Graded Algebraic Theories

Satoshi Kura Sep 8, 2020 @ JSSST 2020

National Institute of Informatics, Tokyo, Japan

The Graduate University for Advanced Studies (SOKENDAI), Kanagawa, Japan

Outline

Algebraic theory for graded monads

- Graded algebraic theory
- Graded Lawvere theory

Application: combining effects

- Sum
- Tensor

Monads

- Semantics for computational effects [Moggi, LICS'89]
- Monads correspond to
 - algebraic theories (= operations + equations)
 - · Lawvere theories [Lawvere, PhD thesis]

Computational effects can be algebraically presented.

• Arity = number of continuations

 $lookup_l(\lambda u.x_u)$ "choose x_u by value u in l"

• Executed from outermost operation

 $\mathsf{update}_{l,v}(\mathsf{lookup}_l(\lambda u.x_u))$

"update l with v and then lookup l"

An example of algebraic theories

$$\begin{array}{lll} \mbox{State monad} & S \rightarrow (-) \times S \\ & \mbox{where} & S = V^{\rm Loc} \end{array}$$

Algebraic theory \mathcal{T}^{st} :

- update_{l,v}: unary
- lookup_l: V-ary
- with several equational axioms
 - e.g. $\mathsf{update}_{l,v}(\mathsf{lookup}_l(\lambda u.x_u)) = x_v$

terms in \mathcal{T}^{st} = stateful computations

Effect systems

- · estimate the scope of computational effects
 - · accessed memory locations
 - · raised exceptions
- · via type systems

 $egin{array}{ccc} \Gammadash M:T(e, au) & \Gamma,x: audash N:T(e',\sigma) \ \hline \Gammadash ext{ let }x ext{ be }M ext{ in }N:T(e\cdot e',\sigma) \end{array}$

• semantics by graded monads [Katsumata, POPL'14]

 $\mathbf{M} = (\mathbf{M}, \mathbf{I}, \otimes)$: monoidal category.

M-graded monad on category C consists of

- a functor $T: \mathbf{M} \to [C, C]$
- two natural transformations

$$\eta: \mathrm{Id}_C o T_I \ \mu_{m_1,m_2}: (T_{m_1}) \circ (T_{m_2}) o T_{m_1 \otimes m_2}$$

• satisfying monad laws.

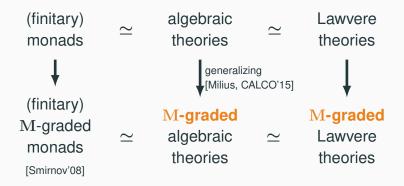
Only a few pieces of work.

- When $\mathbf{M}=(\mathbb{N},0,+)\text{:}$ [Milius, CALCO'15]

Why not consider

- more general M?
 - "subeffecting" $m \leq m'$?
- "graded" Lawvere theories?

Extending algebraic/Lawvere theories



M: small strict monoidal category(M is not necessarily symmetric)

Graded algebraic theories

Graded algebraic theories

$\mathcal{T} = (\Sigma, E)$

- Σ : operations (equipped with grade)
- E: equational axioms

Each operation has its grade $m \in M$.

$$\Sigma = (\Sigma_{n,m})_{n \in \mathbb{N}, m \in \mathbb{M}}$$

Example (graded state monad).grade: accessed memory locations $M = ((2^{Loc}, \subseteq), \emptyset, \cup)$ $\begin{array}{c} \begin{array}{c} \text{operation} & \text{arity} & \text{grade} \\ \hline \text{lookup}_l & V & \{l\} \\ \text{update}_{l,v} & 1 & \{l\} \end{array}$



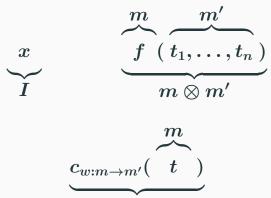
We define the set of terms with grade $m \in \mathrm{M}$ $T^\Sigma_m(X)$

where X is a set of variables.

Three rules of term formation:

- Variables: x
- Operations: $f(t_1,\ldots,t_n)$
- Coercions (or subeffecting): $c_{w:m o m'}(t)$

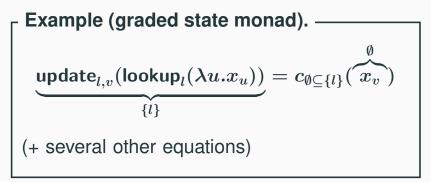
Grade of Terms



m'

Grades of both sides must be equal.

$$E = \left(E_m \ \subseteq \ T_m^\Sigma(X) imes T_m^\Sigma(X)
ight)_{m \in \mathrm{M}}$$



Equivalence

terms	=	computations
with grade $m{m}$		with effect m

Theorem.category ofcategory ofM-graded \simeq algebraic theoriesmonads on Set

¹graded monad T where $T_m:\operatorname{Set} o\operatorname{Set}$ is finitary for each $m\in\operatorname{M}$

Graded Lawvere theories

Independent from specific choice of operations

 because we consider all terms generated by algebraic theories

Independent from base category

 compared to monads, which are defined on one base category

Intuition of Lawvere theories

A Lawvere theory is a category L where

• objects: arities

$$\mathrm{ob}L = \mathbb{N}$$

arrows: terms (modulo equational axioms)

$$n \xrightarrow{(t_1,...,t_k)} k$$

composition: substitution

$$n \xrightarrow{(t_1,...,t_k)} k \xrightarrow{s} 1 \quad = \quad n \xrightarrow{s[t_1/x_1,...,t_k/x_k]} 1$$

Lawvere theories

Definition.

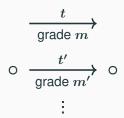
A Lawvere theory is a pair of

- a category L with finite products and
- a strict finite-product preserving identity-on-objects functor $J: \aleph_0^{\text{op}} \to L$.

Here, \aleph_0 is the full subcategory of Set on natural numbers.

Graded Lawvere theories?

Different terms have different grades.



Take hom-objects not from Set but from [M, Set].

How to express the mathematical structure of grades?

$$n \xrightarrow[m]{(t_1,...,t_k)}{m} k \xrightarrow[m']{} 1 \quad = \quad n \xrightarrow[m' \otimes m]{} rac{s[t_1/x_1,...,t_k/x_k]}{m' \otimes m} 1$$

Consider [M, Set]-enriched categories using the Day tensor monoidal structure.

[M, Set]-categories

If C is an [M, Set]-category,

• hom-object: $C(X,Y): \mathbf{M} \to \mathbf{Set}$



• composition:

$$X \xrightarrow{f}_{m} Y \xrightarrow{g}_{m'} Z = X \xrightarrow{g \circ f}_{m' \otimes m} Z$$

Graded Lawvere Theories

Definition.A graded Lawvere theory is a pair of• an [M, Set]-category L with
 $N_{\rm M}^{\rm op}$ -cotensors and• a $N_{\rm M}^{\rm op}$ -cotensor preserving
identity-on-objects functor $J: N_{\rm M}^{\rm op} \to L.$

 N_{M} is the full sub-[M, Set]-category of [M, Set] on $\{n \cdot y(I) \mid n \in \mathbb{N}\}$ (if M is symmetric).

Equivalence

Theorem.

category of		category of
\mathbf{M} -graded	\simeq	finitary ² M-graded
Lawvere theories		monads on Set

 2 graded monad T where $T_m:\operatorname{Set} o\operatorname{Set}$ is finitary for each $m\in\operatorname{M}$ Satoshi Kura (NII, Tokyo)

An application: combining effects

Combining effects

For ordinary algebraic theories [Hyland & Power, TCS'06]

Sum: add operations with no additional equation

$$\mathcal{T}_1 \qquad \mathcal{T}_2 \qquad \mapsto \qquad \mathcal{T}_1 + \mathcal{T}_2$$

Tensor: allow commutation f(g(x)) = g(f(x))between two theories

$$\mathcal{T}_1 \qquad \mathcal{T}_2 \qquad \mapsto \qquad \mathcal{T}_1 \otimes \mathcal{T}_2$$

Combining effects

For graded algebraic theories: Sum M-graded M-graded $\mathcal{T}_1 \quad \mathcal{T}_2 \quad \mapsto \quad \mathcal{T}_1 + \mathcal{T}_2$ Tensor M_1 -graded M_2 -graded $M_1 \times M_2$ -graded $\mathcal{T}_2 \qquad \mapsto \qquad \mathcal{T}_1 \otimes \mathcal{T}_2$ \mathcal{T}_1



Given two \mathbf{M} -graded algebraic theories

$$\mathcal{T} = (\Sigma, E), \qquad \mathcal{T}' = (\Sigma', E')$$

where

- Σ, Σ' : sets of operations
- E, E': sets of equations

The sum is

$$\mathcal{T} + \mathcal{T}' \coloneqq (\Sigma \cup \Sigma', E \cup E')$$

+ graded exception monad finitary monad $T: Set \to Set$ $T^{\mathrm{ex}}: \mathrm{M} \to [\mathrm{Set}, \mathrm{Set}]$ $TT^{\mathrm{ex}}: \mathrm{M} \to [\mathrm{Set}, \mathrm{Set}]$ =Cf. exception monad (-) + Exfinitary monad T= T((-) + Ex)

Tensors in [Hyland & Power, TCS'06]

Tensors are defined by

$$(\Sigma,E)\otimes (\Sigma',E')\coloneqq (\Sigma\cup\Sigma',E\cup E'\cup E_{\otimes})$$

where E_{\otimes} consists of

$$f(\lambda i.g(\lambda j.x_{ij})) = g(\lambda j.f(\lambda i.x_{ij}))$$

for any $f \in \Sigma$ and $g \in \Sigma'$.

Problem of commutativity

 $\begin{array}{ccc} \text{M-graded} & \text{M-graded} \\ \mathcal{T} & \mathcal{T}' & \mapsto & \mathcal{T} \otimes \mathcal{T}' \end{array} (?)$

We cannot define

$$\underbrace{ f \ (\lambda i. \ g \ (\lambda j. x_{ij})) = }_{m} \underbrace{ g \ (\lambda j. x_{ij})) }_{m'} \underbrace{ g \ (\lambda j. \ f \ (\lambda i. x_{ij})) }_{m'} \underbrace{ f \ (\lambda i. x_{ij})) }_{m'}$$

because \mathbf{M} is not necessarily symmetric.

$$m\otimes m'
eq m'\otimes m$$

Tensors

Example: theories for graded state monad Suppose

- \mathcal{T}_1 : 2-graded theory with $\#\mathsf{Loc}=1$
- \mathcal{T}_n : 2^n -graded theory with $\#\mathsf{Loc} = n$
- $2 = \{ \perp \leq \top \}$ and 2^n are join-semilattices

We have

$$\mathcal{T}_n = \underbrace{\mathcal{T}_1 \otimes \cdots \otimes \mathcal{T}_1}_{n ext{-fold tensor}}$$

Conclusions

- Graded algebraic theories: by assigning grades to each term.
- Graded Lawvere theories: by considering enrichment in [M, Set].
- They correspond to graded monads.
- Sums and tensors can be extended to graded algebraic theories.

Term formation

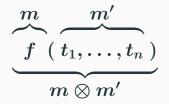
Variables: grade = unit I

$$\frac{x\in X}{x\in T^\Sigma_I(X)}$$

$$\overset{x}{\overbrace{I}}$$

Operations: grade = tensor product $m \otimes m'$

$$egin{array}{ccc} f\in \Sigma_{n,m} & t_1,\ldots,t_n\in T^\Sigma_{m'}(X) \ \hline f(t_1,\ldots,t_n)\in T^\Sigma_{m\otimes m'}(X) \end{array}$$



Term formation

Coercions: grade $m \xrightarrow{w} m'$

$$rac{t\in T^\Sigma_m(X) \quad w:m o m' ext{ in }\mathrm{M}}{c_w(t)\in T^\Sigma_{m'}(X)}$$

