grade  $m \xrightarrow{w} m'$ 

$$rac{t\in T^\Sigma_m(X) \quad w:m o m' ext{ in }\mathrm{M}}{c_w(t)\in T^\Sigma_{m'}(X)}$$

