

Bachelor's Thesis

# A Categorical Approach to L-Convexity (L凸性への圏論的アプローチ)

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## Abstract

We investigate an enriched-categorical approach to a field of discrete mathematics. The main result is a duality theorem between a class of enriched categories (called  $\overline{\mathbb{Z}}$ - or  $\overline{\mathbb{R}}$ -categories) and that of what we call ( $\overline{\mathbb{Z}}$ - or  $\overline{\mathbb{R}}$ -) extended L-convex sets. We introduce extended L-convex sets as variants of certain discrete structures called L-convex sets and L-convex polyhedra, studied in the field of discrete convex analysis. We also introduce homomorphisms between extended L-convex sets. The theorem claims that there is a one to one correspondence (up to isomorphism) between two classes. The thesis also contains an introductory chapter on enriched categories and no categorical knowledge is assumed.



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# Notation

## Proof Trees

Throughout the thesis, we use a proof-tree-like notation as a visually intuitive way of reasoning. The tree

$$\frac{P(x, y, \dots)}{Q(x, y, \dots)}$$

where  $P$  and  $Q$  are formulas possibly with some free variables  $(x, y, \dots)$ , represents the assertion

$$P(x, y, \dots) \iff Q(x, y, \dots)$$

for all possible values of  $x, y, \dots$  in some appropriate domains.

For example, the supremum of a subset  $S$  of a complete lattice  $(L, \sqsubseteq)$  is the unique element  $a \in L$  with the following property:

$$\frac{s \sqsubseteq x \quad (\forall s \in S)}{a \sqsubseteq x}$$

which is to be read as

$$x \text{ is an upper bound of } S \iff a \sqsubseteq x$$

for all  $x \in L$ , or equivalently “ $a$  is the least element among upper bounds of  $S$ .”

Higher trees also appear:

$$\frac{\frac{P(x, y, \dots)}{Q(x, y, \dots)}}{R(x, y, \dots)}$$

Note that any two formulas, especially the top and the bottom ones, are equivalent in a higher tree as well.

## $\lambda$ -Notation

In order to define maps succinctly without giving a name, we exploit the  $\lambda$ -notation borrowed from (typed) lambda calculi. An example of definitions of maps via the  $\lambda$ -notation is the following:

$$\lambda x \in \mathbb{R}. x^2 + 2x.$$

This expression denotes a function which takes an element  $x$  of  $\mathbb{R}$  and returns the value  $x^2 + 2x$ . Therefore if we pass an argument, say, 3, we have:

$$(\lambda x \in \mathbb{R}. x^2 + 2x)(3) = 3^2 + 2 \cdot 3 = 15.$$

## Conventions

To highlight definitions, terms being defined (either explicitly or implicitly) are written in the **boldface** font; the *italic* font is not used for this purpose, and is reserved for emphases.

The ends of definitions and examples are marked by ■, and the ends of proofs by □.



# Chapter 1

## Introduction

### 1.1 Background and Our Results

The study presented in this thesis is first motivated and inspired by a recent paper by Simon Willerton [10]. In this paper, he recognizes that a construction called *directed tight span*, introduced independently by [1] and [4], is an instance of a more general construction known as the *Isbell completion*. In fact, the directed tight span is a result within the theory of metric spaces, whereas the Isbell completion belongs to a discipline called enriched category theory. Why such a link could exist? This is due to an influential 1973 paper [5] of William Lawvere; he remarks that, among others, *enriched categories specialize to metric spaces*.

Let us be more precise. Enriched categories are abstract entities that generalize (ordinary) categories studied in category theory. A general definition of an enriched category involves a parameter  $\mathcal{V}$  called the *enriching category*, itself a category (with some additional structures). Therefore, if we restrict our attention to those enriched categories with a specific enriching category, we obtain a theory to which the theory of enriched categories specializes. For example, if we let  $\mathcal{V} = \mathbf{Set}$ , where  $\mathbf{Set}$  is the category of sets and maps, then it turns out that enriched categories with the enriching category  $\mathbf{Set}$  (called **Set**-categories) are nothing but categories; thus enriched category theory specializes to category theory. What Lawvere observed is, if we set  $\mathcal{V} = \overline{\mathbb{R}}_+$ , a poset of nonnegative real numbers together with an additional element  $\infty$  (for those not familiar with categories: posets are a special kind of categories), then  $\overline{\mathbb{R}}_+$ -categories are a little generalized metric spaces (in a precise statement, every metric space is an  $\overline{\mathbb{R}}_+$ -category). Therefore he established a link between the theory of enriched categories and that of metric spaces. Now it is amazing that many notions of metric spaces, e.g., that of nonexpansive maps, sup-distances on function spaces, the Fréchet embeddings of metric spaces, and directed tight spans, already appear quite canonically in enriched category theory.

The main objective of this thesis is to present an enriched-categorical approach to another seemingly unrelated field: *discrete convex analysis*. The theory of discrete convex analysis (see [8] for details) can be seen as a discrete (or  $\mathbb{Z}$ ) version of convex analysis (based on  $\mathbb{R}$ ), transporting various notions of convex analysis to a discrete setting. The notion of convex sets ( $\subseteq \mathbb{R}^n$ ), for example, thus has its discrete counterparts, notably that of *L-convex sets* (named after lattices) and *M-convex sets* (after matroids), which are certain subsets of  $\mathbb{Z}^n$ . Our main result concerns L-convex sets; rather, their variant what we call  *$\overline{\mathbb{Z}}$ -extended L-convex sets*. By  $\overline{\mathbb{Z}}$  we mean (among several other things) a poset of integers together with two additional elements  $-\infty$  and  $\infty$ . We introduce  $\overline{\mathbb{Z}}$ -extended L-convex sets roughly as certain subsets of  $\overline{\mathbb{Z}}^n$ , or more generally, subsets of  $\overline{\mathbb{Z}}^V$  where  $V$  is some (possibly infinite) set. The axioms we impose for  $\overline{\mathbb{Z}}$ -extended L-convex sets are similar (but definitely not equal) to that for L-convex sets, so we regard  $\overline{\mathbb{Z}}$ -extended

L-convex sets as analogs of L-convex sets. On the other hand, the poset  $\overline{\mathbb{Z}}$  has a natural structure to be an enriching category, and we can consider  $\overline{\mathbb{Z}}$ -categories and the theory of them. Recall the observation of Lawvere that  $\overline{\mathbb{R}}_+$ -categories are like metric spaces. Because the structures of  $\overline{\mathbb{R}}_+$  and  $\overline{\mathbb{Z}}$  are similar, one can say that  $\overline{\mathbb{Z}}$ -categories are like metric spaces as well, although the analogy is even weaker.

Our main result is a correspondence (or duality) between  $\overline{\mathbb{Z}}$ -categories and  $\overline{\mathbb{Z}}$ -extended L-convex sets. The correspondence is established at three levels: first for individual objects, second for maps between objects, and finally for what we call canonical orderings (usually called *natural transformations*) between maps. At the first level, we present a construction that makes a  $\overline{\mathbb{Z}}$ -extended L-convex set out of a  $\overline{\mathbb{Z}}$ -category, and another one that performs the reverse. These two constructions are *inverses* in the sense that if we start from a  $\overline{\mathbb{Z}}$ -category and apply the first construction to get a  $\overline{\mathbb{Z}}$ -extended L-convex set, and then apply the second to obtain another  $\overline{\mathbb{Z}}$ -category, then the resulting one is isomorphic to the  $\overline{\mathbb{Z}}$ -category we started from; and likewise if we start from a  $\overline{\mathbb{Z}}$ -extended L-convex set. Such a result is in fact already established for L-convex sets and distance functions satisfying the triangle inequality, with essentially the same technical contents; see [8, Section 5.3]. However, our result is new in its formulation of the constructions. It turns out that the set  $\mathbb{Z} \cup \{-\infty, \infty\}$ , which is the underlying set of the poset  $\overline{\mathbb{Z}}$ , can naturally be seen both as a  $\overline{\mathbb{Z}}$ -category and as a  $\overline{\mathbb{Z}}$ -extended L-convex set; in either case we denote the resulting entity by  $\overline{\mathbb{Z}}$ . Our constructions explicitly involve the *function space constructions with codomain  $\overline{\mathbb{Z}}$*  (either as a  $\overline{\mathbb{Z}}$ -category or as a  $\overline{\mathbb{Z}}$ -extended L-convex set) as key steps. Therefore in our formulation, maps play a crucial role. Let us discuss them next; the discussion also leads to the second level of the correspondence.

In enriched category theory, there is an established notion of maps between enriched (say,  $\mathcal{V}$ -) categories, called  $\mathcal{V}$ -functors. Thus we adopt  $\overline{\mathbb{Z}}$ -functors as the members of the class of maps between  $\overline{\mathbb{Z}}$ -categories we consider, which are basically nonexpansive (distance-nonincreasing) maps. For  $\overline{\mathbb{Z}}$ -extended L-convex sets, we adopt what we call *homomorphisms* between them as natural maps. The introduction of the notion of homomorphisms is among the contribution of the thesis. Its definition is closely tied to our definition of  $\overline{\mathbb{Z}}$ -extended L-convex sets and seems to be natural. The duality at the second level states that there is a bijection between the set of  $\overline{\mathbb{Z}}$ -functors with specified domain and codomain  $\overline{\mathbb{Z}}$ -categories and that of homomorphisms with the corresponding domain and codomain  $\overline{\mathbb{Z}}$ -extended L-convex sets, where the correspondence is that built in the first level. However, one noteworthy point is that the *directions* of maps reverse; that is, what corresponds to the domain  $\overline{\mathbb{Z}}$ -extended L-convex set is the codomain  $\overline{\mathbb{Z}}$ -category, and the codomain  $\overline{\mathbb{Z}}$ -extended L-convex set the domain  $\overline{\mathbb{Z}}$ -category. This is the reason why we call the whole correspondence a *duality* as well.

The canonical orderings are certain preorder relations on the sets of  $\overline{\mathbb{Z}}$ -functors or homomorphisms with specified domain and codomain. They are specialization of  $\mathcal{V}$ -natural transformations between  $\mathcal{V}$ -functors in enriched category theory (one can interpret homomorphisms as a special kind of  $\overline{\mathbb{Z}}$ -functors). We show that, as a third level of the duality, the correspondence of maps at the second level respects the canonical orderings.

In fact, there is no difficulty to develop an entirely parallel story by replacing  $\mathbb{Z}$  by  $\mathbb{R}$ ; in this case, what correspond to our  $\overline{\mathbb{R}}$ -extended L-convex sets turn out to be *L-convex polyhedra*, which are also studied in discrete convex analysis. Therefore in the later chapters of the thesis, we use the symbol  $\mathbb{K}$  to denote either  $\mathbb{Z}$  or  $\mathbb{R}$  and discuss  $\overline{\mathbb{K}}$ -categories and  $\overline{\mathbb{K}}$ -extended L-convex sets, treating both cases simultaneously.

## 1.2 Chapter Overview

In Chapter 2, we develop the theory of enriched categories, in a simplified form. Although the general theory of enriched categories normally requires acquaintance with basic (ordinary) category theory (as presented in [7]), we avoid those points where such knowledge is compulsory by restricting our interest to the cases where enriching categories are posets. Therefore, the exposition is intended to be so introductory as to be readable without any previous categorical experience. One remarkable point of (enriched) category theory is that many general abstract notions specialize to ones which are well-known inside some particular branch of mathematics. We hope that, with preordered sets and (somewhat generalized variants of) metric spaces as running examples, the chapter serves to convey the reader this fascinating aspect of categories.

Chapter 3 contains our main contribution. First we introduce the notions of  $\overline{\mathbb{K}}$ -extended  $L$ -convex sets and homomorphisms between them. To indicate the underlying categorical viewpoints (in this case, not in the level of (enriched)  $\overline{\mathbb{K}}$ -categories, but in that of the (ordinary, or 2-) category of  $\overline{\mathbb{K}}$ -categories or the category of  $\overline{\mathbb{K}}$ -extended  $L$ -convex sets), we occasionally use the terminology of category theory without giving definitions. However, all statements are translated into elementary terms and therefore the reader can entirely skip these parts. We present the duality theorem, first for individual objects, next for maps between them, and finally for canonical orderings between maps.

Finally, in Chapter 4 we summarize the results of the thesis and briefly indicate ways to future research.

## Chapter 2

# Poset-Enriched Category Theory

In this chapter, we develop the theory of enriched categories, under a crucial assumption that the enriching category is actually a *poset*. This assumption entirely removes the burden of checking the commutativity of (usually large) diagrams in the enriching category, simply because every diagram in a poset commutes. Since all enriching categories we will encounter in this thesis are posets, such a simplified theory suffices. The organization of the chapter roughly follows that of [3], which we also recommend as a text on general enriched category theory.

### 2.1 Enriching Posets

As one needs the notion of fields to define vector spaces, in order to define enriched categories, we need the notion of *enriching categories*, often denoted by the symbol  $\mathcal{V}$ . The fertility of the resulting theory of enriched categories over a particular enriching category  $\mathcal{V}$  depends on how *nice*  $\mathcal{V}$  is, and it turns out that in order to gain a fully fruitful theory,  $\mathcal{V}$  should be a *bicomplete symmetric monoidal closed category*. As promised above, we treat only the cases where  $\mathcal{V}$  is a poset; when this is the case, the definition of bicomplete symmetric monoidal closed categories, rather, *bicomplete symmetric monoidal closed posets* = *symmetric monoidal closed complete lattices*, is given successively as follows:

**Definition 2.1.** A **symmetric monoidal poset (SM-P)**  $\mathcal{V}$  is a triple  $(\mathcal{V}_0, \otimes, e)$  where

- $\mathcal{V}_0 = (\mathcal{V}_0, \sqsubseteq)$  is a poset called the **underlying poset**;
- $\otimes: \mathcal{V}_0 \times \mathcal{V}_0 \longrightarrow \mathcal{V}_0$  is a binary operation on  $\mathcal{V}_0$  called the **tensor product**;
- $e$  is an element of  $\mathcal{V}_0$  called the **unit element**;

such that the following axioms hold:

- (Monotonicity of  $\otimes$ )  $x \sqsubseteq x'$  and  $y \sqsubseteq y'$  imply  $x \otimes y \sqsubseteq x' \otimes y'$  ( $\forall x, x', y, y' \in \mathcal{V}_0$ );
- (Associative law)  $x \otimes (y \otimes z) = (x \otimes y) \otimes z$  ( $\forall x, y, z \in \mathcal{V}_0$ );
- (Unit law for  $\otimes$ )  $e \otimes x = x = x \otimes e$  ( $\forall x \in \mathcal{V}_0$ );
- (Commutative law for  $\otimes$ )  $x \otimes y = y \otimes x$  ( $\forall x, y \in \mathcal{V}_0$ ).

A **symmetric monoidal complete lattice (SM-CL)** is an SM-P whose underlying poset  $\mathcal{V}_0$  is a complete lattice. ■

Thus, an SM-P is nothing but a partially ordered commutative monoid. The name of the operation  $\otimes$ , tensor product, comes from the fact that in some (non-poset) symmetric monoidal categories, the corresponding operation is given by the classical tensor product of e.g., modules.

**Definition 2.2.** A **symmetric closed poset (SC-P)**  $\mathcal{V}$  is a triple  $(\mathcal{V}_0, [-, -], e)$  where

- $\mathcal{V}_0 = (\mathcal{V}_0, \sqsubseteq)$  is a poset called the **underlying poset**;
- $[-, -]: \mathcal{V}_0^{\text{op}} \times \mathcal{V}_0 \rightarrow \mathcal{V}_0$  is a binary operation on  $\mathcal{V}_0$  called the **internal-hom**;
- $e$  is an element of  $\mathcal{V}_0$  called the **unit element**;

such that the following axioms hold:

- (Monotonicity of  $[-, -]$ )  $y \sqsupseteq y'$  and  $z \sqsubseteq z'$  imply  $[y, z] \sqsubseteq [y', z']$  ( $\forall y, y', z, z' \in \mathcal{V}_0$ );
- (Composition law)  $[y, z] \sqsubseteq [[x, y], [x, z]]$  ( $\forall x, y, z \in \mathcal{V}_0$ );
- (Unit law for  $[-, -]$ )  $z = [e, z]$  ( $\forall z \in \mathcal{V}_0$ );
- (Commutative law for  $[-, -]$ )  $x \sqsubseteq [y, z] \iff y \sqsubseteq [x, z]$  ( $\forall x, y, z \in \mathcal{V}_0$ ).

A **symmetric closed complete lattice (SC-CL)** is an SC-P whose underlying poset  $\mathcal{V}_0$  is a complete lattice. ■

Note the sign of an inequality in the monotonicity axiom: the domain of the first argument of  $[-, -]$  is  $\mathcal{V}_0^{\text{op}} = (\mathcal{V}_0, \sqsupseteq)$ , the poset obtained by reversing the order  $\sqsubseteq$  in  $\mathcal{V}_0$ , thus a clause in the antecedent of the monotonicity axiom is  $y \sqsupseteq y'$ , not  $y \sqsubseteq y'$ . We will explain the mysterious name “internal-hom” later.

**Definition 2.3.** A **symmetric monoidal closed poset (SMC-P)**  $\mathcal{V}$  is a quadruple  $(\mathcal{V}_0, \otimes, [-, -], e)$  where

- the triple  $(\mathcal{V}_0, \otimes, e)$  is an SM-P;
- the triple  $(\mathcal{V}_0, [-, -], e)$  is an SC-P;

such that these two structures are related as follows:

$$\text{(Adjointness relation)} \quad x \otimes y \sqsubseteq z \iff x \sqsubseteq [y, z] \quad (\forall x, y, z \in \mathcal{V}_0).$$

A **symmetric monoidal closed complete lattice (SMC-CL)** is an SMC-P whose underlying poset  $\mathcal{V}_0$  is a complete lattice. ■

The adjointness relation, in the proof-tree notation

$$\frac{x \otimes y \sqsubseteq z}{x \sqsubseteq [y, z]}$$

is equivalent to the requirement that the unary operations  $(- \otimes y)$  and  $[y, -]$ , obtained by substituting  $y$ , form a Galois connection with  $(- \otimes y)$  the left (lower) adjoint and  $[y, -]$  the right (upper) adjoint. One can view the commutative law for  $[-, -]$  in a similar way.

Although we presented an SMC-P  $\mathcal{V}$  as a quadruple  $(\mathcal{V}_0, \otimes, [-, -], e)$ , in fact the adjointness relation is so rigid that under which the tensor product  $\otimes$  and the internal-hom  $[-, -]$  determine each other uniquely. In particular, when the underlying poset is a complete lattice, the following holds:

**Proposition 2.4.** *Let  $\mathcal{V}_0 = (\mathcal{V}_0, \sqsubseteq)$  be a complete lattice. Then the following are equivalent:*

- (i)  $(\mathcal{V}_0, \otimes, [-, -], e)$  is an SMC-CL.
- (ii)  $(\mathcal{V}_0, \otimes, e)$  is an SM-CL and for each  $y \in \mathcal{V}_0$ ,  $(- \otimes y)$  preserves suprema.
- (iii)  $(\mathcal{V}_0, [-, -], e)$  is an SC-CL and for each  $y \in \mathcal{V}_0$ ,  $[y, -]$  preserves infima.

Moreover, the data in (ii) or (iii) are sufficient to recover the whole data in (i).

*Proof.* [(i)  $\implies$  (ii)] The adjointness relation implies the following:

$$\frac{\frac{\frac{(\bigvee x_i) \otimes y \sqsubseteq z}{\bigvee x_i \sqsubseteq [y, z]}}{x_i \sqsubseteq [y, z]} (\forall i)}{x_i \otimes y \sqsubseteq z} (\forall i)$$

Thus  $(\bigvee x_i) \otimes y = \bigvee(x_i \otimes y)$ , as required.

[(ii)  $\implies$  (i)] First we claim that in order to make the adjointness relation

$$\frac{x \otimes y \sqsubseteq z}{x \sqsubseteq [y, z]}$$

hold for some binary operation  $[-, -]$ , the value of  $[y, z]$  must be the supremum of  $x$ 's satisfying  $x \otimes y \sqsubseteq z$ , namely,

$$[y, z] = \bigvee \{x \in \mathcal{V}_0 \mid x \otimes y \sqsubseteq z\}. \quad (2.1)$$

Note that the supremum certainly exists since  $\mathcal{V}_0$  is assumed to be a complete lattice. Suppose  $[-, -]$  is a binary operation on  $\mathcal{V}_0$  satisfying the adjointness relation:  $[\bigvee \{x \in \mathcal{V}_0 \mid x \otimes y \sqsubseteq z\} \sqsubseteq [y, z]]$  For every  $x \in \{x \in \mathcal{V}_0 \mid x \otimes y \sqsubseteq z\}$ ,  $x \sqsubseteq [y, z]$  holds by the adjointness relation, so

$$\bigvee \{x \in \mathcal{V}_0 \mid x \otimes y \sqsubseteq z\} \sqsubseteq [y, z].$$

$[y, z] \sqsubseteq \bigvee \{x \in \mathcal{V}_0 \mid x \otimes y \sqsubseteq z\}$   $[y, z] \sqsubseteq [y, z]$  implies by the adjointness relation that  $[y, z] \otimes y \sqsubseteq z$  and hence  $[y, z] \in \{x \in \mathcal{V}_0 \mid x \otimes y \sqsubseteq z\}$ , thus

$$[y, z] \sqsubseteq \bigvee \{x \in \mathcal{V}_0 \mid x \otimes y \sqsubseteq z\}.$$

Therefore (2.1) is the only possible definition for the operation  $[-, -]$ . Conversely, (2.1) implies the adjointness relation:

$[x \otimes y \sqsubseteq z \implies x \sqsubseteq [y, z]]$  Since  $x \otimes y \sqsubseteq z$  implies  $x \in \{x \in \mathcal{V}_0 \mid x \otimes y \sqsubseteq z\}$ ,

$$x \sqsubseteq \bigvee \{x \in \mathcal{V}_0 \mid x \otimes y \sqsubseteq z\} = [y, z].$$

$[x \sqsubseteq [y, z] \implies x \otimes y \sqsubseteq z]$  We first show that  $[y, z] \otimes y \sqsubseteq z$ :

$$\begin{aligned} [y, z] \otimes y &= \bigvee \{x \in \mathcal{V}_0 \mid x \otimes y \sqsubseteq z\} \otimes y \\ &= \bigvee \{x \otimes y \mid x \otimes y \sqsubseteq z\} \\ &\sqsubseteq z. \end{aligned}$$

We used the assumption that  $(- \otimes y)$  preserves suprema. Using this fact and the monotonicity of  $\otimes$ , we conclude

$$x \otimes y \sqsubseteq [y, z] \otimes y \sqsubseteq z.$$

We now check that the operation  $[-, -]$  defined by (2.1) satisfies the axioms required for SC-Ps.

[Monotonicity of  $[-, -]$ ] Suppose  $y \sqsupseteq y'$  and  $z \sqsubseteq z'$  hold. Then,

$$\{x \in \mathcal{V}_0 \mid x \otimes y \sqsubseteq z\} \subseteq \{x \in \mathcal{V}_0 \mid x \otimes y' \sqsubseteq z'\}$$

because  $x \otimes y \sqsubseteq z$  implies  $x \otimes y' \sqsubseteq x \otimes y \sqsubseteq z \sqsubseteq z'$ . Therefore

$$[y, z] = \bigvee \{x \in \mathcal{V}_0 \mid x \otimes y \sqsubseteq z\} \sqsubseteq \bigvee \{x \in \mathcal{V}_0 \mid x \otimes y' \sqsubseteq z'\} = [y', z'],$$

as required.

[Composition law] The claim is  $[y, z] \sqsubseteq [[x, y], [x, z]]$ , namely,

$$\bigvee \{w \in \mathcal{V}_0 \mid w \otimes y \sqsubseteq z\} \sqsubseteq \bigvee \{w \in \mathcal{V}_0 \mid w \otimes [x, y] \sqsubseteq [x, z]\}.$$

Thus it suffices to show

$$\begin{aligned} \{w \in \mathcal{V}_0 \mid w \otimes y \sqsubseteq z\} &\subseteq \{w \in \mathcal{V}_0 \mid w \otimes [x, y] \sqsubseteq [x, z]\} \\ &= \{w \mid w \otimes \bigvee \{v \mid v \otimes x \sqsubseteq y\} \sqsubseteq \bigvee \{v \mid v \otimes x \sqsubseteq z\}\} \\ &= \{w \mid \bigvee \{w \otimes v \mid v \otimes x \sqsubseteq y\} \sqsubseteq \bigvee \{v \mid v \otimes x \sqsubseteq z\}\} \end{aligned}$$

(here we used the assumption that  $(w \otimes -) = (- \otimes w)$  preserves suprema). So let us suppose  $w \otimes y \sqsubseteq z$  and aim to show that

$$\{w \otimes v \mid v \otimes x \sqsubseteq y\} \subseteq \{v \mid v \otimes x \sqsubseteq z\}.$$

This holds because for  $v$  with  $v \otimes x \sqsubseteq y$ ,

$$(w \otimes v) \otimes x = w \otimes (v \otimes x) \sqsubseteq w \otimes y \sqsubseteq z$$

holds.

[Unit law for  $[-, -]$ ] The claim is  $z = [e, z]$ . This reduces to

$$\begin{aligned} z &= \bigvee \{x \in \mathcal{V}_0 \mid x \otimes e \sqsubseteq z\} \\ &= \bigvee \{x \in \mathcal{V}_0 \mid x \sqsubseteq z\}, \end{aligned}$$

a trivial equality.

[Commutative law for  $[-, -]$ ] The claim is  $x \sqsubseteq [y, z] \iff y \sqsubseteq [x, z]$ , which follows from the following proof tree:

$$\frac{\frac{\frac{x \sqsubseteq [y, z]}{\underline{\underline{x \otimes y \sqsubseteq z}}}}{\underline{\underline{y \otimes x \sqsubseteq z}}}}{y \sqsubseteq [x, z]}$$

[(i)  $\implies$  (iii)] The adjointness relation implies the following:

$$\frac{\frac{\frac{x \sqsubseteq [y, \bigwedge z_i]}{\underline{\underline{x \otimes y \sqsubseteq \bigwedge z_i}}}}{\underline{\underline{x \otimes y \sqsubseteq z_i}} \quad (\forall i)}}{x \sqsubseteq [y, z_i] \quad (\forall i)}$$

Thus  $[y, \bigwedge z_i] = \bigwedge [y, z_i]$ , as required.

[(iii)  $\implies$  (i)] First we claim that in order to make the adjointness relation

$$\frac{x \otimes y \sqsubseteq z}{\underline{\underline{x \sqsubseteq [y, z]}}$$

hold for some binary operation  $\otimes$ , the value of  $x \otimes y$  must be the infimum of  $z$ 's satisfying  $x \sqsubseteq [y, z]$ , namely,

$$x \otimes y = \bigwedge \{z \in \mathcal{V}_0 \mid x \sqsubseteq [y, z]\}. \quad (2.2)$$

Note that the infimum certainly exists since  $\mathcal{V}_0$  is assumed to be a complete lattice. Suppose  $\otimes$  is a binary operation on  $\mathcal{V}_0$  satisfying the adjointness relation:  $[x \otimes y \sqsubseteq \bigwedge \{z \in \mathcal{V}_0 \mid x \sqsubseteq [y, z]\}]$  For every  $z \in \{z \in \mathcal{V}_0 \mid x \sqsubseteq [y, z]\}$ ,  $x \otimes y \sqsubseteq z$  holds by the adjointness relation, so

$$x \otimes y \sqsubseteq \bigwedge \{z \in \mathcal{V}_0 \mid x \sqsubseteq [y, z]\}.$$

$[\bigwedge \{z \in \mathcal{V}_0 \mid x \sqsubseteq [y, z]\} \sqsubseteq x \otimes y]$   $x \otimes y \sqsubseteq x \otimes y$  implies by the adjointness relation that  $x \sqsubseteq [y, x \otimes y]$  and hence  $x \otimes y \in \{z \in \mathcal{V}_0 \mid x \sqsubseteq [y, z]\}$ , thus

$$\bigwedge \{z \in \mathcal{V}_0 \mid x \sqsubseteq [y, z]\} \sqsubseteq x \otimes y.$$

Therefore (2.2) is the only possible definition for the operation  $\otimes$ . Conversely, (2.2) implies the adjointness relation:

$[x \sqsubseteq [y, z] \implies x \otimes y \sqsubseteq z]$  Since  $x \sqsubseteq [y, z]$  implies  $z \in \{z \in \mathcal{V}_0 \mid x \sqsubseteq [y, z]\}$ ,

$$x \otimes y = \bigwedge \{z \in \mathcal{V}_0 \mid x \sqsubseteq [y, z]\} \sqsubseteq z.$$

$[x \otimes y \sqsubseteq z \implies x \sqsubseteq [y, z]]$  We first show that  $x \sqsubseteq [y, x \otimes y]$ :

$$\begin{aligned} [y, x \otimes y] &= [y, \bigwedge \{z \in \mathcal{V}_0 \mid x \sqsubseteq [y, z]\}] \\ &= \bigwedge \{[y, z] \mid x \sqsubseteq [y, z]\} \\ &\supseteq x. \end{aligned}$$

We used the assumption that  $[y, -]$  preserves infima. Using this fact and the monotonicity of  $[-, -]$ , we conclude

$$x \sqsubseteq [y, x \otimes y] \sqsubseteq [y, z].$$

We now check that the operation  $\otimes$  defined by (2.2) satisfies the axioms required for SM-Pos.

[Monotonicity of  $\otimes$ ] Suppose  $x \sqsubseteq x'$  and  $y \sqsubseteq y'$  hold. Then,

$$\{z \in \mathcal{V}_0 \mid x' \sqsubseteq [y', z]\} \subseteq \{z \in \mathcal{V}_0 \mid x \sqsubseteq [y, z]\}$$

because  $x' \sqsubseteq [y', z]$  implies  $x \sqsubseteq x' \sqsubseteq [y', z] \sqsubseteq [y, z]$ . Therefore

$$x \otimes y = \bigwedge \{z \in \mathcal{V}_0 \mid x \sqsubseteq [y, z]\} \sqsubseteq \bigwedge \{z \in \mathcal{V}_0 \mid x' \sqsubseteq [y', z]\} = x' \otimes y',$$

as required.

[Associative law] The claim is  $x \otimes (y \otimes z) = (x \otimes y) \otimes z$ , namely,

$$\bigwedge \{w \in \mathcal{V}_0 \mid x \sqsubseteq [y \otimes z, w]\} = \bigwedge \{w \in \mathcal{V}_0 \mid x \otimes y \sqsubseteq [z, w]\}.$$

We prove this by showing  $\{w \in \mathcal{V}_0 \mid x \sqsubseteq [y \otimes z, w]\} = \{w \in \mathcal{V}_0 \mid x \otimes y \sqsubseteq [z, w]\}$ , or more explicitly,

$$\{w \mid x \sqsubseteq [\bigwedge \{v \mid y \sqsubseteq [z, v]\}, w]\} = \{w \mid \bigwedge \{v \mid x \sqsubseteq [y, v]\} \sqsubseteq [z, w]\}. \quad (2.3)$$



Note that by the commutative law for  $[-, -]$ , (2.3) is equivalent to

$$\{w \mid \bigwedge\{v \mid y \sqsubseteq [z, v]\} \sqsubseteq [x, w]\} = \{w \mid z \sqsubseteq [\bigwedge\{v \mid x \sqsubseteq [y, v]\}, w]\}. \quad (2.4)$$

[(LHS) $\sqsubseteq$ (RHS)] We show this in the form of (2.3). It suffices to show

$$\begin{aligned} x \sqsubseteq [\bigwedge\{v \mid y \sqsubseteq [z, v]\}, w] &\implies [z, w] \in \{v \mid x \sqsubseteq [y, v]\} \\ &(\iff x \sqsubseteq [y, [z, w]]) \end{aligned}$$

and this follows from

$$\begin{aligned} [\bigwedge\{v \mid y \sqsubseteq [z, v]\}, w] &\sqsubseteq [[z, \bigwedge\{v \mid y \sqsubseteq [z, v]\}], [z, w]] \\ &= [\bigwedge\{[z, v] \mid y \sqsubseteq [z, v]\}, [z, w]] \\ &\sqsubseteq [y, [z, w]], \end{aligned}$$

where the first inequality is an instance of the composition law, the equality follows from the inf-preserving property of  $[z, -]$ , and the last inequality from  $\bigwedge\{[z, v] \mid y \sqsubseteq [z, v]\} \supseteq y$  and the monotonicity of  $[-, -]$ .

[(LHS) $\supseteq$ (RHS)] In this case, we refer to the form (2.4). Now, a sufficient condition is

$$\begin{aligned} z \sqsubseteq [\bigwedge\{v \mid x \sqsubseteq [y, v]\}, w] &\implies [x, w] \in \{v \mid y \sqsubseteq [z, v]\} \\ &(\iff y \sqsubseteq [z, [x, w]] \\ &\iff z \sqsubseteq [y, [x, w]]) \end{aligned}$$

and this follows from

$$\begin{aligned} [\bigwedge\{v \mid x \sqsubseteq [y, v]\}, w] &\sqsubseteq [[x, \bigwedge\{v \mid x \sqsubseteq [y, v]\}], [x, w]] \\ &= [\bigwedge\{[x, v] \mid x \sqsubseteq [y, v]\}, [x, w]] \\ &= [\bigwedge\{[x, v] \mid y \sqsubseteq [x, v]\}, [x, w]] \\ &\sqsubseteq [y, [x, w]], \end{aligned}$$

where the first inequality is an instance of the composition law, the first equality follows from the inf-preserving property of  $[z, -]$ , and the last inequality from  $\bigwedge\{[x, v] \mid y \sqsubseteq [x, v]\} \supseteq y$  and the monotonicity of  $[-, -]$ .

[Unit law for  $\otimes$ ] The claim is  $e \otimes x = x = x \otimes e$ . Assuming the commutative law for  $\otimes$  proved below, it suffices to show  $x = x \otimes e$ , namely,

$$\begin{aligned} x &= \bigwedge\{z \in \mathcal{V}_0 \mid x \sqsubseteq [e, z]\} \\ &= \bigwedge\{z \in \mathcal{V}_0 \mid x \sqsubseteq z\}, \end{aligned}$$

which obviously holds.

[Commutative law for  $\otimes$ ] The claim is  $x \otimes y = y \otimes x$ , which follows from the following proof tree:

$$\frac{\frac{\frac{x \otimes y \sqsubseteq z}{x \sqsubseteq [y, z]}}{y \sqsubseteq [x, z]}}{y \otimes x \sqsubseteq z}$$

where  $z$  is an arbitrary element of  $\mathcal{V}_0$ . □

By virtue of this proposition, we will usually check only the condition (ii) to prove that a particular system is an SMC-CL. However, we will not ignore internal-hom's, since writing them down explicitly often clarifies the situation.

We complete the limit-preserving properties of  $\otimes$  and  $[-, -]$  of SMC-CLs by the following proposition:

**Proposition 2.5.** *Let  $\mathcal{V} = (\mathcal{V}_0, \otimes, [-, -], e)$  be an SMC-CL. Then the following hold:*

- (i) *For each  $x \in \mathcal{V}_0$ ,  $(x \otimes -)$  preserves suprema.*
- (ii) *For each  $z \in \mathcal{V}_0$ ,  $[-, z]$  turns suprema into infima.*

*Proof.* [(i)] The commutativity of  $\otimes$  and sup-preserving property of  $(- \otimes x)$  imply the following:

$$\frac{\frac{x \otimes (\bigvee y_i) \sqsubseteq z}{(\bigvee y_i) \otimes x \sqsubseteq z}}{\frac{y_i \otimes x \sqsubseteq z \quad (\forall i)}{x \otimes y_i \sqsubseteq z \quad (\forall i)}}$$

Thus  $x \otimes (\bigvee y_i) = \bigvee (x \otimes y_i)$ , as required.

[(ii)] The commutativity of  $[-, -]$  implies the following:

$$\frac{\frac{x \sqsubseteq [(\bigvee y_i), z]}{\bigvee y_i \sqsubseteq [x, z]}}{\frac{y_i \sqsubseteq [x, z] \quad (\forall i)}{x \sqsubseteq [y_i, z] \quad (\forall i)}}$$

Thus  $[(\bigvee y_i), z] = \bigwedge [y_i, z]$ , as required. □

Now we proceed to see some examples. We begin with a toy example.

**Example 2.6** ([5]). The quadruple  $\mathbf{2} = (\{\text{true}, \text{false}\}, \&, \supset, \text{true})$  is defined as follows:

- $\{\text{true}, \text{false}\} = (\{\text{true}, \text{false}\}, \vdash)$  is the two-element poset of truth values ordered by entailment  $\vdash$ ; see the table below for detail (the symbol  $\circ$  denotes that the relation holds, and  $\times$  that the relation does not hold):

$x \vdash y$		$y$	
		true	false
$x$	true	$\circ$	$\times$
	false	$\circ$	$\circ$

- $\&$  is the binary operation defined as the usual conjunction, as in the following table:

$x \& y$		$y$	
		true	false
$x$	true	true	false
	false	false	false

- $\supset$  is the binary operation defined as the usual implication:

$x \supset y$		$y$	
		true	false
$x$	true	true	false
	false	true	true

One can easily verify that  $\mathbf{2}$  is an SMC-CL. ■

The current thesis is largely indebted to William Lawvere's 1973 paper [5]. In this paper, he observed that metric spaces are a special kind of enriched categories, using the following category (in fact, poset)  $\overline{\mathbb{R}}_+$ :

**Example 2.7.** The quadruple  $\overline{\mathbb{R}}_+ = (\mathbb{R}_+ \cup \{\infty\}, +, \dot{-}, 0)$  is defined as follows:

- $\mathbb{R}_+ \cup \{\infty\} = (\mathbb{R}_+ \cup \{\infty\}, \geq)$  is the set of nonnegative real numbers  $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$  with an additional element  $\infty$ , ordered by the *opposite*  $\geq$  of the usual ordering  $\leq$  (extended from  $\mathbb{R}_+$  to  $\mathbb{R}_+ \cup \{\infty\}$  in an obvious way). More explicitly, supposing  $s, t \in \mathbb{R}_+$  (thus we already know whether  $s \geq t$  holds or not), the order is given by the following table:

$x \geq y$		$y$	
		$t$	$\infty$
$x$	$s$	$s \geq t$	$\times$
	$\infty$	$\circ$	$\circ$

- $+$  is the binary operation defined as the natural extension of the usual addition:

$x + y$		$y$	
		$t$	$\infty$
$x$	$s$	$s + t$	$\infty$
	$\infty$	$\infty$	$\infty$

- $\dot{-}$  is the binary operation called the truncated subtraction. To define this, we first set  $t \dot{-} s = \max\{t - s, 0\}$  for  $s, t \in \mathbb{R}_+$ . Then the domain of  $\dot{-}$  is extended to  $\mathbb{R}_+ \cup \{\infty\}$  as follows (note the order of the arguments; we set  $[x, y] = y \dot{-} x$ ):

$y \dot{-} x$		$y$	
		$t$	$\infty$
$x$	$s$	$t \dot{-} s$	$\infty$
	$\infty$	$0$	$0$

Then it turns out that  $\overline{\mathbb{R}}_+$  is an SMC-CL.

The reader may feel that the definition of the extended operations is somewhat arbitrary, especially in the case  $\infty \dot{-} \infty = 0$ . We first note that, contrary to such a view, *in order*  $\overline{\mathbb{R}}_+$  *to be an SMC-CL, the extensions of  $+$  and  $\dot{-}$  are completely determined as soon as we extend the order  $\geq$ .* To see this, recall from Propositions 2.4 and 2.5 that necessary conditions for the quadruple  $\overline{\mathbb{R}}_+ = (\mathbb{R}_+ \cup \{\infty\}, +, \dot{-}, 0)$  to be an SMC-CL are that

$$x + (-) \text{ preserves suprema for each } x \in \mathbb{R}_+ \cup \{\infty\}; \quad (2.5)$$

$$(-) + y \text{ preserves suprema for each } y \in \mathbb{R}_+ \cup \{\infty\}; \quad (2.6)$$

$$(-) \dot{+} x \text{ preserves infima for each } x \in \mathbb{R}_+ \cup \{\infty\}; \quad (2.7)$$

$$y \dot{+} (-) \text{ turns suprema into infima for each } y \in \mathbb{R}_+ \cup \{\infty\}. \quad (2.8)$$

Now that we are working with the opposite ordering  $\geq$ , which means our supremum  $\bigvee$  corresponds to the usual infimum (for which we use a different symbol  $\inf$ ) and dually, we obtain the following for each  $x, y \in \mathbb{R}_+ \cup \{\infty\}$  and  $s \in \mathbb{R}_+$ :

$$x + \infty = x + \inf \emptyset \stackrel{(2.5)}{=} \inf \emptyset = \infty;$$

$$\infty + y = \inf \emptyset + y \stackrel{(2.6)}{=} \inf \emptyset = \infty;$$

$$\infty \dot{+} s = \sup_{t \in \mathbb{R}_+} \{t\} \dot{+} s \stackrel{(2.7)}{=} \sup_{t \in \mathbb{R}_+} \{t \dot{+} s\} = \infty;$$

$$y \dot{+} \infty = y \dot{+} \inf \emptyset \stackrel{(2.8)}{=} \sup \emptyset = 0.$$

These specify the extensions.

We then show that  $\overline{\mathbb{R}}_+$  is indeed an SMC-CL, following the condition (ii) of Proposition 2.4.

[ $\mathbb{R}_+ \cup \{\infty\}$  is a complete lattice] A well-known fact.

[Monotonicity of  $+$ ] Suppose  $x \geq x'$  and  $y \geq y'$  hold. If all of  $x, x', y, y'$  are elements of  $\mathbb{R}_+$ , then  $x + y \geq x' + y'$  holds. Otherwise, either  $x$  or  $y$  is  $\infty$ ; thus  $x + y = \infty$  and again  $x + y \geq x' + y'$  holds.

[Associative law] The claim is  $x + (y + z) = (x + y) + z$ . If all of  $x, y, z$  are elements of  $\mathbb{R}_+$ , the claim reduces to an elementary fact. Otherwise, one of  $x, y, z$  is  $\infty$ , and both of  $x + (y + z)$  and  $(x + y) + z$  are equal to  $\infty$ .

[Unit law] The claim is  $0 + x = x = x + 0$ . If  $x \in \mathbb{R}_+$ , it certainly holds; if not,  $0 + \infty = \infty = \infty + 0$  and again it holds.

[Commutative law] The claim is  $x + y = y + x$ . If  $x, y \in \mathbb{R}_+$ , it holds. Otherwise, both sides are equal to  $\infty$ .

[ $(-) + y$  preserves suprema] If  $y = \infty$ , the function  $(-) + y$  constantly returns  $\infty$ , which is also the least element (the supremum of the empty set). Thus it preserves all suprema.

Suppose  $y \in \mathbb{R}_+$  and take an arbitrary family  $x_i \in \mathbb{R}_+ \cup \{\infty\}$  ( $i \in I$ ). We aim to prove

$$\inf_{i \in I} \{x_i\} + y = \inf_{i \in I} \{x_i + y\}. \quad (2.9)$$

If  $\inf_{i \in I} \{x_i\} = \infty$ , then for all  $i \in I$ ,  $x_i = \infty$ . Therefore for all  $i \in I$ ,  $x_i + y = \infty$  and both sides of (2.9) evaluate to  $\infty$ .

If  $\inf_{i \in I} \{x_i\} \in \mathbb{R}_+$ , then there exists  $i \in I$  with  $x_i \in \mathbb{R}_+$ , and we may simply omit those  $x_i$ 's with  $x_i = \infty$ , obtaining a new family  $x_i \in \mathbb{R}_+$  ( $i \in I \neq \emptyset$ ) of nonnegative real numbers. Now (2.9) essentially reduces to the continuity of  $+$ .

Finally, we show that  $\dot{+}$  is indeed the internal-hom, by checking the adjointness relation.

[ $x + y \geq z \implies x \geq z \dot{+} y$ ] The monotonicity of  $\dot{+}$  is easily verified. Therefore,  $x + y \geq z$  implies  $(x + y) \dot{+} y \geq z \dot{+} y$ . So it suffices to show  $x \geq (x + y) \dot{+} y$ , which follows from the table below:

$(x + y) \dot{-} y$		$y$	
		$t$	$\infty$
$x$	$s$	$s$	$0$
	$\infty$	$\infty$	$0$

$[x \geq z \dot{-} y \implies x + y \geq z]$  By the monotonicity of  $+$ ,  $x \geq z \dot{-} y$  implies  $x + y \geq (z \dot{-} y) + y$ . So it suffices to show  $(z \dot{-} y) + y \geq z$ , which follows from the table below:

$(z \dot{-} y) + y$		$z$	
		$t$	$\infty$
$y$	$s$	$\max\{s, t\}$	$\infty$
	$\infty$	$\infty$	$\infty$

■

Several variants of  $\overline{\mathbb{R}}_+$  follow. First we observe that a similar construction works when  $\mathbb{R}$  is replaced by  $\mathbb{Z}$ .

**Example 2.8.** The quadruple  $\overline{\mathbb{Z}}_+ = (\mathbb{Z}_+ \cup \{\infty\}, +, \dot{-}, 0)$  is defined as follows:

- $\mathbb{Z}_+ \cup \{\infty\} = (\mathbb{Z}_+ \cup \{\infty\}, \geq)$  is the set of nonnegative integers  $\mathbb{Z}_+ = \{x \in \mathbb{Z} \mid x \geq 0\}$  with an additional element  $\infty$ , ordered by  $\geq$ , the order relation similar to that on  $\mathbb{R}_+ \cup \{\infty\}$ . For  $s, t \in \mathbb{Z}_+$ , it is given as follows:

$x \geq y$		$y$	
		$t$	$\infty$
$x$	$s$	$s \geq t$	$\times$
	$\infty$	$\circ$	$\circ$

- $+$  is an extension of the usual addition:

$x + y$		$y$	
		$t$	$\infty$
$x$	$s$	$s + t$	$\infty$
	$\infty$	$\infty$	$\infty$

- $\dot{-}$  is an extension of the operation defined as  $t \dot{-} s = \max\{t - s, 0\}$ :

$y \dot{-} x$		$y$	
		$t$	$\infty$
$x$	$s$	$t \dot{-} s$	$\infty$
	$\infty$	$0$	$0$

$\overline{\mathbb{Z}}_+$  is an SMC-CL; its proof is obtained by a slight modification to that for  $\overline{\mathbb{R}}_+$ . Again we have no choice for extensions of operations  $+$  and  $\dot{-}$ . ■

The reader may notice that the cases for  $\mathbb{Z}$  and for  $\mathbb{R}$  are almost completely parallel; such a phenomenon occurs frequently in what follows. We henceforth let the symbol  $\mathbb{K}$  denote either  $\mathbb{Z}$  or  $\mathbb{R}$  and treat both cases simultaneously whenever convenient. This convention lasts throughout the thesis.

Next example introduces the most important enriching posets in this thesis. The definition when  $\mathbb{K} = \mathbb{R}$  appears in [6].

**Example 2.9.** The quadruple  $\overline{\mathbb{K}} = (\mathbb{K} \cup \{-\infty, \infty\}, +, -, 0)$  is defined as follows:

- $\mathbb{K} \cup \{-\infty, \infty\} = (\mathbb{K} \cup \{-\infty, \infty\}, \geq)$  is the set  $\mathbb{K}$  with two additional elements  $-\infty$  and  $\infty$ , ordered by  $\geq$ . Supposing  $s, t \in \mathbb{K}$ , the relation is specified by the following table:

$x \geq y$		$y$	
		$-\infty$	$\infty$
$-\infty$		○	×
$x$	$s$	○	$s \geq t$
$\infty$		○	○

- $+$  is the binary operation defined as an extension of the usual addition:

$x + y$		$y$	
		$-\infty$	$\infty$
$-\infty$		$-\infty$	$\infty$
$x$	$s$	$-\infty$	$s + t$
$\infty$		$\infty$	$\infty$

- $-$  is the binary operation defined as an extension of the usual subtraction (the order of the arguments is similar to that of the truncated subtraction, namely,  $[x, y] = y - x$ ):

$y - x$		$y$	
		$-\infty$	$\infty$
$-\infty$		$-\infty$	$\infty$
$x$	$s$	$-\infty$	$t - s$
$\infty$		$-\infty$	$-\infty$

We claim that  $\overline{\mathbb{K}}$  is an SMC-CL.

First we see that the extensions of  $+$  and  $-$  are automatically derived by a method similar to one we used in the  $\overline{\mathbb{R}}_+$ -case. Let us write down the results of Propositions 2.4 and 2.5 concerning necessary conditions for the quadruple  $\overline{\mathbb{K}} = (\mathbb{K} \cup \{-\infty, \infty\}, +, -, 0)$  to be an SMC-CL:

$$x + (-) \text{ preserves suprema for each } x \in \mathbb{K} \cup \{-\infty, \infty\}; \quad (2.10)$$

$$(-) + y \text{ preserves suprema for each } y \in \mathbb{K} \cup \{-\infty, \infty\}; \quad (2.11)$$

$$(-) - x \text{ preserves infima for each } x \in \mathbb{K} \cup \{-\infty, \infty\}; \quad (2.12)$$

$$y - (-) \text{ turns suprema into infima for each } y \in \mathbb{K} \cup \{-\infty, \infty\}. \quad (2.13)$$

Since the ordering  $\geq$  is again the reverse of the usual one, we introduce the notation in which we denote suprema and infima with respect to  $\geq$  by  $\bigvee$  and  $\bigwedge$ , whereas infima and suprema with respect to the usual ordering  $\leq$  by  $\inf$  and  $\sup$ , respectively. We obtain, for each  $x, y \in \mathbb{K} \cup \{-\infty, \infty\}$  and  $s, t \in \mathbb{K}$ , the following:

$$x + \infty = x + \inf \emptyset \stackrel{(2.10)}{=} \inf \emptyset = \infty;$$

$$s + (-\infty) = s + \inf_{t \in \mathbb{K}} \{t\} \stackrel{(2.10)}{=} \inf_{t \in \mathbb{K}} \{s + t\} = -\infty; \quad (2.14)$$

$$\infty + y = \inf \emptyset + y \stackrel{(2.11)}{=} \inf \emptyset = \infty;$$

$$(-\infty) + t = \inf_{s \in \mathbb{K}} \{s\} + t \stackrel{(2.11)}{=} \inf_{s \in \mathbb{K}} \{s + t\} = -\infty;$$

$$\infty - s = \sup_{t \in \mathbb{K}} \{t\} - s \stackrel{(2.12)}{=} \sup_{t \in \mathbb{K}} \{t - s\} = \infty; \quad (2.15)$$

$$(-\infty) - x = \sup \emptyset - x \stackrel{(2.12)}{=} \sup \emptyset = -\infty;$$

$$y - \infty = y - \inf \emptyset \stackrel{(2.13)}{=} \sup \emptyset = -\infty;$$

$$t - (-\infty) = t - \inf_{s \in \mathbb{K}} \{s\} \stackrel{(2.13)}{=} \sup_{s \in \mathbb{K}} \{t - s\} = \infty.$$

Finally, we define two remaining cases:

$$(-\infty) + (-\infty) = \inf_{s \in \mathbb{K}} \{s\} + (-\infty) \stackrel{(2.11)}{=} \inf_{s \in \mathbb{K}} \{s + (-\infty)\} \stackrel{(2.14)}{=} \inf_{s \in \mathbb{K}} \{-\infty\} = -\infty;$$

$$\infty - (-\infty) = \infty - \inf_{s \in \mathbb{K}} \{s\} \stackrel{(2.13)}{=} \sup_{s \in \mathbb{K}} \{\infty - s\} \stackrel{(2.15)}{=} \sup_{s \in \mathbb{K}} \{\infty\} = \infty.$$

Now the extensions have completed.

The proof that  $\overline{\mathbb{K}}$  is an SMC-CL is simply a little bigger version of the case analyses we did in Example 2.7.  $\blacksquare$

The last example below is of a bit different flavor:

**Example 2.10.** Let  $L = (L_0, \supset)$  be a complete Heyting algebra, that is, a complete lattice  $L_0 = (L_0, \sqsubseteq)$  equipped with a binary operation  $\supset: L_0^{\text{op}} \times L_0 \rightarrow L_0$  called the implication, with the following property:

$$\frac{x \wedge y \sqsubseteq z}{x \sqsubseteq y \supset z}$$

Then, the quadruple  $L = (L_0, \wedge, \supset, \top)$ , where  $\top$  is the largest element of  $L_0$ , is an SMC-CL. Such SMC-CLs (in which the tensor product is given by the meet  $\wedge$ ) are called Cartesian closed complete lattices, which are nevertheless essentially the same as complete Heyting algebras.

As a special case, we have the following Cartesian closed complete lattices: the quadruple  $\overline{\mathbb{K}}_+^{\text{Cart}} = (\mathbb{K}_+ \cup \{\infty\}, \max, \supset, 0)$ , where  $\mathbb{K}_+ \cup \{\infty\}$  is ordered by  $\geq$ , and  $\supset$  (the order of arguments is  $[x, y] = x \supset y$ ) is defined as

$$x \supset y = \begin{cases} 0 & (x \geq y) \\ y & (x < y). \end{cases}$$

This appears in [5] when  $\mathbb{K} = \mathbb{R}$ . Just as  $\overline{\mathbb{K}}_+$  will be used to generalize metric spaces, we will later use  $\overline{\mathbb{K}}_+^{\text{Cart}}$  to generalize *ultrametric spaces*.  $\blacksquare$

## 2.2 Enriched Categories

Every poset-enriched category has a particular enriching poset  $\mathcal{V}$  which is used to describe its structure. Although we introduced the notion of SMC-CLs in the previous section,

in the early stages of the theory of poset-enriched categories, it suffices to require the enriching poset  $\mathcal{V}$  to be an SM-P.

**Definition 2.11.** Let  $\mathcal{V} = (\mathcal{V}_0, \otimes, e)$  be an SM-P. A  $\mathcal{V}$ -category  $\mathcal{A}$  is a pair  $(\text{Ob}(\mathcal{A}), \text{Hom}_{\mathcal{A}})$  where

- $\text{Ob}(\mathcal{A})$  is the set of **objects** of  $\mathcal{A}$ ;
- $\text{Hom}_{\mathcal{A}}$  is a map which assigns for each pair  $a, b$  of objects of  $\mathcal{A}$ , an element  $\text{Hom}_{\mathcal{A}}(a, b)$  of  $\mathcal{V}_0$  called its **hom-object**;

such that the following axioms hold:

$$\begin{aligned} & \text{(Composition law)} \quad \text{Hom}_{\mathcal{A}}(a, b) \otimes \text{Hom}_{\mathcal{A}}(b, c) \sqsubseteq \text{Hom}_{\mathcal{A}}(a, c) \quad (\forall a, b, c \in \text{Ob}(\mathcal{A})); \\ & \text{(Identity law)} \quad e \sqsubseteq \text{Hom}_{\mathcal{A}}(a, a) \quad (\forall a \in \text{Ob}(\mathcal{A})). \end{aligned}$$

■

The definition is best explained by the various examples given below. Note the apparent conflict of the term “composition law”; an axiom in the definition of SC-Ps also bears the same name. However, we will see later that the composition law for  $\mathcal{V}$  can be seen as an instance of the composition law for  $\mathcal{V}$ -categories, thus justifying the terminology.

**Example 2.12.** A **2-category** is nothing but a *preordered set*. In fact, a **2-category**  $\mathcal{A}$  is a pair  $(\text{Ob}(\mathcal{A}), \preceq_{\mathcal{A}})$  where

- $\text{Ob}(\mathcal{A})$  is a set;
- $\preceq_{\mathcal{A}}$  takes two elements  $a, b$  of  $\text{Ob}(\mathcal{A})$  and returns a truth value, which can be interpreted to denote whether the relation  $(a \preceq_{\mathcal{A}} b)$  holds or not;

such that the following axioms hold:

$$\begin{aligned} & \text{(Composition law)} \quad (a \preceq_{\mathcal{A}} b) \& (b \preceq_{\mathcal{A}} c) \vdash (a \preceq_{\mathcal{A}} c) \quad (\forall a, b, c \in \text{Ob}(\mathcal{A})); \\ & \text{(Identity law)} \quad \text{true} \vdash (a \preceq_{\mathcal{A}} a) \quad (\forall a \in \text{Ob}(\mathcal{A})). \end{aligned}$$

Of course, the composition law requires transitivity of  $\preceq_{\mathcal{A}}$  and the identity law the reflexivity of  $\preceq_{\mathcal{A}}$ . ■

**Example 2.13.** An  $\overline{\mathbb{R}}_+$ -category, or a **Lawvere metric space**,  $\mathcal{A}$  is a pair  $(\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  where

- $\text{Ob}(\mathcal{A})$  is the set of **points** of  $\mathcal{A}$ ;
- $d_{\mathcal{A}}$ , called the **distance function** of  $\mathcal{A}$ , takes two points  $a, b$  of  $\mathcal{A}$  and returns a value  $d_{\mathcal{A}}(a, b) \in \mathbb{R}_+ \cup \{\infty\}$ ;

such that the following axioms hold:

$$\begin{aligned} & \text{(Composition law)} \quad d_{\mathcal{A}}(a, b) + d_{\mathcal{A}}(b, c) \geq d_{\mathcal{A}}(a, c) \quad (\forall a, b, c \in \text{Ob}(\mathcal{A})); \\ & \text{(Identity law)} \quad 0 \geq d_{\mathcal{A}}(a, a) \quad (\forall a \in \text{Ob}(\mathcal{A})). \end{aligned}$$

Since  $0 \geq x$  if and only if  $x = 0$ , the identity law is equivalent to  $d_{\mathcal{A}}(a, a) = 0$ . Lawvere remarked in [5] that every metric space is an  $\overline{\mathbb{R}}_+$ -category. In fact, the composition law is nothing but the *triangle inequality*, and the identity law is also a usual requirement for the distance functions of metric spaces.

Note that the notion of Lawvere metric spaces generalizes that of metric spaces in the following points:

- the distance function  $d_{\mathcal{A}}$  is not necessarily symmetric, i.e., there may exist points  $a, b$  of  $\mathcal{A}$  with  $d_{\mathcal{A}}(a, b) \neq d_{\mathcal{A}}(b, a)$ ;
- the values of distance function can attain  $\infty$ ;



- the points  $a, b$  of  $\mathcal{A}$  with  $d_{\mathcal{A}}(a, b) = 0$ , or even  $d_{\mathcal{A}}(a, b) = d_{\mathcal{A}}(b, a) = 0$ , may be distinct. ■

**Example 2.14.**  $\overline{\mathbb{Z}}_+$ -categories are defined similarly. One can regard them as a generalization of  $\mathbb{Z}_+$ -valued (or discrete) metric spaces. ■

The following classes of enriched categories will be the main field of our study:

**Example 2.15.** A  $\overline{\mathbb{K}}$ -category  $\mathcal{A}$  is a pair  $(\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  where

- $\text{Ob}(\mathcal{A})$  is the set of **points** of  $\mathcal{A}$ ;
- $d_{\mathcal{A}}$ , called the **distance function** of  $\mathcal{A}$ , takes two points  $a, b$  of  $\mathcal{A}$  and returns a value  $d_{\mathcal{A}}(a, b) \in \mathbb{K} \cup \{-\infty, \infty\}$ ;

such that the following axioms hold:

- (Composition law)  $d_{\mathcal{A}}(a, b) + d_{\mathcal{A}}(b, c) \geq d_{\mathcal{A}}(a, c) \quad (\forall a, b, c \in \text{Ob}(\mathcal{A}))$ ;
- (Identity law)  $0 \geq d_{\mathcal{A}}(a, a) \quad (\forall a \in \text{Ob}(\mathcal{A}))$ .

Note that for each  $a \in \text{Ob}(\mathcal{A})$  the composition law yields

$$d_{\mathcal{A}}(a, a) + d_{\mathcal{A}}(a, a) \geq d_{\mathcal{A}}(a, a); \quad (2.16)$$

we can combine this with the identity law and deduce

$$d_{\mathcal{A}}(a, a) = 0 \text{ or } -\infty,$$

because no negative number  $d_{\mathcal{A}}(a, a) \in \mathbb{K}$  satisfies (2.16). ■

**Example 2.16.** A  $\overline{\mathbb{K}}_+^{\text{Cart}}$ -category  $\mathcal{A}$  is a pair  $(\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  where

- $\text{Ob}(\mathcal{A})$  is the set of **points** of  $\mathcal{A}$ ;
- $d_{\mathcal{A}}$  takes two points  $a, b$  of  $\mathcal{A}$  and returns a value  $d_{\mathcal{A}}(a, b) \in \mathbb{K}_+ \cup \{\infty\}$ ;

such that the following axioms hold:

- (Composition law)  $\max\{d_{\mathcal{A}}(a, b), d_{\mathcal{A}}(b, c)\} \geq d_{\mathcal{A}}(a, c) \quad (\forall a, b, c \in \text{Ob}(\mathcal{A}))$ ;
- (Identity law)  $0 \geq d_{\mathcal{A}}(a, a) \quad (\forall a \in \text{Ob}(\mathcal{A}))$ .

Of course, we can rewrite the identity law as  $d_{\mathcal{A}}(a, a) = 0$ . As observed in [5] (when  $\mathbb{K} = \mathbb{R}$ ),  $\overline{\mathbb{K}}_+^{\text{Cart}}$ -categories generalize (discrete or ordinary) ultrametric spaces. ■

## 2.3 Enriched Functors and Canonical Orderings

One can understand the notion of  $\mathcal{V}$ -functors as structure-preserving maps between  $\mathcal{V}$ -categories. In the general enriched category theory (where the enriching category is not necessarily a poset), there is a yet higher notion of  $\mathcal{V}$ -*natural transformations*, which are regarded as *maps between  $\mathcal{V}$ -functors*. In our theory, however, the notion of  $\mathcal{V}$ -natural transformations are largely trivialized, and it only remains as a certain preorder relation among  $\mathcal{V}$ -functors with specified domain and codomain  $\mathcal{V}$ -categories.

**Definition 2.17.** Let  $\mathcal{V} = (\mathcal{V}_0, \otimes, e)$  be an SM-P, and  $\mathcal{A} = (\text{Ob}(\mathcal{A}), \text{Hom}_{\mathcal{A}})$  and  $\mathcal{B} = (\text{Ob}(\mathcal{B}), \text{Hom}_{\mathcal{B}})$  be  $\mathcal{V}$ -categories. A  $\mathcal{V}$ -**functor**

$$F: \mathcal{A} \longrightarrow \mathcal{B}$$

from  $\mathcal{A}$  to  $\mathcal{B}$  is a map

$$\text{Ob}(F): \text{Ob}(\mathcal{A}) \longrightarrow \text{Ob}(\mathcal{B})$$

between the sets of objects with the following property:

$$(\text{Increasing condition}) \quad \text{Hom}_{\mathcal{A}}(a, a') \sqsubseteq \text{Hom}_{\mathcal{B}}(\text{Ob}(F)(a), \text{Ob}(F)(a')) \quad (\forall a, a' \in \text{Ob}(\mathcal{A})).$$

In case a stronger condition

$$\text{Hom}_{\mathcal{A}}(a, a') = \text{Hom}_{\mathcal{B}}(\text{Ob}(F)(a), \text{Ob}(F)(a')) \quad (\forall a, a' \in \text{Ob}(\mathcal{A}))$$

holds,  $F$  is called a **fully faithful**  $\mathcal{V}$ -functor.

$\mathcal{A}$  is called the **domain** of  $F$  and  $\mathcal{B}$  the **codomain** of  $F$ . We denote the set of all  $\mathcal{V}$ -functors from  $\mathcal{A}$  to  $\mathcal{B}$  by  $[\mathcal{A}, \mathcal{B}]_0$ . ■

For brevity, we often denote the values like  $\text{Ob}(F)(a)$  by  $F(a)$ . Clearly,  $\mathcal{V}$ -functors are closed under composition:

**Proposition 2.18.** *Let  $F: \mathcal{A} \longrightarrow \mathcal{B}$  and  $G: \mathcal{B} \longrightarrow \mathcal{C}$  be  $\mathcal{V}$ -functors. Then, the composite map*

$$\text{Ob}(G) \circ \text{Ob}(F): \text{Ob}(\mathcal{A}) \longrightarrow \text{Ob}(\mathcal{B}) \longrightarrow \text{Ob}(\mathcal{C})$$

*defines a  $\mathcal{V}$ -functor  $G \circ F: \mathcal{A} \longrightarrow \mathcal{C}$ .*

*Proof.* For all  $a, a' \in \text{Ob}(\mathcal{A})$ , we have

$$\text{Hom}_{\mathcal{A}}(a, a') \sqsubseteq \text{Hom}_{\mathcal{B}}(F(a), F(a')) \sqsubseteq \text{Hom}_{\mathcal{C}}(G \circ F(a), G \circ F(a')).$$

□

Also, for every  $\mathcal{V}$ -category  $\mathcal{A}$ , the identity map on  $\text{Ob}(\mathcal{A})$  defines a  $\mathcal{V}$ -functor. It is called the **identity  $\mathcal{V}$ -functor** on  $\mathcal{A}$  and denoted by  $1_{\mathcal{A}}$ .

**Definition 2.19.** Let  $\mathcal{V}$  be an SM-P, and  $\mathcal{A}$  and  $\mathcal{B}$  be  $\mathcal{V}$ -categories. A  $\mathcal{V}$ -functor

$$F: \mathcal{A} \longrightarrow \mathcal{B}$$

from  $\mathcal{A}$  to  $\mathcal{B}$  is an **isomorphism** if there exists a  $\mathcal{V}$ -functor (called its **inverse**)

$$G: \mathcal{B} \longrightarrow \mathcal{A}$$

from  $\mathcal{B}$  to  $\mathcal{A}$  such that  $G \circ F = 1_{\mathcal{A}}$  and  $F \circ G = 1_{\mathcal{B}}$  hold.

If there is an isomorphism between  $\mathcal{A}$  and  $\mathcal{B}$ , then they are said to be **isomorphic** and written as  $\mathcal{A} \cong \mathcal{B}$ . ■

**Proposition 2.20.** *A  $\mathcal{V}$ -functor*

$$F: \mathcal{A} \longrightarrow \mathcal{B}$$

*is an isomorphism if and only if it is fully faithful, and the map*

$$\text{Ob}(F): \text{Ob}(\mathcal{A}) \longrightarrow \text{Ob}(\mathcal{B})$$

*between the sets of objects is a bijection.*

*Proof.* If  $F$  is an isomorphism, let  $G$  be its inverse. Then for each pair  $a, a' \in \text{Ob}(\mathcal{A})$ ,

$$\begin{aligned} \text{Hom}_{\mathcal{A}}(a, a') &\sqsubseteq \text{Hom}_{\mathcal{B}}(F(a), F(a')) \\ &\sqsubseteq \text{Hom}_{\mathcal{A}}(G \circ F(a), G \circ F(a')) \\ &= \text{Hom}_{\mathcal{A}}(a, a') \end{aligned}$$

holds and  $F$  is fully faithful. Also,  $\text{Ob}(G)$  witnesses that  $\text{Ob}(F)$  is a bijection as a map.

Suppose then, that  $F$  is fully faithful and  $\text{Ob}(F)$  is a bijection. Let  $\text{Ob}(G): \text{Ob}(\mathcal{B}) \rightarrow \text{Ob}(\mathcal{A})$  be the inverse of  $\text{Ob}(F)$  (as a map). Then, for each pair  $b, b' \in \text{Ob}(\mathcal{B})$ ,

$$\text{Hom}_{\mathcal{B}}(b, b') = \text{Hom}_{\mathcal{B}}(F \circ G(b), F \circ G(b')) = \text{Hom}_{\mathcal{A}}(G(b), G(b'))$$

holds, thus  $G$  is also a (fully faithful)  $\mathcal{V}$ -functor. By construction, it is clear that  $G \circ F = 1_{\mathcal{A}}$  and  $F \circ G = 1_{\mathcal{B}}$  hold.  $\square$

**Definition 2.21.** Let  $\mathcal{V} = (\mathcal{V}_0, \otimes, e)$  be an SM-P, and  $\mathcal{A} = (\text{Ob}(\mathcal{A}), \text{Hom}_{\mathcal{A}})$  and  $\mathcal{B} = (\text{Ob}(\mathcal{B}), \text{Hom}_{\mathcal{B}})$  be  $\mathcal{V}$ -categories. The set  $[\mathcal{A}, \mathcal{B}]_0$  of all  $\mathcal{V}$ -functors from  $\mathcal{A}$  to  $\mathcal{B}$  admits a canonical relation  $\Rightarrow$  defined as follows:

$$F \Rightarrow G \iff e \sqsubseteq \text{Hom}_{\mathcal{B}}(F(a), G(a)) \quad (\forall a \in \text{Ob}(\mathcal{A}));$$

where  $F, G: \mathcal{A} \rightarrow \mathcal{B}$  are  $\mathcal{V}$ -functors. We call  $\Rightarrow$  the **canonical ordering** on  $[\mathcal{A}, \mathcal{B}]_0$ .  $\blacksquare$

**Proposition 2.22.** *The relation  $\Rightarrow$  defined on the set  $[\mathcal{A}, \mathcal{B}]_0$  is a preorder relation.*

*Proof.* [Reflexivity] We wish to prove  $e \sqsubseteq \text{Hom}_{\mathcal{B}}(F(a), F(a))$ , but this is an instance of the identity law for  $\mathcal{B}$ .

[Transitivity] Suppose  $F \Rightarrow G$  and  $G \Rightarrow H$  for  $F, G, H: \mathcal{A} \rightarrow \mathcal{B}$ . Then an application of the composition law for  $\mathcal{B}$  yields

$$\begin{aligned} e &= e \otimes e \\ &\sqsubseteq \text{Hom}_{\mathcal{B}}(F(a), G(a)) \otimes \text{Hom}_{\mathcal{B}}(G(a), H(a)) \\ &\sqsubseteq \text{Hom}_{\mathcal{B}}(F(a), H(a)). \end{aligned}$$

$\square$

Also, the relation  $\Rightarrow$  is preserved under composition of  $\mathcal{V}$ -functors, in the sense of the following proposition:

**Proposition 2.23.** *Let  $F, G: \mathcal{A} \rightarrow \mathcal{B}$  and  $H, K: \mathcal{B} \rightarrow \mathcal{C}$  be  $\mathcal{V}$ -functors such that  $F \Rightarrow G$  and  $H \Rightarrow K$  hold. Then, for the composite  $\mathcal{V}$ -functors  $H \circ F, K \circ G: \mathcal{A} \rightarrow \mathcal{C}$ ,  $H \circ F \Rightarrow K \circ G$  holds.*

*Proof.*

$$\begin{aligned} e &= e \otimes e \\ &\sqsubseteq \text{Hom}_{\mathcal{B}}(F(a), F(a)) \otimes \text{Hom}_{\mathcal{B}}(F(a), G(a)) \\ &\sqsubseteq \text{Hom}_{\mathcal{C}}(H \circ F(a), K \circ F(a)) \otimes \text{Hom}_{\mathcal{C}}(K \circ F(a), K \circ G(a)) \\ &\sqsubseteq \text{Hom}_{\mathcal{C}}(H \circ F(a), K \circ G(a)). \end{aligned}$$

$\square$

Below we see examples of  $\mathcal{V}$ -functors and relation  $\Rightarrow$ .

**Example 2.24.** Let us take  $\mathcal{V} = \mathbf{2}$ . We claim that  $\mathbf{2}$ -functors between  $\mathbf{2}$ -categories are just *monotonic maps*. Indeed, for  $\mathbf{2}$ -categories  $\mathcal{A} = (\text{Ob}(\mathcal{A}), \preceq_{\mathcal{A}})$  and  $\mathcal{B} = (\text{Ob}(\mathcal{B}), \preceq_{\mathcal{B}})$ , a  $\mathbf{2}$ -functor  $F: \mathcal{A} \rightarrow \mathcal{B}$  is a map between the sets of objects such that

$$(\text{Increasing condition}) \quad (a \preceq_{\mathcal{A}} a') \vdash (F(a) \preceq_{\mathcal{B}} F(a')) \quad (\forall a, a' \in \text{Ob}(\mathcal{A}))$$

holds. If  $F$  is moreover fully faithful,

$$(a \preceq_{\mathcal{A}} a') = (F(a) \preceq_{\mathcal{B}} F(a')) \quad (\forall a, a' \in \text{Ob}(\mathcal{A}))$$

holds and  $F$  is nothing but a usual embedding between preorders. The notion of isomorphic  $\mathbf{2}$ -categories agrees with that of isomorphic preorders.

The relation  $\Rightarrow$  turns out to be the usual ordering on the set of monotonic maps:

$$F \Rightarrow G \iff \text{true} \vdash (F(a) \preceq_{\mathcal{B}} G(a)) \quad (\forall a \in \text{Ob}(\mathcal{A})).$$

■

**Example 2.25.** For  $\overline{\mathbb{K}}_+$ -categories, one can think of  $\overline{\mathbb{K}}_+$ -functors as *nonexpansive maps*. Let  $\mathcal{A} = (\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  and  $\mathcal{B} = (\text{Ob}(\mathcal{B}), d_{\mathcal{B}})$  be  $\overline{\mathbb{K}}_+$ -categories. A  $\overline{\mathbb{K}}_+$ -functor  $F: \mathcal{A} \rightarrow \mathcal{B}$  is a map between the sets of points with the following property:

$$(\text{Increasing condition}) \quad d_{\mathcal{A}}(a, a') \geq d_{\mathcal{B}}(F(a), F(a')) \quad (\forall a, a' \in \text{Ob}(\mathcal{A})).$$

If  $F$  is fully faithful,

$$d_{\mathcal{A}}(a, a') = d_{\mathcal{B}}(F(a), F(a')) \quad (\forall a, a' \in \text{Ob}(\mathcal{A}))$$

holds; i.e., in the terminology of metric spaces,  $F$  is an isometry. The notion of isomorphic  $\overline{\mathbb{K}}_+$ -categories specializes to that of isometric metric spaces.

The relation  $\Rightarrow$  is defined as follows:

$$\begin{aligned} F \Rightarrow G &\iff 0 \geq d_{\mathcal{B}}(F(a), G(a)) \quad (\forall a \in \text{Ob}(\mathcal{A})) \\ &\iff d_{\mathcal{B}}(F(a), G(a)) = 0 \quad (\forall a \in \text{Ob}(\mathcal{A})). \end{aligned}$$

■

**Example 2.26.** Let  $\mathcal{A} = (\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  and  $\mathcal{B} = (\text{Ob}(\mathcal{B}), d_{\mathcal{B}})$  be  $\overline{\mathbb{K}}$ -categories. A  $\overline{\mathbb{K}}$ -functor  $F: \mathcal{A} \rightarrow \mathcal{B}$  is a map between the sets of points with the following property:

$$(\text{Increasing condition}) \quad d_{\mathcal{A}}(a, a') \geq d_{\mathcal{B}}(F(a), F(a')) \quad (\forall a, a' \in \text{Ob}(\mathcal{A})).$$

A fully faithful  $\overline{\mathbb{K}}$ -functor is an isometry (in an obvious sense).

The relation  $\Rightarrow$  is defined as follows:

$$F \Rightarrow G \iff 0 \geq d_{\mathcal{B}}(F(a), G(a)) \quad (\forall a \in \text{Ob}(\mathcal{A})).$$

■

**Example 2.27.** Let  $\mathcal{A} = (\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  and  $\mathcal{B} = (\text{Ob}(\mathcal{B}), d_{\mathcal{B}})$  be  $\overline{\mathbb{K}}_+^{\text{Cart}}$ -categories. A  $\overline{\mathbb{K}}_+^{\text{Cart}}$ -functor  $F: \mathcal{A} \rightarrow \mathcal{B}$  is a map between the sets of points with the following property:

$$(\text{Increasing condition}) \quad d_{\mathcal{A}}(a, a') \geq d_{\mathcal{B}}(F(a), F(a')) \quad (\forall a, a' \in \text{Ob}(\mathcal{A})).$$

A fully faithful  $\overline{\mathbb{K}}_+^{\text{Cart}}$ -functor is an isometry.

The relation  $\Rightarrow$  is defined as follows:

$$\begin{aligned} F \Rightarrow G &\iff 0 \geq d_{\mathcal{B}}(F(a), G(a)) \quad (\forall a \in \text{Ob}(\mathcal{A})) \\ &\iff d_{\mathcal{B}}(F(a), G(a)) = 0 \quad (\forall a \in \text{Ob}(\mathcal{A})). \end{aligned}$$

■

## 2.4 The Opposites of $\mathcal{V}$ -Categories

In this section we present a way to construct new  $\mathcal{V}$ -categories out of some already existing  $\mathcal{V}$ -categories.

**Proposition 2.28.** *Let  $\mathcal{V} = (\mathcal{V}_0, \otimes, e)$  be an SM-P, and  $\mathcal{A} = (\text{Ob}(\mathcal{A}), \text{Hom}_{\mathcal{A}})$  be a  $\mathcal{V}$ -category. Then there is a  $\mathcal{V}$ -category denoted by  $\mathcal{A}^{\text{op}} = (\text{Ob}(\mathcal{A}^{\text{op}}), \text{Hom}_{\mathcal{A}^{\text{op}}})$ , where*

- $\text{Ob}(\mathcal{A}^{\text{op}}) = \text{Ob}(\mathcal{A})$ , i.e., the set of objects for  $\mathcal{A}^{\text{op}}$  is identical with that for  $\mathcal{A}$ ;
- the hom-object function  $\text{Hom}_{\mathcal{A}^{\text{op}}}$  is given as

$$\text{Hom}_{\mathcal{A}^{\text{op}}}(a, b) = \text{Hom}_{\mathcal{A}}(b, a).$$

*Proof.* [Composition law] The claim is

$$\text{Hom}_{\mathcal{A}^{\text{op}}}(a, b) \otimes \text{Hom}_{\mathcal{A}^{\text{op}}}(b, c) \sqsubseteq \text{Hom}_{\mathcal{A}^{\text{op}}}(a, c).$$

It is proved as follows:

$$\begin{aligned} \text{Hom}_{\mathcal{A}^{\text{op}}}(a, b) \otimes \text{Hom}_{\mathcal{A}^{\text{op}}}(b, c) &= \text{Hom}_{\mathcal{A}}(b, a) \otimes \text{Hom}_{\mathcal{A}}(c, b) \\ &= \text{Hom}_{\mathcal{A}}(c, b) \otimes \text{Hom}_{\mathcal{A}}(b, a) \\ &\sqsubseteq \text{Hom}_{\mathcal{A}}(c, a) \\ &= \text{Hom}_{\mathcal{A}^{\text{op}}}(a, c). \end{aligned}$$

[Identity law] The claim is  $e \sqsubseteq \text{Hom}_{\mathcal{A}^{\text{op}}}(a, a)$ , an obvious formula. □

**Definition 2.29.** The category  $\mathcal{A}^{\text{op}}$  is called the **opposite category** of  $\mathcal{A}$ . ■

**Example 2.30.** Let  $\mathcal{A} = (\text{Ob}(\mathcal{A}), \preceq_{\mathcal{A}})$  and  $\mathcal{B} = (\text{Ob}(\mathcal{B}), \preceq_{\mathcal{B}})$  be **2**-categories. The opposite category  $\mathcal{A}^{\text{op}} = (\text{Ob}(\mathcal{A}^{\text{op}}), \preceq_{\mathcal{A}^{\text{op}}})$  is the opposite (or dual) of preorder: ■

- $\text{Ob}(\mathcal{A}^{\text{op}}) = \text{Ob}(\mathcal{A})$ ;
- $(a \preceq_{\mathcal{A}^{\text{op}}} b) = (b \preceq_{\mathcal{A}} a)$ .

**Example 2.31.** Let  $\mathcal{A} = (\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  and  $\mathcal{B} = (\text{Ob}(\mathcal{B}), d_{\mathcal{B}})$  be  $\overline{\mathbb{K}}_+$ -categories. The opposite category  $\mathcal{A}^{\text{op}} = (\text{Ob}(\mathcal{A}^{\text{op}}), d_{\mathcal{A}^{\text{op}}})$  is defined as follows: ■

- $\text{Ob}(\mathcal{A}^{\text{op}}) = \text{Ob}(\mathcal{A})$ ;
- $d_{\mathcal{A}^{\text{op}}}(a, b) = d_{\mathcal{A}}(b, a)$ .

Note that since a classical metric space has a symmetric distance function, it is isometric to its opposite. ■

**Example 2.32.** Let  $\mathcal{A} = (\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  and  $\mathcal{B} = (\text{Ob}(\mathcal{B}), d_{\mathcal{B}})$  be  $\overline{\mathbb{K}}$ -categories. The opposite category  $\mathcal{A}^{\text{op}} = (\text{Ob}(\mathcal{A}^{\text{op}}), d_{\mathcal{A}^{\text{op}}})$  is defined as follows: ■

- $\text{Ob}(\mathcal{A}^{\text{op}}) = \text{Ob}(\mathcal{A})$ ;
- $d_{\mathcal{A}^{\text{op}}}(a, b) = d_{\mathcal{A}}(b, a)$ .

**Example 2.33.** Let  $\mathcal{A} = (\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  and  $\mathcal{B} = (\text{Ob}(\mathcal{B}), d_{\mathcal{B}})$  be  $\overline{\mathbb{K}}_+^{\text{Cart}}$ -categories. The opposite category  $\mathcal{A}^{\text{op}} = (\text{Ob}(\mathcal{A}^{\text{op}}), d_{\mathcal{A}^{\text{op}}})$  is defined as follows: ■

- $\text{Ob}(\mathcal{A}^{\text{op}}) = \text{Ob}(\mathcal{A})$ ;
- $d_{\mathcal{A}^{\text{op}}}(a, b) = d_{\mathcal{A}}(b, a)$ .

■

## 2.5 $\mathcal{V}$ as a $\mathcal{V}$ -Category

We henceforth assume that the enriching category  $\mathcal{V}$  is an *SMC-P*. This assumption enables us to endow (the underlying set of)  $\mathcal{V}_0$  with a canonical  $\mathcal{V}$ -category structure:

**Proposition 2.34.** *Let  $\mathcal{V} = (\mathcal{V}_0, \otimes, [-, -], e)$  be an *SMC-P*. Then there is a  $\mathcal{V}$ -category, also denoted by  $\mathcal{V} = (\mathcal{V}_0, [-, -])$ , where*

- $\text{Ob}(\mathcal{V})$  is  $\mathcal{V}_0$ , the underlying set of the poset  $\mathcal{V}_0$ ;
- the hom-object function is the internal-hom operation  $[-, -]$ .

*Proof.* [Composition law] The claim is  $[x, y] \otimes [y, z] \sqsubseteq [x, z]$  for all  $x, y, z \in \mathcal{V}_0$ , but this is equivalent to the composition law for SC-PPs, which we know to hold:

$$\frac{\frac{[x, y] \otimes [y, z] \sqsubseteq [x, z]}{[y, z] \otimes [x, y] \sqsubseteq [x, z]}}{[y, z] \sqsubseteq [[x, y], [x, z]]}$$

Incidentally, this proof tree explains the duplication of the name “composition law.” [Identity law] The claim is  $e \sqsubseteq [z, z]$  for all  $z \in \mathcal{V}_0$ . Using the commutative law for  $[-, -]$ , it suffices to show  $z \sqsubseteq [e, z]$ . But the unit law for  $[-, -]$  states  $z = [e, z]$ , so the proof is done. □

Now the name of the operation “internal-hom” is motivated. In a sense, it acts like  $\text{Hom}$ , and takes values inside  $\mathcal{V}$ . As an aside, one can think of the (external) hom of the poset  $\mathcal{V}_0$  to be that takes two elements  $x, y$  of  $\mathcal{V}_0$  and returns the truth value of  $x \sqsubseteq y$ , which is a value in  $\mathbf{2}$ ; thus (unless  $\mathcal{V} = \mathbf{2}$ ) outside  $\mathcal{V}$ .

Let us take a look at some examples of such canonical  $\mathcal{V}$ -categories.

**Example 2.35.**  $\mathbf{2}_0 = \{\text{true}, \text{false}\}$  becomes a  $\mathbf{2}$ -category (i.e., preordered set) in a canonical way. By the way,  $\mathbf{2}_0$  is defined to be the underlying poset of  $\mathbf{2}$ , hence is already a preordered set. In fact, the  $\mathbf{2}$ -category  $\mathbf{2}$  is isomorphic to the *underlying poset*  $\mathbf{2}_0$  of the SMC-CL  $\mathbf{2}$ : one can see this by observing that the tables for  $\vdash$  and for  $\supset$  in Example 2.6 carry the same pattern. ■

**Example 2.36.** The set  $\mathbb{K}_+ \cup \{\infty\}$  becomes a  $\overline{\mathbb{K}}_+$ -category by the following distance function:

$$d_{\overline{\mathbb{K}}_+}(x, y) = y \dot{-} x.$$

Therefore, if  $x \leq y$  the distance to climb from  $x$  to  $y$  is the same as their the difference, but if  $x \geq y$ , we descend and the corresponding distance is 0.

Note that the  $\overline{\mathbb{K}}_+$ -category  $\overline{\mathbb{K}}_+$  is *not* an ordinary (discrete) metric space; among others, its distance function is highly asymmetric. ■

**Example 2.37.** The set  $\mathbb{K} \cup \{-\infty, \infty\}$  becomes a  $\overline{\mathbb{K}}$ -category by the following distance function:

$$d_{\overline{\mathbb{K}}}(x, y) = y - x.$$

Recall that in any  $\overline{\mathbb{K}}$ -category  $\mathcal{A}$

$$d_{\mathcal{A}}(a, a) = 0 \text{ or } -\infty$$

holds.  $\overline{\mathbb{K}}$  as a  $\overline{\mathbb{K}}$ -category indicates that the distances between the same points may take both 0 and  $-\infty$  in one  $\overline{\mathbb{K}}$ -category; in fact, if  $s \in \mathbb{K}$ ,

$$\begin{aligned} d_{\overline{\mathbb{K}}}(s, s) &= 0, \text{ and} \\ d_{\overline{\mathbb{K}}}(-\infty, -\infty) &= d_{\overline{\mathbb{K}}}(\infty, \infty) = -\infty \end{aligned}$$

hold. ■

**Example 2.38.** The set  $\mathbb{K}_+ \cup \{\infty\}$  becomes a  $\overline{\mathbb{K}}_+^{\text{Cart}}$ -category by the following distance function:

$$\begin{aligned} d_{\overline{\mathbb{K}}_+^{\text{Cart}}}(x, y) &= x \supset y \\ &= \begin{cases} 0 & (x \geq y) \\ y & (x < y). \end{cases} \end{aligned}$$
■

## 2.6 Functor Categories and the Yoneda Embedding

Now the theory has reached the stage where we perform some limiting processes within  $\mathcal{V}_0$ . Therefore, we finally assume that our enriching category  $\mathcal{V}$  is an *SMC-CL*. We begin this section with the definition of a  $\mathcal{V}$ -category of all  $\mathcal{V}$ -functors with specified domain and codomain, which is intuitively seen as an analog of a function space in some sense.

**Proposition 2.39.** *Let  $\mathcal{V} = (\mathcal{V}_0, \otimes, [-, -], e)$  be an *SMC-CL*, and  $\mathcal{A} = (\text{Ob}(\mathcal{A}), \text{Hom}_{\mathcal{A}})$  and  $\mathcal{B} = (\text{Ob}(\mathcal{B}), \text{Hom}_{\mathcal{B}})$  be  $\mathcal{V}$ -categories. Then there is a  $\mathcal{V}$ -category denoted by  $\text{Fun}(\mathcal{A}, \mathcal{B}) = ([\mathcal{A}, \mathcal{B}]_0, \text{Hom}_{\text{Fun}(\mathcal{A}, \mathcal{B})})$ , where*

- $\text{Ob}(\text{Fun}(\mathcal{A}, \mathcal{B}))$  is  $[\mathcal{A}, \mathcal{B}]_0$ , the set of all  $\mathcal{V}$ -functors from  $\mathcal{A}$  to  $\mathcal{B}$ ;
- the hom-object function  $\text{Hom}_{\text{Fun}(\mathcal{A}, \mathcal{B})}$  is given as

$$\text{Hom}_{\text{Fun}(\mathcal{A}, \mathcal{B})}(F, G) = \bigwedge_{a \in \text{Ob}(\mathcal{A})} \{\text{Hom}_{\mathcal{B}}(F(a), G(a))\},$$

where the infimum is taken in the complete lattice  $\mathcal{V}_0$ .

*Proof.* [Composition law] The claim is

$$\text{Hom}_{\text{Fun}(\mathcal{A}, \mathcal{B})}(F, G) \otimes \text{Hom}_{\text{Fun}(\mathcal{A}, \mathcal{B})}(G, H) \sqsubseteq \text{Hom}_{\text{Fun}(\mathcal{A}, \mathcal{B})}(F, H).$$

For each  $a \in \text{Ob}(\mathcal{A})$ , we have

$$\begin{aligned} &\text{Hom}_{\text{Fun}(\mathcal{A}, \mathcal{B})}(F, G) \otimes \text{Hom}_{\text{Fun}(\mathcal{A}, \mathcal{B})}(G, H) \\ &= \bigwedge_{a' \in \text{Ob}(\mathcal{A})} \{\text{Hom}_{\mathcal{B}}(F(a'), G(a'))\} \otimes \bigwedge_{a' \in \text{Ob}(\mathcal{A})} \{\text{Hom}_{\mathcal{B}}(G(a'), H(a'))\} \\ &\sqsubseteq \text{Hom}_{\mathcal{B}}(F(a), G(a)) \otimes \text{Hom}_{\mathcal{B}}(G(a), H(a)) \\ &\sqsubseteq \text{Hom}_{\mathcal{B}}(F(a), H(a)). \end{aligned}$$

Taking the infimum with respect to  $a$ , we obtain the required result.

[Identity law] The claim is  $e \sqsubseteq \text{Hom}_{\text{Fun}(\mathcal{A}, \mathcal{B})}(F, F)$ . The identity law for  $\mathcal{B}$  assures

$$e \sqsubseteq \text{Hom}_{\mathcal{B}}(F(a), F(a)),$$

and the inequality is preserved after we take the infimum with respect to  $a$ .  $\square$

**Definition 2.40.** The  $\mathcal{V}$ -category  $\text{Fun}(\mathcal{A}, \mathcal{B})$  is called a **functor category**,  $\mathcal{A}$  its **domain** and  $\mathcal{B}$  its **codomain**.  $\blacksquare$

Given a  $\mathcal{V}$ -category  $\mathcal{A}$ , there are many functor categories associated with  $\mathcal{A}$ . But two of them are of special importance; they are called the *presheaf category* and the *op-copresheaf category* of  $\mathcal{A}$ .

**Definition 2.41.** Let  $\mathcal{V} = (\mathcal{V}_0, \otimes, [-, -], e)$  be an SMC-CL, and  $\mathcal{A} = (\text{Ob}(\mathcal{A}), \text{Hom}_{\mathcal{A}})$  be a  $\mathcal{V}$ -category. A **presheaf**  $P$  on  $\mathcal{A}$  is a  $\mathcal{V}$ -functor  $P: \mathcal{A}^{\text{op}} \rightarrow \mathcal{V}$ , that is, a map  $\text{Ob}(P): \text{Ob}(\mathcal{A}) \rightarrow \mathcal{V}_0$  satisfying

$$\text{Hom}_{\mathcal{A}}(a, b) \sqsubseteq [P(b), P(a)]$$

for every  $a, b \in \text{Ob}(\mathcal{A})$ . A **copresheaf**  $Q$  on  $\mathcal{A}$  is a  $\mathcal{V}$ -functor  $Q: \mathcal{A} \rightarrow \mathcal{V}$ , that is, a map  $\text{Ob}(Q): \text{Ob}(\mathcal{A}) \rightarrow \mathcal{V}_0$  satisfying

$$\text{Hom}_{\mathcal{A}}(a, b) \sqsubseteq [Q(a), Q(b)]$$

for every  $a, b \in \text{Ob}(\mathcal{A})$ .

We call the functor category  $\text{Fun}(\mathcal{A}^{\text{op}}, \mathcal{V})$  the **presheaf category** of  $\mathcal{A}$ , and the category  $\text{Fun}(\mathcal{A}, \mathcal{V})^{\text{op}}$  the **op-copresheaf category** of  $\mathcal{A}$ .  $\blacksquare$

Presheaf categories and op-copresheaf categories are important because the original category embeds into them canonically. This is called the Yoneda embedding theorem; but before we present and prove it, let us see some examples of functor categories and presheaf or op-copresheaf categories:

**Example 2.42.** Let  $\mathcal{A} = (\text{Ob}(\mathcal{A}), \preceq_{\mathcal{A}})$  and  $\mathcal{B} = (\text{Ob}(\mathcal{B}), \preceq_{\mathcal{B}})$  be  $\mathbf{2}$ -categories. The functor category  $\text{Fun}(\mathcal{A}, \mathcal{B}) = ([\mathcal{A}, \mathcal{B}]_0, \preceq_{\text{Fun}(\mathcal{A}, \mathcal{B})})$  is given as follows, noting that  $\bigwedge$  in  $\mathbf{2}_0$  corresponds to “for all”:

- $[\mathcal{A}, \mathcal{B}]_0$  is the set of all monotonic maps from  $\mathcal{A}$  to  $\mathcal{B}$ ;
- $(F \preceq_{\text{Fun}(\mathcal{A}, \mathcal{B})} G) = \text{true} \iff (F(a) \preceq_{\mathcal{B}} G(a)) = \text{true} \text{ for all } a \in \text{Ob}(\mathcal{A})$ .

Thus, as a binary relation on the set  $[\mathcal{A}, \mathcal{B}]_0$ ,  $\preceq_{\text{Fun}(\mathcal{A}, \mathcal{B})}$  coincides with the canonical ordering  $\Rightarrow$ .

A presheaf  $P$  on  $\mathcal{A}$  is a monotonic map  $P: \mathcal{A}^{\text{op}} \rightarrow \mathbf{2}$ , and by taking the inverse image of  $\{\text{true}\} \subseteq \mathbf{2}_0$ , it can be seen as a *lower set* of  $\mathcal{A}$  (a lower set  $S$  of a preorder  $(A, \preceq)$  is a subset  $S \subseteq A$  such that  $a \in S$  and  $b \preceq a$  imply  $b \in S$ ). In fact, presheaves on  $\mathcal{A}$  correspond one-to-one with lower sets of  $\mathcal{A}$ . Moreover, the ordering  $\preceq_{\text{Fun}(\mathcal{A}^{\text{op}}, \mathbf{2})}$  on the presheaf category coincides with the inclusion ordering  $\subseteq$  on the set of lower sets: for all presheaves  $P_1, P_2$  on  $\mathcal{A}$ ,

$$P_1 \preceq_{\text{Fun}(\mathcal{A}^{\text{op}}, \mathbf{2})} P_2 \iff \text{Ob}(P_1)^{-1}(\{\text{true}\}) \subseteq \text{Ob}(P_2)^{-1}(\{\text{true}\})$$

hold.

A copresheaf  $Q$  on  $\mathcal{A}$  is a monotonic map  $Q: \mathcal{A} \rightarrow \mathbf{2}$ . If we take the inverse image of  $\{\text{true}\} \subseteq \mathbf{2}_0$ , we obtain an *upper set* of  $\mathcal{A}$ . Copresheaves on  $\mathcal{A}$  correspond one-to-one with upper sets of  $\mathcal{A}$ ; also, the ordering  $\preceq_{\text{Fun}(\mathcal{A}, \mathbf{2})^{\text{op}}} = \preceq_{\text{Fun}(\mathcal{A}, \mathbf{2})}$  on the op-copresheaf category



coincides with the opposite  $\supseteq$  of the inclusion ordering  $\subseteq$  on the set of upper sets: for all copresheaves  $Q_1, Q_2$  on  $\mathcal{A}$ ,

$$Q_1 \preceq_{\text{Fun}(\mathcal{A}, \mathbf{2})^{\text{op}}} Q_2 \iff \text{Ob}(Q_1)^{-1}(\{\text{true}\}) \supseteq \text{Ob}(Q_2)^{-1}(\{\text{true}\})$$

hold.

To summarize, the presheaf category  $\text{Fun}(\mathcal{A}^{\text{op}}, \mathbf{2})$  of  $\mathcal{A}$  can naturally be seen as the pre-ordered set (in fact, poset) of lower sets of  $\mathcal{A}$ , and the op-copresheaf category  $\text{Fun}(\mathcal{A}, \mathbf{2})^{\text{op}}$  as that of upper sets of  $\mathcal{A}$ .  $\blacksquare$

**Example 2.43.** Let  $\mathcal{A} = (\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  and  $\mathcal{B} = (\text{Ob}(\mathcal{B}), d_{\mathcal{B}})$  be  $\overline{\mathbb{K}}_+$ -categories. The functor category  $\text{Fun}(\mathcal{A}, \mathcal{B}) = ([\mathcal{A}, \mathcal{B}]_0, d_{\text{Fun}(\mathcal{A}, \mathcal{B})})$  is given as follows, noting that  $\wedge$  in  $\mathbb{K}_+ \cup \{\infty\}$  corresponds to (the usual) sup:

- $[\mathcal{A}, \mathcal{B}]_0$  is the set of all nonexpansive maps from  $\mathcal{A}$  to  $\mathcal{B}$ ;
- $d_{\text{Fun}(\mathcal{A}, \mathcal{B})}(F, G) = \sup_{a \in \text{Ob}(\mathcal{A})} \{d_{\mathcal{B}}(F(a), G(a))\}$ .

Therefore, in the current setting, the natural distance on a function space turns out to be the sup-distance. Since we adjoined  $\infty$  and turned  $\mathbb{K}_+$  into a complete lattice, we can get rid of additional assumptions such as the compactness of  $\mathcal{A}$  or the boundedness of  $\mathcal{B}$ , the usual requirements when performing the sup-distance construction in the theory of classical metric spaces.

A presheaf  $P$  on  $\mathcal{A}$  is a nonexpansive map  $P: \mathcal{A}^{\text{op}} \rightarrow \overline{\mathbb{K}}_+$ , namely, a map  $\text{Ob}(P): \text{Ob}(\mathcal{A}) \rightarrow \mathbb{K}_+ \cup \{\infty\}$  satisfying

$$d_{\mathcal{A}}(a, b) \geq P(a) \dot{-} P(b)$$

for all  $a, b \in \text{Ob}(\mathcal{A})$ , and can be seen as a scalar-valued function on  $\mathcal{A}$ . The distance between two presheaves  $P_1$  and  $P_2$  is given by

$$d_{\text{Fun}(\mathcal{A}^{\text{op}}, \overline{\mathbb{K}}_+)}(P_1, P_2) = \sup_{a \in \text{Ob}(\mathcal{A})} \{P_2(a) \dot{-} P_1(a)\}.$$

The canonical ordering  $\Rightarrow$  is defined as

$$\begin{aligned} P_1 \Rightarrow P_2 &\iff 0 \geq P_2(a) \dot{-} P_1(a) \quad (\forall a \in \text{Ob}(\mathcal{A})) \\ &\iff P_2(a) \dot{-} P_1(a) = 0 \quad (\forall a \in \text{Ob}(\mathcal{A})) \\ &\iff d_{\text{Fun}(\mathcal{A}^{\text{op}}, \overline{\mathbb{K}}_+)}(P_1, P_2) = 0 \\ &\iff P_1(a) \geq P_2(a) \quad (\forall a \in \text{Ob}(\mathcal{A})). \end{aligned}$$

A copresheaf  $Q$  on  $\mathcal{A}$  is a nonexpansive map  $Q: \mathcal{A} \rightarrow \overline{\mathbb{K}}_+$ , namely, a map  $\text{Ob}(Q): \text{Ob}(\mathcal{A}) \rightarrow \mathbb{K}_+ \cup \{\infty\}$  satisfying

$$d_{\mathcal{A}}(a, b) \geq Q(b) \dot{-} Q(a)$$

for all  $a, b \in \text{Ob}(\mathcal{A})$ , and forms another natural class of scalar-valued functions on  $\mathcal{A}$ . The distance between two copresheaves  $Q_1$  and  $Q_2$  in the op-copresheaf category  $\text{Fun}(\mathcal{A}, \overline{\mathbb{K}}_+)^{\text{op}}$  is given by

$$d_{\text{Fun}(\mathcal{A}, \overline{\mathbb{K}}_+)^{\text{op}}}(Q_1, Q_2) = \sup_{a \in \text{Ob}(\mathcal{A})} \{Q_1(a) \dot{-} Q_2(a)\}.$$

As  $\overline{\mathbb{K}}_+$ -functors, the canonical ordering  $\Rightarrow$  on copresheaves is determined by conditions similar to that for presheaves.  $\blacksquare$

**Example 2.44.** Let  $\mathcal{A} = (\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  and  $\mathcal{B} = (\text{Ob}(\mathcal{B}), d_{\mathcal{B}})$  be  $\overline{\mathbb{K}}$ -categories. The functor category  $\text{Fun}(\mathcal{A}, \mathcal{B}) = ([\mathcal{A}, \mathcal{B}]_0, d_{\text{Fun}(\mathcal{A}, \mathcal{B})})$  is given as follows:

- $[\mathcal{A}, \mathcal{B}]_0$  is the set of all nonexpansive maps from  $\mathcal{A}$  to  $\mathcal{B}$ ;
- $d_{\text{Fun}(\mathcal{A}, \mathcal{B})}(F, G) = \sup_{a \in \text{Ob}(\mathcal{A})} \{d_{\mathcal{B}}(F(a), G(a))\}$ .

A presheaf  $P$  on  $\mathcal{A}$  is a nonexpansive map  $P: \mathcal{A}^{\text{op}} \rightarrow \overline{\mathbb{K}}$ , namely, a map  $\text{Ob}(P): \text{Ob}(\mathcal{A}) \rightarrow \mathbb{K} \cup \{-\infty, \infty\}$  satisfying

$$d_{\mathcal{A}}(a, b) \geq P(a) - P(b)$$

for all  $a, b \in \text{Ob}(\mathcal{A})$ . The distance between two presheaves  $P_1$  and  $P_2$  is given by

$$d_{\text{Fun}(\mathcal{A}^{\text{op}}, \overline{\mathbb{K}})}(P_1, P_2) = \sup_{a \in \text{Ob}(\mathcal{A})} \{P_2(a) - P_1(a)\}.$$

The canonical ordering  $\Rightarrow$  is defined as

$$\begin{aligned} P_1 \Rightarrow P_2 &\iff 0 \geq P_2(a) - P_1(a) \quad (\forall a \in \text{Ob}(\mathcal{A})) \\ &\iff d_{\text{Fun}(\mathcal{A}^{\text{op}}, \overline{\mathbb{K}})}(P_1, P_2) \leq 0 \\ &\iff P_1(a) \geq P_2(a) \quad (\forall a \in \text{Ob}(\mathcal{A})). \end{aligned}$$

A copresheaf  $Q$  on  $\mathcal{A}$  is a nonexpansive map  $Q: \mathcal{A} \rightarrow \overline{\mathbb{K}}$ , namely, a map  $\text{Ob}(Q): \text{Ob}(\mathcal{A}) \rightarrow \mathbb{K} \cup \{-\infty, \infty\}$  satisfying

$$d_{\mathcal{A}}(a, b) \geq Q(b) - Q(a)$$

for all  $a, b \in \text{Ob}(\mathcal{A})$ . The distance between two copresheaves  $Q_1$  and  $Q_2$  in the op-copresheaf category  $\text{Fun}(\mathcal{A}, \overline{\mathbb{K}})^{\text{op}}$  is given by

$$d_{\text{Fun}(\mathcal{A}, \overline{\mathbb{K}})^{\text{op}}}(Q_1, Q_2) = \sup_{a \in \text{Ob}(\mathcal{A})} \{Q_1(a) - Q_2(a)\}.$$

■

**Example 2.45.** Let  $\mathcal{A} = (\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  and  $\mathcal{B} = (\text{Ob}(\mathcal{B}), d_{\mathcal{B}})$  be  $\overline{\mathbb{K}}_+^{\text{Cart}}$ -categories. The functor category  $\text{Fun}(\mathcal{A}, \mathcal{B}) = ([\mathcal{A}, \mathcal{B}]_0, d_{\text{Fun}(\mathcal{A}, \mathcal{B})})$  is given as follows:

- $[\mathcal{A}, \mathcal{B}]_0$  is the set of all nonexpansive maps from  $\mathcal{A}$  to  $\mathcal{B}$ ;
- $d_{\text{Fun}(\mathcal{A}, \mathcal{B})}(F, G) = \sup_{a \in \text{Ob}(\mathcal{A})} \{d_{\mathcal{B}}(F(a), G(a))\}$ .

A presheaf  $P$  on  $\mathcal{A}$  is a nonexpansive map  $P: \mathcal{A}^{\text{op}} \rightarrow \overline{\mathbb{K}}_+^{\text{Cart}}$ , namely, a map  $\text{Ob}(P): \text{Ob}(\mathcal{A}) \rightarrow \mathbb{K}_+ \cup \{\infty\}$  satisfying

$$\begin{aligned} d_{\mathcal{A}}(a, b) &\geq P(b) \supset P(a) \\ &= \begin{cases} 0 & (P(b) \geq P(a)) \\ P(a) & (P(b) < P(a)), \end{cases} \end{aligned}$$

or equivalently,

$$P(b) < P(a) \implies d_{\mathcal{A}}(a, b) \geq P(a)$$

for all  $a, b \in \text{Ob}(\mathcal{A})$ . The distance between two presheaves  $P_1$  and  $P_2$  is given by

$$\begin{aligned} d_{\text{Fun}(\mathcal{A}^{\text{op}}, \overline{\mathbb{K}}_+^{\text{Cart}})}(P_1, P_2) &= \sup_{a \in \text{Ob}(\mathcal{A})} \{P_1(a) \supset P_2(a)\} \\ &= \sup_{\substack{a \in \text{Ob}(\mathcal{A}) \\ P_1(a) < P_2(a)}} \{P_2(a)\}. \end{aligned}$$

The canonical ordering  $\Rightarrow$  is defined as

$$\begin{aligned} P_1 \Rightarrow P_2 &\iff 0 \geq P_1(a) \supset P_2(a) \quad (\forall a \in \text{Ob}(\mathcal{A})) \\ &\iff P_1(a) \supset P_2(a) = 0 \quad (\forall a \in \text{Ob}(\mathcal{A})) \\ &\iff d_{\text{Fun}(\mathcal{A}^{\text{op}}, \overline{\mathbb{K}}_+^{\text{Cart}})}(P_1, P_2) = 0 \\ &\iff P_1(a) \geq P_2(a) \quad (\forall a \in \text{Ob}(\mathcal{A})). \end{aligned}$$

A copresheaf  $Q$  on  $\mathcal{A}$  is a nonexpansive map  $Q: \mathcal{A} \rightarrow \overline{\mathbb{K}}_+^{\text{Cart}}$ , namely, a map  $\text{Ob}(Q): \text{Ob}(\mathcal{A}) \rightarrow \mathbb{K}_+ \cup \{\infty\}$  satisfying

$$\begin{aligned} d_{\mathcal{A}}(a, b) \geq Q(a) \supset Q(b) \\ = \begin{cases} 0 & (Q(a) \geq Q(b)) \\ Q(b) & (Q(a) < Q(b)), \end{cases} \end{aligned}$$

or equivalently,

$$Q(a) < Q(b) \implies d_{\mathcal{A}}(a, b) \geq Q(b)$$

for all  $a, b \in \text{Ob}(\mathcal{A})$ . The distance between two copresheaves  $Q_1$  and  $Q_2$  in the op-copresheaf category  $\text{Fun}(\mathcal{A}, \overline{\mathbb{K}}_+^{\text{Cart}})^{\text{op}}$  is given by

$$\begin{aligned} d_{\text{Fun}(\mathcal{A}, \overline{\mathbb{K}}_+^{\text{Cart}})^{\text{op}}}(Q_1, Q_2) &= \sup_{a \in \text{Ob}(\mathcal{A})} \{Q_2(a) \supset Q_1(a)\} \\ &= \sup_{\substack{a \in \text{Ob}(\mathcal{A}) \\ Q_2(a) < Q_1(a)}} \{Q_1(a)\}. \end{aligned}$$

■

Now we present the Yoneda embedding theorem:

**Theorem 2.46.** *Let  $\mathcal{V} = (\mathcal{V}_0, \otimes, [-, -], e)$  be an SMC-CL, and  $\mathcal{A} = (\text{Ob}(\mathcal{A}), \text{Hom}_{\mathcal{A}})$  be a  $\mathcal{V}$ -category. Then there are fully faithful  $\mathcal{V}$ -functors  $Y$  and  $\overline{Y}$ , defined as follows:*

$$\begin{aligned} Y: \mathcal{A} &\longrightarrow \text{Fun}(\mathcal{A}^{\text{op}}, \mathcal{V}), & \text{Ob}(Y(b)) &= \lambda a \in \text{Ob}(\mathcal{A}^{\text{op}}). \text{Hom}_{\mathcal{A}}(a, b) \quad (\forall b \in \text{Ob}(\mathcal{A})); \\ \overline{Y}: \mathcal{A} &\longrightarrow \text{Fun}(\mathcal{A}, \mathcal{V})^{\text{op}}, & \text{Ob}(\overline{Y}(a)) &= \lambda b \in \text{Ob}(\mathcal{A}). \text{Hom}_{\mathcal{A}}(a, b) \quad (\forall a \in \text{Ob}(\mathcal{A}^{\text{op}})). \end{aligned}$$

*Proof.* We first show that the values  $Y(b)$  and  $\overline{Y}(a)$  are indeed a presheaf or a copresheaf, respectively.

[ $Y(b)$  is a presheaf on  $\mathcal{A}$ ] The claim is

$$\begin{aligned} \text{Hom}_{\mathcal{A}}(a, a') &\sqsubseteq [Y(b)(a'), Y(b)(a)] \\ &= [\text{Hom}_{\mathcal{A}}(a', b), \text{Hom}_{\mathcal{A}}(a, b)], \end{aligned}$$

and this is equivalent to an instance of the composition law for  $\mathcal{A}$ :

$$\frac{\text{Hom}_{\mathcal{A}}(a, a') \sqsubseteq [\text{Hom}_{\mathcal{A}}(a', b), \text{Hom}_{\mathcal{A}}(a, b)]}{\text{Hom}_{\mathcal{A}}(a, a') \otimes \text{Hom}_{\mathcal{A}}(a', b) \sqsubseteq \text{Hom}_{\mathcal{A}}(a, b)}$$

[ $\overline{Y}(a)$  is a copresheaf on  $\mathcal{A}$ ] The claim is

$$\begin{aligned} \text{Hom}_{\mathcal{A}}(b, b') &\sqsubseteq [\overline{Y}(a)(b), \overline{Y}(a)(b')] \\ &= [\text{Hom}_{\mathcal{A}}(a, b), \text{Hom}_{\mathcal{A}}(a, b')], \end{aligned}$$

and this is equivalent to an instance of the composition law for  $\mathcal{A}$ :

$$\begin{array}{c} \text{Hom}_{\mathcal{A}}(b, b') \sqsubseteq [\text{Hom}_{\mathcal{A}}(a, b), \text{Hom}_{\mathcal{A}}(a, b')] \\ \hline \text{Hom}_{\mathcal{A}}(b, b') \otimes \text{Hom}_{\mathcal{A}}(a, b) \sqsubseteq \text{Hom}_{\mathcal{A}}(a, b') \\ \hline \text{Hom}_{\mathcal{A}}(a, b) \otimes \text{Hom}_{\mathcal{A}}(b, b') \sqsubseteq \text{Hom}_{\mathcal{A}}(a, b') \end{array}$$

Next we show that  $Y$  and  $\bar{Y}$  are fully faithful  $\mathcal{V}$ -functors:

[ $Y$  is a fully faithful  $\mathcal{V}$ -functor] The claim is

$$\begin{aligned} \text{Hom}_{\mathcal{A}}(b, b') &= \text{Hom}_{\text{Fun}(\mathcal{A}^{\text{op}}, \mathcal{V})}(Y(b), Y(b')) \\ &= \bigwedge_{a \in \text{Ob}(\mathcal{A})} \{[Y(b)(a), Y(b')(a)]\} \\ &= \bigwedge_{a \in \text{Ob}(\mathcal{A})} \{[\text{Hom}_{\mathcal{A}}(a, b), \text{Hom}_{\mathcal{A}}(a, b')]\}. \end{aligned}$$

We prove this by showing the two inequalities:

$[\text{Hom}_{\mathcal{A}}(b, b') \sqsubseteq \bigwedge_{a \in \text{Ob}(\mathcal{A})} \{[\text{Hom}_{\mathcal{A}}(a, b), \text{Hom}_{\mathcal{A}}(a, b')]\}]$  It is equivalent to

$$\text{Hom}_{\mathcal{A}}(b, b') \sqsubseteq [\text{Hom}_{\mathcal{A}}(a, b), \text{Hom}_{\mathcal{A}}(a, b')] \quad (\forall a \in \text{Ob}(\mathcal{A}))$$

and its proof for some fixed  $a \in \text{Ob}(\mathcal{A})$  is already given above when we prove that  $\bar{Y}(a)$  is a copresheaf on  $\mathcal{A}$ .

$[\bigwedge_{a \in \text{Ob}(\mathcal{A})} \{[\text{Hom}_{\mathcal{A}}(a, b), \text{Hom}_{\mathcal{A}}(a, b')]\} \sqsubseteq \text{Hom}_{\mathcal{A}}(b, b')]$  First note that

$$\bigwedge_{a \in \text{Ob}(\mathcal{A})} \{[\text{Hom}_{\mathcal{A}}(a, b), \text{Hom}_{\mathcal{A}}(a, b')]\} \sqsubseteq [\text{Hom}_{\mathcal{A}}(b, b), \text{Hom}_{\mathcal{A}}(b, b')]$$

holds. Using an instance of the identity law for  $\mathcal{A}$ ,  $e \sqsubseteq \text{Hom}_{\mathcal{A}}(b, b)$ , we have the following:

$$\begin{aligned} [\text{Hom}_{\mathcal{A}}(b, b), \text{Hom}_{\mathcal{A}}(b, b')] &\sqsubseteq [e, \text{Hom}_{\mathcal{A}}(b, b')] \\ &= \text{Hom}_{\mathcal{A}}(b, b'). \end{aligned}$$

Therefore the claim follows.

[ $\bar{Y}$  is a fully faithful  $\mathcal{V}$ -functor] The claim is

$$\begin{aligned} \text{Hom}_{\mathcal{A}}(a, a') &= \text{Hom}_{\text{Fun}(\mathcal{A}, \mathcal{V})}(\bar{Y}(a'), \bar{Y}(a)) \\ &= \bigwedge_{b \in \text{Ob}(\mathcal{A})} \{[\bar{Y}(a')(b), \bar{Y}(a)(b)]\} \\ &= \bigwedge_{b \in \text{Ob}(\mathcal{A})} \{[\text{Hom}_{\mathcal{A}}(a', b), \text{Hom}_{\mathcal{A}}(a, b)]\}. \end{aligned}$$

We prove this by showing the two inequalities:

$[\text{Hom}_{\mathcal{A}}(a, a') \sqsubseteq \bigwedge_{b \in \text{Ob}(\mathcal{A})} \{[\text{Hom}_{\mathcal{A}}(a', b), \text{Hom}_{\mathcal{A}}(a, b)]\}]$  It is equivalent to

$$\text{Hom}_{\mathcal{A}}(a, a') \sqsubseteq [\text{Hom}_{\mathcal{A}}(a', b), \text{Hom}_{\mathcal{A}}(a, b)] \quad (\forall b \in \text{Ob}(\mathcal{A}))$$

and its proof for some fixed  $b \in \text{Ob}(\mathcal{A})$  is already given above when we prove that  $Y(b)$  is a presheaf on  $\mathcal{A}$ .

$[\bigwedge_{b \in \text{Ob}(\mathcal{A})} \{[\text{Hom}_{\mathcal{A}}(a', b), \text{Hom}_{\mathcal{A}}(a, b)]\} \sqsubseteq \text{Hom}_{\mathcal{A}}(a, a')]$  First note that

$$\bigwedge_{b \in \text{Ob}(\mathcal{A})} \{[\text{Hom}_{\mathcal{A}}(a', b), \text{Hom}_{\mathcal{A}}(a, b)]\} \sqsubseteq [\text{Hom}_{\mathcal{A}}(a', a'), \text{Hom}_{\mathcal{A}}(a, a')]$$

holds. Using an instance of the identity law for  $\mathcal{A}$ ,  $e \sqsubseteq \text{Hom}_{\mathcal{A}}(a', a')$ , we have the following:

$$\begin{aligned} [\text{Hom}_{\mathcal{A}}(a', a'), \text{Hom}_{\mathcal{A}}(a, a')] &\sqsubseteq [e, \text{Hom}_{\mathcal{A}}(a, a')] \\ &= \text{Hom}_{\mathcal{A}}(a, a'). \end{aligned}$$

Therefore the claim follows. □

**Definition 2.47.** The  $\mathcal{V}$ -functor  $Y$  is called the **Yoneda embedding**, and  $\bar{Y}$  the **co-Yoneda embedding**. ■

We see some examples of the (co-)Yoneda embeddings.

**Example 2.48.** Let  $\mathcal{A} = (\text{Ob}(\mathcal{A}), \preceq_{\mathcal{A}})$  be a **2**-category. Under the identification of presheaves on  $\mathcal{A}$  and lower sets of  $\mathcal{A}$ , the Yoneda embedding turns out to be the well-known *principal ideal* operation  $\downarrow(-)$ :

$$\begin{aligned} \text{Ob}(Y(b))^{-1}(\{\text{true}\}) &= \downarrow(b) \\ &= \{a \in \text{Ob}(\mathcal{A}) \mid a \preceq_{\mathcal{A}} b\}. \end{aligned}$$

Dually, the co-Yoneda embedding corresponds to the *principal filter* operation  $\uparrow(-)$ :

$$\begin{aligned} \text{Ob}(\bar{Y}(a))^{-1}(\{\text{true}\}) &= \uparrow(a) \\ &= \{b \in \text{Ob}(\mathcal{A}) \mid a \preceq_{\mathcal{A}} b\}. \end{aligned}$$

Therefore, the Yoneda embedding theorem specializes to the statement that *every pre-ordered set embeds into the poset of its lower subsets (ordered by the inclusion) and the poset of its upper subsets (ordered by the opposite of the inclusion)*:

$$\begin{aligned} b \preceq_{\mathcal{A}} b' &\iff \downarrow(b) \subseteq \downarrow(b'); \\ a \preceq_{\mathcal{A}} a' &\iff \uparrow(a) \supseteq \uparrow(a'). \end{aligned}$$

But note that such *embeddings* are not injective unless the preorder is an order. ■

**Example 2.49.** Let  $\mathcal{A} = (\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  be a  $\overline{\mathbb{K}}_+$ -category. The Yoneda embedding  $Y$  sends a point  $b$  of  $\mathcal{A}$  to the “distance to  $b$ ” function:

$$\text{Ob}(Y(b)) = \lambda a \in \text{Ob}(\mathcal{A}^{\text{op}}). d_{\mathcal{A}}(a, b).$$

The co-Yoneda embedding  $\bar{Y}$  sends a point  $a$  of  $\mathcal{A}$  to the “distance from  $a$ ” function:

$$\text{Ob}(\bar{Y}(a)) = \lambda b \in \text{Ob}(\mathcal{A}). d_{\mathcal{A}}(a, b).$$

The Yoneda embedding theorem reads:

$$\begin{aligned} d_{\mathcal{A}}(b, b') &= d_{\text{Fun}(\mathcal{A}^{\text{op}}, \overline{\mathbb{K}}_+)}(Y(b), Y(b')); \\ d_{\mathcal{A}}(a, a') &= d_{\text{Fun}(\mathcal{A}, \overline{\mathbb{K}}_+)^{\text{op}}}(\bar{Y}(a), \bar{Y}(a')). \end{aligned}$$

If  $\mathcal{A}$  is a classical metric space, then by symmetry  $Y$  and  $\bar{Y}$  coincide and turn out to be the so-called *Fréchet embedding*. ■

**Example 2.50.** Let  $\mathcal{A} = (\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  be a  $\overline{\mathbb{K}}$ -category. We have

$$\text{Ob}(Y(b)) = \lambda a \in \text{Ob}(\mathcal{A}^{\text{op}}). d_{\mathcal{A}}(a, b), \quad \text{Ob}(\overline{Y}(a)) = \lambda b \in \text{Ob}(\mathcal{A}). d_{\mathcal{A}}(a, b)$$

and

$$d_{\mathcal{A}}(b, b') = d_{\text{Fun}(\mathcal{A}^{\text{op}}, \overline{\mathbb{K}})}(Y(b), Y(b')), \quad d_{\mathcal{A}}(a, a') = d_{\text{Fun}(\mathcal{A}, \overline{\mathbb{K}})^{\text{op}}}(\overline{Y}(a), \overline{Y}(a')).$$

■

**Example 2.51.** Let  $\mathcal{A} = (\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  be a  $\overline{\mathbb{K}}_+^{\text{Cart}}$ -category. We have

$$\text{Ob}(Y(b)) = \lambda a \in \text{Ob}(\mathcal{A}^{\text{op}}). d_{\mathcal{A}}(a, b), \quad \text{Ob}(\overline{Y}(a)) = \lambda b \in \text{Ob}(\mathcal{A}). d_{\mathcal{A}}(a, b)$$

and

$$d_{\mathcal{A}}(b, b') = d_{\text{Fun}(\mathcal{A}^{\text{op}}, \overline{\mathbb{K}}_+^{\text{Cart}})}(Y(b), Y(b')), \quad d_{\mathcal{A}}(a, a') = d_{\text{Fun}(\mathcal{A}, \overline{\mathbb{K}}_+^{\text{Cart}})^{\text{op}}}(\overline{Y}(a), \overline{Y}(a')).$$

■

## Chapter 3

# $\overline{\mathbb{K}}$ -Categories and $\overline{\mathbb{K}}$ -Extended L-Convex Sets

We are now ready to show the main result, a correspondence between  $\overline{\mathbb{K}}$ -categories and  $\overline{\mathbb{K}}$ -extended L-convex sets, which are variants of L-convex sets or L-convex polyhedra of discrete convex analysis. A chief difference between  $\overline{\mathbb{K}}$ -extended L-convex sets and L-convex sets (corresponds to the case  $\mathbb{K} = \mathbb{Z}$ ) or L-convex polyhedra ( $\mathbb{K} = \mathbb{R}$ ) is that the former have the ambient sets of the form  $\overline{\mathbb{K}}^V$  whereas the latter two  $\mathbb{Z}^V$  or  $\mathbb{R}^V$  ( $V$  is some set). In this chapter, we introduce  $\overline{\mathbb{K}}$ -extended L-convex sets and homomorphisms between them, and exploit these to show a duality theorem between  $\overline{\mathbb{K}}$ -categories and  $\overline{\mathbb{K}}$ -extended L-convex sets. The theorem rests on the fact that the set  $\mathbb{K} \cup \{-\infty, \infty\}$  can canonically be seen both as a  $\overline{\mathbb{K}}$ -category and as a  $\overline{\mathbb{K}}$ -extended L-convex set, and has two faces. Using this fact, both directions of the constructions in the duality (namely, from  $\overline{\mathbb{K}}$ -categories to  $\overline{\mathbb{K}}$ -extended L-convex sets, and conversely) are realized as that of forming the function spaces with codomain  $\overline{\mathbb{K}}$  (either as a  $\overline{\mathbb{K}}$ -category or as a  $\overline{\mathbb{K}}$ -extended L-convex set). As a byproduct, we remark that (the underlying sets of)  $\overline{\mathbb{K}}$ -extended L-convex sets, seen as  $\overline{\mathbb{K}}$ -categories with distance functions the restrictions of that for  $\overline{\mathbb{K}}^V$  with sup-distances, are nothing but  $\overline{\mathbb{K}}$ -presheaf categories. This result gives rise to a claim that (variants of) *L-convex sets arise canonically in the enriched-categorical setting*, suggesting the possibility of a categorical approach to discrete convex analysis.

### 3.1 $\overline{\mathbb{K}}$ -Extended L-Convex Sets

We begin with the definition of  $\overline{\mathbb{K}}$ -extended L-convex sets (recall that  $\mathbb{K}$  denotes either  $\mathbb{Z}$  or  $\mathbb{R}$ ).

**Definition 3.1.** A  $\overline{\mathbb{K}}$ -extended L-convex set  $\mathbf{D}$  is a pair  $(\text{Ind}(\mathbf{D}), \mathbf{D}_0)$  where

- $\text{Ind}(\mathbf{D})$  is a set called the **index set** of  $\mathbf{D}$ ;
- $\mathbf{D}_0$  is a subset of  $\overline{\mathbb{K}}^{\text{Ind}(\mathbf{D})}$  called the **underlying set** of  $\mathbf{D}$ ;

such that the following axioms hold:

- (Order completeness)  $\bigvee_{p \in S} p, \bigwedge_{p \in S} p \in \mathbf{D}_0 \quad (\forall S \subseteq \mathbf{D}_0)$ ;
- (Weight completeness)  $p + \alpha \cdot \mathbf{1} \in \mathbf{D}_0 \quad (\forall p \in \mathbf{D}_0, \forall \alpha \in \mathbb{K} \cup \{-\infty, \infty\})$ .

■

Let us clarify the notations.  $\overline{\mathbb{K}}^{\text{Ind}(\mathbf{D})}$  denotes the set of all maps from the set  $\text{Ind}(\mathbf{D})$  to

$\mathbb{K} \cup \{-\infty, \infty\}$ . The supremum  $\bigvee$  and the infimum  $\bigwedge$  in  $\overline{\mathbb{K}}^{\text{Ind}(\mathbf{D})}$  are taken coordinate-wise:

$$\left(\bigvee_{p \in S} p\right)(v) = \bigvee_{p \in S} (p(v)), \quad \left(\bigwedge_{p \in S} p\right)(v) = \bigwedge_{p \in S} (p(v)) \quad (\forall v \in \text{Ind}(\mathbf{D})),$$

where the supremum and the infimum in the right hand sides are taken in the set  $\mathbb{K} \cup \{-\infty, \infty\}$  using the order  $\geq$  (recall that since  $\geq$  is the opposite of the usual ordering  $\leq$ , the supremum  $\bigvee$  here corresponds to the usual  $\inf$ , and the infimum  $\bigwedge$  the usual  $\sup$ ; we continue to use our previous notational conventions  $\bigvee = \inf$ ,  $\bigwedge = \sup$ ).  $\alpha \cdot \mathbf{1}$  represents an element of  $\overline{\mathbb{K}}^{\text{Ind}(\mathbf{D})}$  which is the “all- $\alpha$  vector” or the constant map with the unique value  $\alpha$ :

$$(\alpha \cdot \mathbf{1})(v) = \alpha \quad (\forall v \in \text{Ind}(\mathbf{D})).$$

The operation  $+$  on  $\overline{\mathbb{K}}^{\text{Ind}(\mathbf{D})}$  is defined coordinate-wise using the extended addition of Example 2.9:

$$(p + q)(v) = p(v) + q(v) \quad (\forall v \in \text{Ind}(\mathbf{D})).$$

One can define the operation  $-$  on  $\overline{\mathbb{K}}^{\text{Ind}(\mathbf{D})}$  in the similar way. The name “weight completeness” has its origin in the fact that together with the order completeness, it assures the existence of all *weighted limits* and *weighted colimits*, which are fundamental notions of enriched category theory.

First we check that the  $-$  version of the weight completeness condition comes for free.

**Proposition 3.2.** *Let  $\mathbf{D} = (\text{Ind}(\mathbf{D}), \mathbf{D}_0)$  be a  $\overline{\mathbb{K}}$ -extended L-convex set. Then the following hold:*

$$p - \alpha \cdot \mathbf{1} \in \mathbf{D}_0 \quad (\forall p \in \mathbf{D}_0, \forall \alpha \in \mathbb{K} \cup \{-\infty, \infty\}). \quad (3.1)$$

*Proof.* When  $\alpha \in \mathbb{K}$ , then clearly

$$p - \alpha \cdot \mathbf{1} = p + (-\alpha) \cdot \mathbf{1}.$$

If  $\alpha = -\infty$ , then

$$(p - (-\infty) \cdot \mathbf{1})(v) = \begin{cases} -\infty & (p(v) = -\infty) \\ \infty & (\text{otherwise}) \end{cases} \quad (\forall v \in \text{Ind}(\mathbf{D})),$$

and this is realized as

$$\bigwedge_{\alpha \in \mathbb{K}} (p + \alpha \cdot \mathbf{1}) = \sup_{\alpha \in \mathbb{K}} \{p + \alpha \cdot \mathbf{1}\}.$$

Finally, if  $\alpha = \infty$ , then

$$(p - \infty \cdot \mathbf{1})(v) = -\infty \quad (\forall v \in \text{Ind}(\mathbf{D}))$$

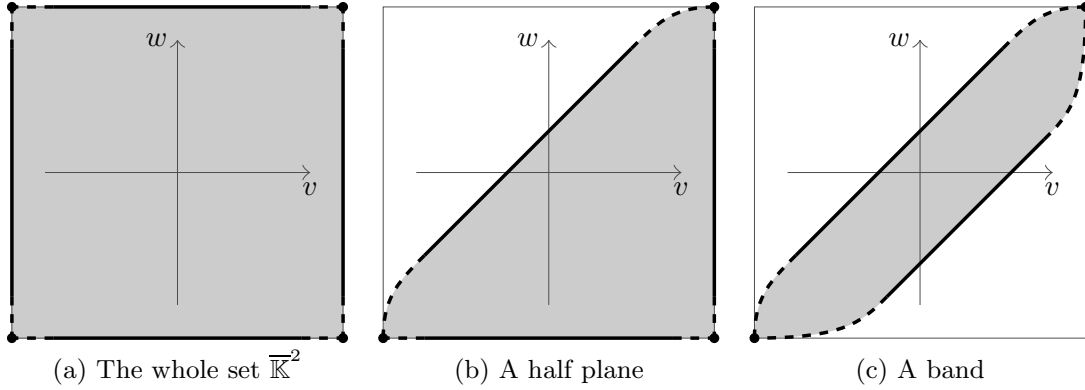
and this is written as

$$\bigwedge \emptyset = \sup \emptyset.$$

Hence, in either case  $(p - \alpha \cdot \mathbf{1}) \in \mathbf{D}_0$  by the order and weight completeness of  $\mathbf{D}$ .  $\square$

In what follows, we refer to the condition (3.1) also as the weight completeness of  $\mathbf{D}$ . The definition of  $\overline{\mathbb{K}}$ -extended L-convex sets slightly differs from that of L-convex sets (when  $\mathbb{K} = \mathbb{Z}$ ) or L-convex polyhedra (when  $\mathbb{K} = \mathbb{R}$ ). For comparison, below we present the definitions of them.



Fig. 3.1: The underlying sets of  $\overline{\mathbb{K}}$ -extended L-convex sets

**Definition 3.3** ([8]). Let  $V$  be a *finite* set. A subset  $D \subseteq \mathbb{Z}^V$  is an **L-convex set** if the following hold:

- (Nonemptiness)  $D \neq \emptyset$ ;
- (SBS[ $\mathbb{Z}$ ])  $p \wedge q, p \vee q \in D \quad (\forall p, q \in D)$ ;
- (TRS[ $\mathbb{Z}$ ])  $p + \alpha \cdot \mathbf{1} \in D \quad (\forall p \in D, \forall \alpha \in \mathbb{Z})$ .

■

**Definition 3.4** ([8]). Let  $V$  be a *finite* set. A subset  $D \subseteq \mathbb{R}^V$  is an **L-convex polyhedron** if the following hold:

- (Nonemptiness)  $D \neq \emptyset$ ;
- (Closedness)  $D$  is a closed subset of  $\mathbb{R}^V$ ;
- (SBS[ $\mathbb{R}$ ])  $p \wedge q, p \vee q \in D \quad (\forall p, q \in D)$ ;
- (TRS[ $\mathbb{R}$ ])  $p + \alpha \cdot \mathbf{1} \in D \quad (\forall p \in D, \forall \alpha \in \mathbb{R})$ .

■

We note that the order completeness condition entails the nonemptiness condition; just take  $S = \emptyset$ . Since we impose on  $\overline{\mathbb{K}}$ -extended L-convex sets the order completeness condition stronger than its binary versions (SBS[ $\mathbb{Z}$ ] or SBS[ $\mathbb{R}$ ]), the index sets are no longer limited to be finite. However, we adopt the order completeness condition not just because we can treat infinite index sets; for our purpose, the binary versions do not work even when  $\text{Ind}(\mathbf{D})$  is finite.

Let us illustrate the point by observing examples of  $\overline{\mathbb{K}}$ -extended L-convex sets and a non-example we wish to exclude. Fig. 3.1 presents examples of (the underlying sets of) typical  $\overline{\mathbb{K}}$ -extended L-convex sets with two-element index sets. We first explain the intended meaning of the figures using Fig. 3.1c; let  $\mathbf{D}$  be the  $\overline{\mathbb{K}}$ -extended L-convex set drawn on it. Since the ambient set of  $\mathbf{D}_0$  is  $\overline{\mathbb{K}}^{\text{Ind}(\mathbf{D})} \cong \overline{\mathbb{K}}^2$ , there are points whose coordinates include  $-\infty$  or  $\infty$ ; the outer frame of the figure represents these points. The subset  $\mathbf{D}_0$  is represented by the gray region (when  $\mathbb{K} = \mathbb{Z}$ , the region should be interpreted as a set of lattice points). The boundaries of  $\mathbf{D}_0$  is two “lines of gradient 1”; although they are straight, they pass the points  $(-\infty, -\infty)$  (at the lower left) and  $(\infty, \infty)$  (the upper right), and to indicate this fact we bend the lines in the margin, where they are also dashed. We place two black dots at the lower left and the upper right to indicate  $(-\infty, -\infty) \in \mathbf{D}_0$  and  $(\infty, \infty) \in \mathbf{D}_0$ , respectively. In Fig. 3.1b, parts of the frame, namely the bottom and right lines, are drawn boldly. This represents that points of the forms respectively  $(s, -\infty)$  and

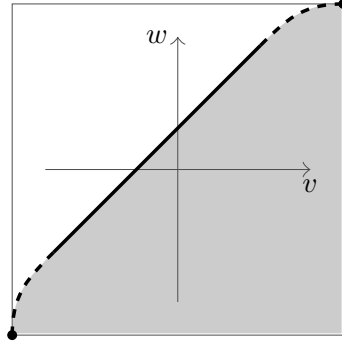


Fig. 3.2: Fig. 3.1b without the bottom and right lines

$(\infty, t)$  ( $s, t \in \mathbb{K}$ ), are in the underlying set. Later we show that  $\overline{\mathbb{K}}$ -extended L-convex sets with two-element index sets correspond to  $\overline{\mathbb{K}}$ -categories with two points, and as an illustration present the correspondence at the level of shapes, with a more exhaustive enumeration.

An example of subsets of  $\overline{\mathbb{K}}^2$  which we want to exclude from the underlying sets of  $\overline{\mathbb{K}}$ -extended L-convex sets with two-element index sets is drawn in Fig. 3.2. This set satisfies the weight completeness condition and the binary version of the order completeness condition. These weaker conditions fail to assure e.g., if  $(0, 0), (1, 0), (2, 0), \dots$  are in the set then so is  $(\infty, 0)$ , and for such entities we cannot establish a nice connection with  $\overline{\mathbb{K}}$ -categories.

Another, more concrete example of  $\overline{\mathbb{K}}$ -extended L-convex sets is the following. It will play a crucial role in the formulation of the duality theorems.

**Example 3.5.** The pair  $\overline{\mathbb{K}} = (\{*\}, \mathbb{K} \cup \{-\infty, \infty\})$ , where  $\{*\}$  is a singleton and  $\mathbb{K} \cup \{-\infty, \infty\}$  is a (non-proper) subset of  $\overline{\mathbb{K}}^{\{*\}}$ , is clearly a  $\overline{\mathbb{K}}$ -extended L-convex set. ■

## 3.2 Homomorphisms of $\overline{\mathbb{K}}$ -Extended L-Convex Sets

**Definition 3.6.** Let  $\mathbf{D} = (\text{Ind}(\mathbf{D}), \mathbf{D}_0)$  and  $\mathbf{E} = (\text{Ind}(\mathbf{E}), \mathbf{E}_0)$  be  $\overline{\mathbb{K}}$ -extended L-convex sets. A **homomorphism**

$$\Phi: \mathbf{D} \longrightarrow \mathbf{E}$$

from  $\mathbf{D}$  to  $\mathbf{E}$  is a map

$$\text{Ind}(\Phi): \text{Ind}(\mathbf{E}) \longrightarrow \text{Ind}(\mathbf{D})$$

from the index set of  $\mathbf{E}$  to that of  $\mathbf{D}$ , such that for all  $p \in \mathbf{D}_0$ , the element  $\Phi_0(p) \in \overline{\mathbb{K}}^{\text{Ind}(\mathbf{E})}$  given by

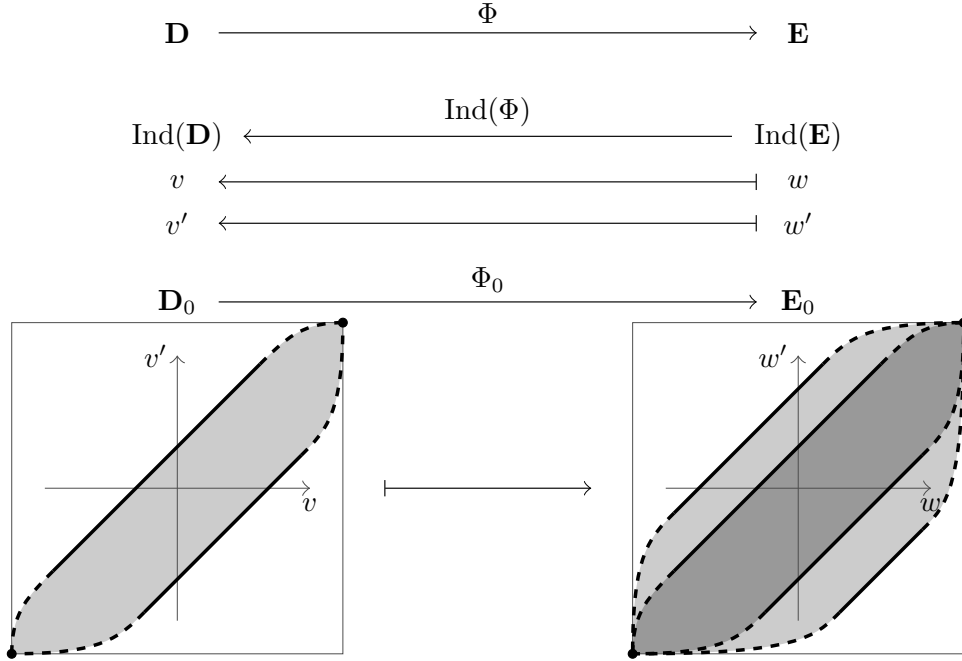
$$\Phi_0(p) = p \circ \text{Ind}(\Phi): \text{Ind}(\mathbf{E}) \xrightarrow{\text{Ind}(\Phi)} \text{Ind}(\mathbf{D}) \xrightarrow{p} \mathbb{K} \cup \{-\infty, \infty\}$$

is an element of  $\mathbf{E}_0$ .

$\mathbf{D}$  is called the **domain** of  $\Phi$  and  $\mathbf{E}$  the **codomain** of  $\Phi$ . We denote the set of all homomorphisms from  $\mathbf{D}$  to  $\mathbf{E}$  by  $\llbracket \mathbf{D}, \mathbf{E} \rrbracket_0$ . ■

The condition for a map  $\text{Ind}(\Phi): \text{Ind}(\mathbf{E}) \longrightarrow \text{Ind}(\mathbf{D})$  to define a homomorphism of  $\overline{\mathbb{K}}$ -extended L-convex sets assures that we can always construct a map (called the **underlying map** of  $\Phi$ )

$$\Phi_0: \mathbf{D}_0 \longrightarrow \mathbf{E}_0$$


 Fig. 3.3: A homomorphism between  $\overline{\mathbb{K}}$ -extended L-convex sets

from the underlying set of  $\mathbf{D}$  to that of  $\mathbf{E}$  (the same direction as the homomorphism  $\Phi$ ). Therefore one can intuitively think of a homomorphism as *a map between the underlying sets with a special property*, namely that all it does is to relabel the indices in some fixed manner (determined by  $\text{Ind}(\Phi)$ ). However, note that the *equality* between homomorphisms  $\Phi, \Psi: \mathbf{D} \rightarrow \mathbf{E}$  are defined as that between the maps  $\text{Ind}(\Phi), \text{Ind}(\Psi): \text{Ind}(\mathbf{E}) \rightarrow \text{Ind}(\mathbf{D})$  of the index sets and not just between the underlying maps  $\Phi_0, \Psi_0: \mathbf{D}_0 \rightarrow \mathbf{E}_0$  (there do exist different homomorphisms with a common underlying map). Fig. 3.3 illustrates an example of homomorphisms between  $\overline{\mathbb{K}}$ -extended L-convex sets with two-element index sets. In this example,  $\Phi$  is given by  $\text{Ind}(\Phi): \{w, w'\} \rightarrow \{v, v'\}$  with  $\text{Ind}(\Phi)(w) = v$  and  $\text{Ind}(\Phi)(w') = v'$ . Correspondingly, we find that  $\Phi_0: \mathbf{D}_0 \rightarrow \mathbf{E}_0$  embeds the “thin band”  $\mathbf{D}_0$  into the “thick band”  $\mathbf{E}_0$  (the dark region in  $\mathbf{E}_0$  represents the image of  $\mathbf{D}_0$  through the map  $\Phi_0$ ). Observe that unlike  $\text{Ind}(\Phi)$ ,  $\text{Ind}(\Phi)^{-1}: \{v, v'\} \rightarrow \{w, w'\}$  does not define a homomorphism, because  $\mathbf{E}_0$  does not fit in  $\mathbf{D}_0$ .

We note that homomorphisms include identity maps and are closed under composition (therefore,  $\overline{\mathbb{K}}$ -extended L-convex sets together with their homomorphisms, form a category). Using these facts, we define the notion of isomorphisms between  $\overline{\mathbb{K}}$ -extended L-convex sets as follows:

**Definition 3.7.** Let  $\mathbf{D}$  and  $\mathbf{E}$  be  $\overline{\mathbb{K}}$ -extended L-convex sets. A homomorphism

$$\Phi: \mathbf{D} \rightarrow \mathbf{E}$$

from  $\mathbf{D}$  to  $\mathbf{E}$  is an **isomorphism** if there exists a homomorphism (called its **inverse**)

$$\Psi: \mathbf{E} \rightarrow \mathbf{D}$$

from  $\mathbf{E}$  to  $\mathbf{D}$  such that  $\Psi \circ \Phi = 1_{\mathbf{D}}$  and  $\Phi \circ \Psi = 1_{\mathbf{E}}$  hold, where  $1_{\mathbf{D}}$  and  $1_{\mathbf{E}}$  are the identity homomorphisms on  $\mathbf{D}$  and  $\mathbf{E}$ , respectively.

If there is an isomorphism between  $\mathbf{D}$  and  $\mathbf{E}$ , then they are said to be **isomorphic** and written as  $\mathbf{D} \cong \mathbf{E}$ . ■

In other words,  $\Psi: \mathbf{E} \rightarrow \mathbf{D}$  is the inverse of  $\Phi: \mathbf{D} \rightarrow \mathbf{E}$  if and only if the map  $\text{Ind}(\Psi): \text{Ind}(\mathbf{D}) \rightarrow \text{Ind}(\mathbf{E})$  is the inverse of  $\text{Ind}(\Phi): \text{Ind}(\mathbf{E}) \rightarrow \text{Ind}(\mathbf{D})$ .

In Definition 2.21 we defined the canonical ordering  $\Rightarrow$  on the set  $[\mathcal{A}, \mathcal{B}]_0$  of all  $\mathcal{V}$ -functors from a  $\mathcal{V}$ -category  $\mathcal{A}$  to  $\mathcal{B}$ , hence also for the special case of  $\mathcal{V} = \overline{\mathbb{K}}$ . Now we define a similar relation (also denoted by  $\Rightarrow$ ) on the set  $[\mathbf{D}, \mathbf{E}]_0$  of homomorphisms.

**Definition 3.8.** Let  $\mathbf{D} = (\text{Ind}(\mathbf{D}), \mathbf{D}_0)$  and  $\mathbf{E} = (\text{Ind}(\mathbf{E}), \mathbf{E}_0)$  be  $\overline{\mathbb{K}}$ -extended L-convex sets. The set  $[\mathbf{D}, \mathbf{E}]$  of all homomorphisms from  $\mathbf{D}$  to  $\mathbf{E}$  admits a canonical relation  $\Rightarrow$  defined as follows:

$$\Phi \Rightarrow \Psi \iff \Phi_0(p) \geq \Psi_0(p) \quad (\forall p \in \mathbf{D}_0),$$

where  $\Phi, \Psi: \mathbf{D} \rightarrow \mathbf{E}$  are homomorphisms and  $\geq$  is the coordinate-wise order on  $\mathbf{E}_0$ . We call  $\Rightarrow$  the **canonical ordering** on  $[\mathbf{D}, \mathbf{E}]_0$ .  $\blacksquare$

Note that we can equip the underlying set  $\mathbf{D}_0$  with a distance function  $d_{\mathbf{D}_0}$  defined by the restriction of the sup-distance on  $\overline{\mathbb{K}}^{\text{Ind}(\mathbf{D})}$ . Under these distances, one can show that the underlying map  $\Phi_0$  of every homomorphism  $\Phi: \mathbf{D} \rightarrow \mathbf{E}$  is nonexpansive, hence defines a  $\overline{\mathbb{K}}$ -functor. Therefore, the notion of the canonical ordering  $\Rightarrow$  for  $\overline{\mathbb{K}}$ -functors restricts to that for the underlying maps of homomorphisms. The above definition coincides with the result of this approach. Hence  $\Rightarrow$  for homomorphisms is also a preorder, and is preserved under composition (note that  $(\Psi \circ \Phi)_0 = \Psi_0 \circ \Phi_0$ ).

### 3.3 Duality Theorems

In this section we show the main result of the thesis, a duality theorem between  $\overline{\mathbb{K}}$ -categories and  $\overline{\mathbb{K}}$ -extended L-convex sets. First we establish the duality at the level of individual objects; the resulting Theorem 3.9 states that there is a one-to-one correspondence between the isomorphism class of  $\overline{\mathbb{K}}$ -categories and that of  $\overline{\mathbb{K}}$ -extended L-convex sets. In fact, a similar result for L-convex sets and L-convex polyhedra is already known via almost parallel constructions in the context of discrete convex analysis. However, there is an essentially novel aspect (conceptually, rather than technically) in our approach; we employ the notions of  $\overline{\mathbb{K}}$ -functors and homomorphisms of  $\overline{\mathbb{K}}$ -extended L-convex sets in the constructions of the duality, and realize them as the function space constructions with codomain  $\overline{\mathbb{K}}$ .

By introducing homomorphisms, we can compare these “structure-preserving maps” of  $\overline{\mathbb{K}}$ -extended L-convex sets with  $\overline{\mathbb{K}}$ -functors, the equally natural maps between  $\overline{\mathbb{K}}$ -categories. We can describe the duality at the level of maps as well; we obtain Theorem 3.13 as a result. In categorical terminology, we establish a *dual equivalence* between the category of  $\overline{\mathbb{K}}$ -categories and the category of  $\overline{\mathbb{K}}$ -extended L-convex sets.

Finally, we conclude the duality in Theorem 3.16 by showing that there is also a correspondence between the canonical orderings  $\Rightarrow$  on  $\overline{\mathbb{K}}$ -functors and on homomorphisms, hence at the level of canonical orderings. In fact, the categories of  $\overline{\mathbb{K}}$ -categories and  $\overline{\mathbb{K}}$ -extended L-convex sets are (strict) 2-categories (do not confuse with **2**-categories, which are just preordered sets) by the canonical orderings  $\Rightarrow$ . What we show is that our construction gives a (strict) *2-equivalence of 2-categories*.

If we denote the constructions for objects by  $[-, \overline{\mathbb{K}}]$  (from a  $\overline{\mathbb{K}}$ -category to a  $\overline{\mathbb{K}}$ -extended L-convex set) and  $[\overline{\mathbb{K}}, -]$  (from a  $\overline{\mathbb{K}}$ -extended L-convex set to a  $\overline{\mathbb{K}}$ -category), the duality at the object-level is expressed as follows:

**Theorem 3.9.** *Let  $\mathcal{A} = (\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  be a  $\overline{\mathbb{K}}$ -category and  $\mathbf{D} = (\text{Ind}(\mathbf{D}), \mathbf{D}_0)$  be a  $\overline{\mathbb{K}}$ -extended L-convex set. Then the following hold:*

- (i)  $\mathcal{A} \cong \llbracket [\mathcal{A}, \overline{\mathbb{K}}], \overline{\mathbb{K}} \rrbracket$ .  
(ii)  $\mathbf{D} \cong \llbracket [\mathbf{D}, \overline{\mathbb{K}}], \overline{\mathbb{K}} \rrbracket$ .

Recall that  $\cong$  means “is isomorphic to.” Let us define the constructions and prove that they certainly work.

**Lemma 3.10.** *Let  $\mathcal{A} = (\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  be a  $\overline{\mathbb{K}}$ -category. Then the pair  $[\mathcal{A}, \overline{\mathbb{K}}] = (\text{Ob}(\mathcal{A}), [\mathcal{A}, \overline{\mathbb{K}}]_0)$ , where  $[\mathcal{A}, \overline{\mathbb{K}}]_0$  is the set of all nonexpansive maps from  $\mathcal{A}$  to  $\overline{\mathbb{K}}$  (as a  $\overline{\mathbb{K}}$ -category), seen as a subset of  $\overline{\mathbb{K}}^{\text{Ob}(\mathcal{A})}$ , is a  $\overline{\mathbb{K}}$ -extended  $L$ -convex set.*

*Proof.* [Order completeness] Let  $S$  be a set of nonexpansive maps from  $\mathcal{A}$  to  $\overline{\mathbb{K}}$ . The claim is that  $\bigvee_{p \in S} p, \bigwedge_{p \in S} p$  are again nonexpansive maps from  $\mathcal{A}$  to  $\overline{\mathbb{K}}$ . In other words, that

$$d_{\mathcal{A}}(a, b) \geq d_{\overline{\mathbb{K}}} \left( \bigvee_{p \in S} p(a), \bigvee_{p \in S} p(b) \right) = \inf_{p \in S} \{p(b)\} - \inf_{p \in S} \{p(a)\}, \quad (3.2)$$

$$d_{\mathcal{A}}(a, b) \geq d_{\overline{\mathbb{K}}} \left( \bigwedge_{p \in S} p(a), \bigwedge_{p \in S} p(b) \right) = \sup_{p \in S} \{p(b)\} - \sup_{p \in S} \{p(a)\} \quad (3.3)$$

hold for all  $a, b \in \text{Ob}(\mathcal{A})$ .

[(3.2)] For each fixed  $p' \in S$ , the increasing condition implies

$$d_{\mathcal{A}}(a, b) \geq p'(b) - p'(a) \geq \inf_{p \in S} \{p(b)\} - p'(a).$$

By taking sup with respect to  $p' \in S$ , we have

$$d_{\mathcal{A}}(a, b) \geq \sup_{p' \in S} \left\{ \inf_{p \in S} \{p(b)\} - p'(a) \right\} = \inf_{p \in S} \{p(b)\} - \inf_{p \in S} \{p(a)\},$$

where the equality follows from an instance of a statement in Proposition 2.5 that

$$\inf_{p \in S} \{p(b)\} - (-)$$

turns suprema (inf's) into infima (sup's).

[(3.3)] For each fixed  $p' \in S$ , the increasing condition implies

$$d_{\mathcal{A}}(a, b) \geq p'(b) - p'(a) \geq p'(b) - \sup_{p \in S} \{p(a)\}.$$

By taking sup with respect to  $p' \in S$ , we have

$$d_{\mathcal{A}}(a, b) \geq \sup_{p' \in S} \left\{ p'(b) - \sup_{p \in S} \{p(a)\} \right\} = \sup_{p \in S} \{p(b)\} - \sup_{p \in S} \{p(a)\},$$

where the equality follows from an instance of a statement in Proposition 2.4 that

$$(-) - \sup_{p \in S} \{p(a)\}$$

preserves infima (sup's).

[Weight completeness (for +)] Let  $p$  be a nonexpansive map from  $\mathcal{A}$  to  $\overline{\mathbb{K}}$  and  $\alpha \in \mathbb{K} \cup \{-\infty, \infty\}$ . The claim is that  $p + \alpha \cdot \mathbf{1}$  is again a nonexpansive map from  $\mathcal{A}$  to  $\overline{\mathbb{K}}$ . In other words, that

$$d_{\mathcal{A}}(a, b) \geq d_{\overline{\mathbb{K}}}(p(a) + \alpha, p(b) + \alpha) = (p(b) + \alpha) - (p(a) + \alpha)$$

hold for all  $a, b \in \text{Ob}(\mathcal{A})$ . We use the increasing condition for  $p$

$$d_{\mathcal{A}}(a, b) \geq p(b) - p(a)$$

and show

$$p(b) - p(a) \geq (p(b) + \alpha) - (p(a) + \alpha),$$

which has an equivalent formula:

$$\frac{\frac{p(b) - p(a) \geq (p(b) + \alpha) - (p(a) + \alpha)}{(p(b) - p(a)) + (p(a) + \alpha) \geq p(b) + \alpha}}{((p(b) - p(a)) + p(a)) + \alpha \geq p(b) + \alpha}$$

Therefore it suffices to prove  $(p(b) - p(a)) + p(a) \geq p(b)$ , as follows:

$$\frac{(p(b) - p(a)) + p(a) \geq p(b)}{p(b) - p(a) \geq p(b) - p(a)}$$

□

**Lemma 3.11.** *Let  $\mathbf{D} = (\text{Ind}(\mathbf{D}), \mathbf{D}_0)$  be a  $\overline{\mathbb{K}}$ -extended L-convex set. Then the pair  $\llbracket \mathbf{D}, \overline{\mathbb{K}} \rrbracket = (\llbracket \mathbf{D}, \overline{\mathbb{K}} \rrbracket_0, d_{\llbracket \mathbf{D}, \overline{\mathbb{K}} \rrbracket})$ , where  $\llbracket \mathbf{D}, \overline{\mathbb{K}} \rrbracket_0$  is the set of all homomorphisms from  $\mathbf{D}$  to  $\overline{\mathbb{K}}$  (as a  $\overline{\mathbb{K}}$ -extended L-convex set) and  $d_{\llbracket \mathbf{D}, \overline{\mathbb{K}} \rrbracket}$  is the distance function on it defined by the sup-distance between the underlying maps, is a  $\overline{\mathbb{K}}$ -category.*

*Proof.* First note that elements of  $\llbracket \mathbf{D}, \overline{\mathbb{K}} \rrbracket_0$  can canonically be identified with that of  $\text{Ind}(\mathbf{D})$ , via a bijection  $\pi_{\mathbf{D}}$  defined as follows:

$$\pi_{\mathbf{D}}: \text{Ind}(\mathbf{D}) \longrightarrow \llbracket \mathbf{D}, \overline{\mathbb{K}} \rrbracket_0, \quad \pi_{\mathbf{D}}(v) = \pi_v \quad (\forall v \in \text{Ind}(\mathbf{D})),$$

where

$$\text{Ind}(\pi_v): \{*\} \longrightarrow \text{Ind}(\mathbf{D}), \quad \text{Ind}(\pi_v)(*) = v.$$

The underlying maps of  $\pi_v$ 's are given as the “projection maps”:

$$(\pi_v)_0: \mathbf{D}_0 \longrightarrow \mathbb{K} \cup \{-\infty, \infty\} \subseteq \overline{\mathbb{K}}^{\{*\}} \quad (\pi_v)_0(p) = p(v).$$

See Fig. 3.4 for an illustration. Thus we may write  $\llbracket \mathbf{D}, \overline{\mathbb{K}} \rrbracket_0 = \{\pi_v\}_{v \in \text{Ind}(\mathbf{D})}$ . Also note that

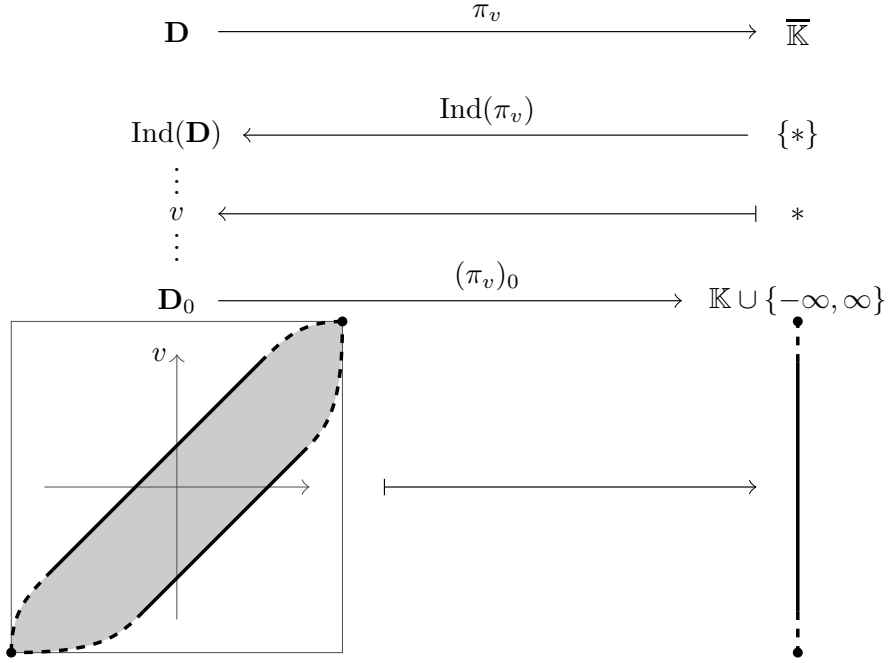
$$\begin{aligned} d_{\llbracket \mathbf{D}, \overline{\mathbb{K}} \rrbracket}(\pi_v, \pi_w) &= \sup_{p \in \mathbf{D}_0} \{d_{\overline{\mathbb{K}}}((\pi_v)_0(p), (\pi_w)_0(p))\} \\ &= \sup_{p \in \mathbf{D}_0} \{d_{\overline{\mathbb{K}}}(p(v), p(w))\} \\ &= \sup_{p \in \mathbf{D}_0} \{p(w) - p(v)\}. \end{aligned} \tag{3.4}$$

[Composition law] The claim is

$$d_{\llbracket \mathbf{D}, \overline{\mathbb{K}} \rrbracket}(\pi_u, \pi_v) + d_{\llbracket \mathbf{D}, \overline{\mathbb{K}} \rrbracket}(\pi_v, \pi_w) \geq d_{\llbracket \mathbf{D}, \overline{\mathbb{K}} \rrbracket}(\pi_u, \pi_w),$$

or by (3.4), equivalently,

$$\sup_{p \in \mathbf{D}_0} \{p(v) - p(u)\} + \sup_{p \in \mathbf{D}_0} \{p(w) - p(v)\} \geq \sup_{p \in \mathbf{D}_0} \{p(w) - p(u)\}. \tag{3.5}$$


 Fig. 3.4: The homomorphism  $\pi_v$ 

We first prove

$$(p(v) - p(u)) + (p(w) - p(v)) \geq p(w) - p(u) \quad (3.6)$$

for each  $p \in \mathbf{D}_0$ , as follows:

$$\begin{array}{c} (p(v) - p(u)) + (p(w) - p(v)) \geq p(w) - p(u) \\ \hline (p(w) - p(v)) + (p(v) - p(u)) \geq p(w) - p(u) \\ \hline p(w) - p(v) \geq (p(w) - p(u)) - (p(v) - p(u)) \end{array}$$

where the bottom expression is an instance of the composition law for  $-$ . Thus, by taking sup in (3.6) with respect to  $p$ , we have

$$\sup_{p \in \mathbf{D}_0} \{(p(v) - p(u)) + (p(w) - p(v))\} \geq \sup_{p \in \mathbf{D}_0} \{p(w) - p(u)\}. \quad (3.7)$$

On the other hand, we have

$$\sup_{p \in \mathbf{D}_0} \{p(v) - p(u)\} + \sup_{p \in \mathbf{D}_0} \{p(w) - p(v)\} \geq (p'(v) - p'(u)) + (p'(w) - p'(v))$$

for each  $p' \in \mathbf{D}_0$ , and therefore

$$\sup_{p \in \mathbf{D}_0} \{p(v) - p(u)\} + \sup_{p \in \mathbf{D}_0} \{p(w) - p(v)\} \geq \sup_{p \in \mathbf{D}_0} \{(p(v) - p(u)) + (p(w) - p(v))\}.$$

We combine this with (3.7) and obtain (3.5).

[Identity law] The claim is

$$0 \geq d_{[\mathbf{D}, \overline{\mathbb{K}}]}(\pi_v, \pi_v),$$

or by (3.4), equivalently,

$$0 \geq \sup_{p \in \mathbf{D}_0} \{p(v) - p(v)\}. \quad (3.8)$$

First note that for each  $p \in \mathbf{D}_0$

$$0 \geq p(v) - p(v) \quad (3.9)$$

holds, as proved below:

$$\frac{0 \geq p(v) - p(v)}{p(v) \geq p(v)}$$

Taking sup in (3.9) with respect to  $p$  yields (3.8). □

Now all the symbols are defined and we are ready to prove Theorem 3.9:

*Proof of Theorem 3.9.* [(i)] Since  $[\mathcal{A}, \overline{\mathbb{K}}] = (\text{Ob}(\mathcal{A}), [\mathcal{A}, \overline{\mathbb{K}}]_0)$ ,  $[[\mathcal{A}, \overline{\mathbb{K}}], \overline{\mathbb{K}}]_0 = \{\pi_a\}_{a \in \text{Ob}(\mathcal{A})}$ . We claim that

$$\eta_{\mathcal{A}}: \mathcal{A} \longrightarrow [[\mathcal{A}, \overline{\mathbb{K}}], \overline{\mathbb{K}}]$$

given by

$$\text{Ob}(\eta_{\mathcal{A}}): \text{Ob}(\mathcal{A}) \longrightarrow [[\mathcal{A}, \overline{\mathbb{K}}], \overline{\mathbb{K}}]_0, \quad \text{Ob}(\eta_{\mathcal{A}})(a) = \eta_{\mathcal{A}}(a) = \pi_a \quad (\forall a \in \text{Ob}(\mathcal{A}))$$

as a map between the sets of points, is an isomorphism (note that  $\text{Ob}(\eta_{\mathcal{A}}) = \pi_{[\mathcal{A}, \overline{\mathbb{K}}]}$ , where  $\pi_{[\mathcal{A}, \overline{\mathbb{K}}]}$  is the bijection defined in Lemma 3.11). By Proposition 2.20, it suffices to show that  $\eta_{\mathcal{A}}$  is a fully faithful  $\overline{\mathbb{K}}$ -functor (because  $\text{Ob}(\eta_{\mathcal{A}})$  is certainly bijective). Therefore, we aim to show

$$d_{\mathcal{A}}(a, b) = d_{[[\mathcal{A}, \overline{\mathbb{K}}], \overline{\mathbb{K}}]}(\pi_a, \pi_b),$$

or equivalently,

$$d_{\mathcal{A}}(a, b) = \sup_{p \in [\mathcal{A}, \overline{\mathbb{K}}]_0} \{p(b) - p(a)\}.$$

$[d_{\mathcal{A}}(a, b) \leq \sup_{p \in [\mathcal{A}, \overline{\mathbb{K}}]_0} \{p(b) - p(a)\}]$  Recall that  $\overline{Y}(a)$  defined by

$$\text{Ob}(\overline{Y}(a)) = \lambda b \in \text{Ob}(\mathcal{A}). \text{Hom}_{\mathcal{A}}(a, b)$$

is an element of  $[\mathcal{A}, \overline{\mathbb{K}}]_0$ . Therefore,

$$d_{\mathcal{A}}(a, b) - d_{\mathcal{A}}(a, a) = \overline{Y}(a)(b) - \overline{Y}(a)(a) \leq \sup_{p \in [\mathcal{A}, \overline{\mathbb{K}}]_0} \{p(b) - p(a)\}$$

and it suffices to show

$$d_{\mathcal{A}}(a, b) \geq d_{\mathcal{A}}(a, b) - d_{\mathcal{A}}(a, a),$$

but this follows from the identity law  $0 \geq d_{\mathcal{A}}(a, a)$  for  $\mathcal{A}$  and the unit law and the monotonicity of  $-$ .



$[d_{\mathcal{A}}(a, b) \geq \sup_{p \in [\mathcal{A}, \overline{\mathbb{K}}]_0} \{p(b) - p(a)\}]$  It suffices to show

$$d_{\mathcal{A}}(a, b) \geq p(b) - p(a)$$

for each  $p \in [\mathcal{A}, \overline{\mathbb{K}}]_0$ ; but this is nothing but the increasing condition for  $p$ .

[(ii)] Recall that  $[[\mathbf{D}, \overline{\mathbb{K}}]] = (\{\pi_v\}_{v \in \text{Ind}(\mathbf{D})}, d_{[[\mathbf{D}, \overline{\mathbb{K}}]]})$ , where

$$d_{[[\mathbf{D}, \overline{\mathbb{K}}]]}(\pi_v, \pi_w) = \sup_{p \in \mathbf{D}_0} \{p(w) - p(v)\}.$$

Now, the  $\overline{\mathbb{K}}$ -extended L-convex set  $[[[\mathbf{D}, \overline{\mathbb{K}}], \overline{\mathbb{K}}]] = (\{\pi_v\}_{v \in \text{Ind}(\mathbf{D})}, [[[\mathbf{D}, \overline{\mathbb{K}}], \overline{\mathbb{K}}]]_0)$  is defined as follows: for all  $q \in \overline{\mathbb{K}}^{\{\pi_v\}_{v \in \text{Ind}(\mathbf{D})}}$ ,

$$q \in [[[\mathbf{D}, \overline{\mathbb{K}}], \overline{\mathbb{K}}]]_0 \iff d_{[[\mathbf{D}, \overline{\mathbb{K}}]]}(\pi_v, \pi_w) \geq q(\pi_w) - q(\pi_v) \quad (\forall \pi_v, \pi_w \in \{\pi_v\}_{v \in \text{Ind}(\mathbf{D})}).$$

We prove that bijections

$$\begin{aligned} \text{Ind}(\varepsilon_{\mathbf{D}}): \text{Ind}(\mathbf{D}) &\longrightarrow \{\pi_v\}_{v \in \text{Ind}(\mathbf{D})}, & \text{Ind}(\varepsilon_{\mathbf{D}})(v) &= \pi_v \quad (\forall v \in \text{Ind}(\mathbf{D})), \\ \text{Ind}(\varepsilon'_{\mathbf{D}}): \{\pi_v\}_{v \in \text{Ind}(\mathbf{D})} &\longrightarrow \text{Ind}(\mathbf{D}), & \text{Ind}(\varepsilon'_{\mathbf{D}})(\pi_v) &= v \quad (\forall \pi_v \in \{\pi_v\}_{v \in \text{Ind}(\mathbf{D})}) \end{aligned}$$

define homomorphisms (note that  $\text{Ind}(\varepsilon_{\mathbf{D}}) = \pi_{\mathbf{D}}$ , where  $\pi_{\mathbf{D}}$  is the bijection defined in the proof of Lemma 3.11); in other words, there are maps

$$\begin{aligned} (\varepsilon_{\mathbf{D}})_0: [[[\mathbf{D}, \overline{\mathbb{K}}], \overline{\mathbb{K}}]]_0 &\longrightarrow \mathbf{D}_0, & (\varepsilon_{\mathbf{D}})_0(q) &= \lambda v \in \text{Ind}(\mathbf{D}). q(\pi_v) \quad (\forall q \in [[[\mathbf{D}, \overline{\mathbb{K}}], \overline{\mathbb{K}}]]_0), \\ (\varepsilon'_{\mathbf{D}})_0: \mathbf{D}_0 &\longrightarrow [[[\mathbf{D}, \overline{\mathbb{K}}], \overline{\mathbb{K}}]]_0, & (\varepsilon'_{\mathbf{D}})_0(p) &= \lambda \pi_v \in \{\pi_v\}_{v \in \text{Ind}(\mathbf{D})}. p(v) \quad (\forall p \in \mathbf{D}_0). \end{aligned}$$

To be more specific about the point, we have to verify that the images of the above maps are contained in the codomains.

$[(\varepsilon_{\mathbf{D}})_0([[[\mathbf{D}, \overline{\mathbb{K}}], \overline{\mathbb{K}}]]_0) \subseteq \mathbf{D}_0]$  First we prove that for all images  $\overline{Y}(\pi_v) \in [[[\mathbf{D}, \overline{\mathbb{K}}], \overline{\mathbb{K}}]]_0$  of the co-Yoneda embedding,  $(\varepsilon_{\mathbf{D}})_0(\text{Ob}(\overline{Y}(\pi_v)))$ , abbreviated as  $(\varepsilon_{\mathbf{D}})_0(\overline{Y}(\pi_v))$ , is an element of  $\mathbf{D}_0$ . To see this, observe that

$$\begin{aligned} (\varepsilon_{\mathbf{D}})_0(\overline{Y}(\pi_v)) &= \lambda w \in \text{Ind}(\mathbf{D}). d_{[[\mathbf{D}, \overline{\mathbb{K}}]]}(\pi_v, \pi_w) \\ &= \lambda w \in \text{Ind}(\mathbf{D}). \sup_{p \in \mathbf{D}_0} \{p(w) - p(v)\}, \end{aligned}$$

in other words,

$$\begin{aligned} (\varepsilon_{\mathbf{D}})_0(\overline{Y}(\pi_v)) &= \sup_{p \in \mathbf{D}_0} \{p - p(v) \cdot \mathbf{1}\} \\ &= \bigwedge_{p \in \mathbf{D}_0} \{p - p(v) \cdot \mathbf{1}\} \end{aligned}$$

holds. So the weight and order completeness conditions for  $\mathbf{D}$  assure  $(\varepsilon_{\mathbf{D}})_0(\overline{Y}(\pi_v)) \in \mathbf{D}_0$ .

Now take arbitrary  $q \in [[[\mathbf{D}, \overline{\mathbb{K}}], \overline{\mathbb{K}}]]_0$ . We have

$$d_{[[\mathbf{D}, \overline{\mathbb{K}}]]}(\pi_v, \pi_w) \geq q(\pi_w) - q(\pi_v) \quad (\forall \pi_v, \pi_w \in \{\pi_v\}_{v \in \text{Ind}(\mathbf{D})}).$$

The adjointness relation (for the SMC-CL  $\overline{\mathbb{K}}$ ) yields

$$d_{[[\mathbf{D}, \overline{\mathbb{K}}]]}(\pi_v, \pi_w) + q(\pi_v) \geq q(\pi_w) \quad (\forall \pi_v, \pi_w \in \{\pi_v\}_{v \in \text{Ind}(\mathbf{D})}),$$

so, with respect to the coordinate-wise order on  $\overline{\mathbb{K}}^{\text{Ind}(\mathbf{D})}$  (also denoted by  $\geq$ ), we obtain

$$(\varepsilon_{\mathbf{D}})_0(\overline{Y}(\pi_v)) + q(\pi_v) \cdot \mathbf{1} \geq (\varepsilon_{\mathbf{D}})_0(q) \quad (\forall v \in \text{Ind}(\mathbf{D})). \quad (3.10)$$

Since by the weight completeness  $(\varepsilon_{\mathbf{D}})_0(\overline{Y}(\pi_v)) + q(\pi_v) \cdot \mathbf{1}$  is an element of  $\mathbf{D}_0$ , so is

$$q' = \inf_{v \in \text{Ind}(\mathbf{D})} \{(\varepsilon_{\mathbf{D}})_0(\overline{Y}(\pi_v)) + q(\pi_v) \cdot \mathbf{1}\} = \bigvee_{v \in \text{Ind}(\mathbf{D})} \{(\varepsilon_{\mathbf{D}})_0(\overline{Y}(\pi_v)) + q(\pi_v) \cdot \mathbf{1}\}$$

by the order completeness. We claim that  $q' = (\varepsilon_{\mathbf{D}})_0(q)$ . Because the definition of  $q'$  and (3.10) imply  $q' \geq (\varepsilon_{\mathbf{D}})_0(q)$ , it suffices to show that for each  $v \in \text{Ind}(\mathbf{D})$ ,  $q'(v) \leq (\varepsilon_{\mathbf{D}})_0(q)(v) = q(\pi_v)$ . This is proved as follows:

$$\begin{aligned} q'(v) &= \left( \inf_{v \in \text{Ind}(\mathbf{D})} \{(\varepsilon_{\mathbf{D}})_0(\overline{Y}(\pi_v)) + q(\pi_v) \cdot \mathbf{1}\} \right)(v) \\ &\leq ((\varepsilon_{\mathbf{D}})_0(\overline{Y}(\pi_v)) + q(\pi_v) \cdot \mathbf{1})(v) \\ &= (\varepsilon_{\mathbf{D}})_0(\overline{Y}(\pi_v))(v) + (q(\pi_v) \cdot \mathbf{1})(v) \\ &= d_{[\mathbf{D}, \overline{\mathbb{K}}]}(\pi_v, \pi_v) + q(\pi_v) \\ &\leq 0 + q(\pi_v) \\ &= q(\pi_v). \end{aligned}$$

$[(\varepsilon'_{\mathbf{D}})_0(\mathbf{D}_0) \subseteq [[\mathbf{D}, \overline{\mathbb{K}}], \overline{\mathbb{K}}]_0]$  The condition for  $(\varepsilon'_{\mathbf{D}})_0(p)$  to be an element of  $[[\mathbf{D}, \overline{\mathbb{K}}], \overline{\mathbb{K}}]_0$  is that

$$d_{[\mathbf{D}, \overline{\mathbb{K}}]}(\pi_v, \pi_w) \geq (\varepsilon'_{\mathbf{D}})_0(p)(\pi_w) - (\varepsilon'_{\mathbf{D}})_0(p)(\pi_v) \quad (\forall \pi_v, \pi_w \in \{\pi_v\}_{v \in \text{Ind}(\mathbf{D})}),$$

which is equivalent to

$$\sup_{p \in \mathbf{D}_0} \{p(w) - p(v)\} \geq p(w) - p(v) \quad (\forall v, w \in \text{Ind}(\mathbf{D})),$$

an obvious inequality.

Therefore, we have two homomorphisms

$$\begin{aligned} \varepsilon_{\mathbf{D}}: [[\mathbf{D}, \overline{\mathbb{K}}], \overline{\mathbb{K}}] &\longrightarrow \mathbf{D}, \\ \varepsilon'_{\mathbf{D}}: \mathbf{D} &\longrightarrow [[\mathbf{D}, \overline{\mathbb{K}}], \overline{\mathbb{K}}]. \end{aligned}$$

Clearly  $\text{Ind}(\varepsilon'_{\mathbf{D}}) \circ \text{Ind}(\varepsilon_{\mathbf{D}}) = 1_{\text{Ind}(\mathbf{D})}$  and  $\text{Ind}(\varepsilon_{\mathbf{D}}) \circ \text{Ind}(\varepsilon'_{\mathbf{D}}) = 1_{\{\pi_v\}_{v \in \text{Ind}(\mathbf{D})}}$  hold, and they are isomorphisms.  $\square$

Theorem 3.9 immediately assures that the underlying sets of  $\overline{\mathbb{K}}$ -extended L-convex sets are, in a sense, nothing but the sets of points of presheaf categories for some  $\overline{\mathbb{K}}$ -categories.

**Corollary 3.12.** *For any  $\overline{\mathbb{K}}$ -extended L-convex set  $\mathbf{D} = (\text{Ind}(\mathbf{D}), \mathbf{D}_0)$ , the pair  $\mathcal{D} = (\mathbf{D}_0, d_{\mathbf{D}_0})$  where  $d_{\mathbf{D}_0}$  is a distance function on  $\mathbf{D}_0$  given as the restriction of the sup-distance on  $\overline{\mathbb{K}}^{\text{Ind}(\mathbf{D})}$ , is isomorphic to the presheaf category  $\text{Fun}([\mathbf{D}, \overline{\mathbb{K}}], \overline{\mathbb{K}})$  of the  $\overline{\mathbb{K}}$ -category  $[\mathbf{D}, \overline{\mathbb{K}}]^{\text{op}}$ . Conversely, the presheaf category  $\text{Fun}(\mathcal{A}^{\text{op}}, \overline{\mathbb{K}}) = ([\mathcal{A}^{\text{op}}, \overline{\mathbb{K}}]_0, d_{[\mathcal{A}^{\text{op}}, \overline{\mathbb{K}}]_0})$  of any  $\overline{\mathbb{K}}$ -category  $\mathcal{A} = (\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  gives rise to a  $\overline{\mathbb{K}}$ -extended L-convex set  $[\mathcal{A}^{\text{op}}, \overline{\mathbb{K}}] = (\text{Ob}(\mathcal{A}), [\mathcal{A}^{\text{op}}, \overline{\mathbb{K}}]_0)$ .*

Let us apply Theorem 3.9 to  $\overline{\mathbb{K}}$ -extended L-convex sets with two-element index sets and see how the shapes of the underlying sets relate to the distances of the corresponding two-point  $\overline{\mathbb{K}}$ -categories. We obtain Fig. 3.5; in each figure, the lower half part is a picture of the underlying set  $\mathbf{D}_0$  of some  $\overline{\mathbb{K}}$ -extended L-convex set  $\mathbf{D}$ , and above it we show the  $\overline{\mathbb{K}}$ -category  $[[\mathbf{D}, \overline{\mathbb{K}}]]$  so that the number on the arrow from a node  $a$  to a node  $b$  represents the distance  $d_{[[\mathbf{D}, \overline{\mathbb{K}}]]}(a, b)$ . In fact, these ten figures enumerate all the possible shapes of  $\overline{\mathbb{K}}$ -extended L-convex sets with two-element index sets, and consequently also gives a complete classification of  $\overline{\mathbb{K}}$ -categories with two points.

We then show the duality for maps. We denote the constructions for maps by the same symbols  $[-, \overline{\mathbb{K}}]$  and  $[[-, \overline{\mathbb{K}}]]$ . The theorem reads:

**Theorem 3.13.** *Let  $\mathcal{A} = (\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  and  $\mathcal{B} = (\text{Ob}(\mathcal{B}), d_{\mathcal{B}})$  be  $\overline{\mathbb{K}}$ -categories, and  $\mathbf{D} = (\text{Ind}(\mathbf{D}), \mathbf{D}_0)$  and  $\mathbf{E} = (\text{Ind}(\mathbf{E}), \mathbf{E}_0)$  be  $\overline{\mathbb{K}}$ -extended L-convex sets. Then the following hold:*

- (i) *The map  $[-, \overline{\mathbb{K}}]: [\mathcal{A}, \mathcal{B}]_0 \rightarrow [[[\mathcal{B}, \overline{\mathbb{K}}], [\mathcal{A}, \overline{\mathbb{K}}]]_0$  is bijective.*
- (ii) *The map  $[[-, \overline{\mathbb{K}}]]: [[\mathbf{D}, \mathbf{E}]]_0 \rightarrow [[[[\mathbf{E}, \overline{\mathbb{K}}]], [[\mathbf{D}, \overline{\mathbb{K}}]]]_0$  is bijective.*

Note that the directions of maps reverse when we apply the constructions. The details of these constructions are given by the following lemmas:

**Lemma 3.14.** *Let  $\mathcal{A} = (\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  and  $\mathcal{B} = (\text{Ob}(\mathcal{B}), d_{\mathcal{B}})$  be  $\overline{\mathbb{K}}$ -categories and*

$$F: \mathcal{A} \longrightarrow \mathcal{B}$$

*be a  $\overline{\mathbb{K}}$ -functor. Then there is a homomorphism of  $\overline{\mathbb{K}}$ -extended L-convex sets*

$$[F, \overline{\mathbb{K}}]: [\mathcal{B}, \overline{\mathbb{K}}] \longrightarrow [\mathcal{A}, \overline{\mathbb{K}}],$$

*defined as  $\text{Ind}([F, \overline{\mathbb{K}}]) = \text{Ob}(F)$ .*

*Proof.* Since  $[\mathcal{B}, \overline{\mathbb{K}}] = (\text{Ob}(\mathcal{B}), [\mathcal{B}, \overline{\mathbb{K}}]_0)$  and  $[\mathcal{A}, \overline{\mathbb{K}}] = (\text{Ob}(\mathcal{A}), [\mathcal{A}, \overline{\mathbb{K}}]_0)$ , a homomorphism from  $[\mathcal{B}, \overline{\mathbb{K}}]$  to  $[\mathcal{A}, \overline{\mathbb{K}}]$  is given as a map from  $\text{Ind}([\mathcal{A}, \overline{\mathbb{K}}]) = \text{Ob}(\mathcal{A})$  to  $\text{Ind}([\mathcal{B}, \overline{\mathbb{K}}]) = \text{Ob}(\mathcal{B})$ , such as  $\text{Ob}(F)$ ; so at least the type matches. What we have to check is that for all  $p \in [\mathcal{B}, \overline{\mathbb{K}}]_0$ ,

$$p \circ \text{Ob}(F): \text{Ob}(\mathcal{A}) \xrightarrow{\text{Ob}(F)} \text{Ob}(\mathcal{B}) \xrightarrow{p} \mathbb{K} \cup \{-\infty, \infty\} \quad (3.11)$$

is an element of  $[\mathcal{A}, \overline{\mathbb{K}}]_0$ , i.e., nonexpansive. But (3.11) is a composition of nonexpansive maps and so nonexpansive as well.  $\square$

**Lemma 3.15.** *Let  $\mathbf{D} = (\text{Ind}(\mathbf{D}), \mathbf{D}_0)$  and  $\mathbf{E} = (\text{Ind}(\mathbf{E}), \mathbf{E}_0)$  be  $\overline{\mathbb{K}}$ -extended L-convex sets and*

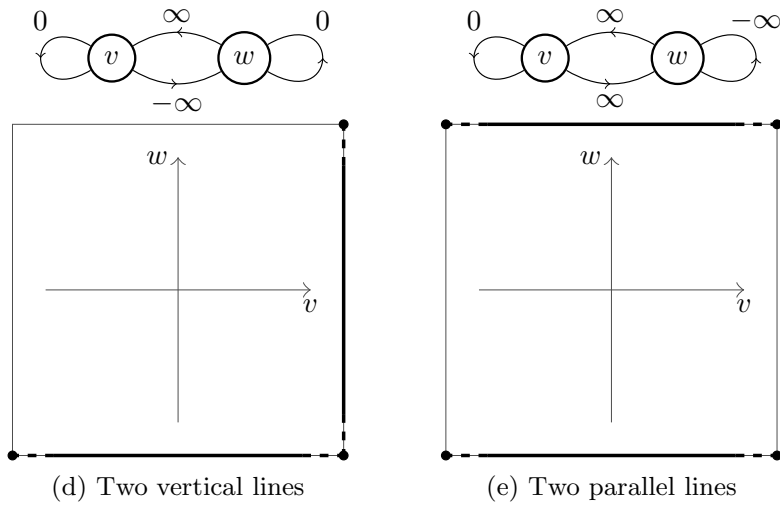
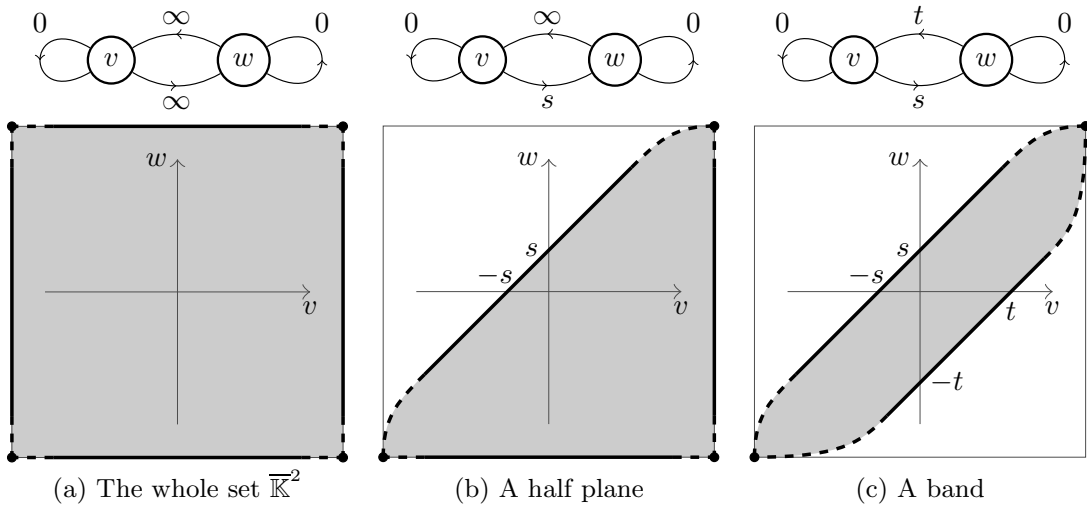
$$\Phi: \mathbf{D} \longrightarrow \mathbf{E}$$

*be a homomorphism. Then there is a  $\overline{\mathbb{K}}$ -functor*

$$[[\Phi, \overline{\mathbb{K}}]]: [[\mathbf{E}, \overline{\mathbb{K}}]] \longrightarrow [[\mathbf{D}, \overline{\mathbb{K}}]],$$

*defined as  $\text{Ob}([[ \Phi, \overline{\mathbb{K}} ]]) = \pi_{\mathbf{D}} \circ \text{Ind}(\Phi) \circ \pi_{\mathbf{E}}^{-1}$ , where  $\pi_{\mathbf{D}}$  and  $\pi_{\mathbf{E}}$  are the bijections defined in the proof of Lemma 3.11.*

*Proof.* Since  $[[\mathbf{E}, \overline{\mathbb{K}}]] = ([[ \mathbf{E}, \overline{\mathbb{K}} ]], d_{[[ \mathbf{E}, \overline{\mathbb{K}} ]]])$  and  $[[\mathbf{D}, \overline{\mathbb{K}}]] = ([[ \mathbf{D}, \overline{\mathbb{K}} ]], d_{[[ \mathbf{D}, \overline{\mathbb{K}} ]]])$ , a  $\overline{\mathbb{K}}$ -functor from  $[[\mathbf{E}, \overline{\mathbb{K}}]]$  to  $[[\mathbf{D}, \overline{\mathbb{K}}]]$  is given as a map from  $\text{Ob}([[ \mathbf{E}, \overline{\mathbb{K}} ]]) = [[ \mathbf{E}, \overline{\mathbb{K}} ]]_0$  to  $\text{Ob}([[ \mathbf{D}, \overline{\mathbb{K}} ]]) = [[ \mathbf{D}, \overline{\mathbb{K}} ]]_0$ , such as  $\text{Ob}([[ \Phi, \overline{\mathbb{K}} ]]) = \pi_{\mathbf{D}} \circ \text{Ind}(\Phi) \circ \pi_{\mathbf{E}}^{-1}$ :



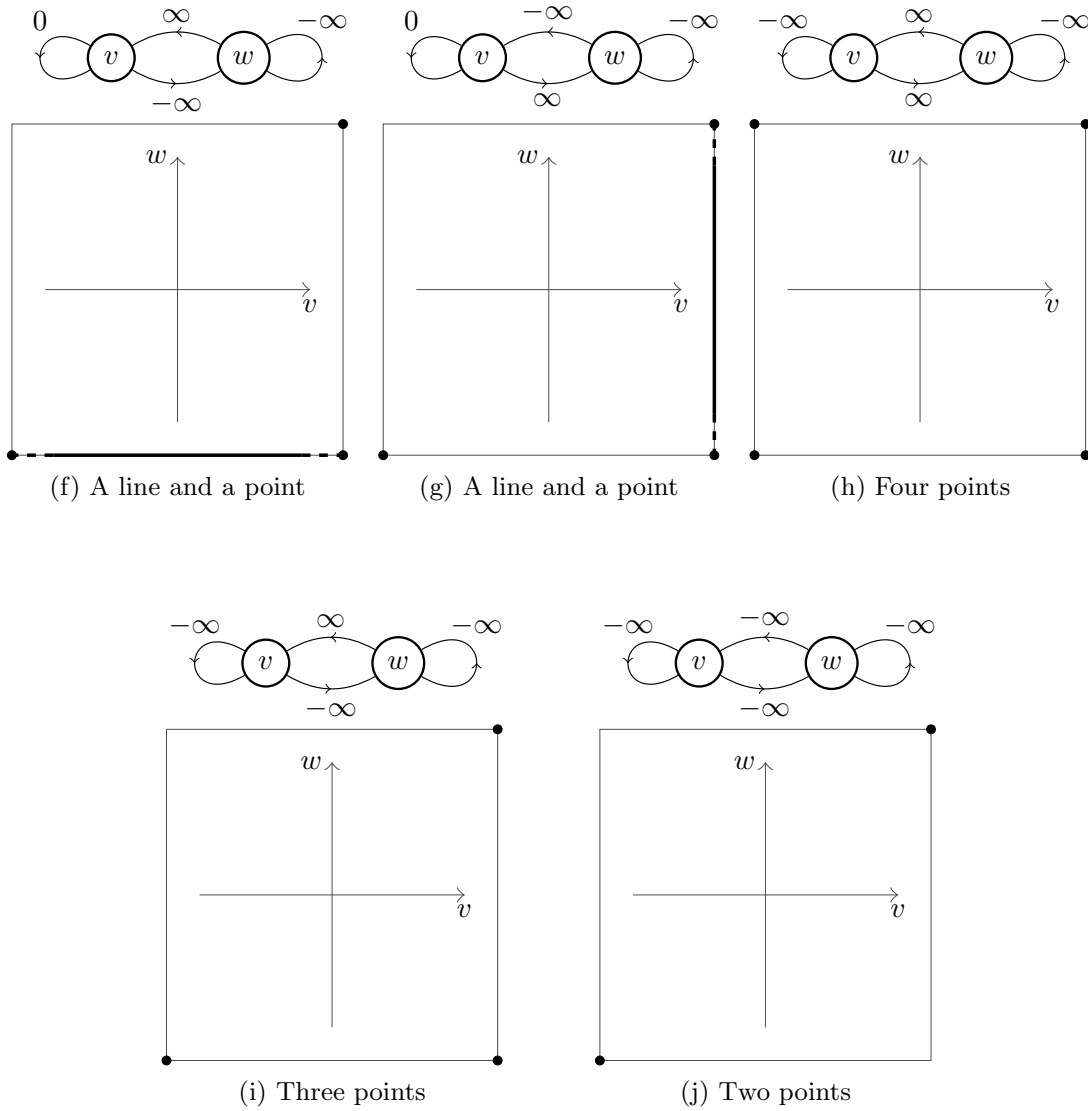


Fig. 3.5:  $\overline{\mathbb{K}}$ -extended L-convex sets and  $\overline{\mathbb{K}}$ -categories ( $s, t \in \mathbb{K}, s + t \geq 0$ )

$$\begin{array}{ccc}
 \text{Ind}(\mathbf{E}) & \xrightarrow{\pi_{\mathbf{E}}} & \llbracket \mathbf{E}, \overline{\mathbb{K}} \rrbracket_0 \\
 \text{Ind}(\Phi) \downarrow & & \downarrow \text{Ob}(\llbracket \Phi, \overline{\mathbb{K}} \rrbracket) \\
 \text{Ind}(\mathbf{D}) & \xrightarrow{\pi_{\mathbf{D}}} & \llbracket \mathbf{D}, \overline{\mathbb{K}} \rrbracket_0
 \end{array}$$

We adopt the notation  $\llbracket \mathbf{E}, \overline{\mathbb{K}} \rrbracket_0 = \{\pi_w\}_{w \in \text{Ind}(\mathbf{E})}$  and  $\llbracket \mathbf{D}, \overline{\mathbb{K}} \rrbracket_0 = \{\pi_v\}_{v \in \text{Ind}(\mathbf{D})}$ ; then

$$\text{Ob}(\llbracket \Phi, \overline{\mathbb{K}} \rrbracket)(\pi_w) = \pi_{\text{Ind}(\Phi)(w)} \quad (\forall \pi_w \in \{\pi_w\}_{w \in \text{Ind}(\mathbf{E})}).$$

We aim to show that

$$d_{\llbracket \mathbf{E}, \overline{\mathbb{K}} \rrbracket}(\pi_w, \pi_{w'}) \geq d_{\llbracket \mathbf{D}, \overline{\mathbb{K}} \rrbracket}(\pi_{\text{Ind}(\Phi)(w)}, \pi_{\text{Ind}(\Phi)(w')}) \quad (\forall \pi_w, \pi_{w'} \in \{\pi_w\}_{w \in \text{Ind}(\mathbf{E})}),$$

or equivalently (by (3.4) in the proof of Lemma 3.11),

$$\sup_{q \in \mathbf{E}_0} \{q(w') - q(w)\} \geq \sup_{p \in \mathbf{D}_0} \{p \circ \text{Ind}(\Phi)(w') - p \circ \text{Ind}(\Phi)(w)\} \quad (\forall w, w' \in \text{Ind}(\mathbf{E})).$$

Now recall that the condition for  $\text{Ind}(\Phi)$  to define a homomorphism is that for all  $p \in \mathbf{D}_0$ ,  $p \circ \text{Ind}(\Phi) \in \mathbf{E}_0$ ; hence

$$\{q(w') - q(w) \mid q \in \mathbf{E}_0\} \supseteq \{p \circ \text{Ind}(\Phi)(w') - p \circ \text{Ind}(\Phi)(w) \mid p \in \mathbf{D}_0\} \quad (\forall w, w' \in \text{Ind}(\mathbf{E}))$$

and the proof is done.  $\square$

Theorem 3.13 is proved as follows:

*Proof of Theorem 3.13.* [(i)] Since for each  $\overline{\mathbb{K}}$ -functor  $F: \mathcal{A} \rightarrow \mathcal{B}$ ,  $[F, \overline{\mathbb{K}}]$  is defined as

$$\text{Ind}([F, \overline{\mathbb{K}}]) = \text{Ob}(F),$$

$[-, \overline{\mathbb{K}}]$  is clearly injective.

To prove surjectivity of  $[-, \overline{\mathbb{K}}]$ , take an arbitrary homomorphism

$$\Phi: [\mathcal{B}, \overline{\mathbb{K}}] \rightarrow [\mathcal{A}, \overline{\mathbb{K}}].$$

Let  $F_\Phi = \eta_{\mathcal{B}}^{-1} \circ [\Phi, \overline{\mathbb{K}}] \circ \eta_{\mathcal{A}}$ , where  $\overline{\mathbb{K}}$ -functors  $\eta_{\mathcal{A}}$  and  $\eta_{\mathcal{B}}$  are the isomorphisms defined in the proof of Theorem 3.9 (i):

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{\eta_{\mathcal{A}}} & \llbracket [\mathcal{A}, \overline{\mathbb{K}}], \overline{\mathbb{K}} \rrbracket \\
 F_\Phi \downarrow & & \downarrow [\Phi, \overline{\mathbb{K}}] \\
 \mathcal{B} & \xrightarrow{\eta_{\mathcal{B}}} & \llbracket [\mathcal{B}, \overline{\mathbb{K}}], \overline{\mathbb{K}} \rrbracket
 \end{array}$$

We claim that  $[F_\Phi, \overline{\mathbb{K}}] = \Phi$ . Recall  $\text{Ob}(\llbracket [\mathcal{A}, \overline{\mathbb{K}}], \overline{\mathbb{K}} \rrbracket) = \llbracket [\mathcal{A}, \overline{\mathbb{K}}], \overline{\mathbb{K}} \rrbracket_0 = \{\pi_a\}_{a \in \text{Ob}(\mathcal{A})}$  and

$$\text{Ob}(\eta_{\mathcal{A}}) = \pi_{[\mathcal{A}, \overline{\mathbb{K}}]} = \lambda a \in \text{Ob}(\mathcal{A}). \pi_a: \text{Ob}(\mathcal{A}) \rightarrow \text{Ob}(\llbracket [\mathcal{A}, \overline{\mathbb{K}}], \overline{\mathbb{K}} \rrbracket).$$

Now we can conclude that

$$\begin{aligned}
\text{Ind}([F_\Phi, \overline{\mathbb{K}}]) &= \text{Ob}(F_\Phi) \\
&= \text{Ob}(\eta_{\mathcal{B}})^{-1} \circ \text{Ob}([\Phi, \overline{\mathbb{K}}]) \circ \text{Ob}(\eta_{\mathcal{A}}) \\
&= \text{Ob}(\eta_{\mathcal{B}})^{-1} \circ \pi_{[\mathcal{B}, \overline{\mathbb{K}}]} \circ \text{Ind}(\Phi) \circ \pi_{[\mathcal{A}, \overline{\mathbb{K}}]}^{-1} \circ \text{Ob}(\eta_{\mathcal{A}}) \\
&= 1_{\text{Ob}(\mathcal{B})} \circ \text{Ind}(\Phi) \circ 1_{\text{Ob}(\mathcal{A})} \\
&= \text{Ind}(\Phi),
\end{aligned}$$

hence  $[F_\Phi, \overline{\mathbb{K}}] = \Phi$ .

[(ii)] Since for each homomorphism  $\Phi: \mathbf{D} \rightarrow \mathbf{E}$ ,  $[\Phi, \overline{\mathbb{K}}]$  is defined as

$$\text{Ob}([\Phi, \overline{\mathbb{K}}]) = \pi_{\mathbf{D}} \circ \text{Ind}(\Phi) \circ \pi_{\mathbf{E}}^{-1},$$

where  $\pi_{\mathbf{D}}$  and  $\pi_{\mathbf{E}}$  are fixed bijections,  $[-, \overline{\mathbb{K}}]$  is clearly injective.

To prove surjectivity of  $[-, \overline{\mathbb{K}}]$ , take an arbitrary  $\overline{\mathbb{K}}$ -functor

$$F: [\mathbf{E}, \overline{\mathbb{K}}] \rightarrow [\mathbf{D}, \overline{\mathbb{K}}].$$

Let  $\Phi_F = \varepsilon_{\mathbf{E}} \circ [F, \overline{\mathbb{K}}] \circ \varepsilon_{\mathbf{D}}^{-1}$ , or equivalently,  $\text{Ind}(\Phi_F) = \text{Ind}(\varepsilon_{\mathbf{D}})^{-1} \circ \text{Ind}([F, \overline{\mathbb{K}}]) \circ \text{Ind}(\varepsilon_{\mathbf{E}})$ , where homomorphisms  $\varepsilon_{\mathbf{D}}$  and  $\varepsilon_{\mathbf{E}}$  are the isomorphisms defined in the proof of Theorem 3.9 (ii):

$$\begin{array}{ccc}
[[\mathbf{D}, \overline{\mathbb{K}}], \overline{\mathbb{K}}] & \xrightarrow{\varepsilon_{\mathbf{D}}} & \mathbf{D} \\
\downarrow [F, \overline{\mathbb{K}}] & & \downarrow \Phi_F \\
[[\mathbf{E}, \overline{\mathbb{K}}], \overline{\mathbb{K}}] & \xrightarrow{\varepsilon_{\mathbf{E}}} & \mathbf{E}
\end{array}$$

We claim that  $[\Phi_F, \overline{\mathbb{K}}] = F$ . Recall  $\text{Ind}([[ \mathbf{D}, \overline{\mathbb{K}} ], \overline{\mathbb{K}}]) = \text{Ob}([\mathbf{D}, \overline{\mathbb{K}}]) = [\mathbf{D}, \overline{\mathbb{K}}]_0 = \{\pi_v\}_{v \in \text{Ind}(\mathbf{D})}$  and

$$\text{Ind}(\varepsilon_{\mathbf{D}}) = \pi_{\mathbf{D}} = \lambda v \in \text{Ind}(\mathbf{D}). \pi_v: \text{Ind}(\mathbf{D}) \rightarrow \text{Ind}([[ \mathbf{D}, \overline{\mathbb{K}} ], \overline{\mathbb{K}}]).$$

Now we can conclude that

$$\begin{aligned}
\text{Ob}([\Phi_F, \overline{\mathbb{K}}]) &= \pi_{\mathbf{D}} \circ \text{Ind}(\Phi_F) \circ \pi_{\mathbf{E}}^{-1} \\
&= \pi_{\mathbf{D}} \circ \text{Ind}(\varepsilon_{\mathbf{D}})^{-1} \circ \text{Ind}([F, \overline{\mathbb{K}}]) \circ \text{Ind}(\varepsilon_{\mathbf{E}}) \circ \pi_{\mathbf{E}}^{-1} \\
&= 1_{\text{Ind}([[ \mathbf{D}, \overline{\mathbb{K}} ], \overline{\mathbb{K}}])} \circ \text{Ind}([F, \overline{\mathbb{K}}]) \circ 1_{\text{Ind}([[ \mathbf{E}, \overline{\mathbb{K}} ], \overline{\mathbb{K}}])} \\
&= \text{Ind}([F, \overline{\mathbb{K}}]) \\
&= \text{Ob}(F),
\end{aligned}$$

hence  $[\Phi_F, \overline{\mathbb{K}}] = F$ . □

Finally, the duality for canonical orderings:

**Theorem 3.16.** *Let  $\mathcal{A} = (\text{Ob}(\mathcal{A}), d_{\mathcal{A}})$  and  $\mathcal{B} = (\text{Ob}(\mathcal{B}), d_{\mathcal{B}})$  be  $\overline{\mathbb{K}}$ -categories,  $\mathbf{D} = (\text{Ind}(\mathbf{D}), \mathbf{D}_0)$  and  $\mathbf{E} = (\text{Ind}(\mathbf{E}), \mathbf{E}_0)$  be  $\overline{\mathbb{K}}$ -extended  $L$ -convex sets,  $F, G: \mathcal{A} \rightarrow \mathcal{B}$  be  $\overline{\mathbb{K}}$ -functors, and  $\Phi, \Psi: \mathbf{D} \rightarrow \mathbf{E}$  be homomorphisms. Then the following hold:*

- (i)  $F \Rightarrow G$  if and only if  $[F, \overline{\mathbb{K}}] \Rightarrow [G, \overline{\mathbb{K}}]$ .  
 (ii)  $\Phi \Rightarrow \Psi$  if and only if  $[[\Phi, \overline{\mathbb{K}}], \overline{\mathbb{K}}] \Rightarrow [[\Psi, \overline{\mathbb{K}}], \overline{\mathbb{K}}]$ .

*Proof.* [(i), the “only if” part] Since  $[F, \overline{\mathbb{K}}], [G, \overline{\mathbb{K}}]: [\mathcal{B}, \overline{\mathbb{K}}] \rightarrow [\mathcal{A}, \overline{\mathbb{K}}]$ , we aim to prove that for all  $q \in [\mathcal{B}, \overline{\mathbb{K}}]_0$ ,

$$q \circ \text{Ob}(F)(a) \geq q \circ \text{Ob}(G)(a) \quad (\forall a \in \text{Ob}(\mathcal{A})).$$

In fact, when we view  $q$  as a  $\overline{\mathbb{K}}$ -functor, it is equivalent to  $q \circ F \Rightarrow q \circ G$ , which follows from the fact that  $\Rightarrow$  on  $\overline{\mathbb{K}}$ -functors is reflexive and preserved under composition.

[(ii), the “only if” part] Since  $[[\Phi, \overline{\mathbb{K}}], \overline{\mathbb{K}}], [[\Psi, \overline{\mathbb{K}}], \overline{\mathbb{K}}]: [\mathbf{E}, \overline{\mathbb{K}}] \rightarrow [\mathbf{D}, \overline{\mathbb{K}}]$ , we aim to prove that for all  $\pi_w \in [\mathbf{E}, \overline{\mathbb{K}}]$ ,

$$d_{[\mathbf{D}, \overline{\mathbb{K}}]}(\pi_{\text{Ind}(\Phi)(w)}, \pi_{\text{Ind}(\Psi)(w)}) = \sup_{p \in \mathbf{D}_0} \{\Psi_0(p)(w) - \Phi_0(p)(w)\} \leq 0.$$

Because  $0 \geq \Psi_0(p)(w) - \Phi_0(p)(w) \iff \Phi_0(p)(w) \geq \Psi_0(p)(w)$  by

$$\frac{0 \geq \Psi_0(p)(w) - \Phi_0(p)(w)}{\Phi_0(p)(w) \geq \Psi_0(p)(w)}$$

it suffices to show that

$$\Phi_0(p) \geq \Psi_0(p)$$

holds in  $\mathbf{E}_0$  for all  $p \in \mathbf{D}_0$ , but this is nothing but the condition for  $\Phi \Rightarrow \Psi$ .

[(i), the “if” part] The “only if” part of (ii) implies that  $[[F, \overline{\mathbb{K}}], \overline{\mathbb{K}}] \Rightarrow [[G, \overline{\mathbb{K}}], \overline{\mathbb{K}}]$  holds. By the proof of Theorem 3.13 (i), we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\eta_{\mathcal{A}}} & [[\mathcal{A}, \overline{\mathbb{K}}], \overline{\mathbb{K}}] \\ \downarrow F & & \downarrow [[F, \overline{\mathbb{K}}], \overline{\mathbb{K}}] \\ \mathcal{B} & \xrightarrow{\eta_{\mathcal{B}}} & [[\mathcal{B}, \overline{\mathbb{K}}], \overline{\mathbb{K}}] \end{array}$$

and similarly for  $G$ . In equations,

$$\begin{aligned} F &= \eta_{\mathcal{B}}^{-1} \circ [[F, \overline{\mathbb{K}}], \overline{\mathbb{K}}] \circ \eta_{\mathcal{A}}, \\ G &= \eta_{\mathcal{B}}^{-1} \circ [[G, \overline{\mathbb{K}}], \overline{\mathbb{K}}] \circ \eta_{\mathcal{A}}. \end{aligned}$$

Since  $\eta_{\mathcal{A}}, \eta_{\mathcal{B}}$  are isomorphisms, we have

$$\begin{aligned} d_{\mathcal{B}}(F(a), G(a)) &= d_{\mathcal{B}}(\eta_{\mathcal{B}}^{-1} \circ [[F, \overline{\mathbb{K}}], \overline{\mathbb{K}}] \circ \eta_{\mathcal{A}}(a), \eta_{\mathcal{B}}^{-1} \circ [[G, \overline{\mathbb{K}}], \overline{\mathbb{K}}] \circ \eta_{\mathcal{A}}(a)) \\ &= d_{\mathcal{B}}(\eta_{\mathcal{B}}^{-1} \circ [[F, \overline{\mathbb{K}}], \overline{\mathbb{K}}](\pi_a), \eta_{\mathcal{B}}^{-1} \circ [[G, \overline{\mathbb{K}}], \overline{\mathbb{K}}](\pi_a)) \\ &= d_{[[\mathcal{B}, \overline{\mathbb{K}}], \overline{\mathbb{K}}]}([[F, \overline{\mathbb{K}}], \overline{\mathbb{K}}](\pi_a), [[G, \overline{\mathbb{K}}], \overline{\mathbb{K}}](\pi_a)) \\ &\leq 0 \end{aligned}$$

for all  $a \in \text{Ob}(\mathcal{A})$ .

[(ii), the “if” part] The “only if” part of (i) implies that  $[[\Phi, \overline{\mathbb{K}}], \overline{\mathbb{K}}] \Rightarrow [[\Psi, \overline{\mathbb{K}}], \overline{\mathbb{K}}]$  holds. By the proof of Theorem 3.13 (ii), we have the following commutative diagram:



$$\begin{array}{ccc}
[[\mathbf{D}, \overline{\mathbb{K}}], \overline{\mathbb{K}}] & \xrightarrow{\varepsilon_{\mathbf{D}}} & \mathbf{D} \\
\downarrow & & \downarrow \Phi \\
[[\mathbf{E}, \overline{\mathbb{K}}], \overline{\mathbb{K}}] & \xrightarrow{\varepsilon_{\mathbf{E}}} & \mathbf{E}
\end{array}$$

and similarly for  $\Psi$ . In equations,

$$\begin{aligned}
\Phi &= \varepsilon_{\mathbf{E}} \circ [[\Phi, \overline{\mathbb{K}}], \overline{\mathbb{K}}] \circ \varepsilon_{\mathbf{D}}^{-1}, \\
\Psi &= \varepsilon_{\mathbf{E}} \circ [[\Psi, \overline{\mathbb{K}}], \overline{\mathbb{K}}] \circ \varepsilon_{\mathbf{D}}^{-1}.
\end{aligned}$$

We have

$$\begin{aligned}
\Phi_0(p)(w) &= (\varepsilon_{\mathbf{E}})_0 \circ [[\Phi, \overline{\mathbb{K}}], \overline{\mathbb{K}}]_0 \circ (\varepsilon_{\mathbf{D}}^{-1})_0(p)(w) \\
&= (\varepsilon_{\mathbf{E}})_0 \left( [[\Phi, \overline{\mathbb{K}}], \overline{\mathbb{K}}]_0 \circ (\varepsilon_{\mathbf{D}}^{-1})_0(p) \right)(w) \\
&= (\lambda w' \in \text{Ind}(\mathbf{E}). [[\Phi, \overline{\mathbb{K}}], \overline{\mathbb{K}}]_0 \circ (\varepsilon_{\mathbf{D}}^{-1})_0(p)(\pi_{w'}))(w) \\
&= [[\Phi, \overline{\mathbb{K}}], \overline{\mathbb{K}}]_0 \circ (\varepsilon_{\mathbf{D}}^{-1})_0(p)(\pi_w) \\
&= [[\Phi, \overline{\mathbb{K}}], \overline{\mathbb{K}}]_0 \left( (\varepsilon_{\mathbf{D}}^{-1})_0(p) \right)(\pi_w)
\end{aligned}$$

and

$$\Psi_0(p)(w) = [[\Psi, \overline{\mathbb{K}}], \overline{\mathbb{K}}]_0 \left( (\varepsilon_{\mathbf{D}}^{-1})_0(p) \right)(\pi_w)$$

for all  $p \in \mathbf{D}_0$  and  $w \in \text{Ind}(\mathbf{E})$ . Since  $(\varepsilon_{\mathbf{D}}^{-1})_0(p) \in [[\mathbf{D}, \overline{\mathbb{K}}], \overline{\mathbb{K}}]_0$ , we have

$$[[\Phi, \overline{\mathbb{K}}], \overline{\mathbb{K}}]_0 \left( (\varepsilon_{\mathbf{D}}^{-1})_0(p) \right)(\pi_w) \geq [[\Psi, \overline{\mathbb{K}}], \overline{\mathbb{K}}]_0 \left( (\varepsilon_{\mathbf{D}}^{-1})_0(p) \right)(\pi_w).$$

Therefore we obtain  $\Phi(p)(w) \geq \Psi(p)(w)$  for all  $p \in \mathbf{D}_0$  and  $w \in \text{Ind}(\mathbf{E})$ ; so  $\Phi(p) \geq \Psi(p)$  for all  $p \in \mathbf{D}_0$ ; so  $\Phi \Rightarrow \Psi$ . □

## Chapter 4

# Conclusion

### 4.1 Summary and Concluding Remarks

We introduced  $\overline{\mathbb{K}}$ -extended L-convex sets, homomorphisms and canonical orderings (between homomorphisms), and established the correspondence of them to entities of enriched-categorical origin;  $\overline{\mathbb{K}}$ -categories,  $\overline{\mathbb{K}}$ -functors and canonical orderings (between  $\overline{\mathbb{K}}$ -functors). The whole correspondence between the theory of  $\overline{\mathbb{K}}$ -extended L-convex sets and that of  $\overline{\mathbb{K}}$ -categories is so harmonious that one may dare to say that these two theories are in fact the *identical* one in different guises.

The formulation of a duality by the function space (or “hom”) with a fixed codomain living in two worlds (such an object is called e.g., a *Janusian object*, after Janus, a god in Roman mythology usually depicted as having two faces), as in ours, is a common one. A classical example is the *Stone duality*; see [2] for details. [9] is an attempt at a general theory of such dualities. One can also interpret Birkhoff’s representation theorem stating correspondence between finite distributive lattices and finite posets, or its infinite generalization [11], in a similar fashion; in fact, it is possible to regard them as the **2**-version of our result (concerning  $\overline{\mathbb{K}}$ ). In particular, we note that from an enriched-categorical viewpoint, finite distributive lattices are nothing but finite **2**-presheaf categories.

### 4.2 Further Work

Finally, we conclude the thesis by mentioning possible directions of further study. An obvious problem is generalization of our duality theorem (on  $\overline{\mathbb{K}}$ ) to other enriching posets (or categories)  $\mathcal{V}$ . It may also be fruitful to study on another particular enriching poset  $\mathcal{V}$  and on  $\mathcal{V}$ -categories;  $\overline{\mathbb{K}}_+^{\text{Cart}}$  is an attractive candidate. Although some results, e.g., Corollary 3.12, seem to suggest a strong link between enriched category theory and discrete convex analysis, we could not find any application which utilizes the link effectively. A full-scale approach to discrete convex analysis from a categorical viewpoint is hoped for.

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