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# Part III :

# Categorical SOS and Bialgebras

## Plan (Tentative)

- concurrency, process theory
- SOS
- compositionality : opr. sem. w/ den. reasoning

## References

Bartek Klin, TCS 2011

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# §3.1 Introduction

**Computer Science**

finitary formalism,
representing
(possibly) infinitary behaviors

E.g.

- a while program, (imperative) ← finite string
  and its execution ← possibly non-term
- a DFA ← finite
  and its accepted language ← infinite, w/ arbitrary long words

- A Büchi autom. and the $\omega$-language it recognizes

$$\varphi$$

$$\boxed{A \subseteq L^{\omega}}$$

- A higher-order recursion scheme and the tree language it produces

- (... and of course)

  a $\lambda$-term and its reduction sequence

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# A central question

- Given a finitary representation (a program, an autom., ...), what is its (infinitary) behavior?

Now notice that this question is ill-formulated ... We mortal humans are incapable of writing down infinite behaviors ...

# The central question, refined and often asked

1. Given two presentations $M, N$, do they exhibit the same behavior?

( $M =_{\beta} N$, $[\![M]\!] = [\![N]\!]$, $M \simeq_{CTxT} N$, $M \approx N$ (bisim) , ... )

[ Foundation of program transformation, automata minimization, etc.

same job, more efficiently

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2 Given a presentation M, does its behavior satisfy a given specification P ?

e.g. P is

- strongly normalizing (termination)
- never produces a label a ($G \neg a$)
- after a occurs, b occurs eventually ($G(a \supset F b)$)

specification in modal (temporal) logic

Anyway To answer such questions, we need a mathematical def. of the behavior of a presentation M.

This is what SEMANTICS (in CS) is about.

"What is its 'meaning'?"

Two common styles of semantics:

- denotational sem.
  mathematical / algebraic, abstract, easy to reason with
- operational sem.
  concrete, akin to actual implem.

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cf. Winskel, Formal Semantics of Prog. Languages (MIT Press)

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The distinction is not clear-cut ...

[e.g.

- Is game semantics den. or opr.?
- In what follows, we see
  initial alg. sem. ($\doteqdot$ den.)
  final coalg. sem. ($\doteqdot$ opr.)
  coincide in lucky situations

[Hoare, in early years]

"Once a denotational model is available, (nasty) operational models should immediately be thrown away"

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# Structural Operational Semantics (SOS)

aims at bringing a mathematical order to operational semantics.

- [Plotkin '81] First appearance
- Used e.g. for def. of the (opr.) sem. of ML, but

  i.e. language specification
- is much more widely used for process calculi

  CCS, CSP, ACP, π-cal., ...
  simple prog. lang. for concurrent systems/processes
- [Turi, Plotkin '97] Categorical formulation via alg. & coalg.

goal of this part

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# SOS The first example

- The process algebra:

$$p ::= 0 \mid a.p \mid p+p \mid p \parallel p$$

Termination

action prefix ($a \in L$) "do a and then do p"

non deterministic choice "choose one and do it"

parallel composition "do both in a concurrent manner"

One can also include recursive definitions
⇒ infinitary behaviors

For simplicity we don't do that now

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We define the (operational) sem. for this process alg. using

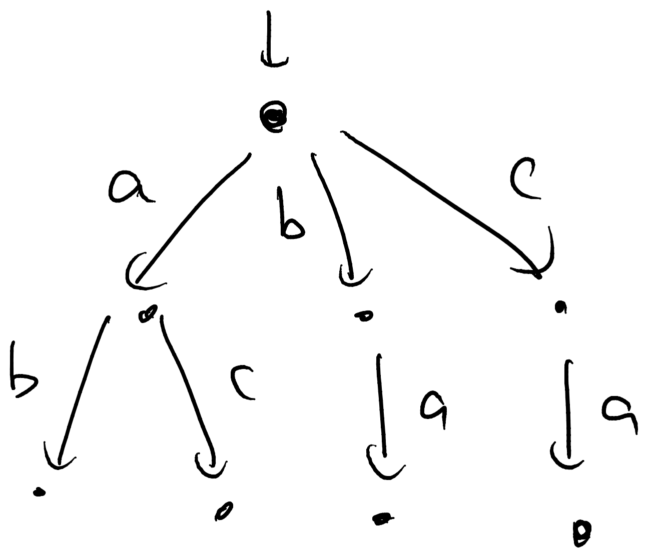
LTS (labeled transition sys., $\mathcal{P}_{fin}(L \times X)$ coalgebra
$$\begin{array}{c} \mathcal{P}_{fin}(L\times X) \\ \uparrow \\ X \end{array}$$
).

e.g.

$$[\![\, a.0 \parallel (b.0 + c.0) \,]\!]$$

= "the meaning of —"

:=



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$\boxed{Q}$ How to define $[\![ - ]\!]$ in a math. rigorous manner?

The SOS answer is as follows.

$\boxed{1}$ You specify the 'meaning' of process operators ($a.\_$, $+$, $\|$, ...) by means of <u>SOS rules</u>

$$\frac{}{a.x \xrightarrow{a} x}\ (\mathrm{Act})$$

$$\frac{x \xrightarrow{a} x'}{x+y \xrightarrow{a} x'}\ (+\text{-}L) \qquad \frac{y \xrightarrow{a} y'}{x+y \xrightarrow{a} y'}\ (+\text{-}R)$$

$$\frac{x \xrightarrow{a} x'}{x\|y \xrightarrow{a} x'\|y}\ (\|\text{-}L) \qquad \frac{y \xrightarrow{a} y'}{x\|y \xrightarrow{a} x\|y'}\ (\|\text{-}R)$$

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**2** Using these rules, you derive transitions between terms

short for $c.0$

$$\begin{tikzcd}
& a \| (b+c) \arrow[dl, "a"'] \arrow[d, "b"] \arrow[dr, "c"] & \\
0 \| b+c \arrow[dr, bend left, "c"] \arrow[dr, bend right, "b"'] & a \| 0 \arrow[d, "a"] & a \| 0 \arrow[dl, "a"] \\
& 0 \| 0 &
\end{tikzcd}$$

$$\dfrac{\dfrac{\dfrac{}{c \xrightarrow{c} 0}\,(Act)}{b+c \xrightarrow{c} 0}\,(+\text{-}R)}{a \| (b+c) \xrightarrow{c} a \| 0}\,(\|\text{-}R)$$

$$\dfrac{\dfrac{}{a \xrightarrow{a} 0}\,(Act)}{a \| b+c \xrightarrow{a} 0 \| b+c}\,(\|\text{-}L)$$

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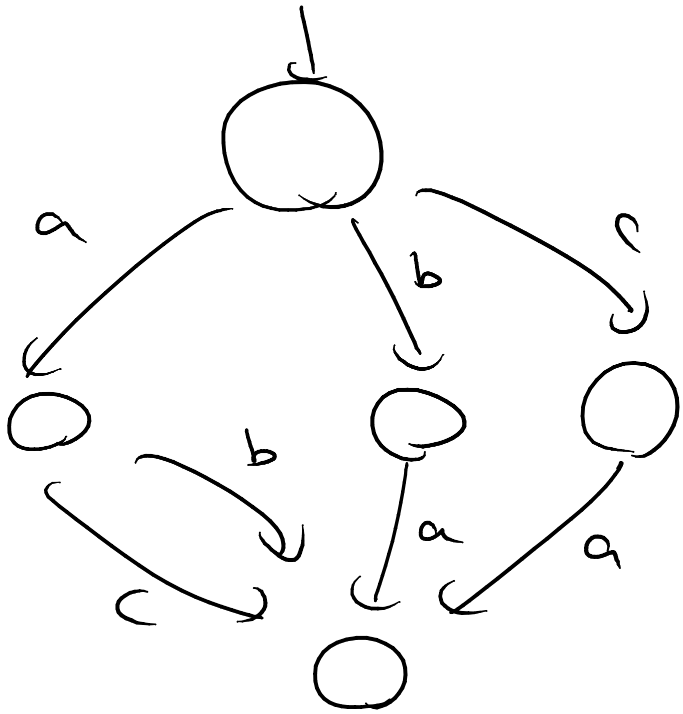
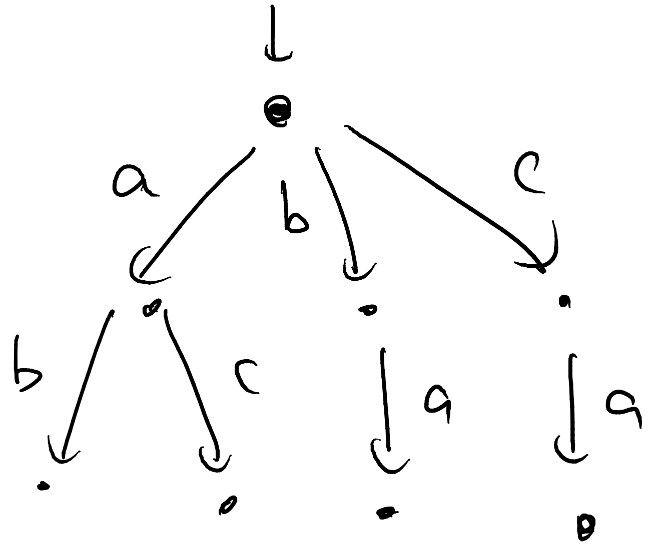
# $\boxed{3}$ The resulting LTS




is bisimilar to the one 3 pages ago:



Moreover, a desirable property:

**Compositionality**

(Modularity, 「要素還元性」
"Bisimilarity is a congruence")

For each process opr. $\sigma$,
$[\![ t_i ]\!]$ and $[\![ s_i ]\!]$ are bisimilar

"equivalence of LTSs"

$\Longrightarrow$ $[\![ \sigma(t_1, \ldots, t_n) ]\!]$ and

$[\![ \sigma(s_1, \ldots, s_n) ]\!]$ are bisimilar.

- Enables algebraic reasoning:

$$\frac{t_1 \sim s_1 \quad \ldots \quad t_n \sim s_n}{\sigma(t_1, \ldots, t_n) \sim \sigma(s_1, \ldots, s_n)}$$

denotes bisim. LTSs

- Replaceability, maintainability, ...
- Typical property of denotational semantics

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**Fact** There are so-called syntactic formats (GSOS, tyft, De Simone, ...) w/ following (meta) results:

templates for SOS rules

- If all the SOS rules adhere to the format,
- Then the induced $[\![-]\!]$ always is compositional.

Such a metaresult (and discovery of such a syntactic format) will be the goal of the categorical/bialgebraic development.

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We notice strong (co)algebraic flavor in SOS:

- {process terms} ∋ a || (b+c), ...
  is an initial algebra
- An LTS is a coalgebra
- "... is bisimilar" → coinduction
- $\dfrac{\dfrac{}{a \xrightarrow{a} 0}\,(\mathrm{Act})}{a \,||\, b+c \xrightarrow{a} 0 \,||\, b+c}\,(||\text{-L})$ : inductive flavor

⇒ Bialgebraic modeling!

BTW Process alg., concurrency

Their significance:

- Nowadays few computational tasks are sequential; most are parallel
  * the Internet
  * a multicore processor
  * HPC
- Parallelism / concurrency results in vast complexity
  * Nondeterminism is inevitable: "who goes first?"
  * $n$ computing units $\Rightarrow$ $\exp(n)$ complexity

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# §3.2 Bialgebraic Modeling: The Simple Setting

Here we present an (even simpler) example of SOS via bialgebras.

(Following [Klin, TCS 2011])

<u>We fix:</u>

- $\mathbb{V}$ar, a countable set of <u>metavariables</u>
- $\Sigma$: an algebraic signature (identified with $\Sigma : \mathbf{Sets} \to \mathbf{Sets}$, $X \mapsto \coprod_{\sigma \in \Sigma} X^{|\sigma|}$)
- $L$, a set of <u>labels</u>

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We consider the process alg. (i.e. a simple progr. lang.)

det. by $\Sigma$

that is for expressing $L$-streams

$$a_0 a_1 a_2 \ldots \in L^\omega .$$

<u>Notice that</u>

- $T_\Sigma 0 = \{ \Sigma\text{-terms with no variables} \}$

carries an initial algebra

$$\begin{array}{c} \Sigma(T_\Sigma 0) \\ \cong \big\downarrow \text{init.} \\ T_\Sigma 0 \end{array}$$

- An $(L\times\_)$-coalgebra $\begin{array}{c} L\times X \\ c\uparrow \\ X \end{array}$ is a <u>stream automaton</u>; and its state $x\in X$ induces an $L$-stream $a_0a_1\ldots\in L^\omega$ <u>by coinduction</u>;

$$\begin{tikzcd}
L\times X \arrow[r, dashed] & L\times L^\omega \\
X \arrow[u, "c"] \arrow[r, dashed, "beh(c)"'] & L^\omega \arrow[u, "\cong"', "final"] \\
x \arrow[r, mapsto] & (beh(c))(x)
\end{tikzcd}$$

<u>Our goal</u> Operational semantics of this process alg., that is (prog. lang.)

$$\begin{tikzcd}
T_\Sigma 0 \arrow[r, "{[\![-]\!]}"] & L^\omega \\
\text{(process term)} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \text{(stream)} \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

We could use either

$$\begin{tikzcd}
\Sigma(T_\Sigma 0) \arrow[d, "\mathrm{init}", "\cong"'] & \\
T_\Sigma 0 \arrow[r] & L^\omega \arrow[d, "\mathrm{final}", "\cong"'] \\
& L \times L^\omega
\end{tikzcd}$$

As before, we start with SOS rules, now subject to a certain syntactic format:

Def. A <u>simple stream SOS rule</u> is

$$\frac{x_1 \xrightarrow{a_1} x_1' \quad \cdots \quad x_n \xrightarrow{a_n} x_n'}{f(x_1, \ldots, x_n) \xrightarrow{b} g(y_1, \ldots, y_m)}$$

where

- $f \in \Sigma_n$, $g \in \Sigma_m$ (operations)
- $x_1, \ldots, x_n, x_1', \ldots, x_n' \in \mathbb{V}\mathrm{ar}$
- $y_j \in \{x_1', \ldots, x_n'\}$ for $j \in [1, m]$
- $b, a_1, \ldots, a_n \in L$.

More precisely: a simple str. SOS rule is

$$R = (f, g, (a_1, \dots, a_n), b, \theta)$$

where $\theta : m \longrightarrow n$ is a function.

**Def.** A <u>simple stream SOS specification</u> for $\Sigma$ is a set $\Lambda$ of stream SOS rules s.t.

for each $f \in \Sigma_n$ and each $a_1, \dots, a_n \in L$, there is exactly one rule in $\Lambda$ of the form

$$\frac{\bigcirc \xrightarrow{a_1} \bigcirc \quad \cdots \quad \bigcirc \xrightarrow{a_n} \bigcirc}{f(\bigcirc, \dots, \bigcirc) \longrightarrow \bigcirc}$$

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**Example** · $L = \{a, b\}$

- $\Sigma_0 = \{\underline{c_a}, c_b\}$ $\quad \Sigma_2 = \{alt\}$

"constantly $a$"

$\Sigma_1 = \Sigma_3 = \Sigma_4 = \cdots = 0$

- $\Lambda$ consists of

$$\frac{}{c_a \xrightarrow{a} c_a} \qquad \frac{}{c_b \xrightarrow{b} c_b}$$

$$\frac{x_1 \xrightarrow{l_1} x_1' \qquad x_2 \xrightarrow{l_2} x_2'}{alt(x_1, x_2) \xrightarrow{l_1} alt(x_2', x_1')} \quad \left(\begin{array}{l}\text{for each} \\ l_1, l_2 \\ \in L\end{array}\right)$$

Then $\Lambda$ is a simple str. SOS specif. for $\Sigma$.

The current syntactic format is very much restrictive, e.g.

$$\frac{x_1 \xrightarrow{l_1} x_1'}{\mathrm{zip}(x_1, x_2) \xrightarrow{l_1} \mathrm{zip}(x_2, x_1')}$$

does not satisfy the restriction.

$\mathrm{zip}(a_0 a_1 \ldots, b_0 b_1 \ldots) = a_0 b_0 a_1 b_1 \ldots$

**Exercise** What is the intention of the operator zip?

**Goals**
- Derive $[\![-]\!] : T_\Sigma 0 \to L^\omega$
- Show $[\![-]\!]$ is compositional

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Crucial observation:

a simple str. SOS specification $\Lambda$

a natural transformation

"map of functors"

$$\Sigma F \Rightarrow F \Sigma$$

(where $F = L \times \_$ )

More generally:

A spec. subj. to a certain syntactic format

a natural transformation

$$\Sigma F \Rightarrow F \Sigma$$

- for many different $F$
- this can be more complex (later)

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... finally we need to introduce nat. trans.!

<u>Def.</u> Let $\mathbb{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{D}$ be functors.

A <u>natural transformation</u>

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathbb{D} \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

In the old age it was sometimes written

$\alpha : F \dot{\to} G : \mathbb{C} \to \mathbb{D}$

is a family

$$\left\{ FX \xrightarrow{\alpha_X} GX \right\}_{X \in \mathbb{C}}$$

$\in \mathbb{D}$

of $\mathbb{D}$-arrows.

$\alpha_X$ : $\alpha$'s <u>component</u> at $X \in \mathbb{C}$

subject to the <u>naturality condition</u>

$\mathbb{C}$

$$X \xrightarrow{f} Y$$

$\mathbb{D}$

$$\begin{tikzcd}
FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] \arrow[dr, phantom, "//"] & GX \arrow[d, "Gf"] \\
FY \arrow[r, "\alpha_Y"'] & GY
\end{tikzcd}$$

---

Before exhibiting

<u>a simple str. SOS specification $\Lambda$</u>

a natural transformation

$$\Sigma F \Rightarrow F\Sigma$$

$$\left(\begin{array}{l}\text{where}\\ F = L \times \_\end{array}\right)$$

We see why $\Sigma F \Rightarrow F\Sigma$ is useful

in the current setting.

Assume we have obtained

$$\Sigma F \overset{\lambda}{\Longrightarrow} F\Sigma . \qquad (F = L \times \_)$$

Then:

- $\begin{array}{c}\Sigma(T_\Sigma 0)\\ \cong\big\downarrow \text{init.}\\ T_\Sigma 0\end{array}$ has an F-coalg. structure, by

$$\begin{tikzcd}
\Sigma(T_\Sigma 0) \arrow[r, dashed] \arrow[dd, "\text{init.}"] & \Sigma(F(T_\Sigma 0)) \arrow[d, "\lambda_{T_\Sigma 0}"] \\
 & F(\Sigma(T_\Sigma 0)) \arrow[d, "F(\text{init.})"] \\
T_\Sigma 0 \arrow[r, dashed] & F(T_\Sigma 0)
\end{tikzcd}$$

!!

- Which can be used in

$$\begin{tikzcd}
F(T_\Sigma 0) \arrow[r, dashed] & F(L^\omega) \\
T_\Sigma 0 \arrow[u] \arrow[r, dashed, "{[\![ - ]\!]}"] & L^\omega \arrow[u, "\text{find}"']
\end{tikzcd}$$

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(more on this is coming later)

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**Prop.**

$$\frac{\text{a simple str. SOS specification } \Lambda}{\overline{\phantom{x}}}$$

a simple str. SOS specification $\Lambda$

---

a natural transformation

$$\Sigma F \Rightarrow F\Sigma$$

(where $F = L \times \_$)

**Proof.**

[I] $\dfrac{\Sigma F \Rightarrow F\Sigma}{\coprod_{f \in \Sigma} (F(-))^{|f|} \Rightarrow F\Sigma}$

$$\overline{\overline{\phantom{xx}}}$$

$(F(-))^{|f|} \Rightarrow F\Sigma$ for each $f \in \Sigma$

$$\underbrace{(F(-))^{|f|}}_{= (L \times (-))^{|f|}} \Rightarrow \underbrace{F\Sigma}_{= L \times \left(\coprod_{h \in \Sigma} (-)^{|h|}\right)}$$

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We define such functions by (Let $f \in \Sigma_n$)

$$(L\times S)^n \longrightarrow L \times \left(\coprod_{h\in\Sigma} S^{|h|}\right)$$

$$(a_1,s_1,\dots,a_n,s_n) \longmapsto \left(a,\ k_g(s_{i_1},\dots,s_{i_m})\right)$$

where the rule in $\Delta$ corresponding to

$$\frac{\bigcirc \xrightarrow{a_1} \bigcirc \quad \cdots \quad \bigcirc \xrightarrow{a_n} \bigcirc}{f(\bigcirc,\dots,\bigcirc) \longrightarrow \bigcirc}$$

is

$$\frac{x_1 \xrightarrow{a_1} x_1' \quad \cdots \quad x_n \xrightarrow{a_n} x_n'}{f(x_1,\dots,x_n) \xrightarrow{a} g(x'_{i_1},\dots,x'_{i_m})}$$

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We need to check naturality:

$$\begin{array}{lccc} S & (L\times S)^n & \longrightarrow & L\times\left(\coprod_{h\in\Sigma} S^{|h|}\right) \\ \downarrow f & \downarrow (L\times f)^n & & \downarrow \\ S' & (L\times S')^n & \longrightarrow & L\times\left(\coprod_{h\in\Sigma} (S')^{|h|}\right) \end{array}$$

which is easy.

[↑] Given a natural transf.

$$\frac{\dfrac{\Sigma F \Rightarrow F\Sigma}{\coprod_{f\in\Sigma}\left(F(-)\right)^{|f|} \Rightarrow F\Sigma}}{\underbrace{\left(F(-)\right)^{|f|}}_{\left(L\times(-)\right)^{|f|}} \Rightarrow \underbrace{F\Sigma}_{L\times\left(\coprod_{h\in\Sigma}(-)^{|h|}\right)} \quad \text{for each } f\in\Sigma}$$

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We fix $f \in \Sigma$,
$a_1, \ldots, a_{|f|} \in L$.

Take its component at $X' := \{x'_1, \ldots, x'_{|f|}\}$:

$$(L \times X')^{|f|} \xrightarrow{\phantom{kkk}} L \times \coprod_{h \in \Sigma} (X')^{|h|}$$

we denote this by $k$

and consider

$$k\left( (a_1, x'_1), \ldots, (a_{|f|}, x'_{|f|}) \right)$$

$$=: \left( a,\ \kappa g(x'_{i_1}, \ldots, x'_{i_{|g|}}) \right)$$

From this we define a rule

$$\frac{x_1 \xrightarrow{a_1} x_1' \quad \ldots \quad x_n \xrightarrow{a_n} x_n'}{f(x_1, \ldots, x_n) \xrightarrow{a} g(x'_{i_1}, \ldots, x'_{i_{|g|}})}$$

We do this for each $f$, $a_1, \ldots, a_{|f|}$
and define a simple str. SOS spec.

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It is not hard to see that [I] and [J] are converse to each other $\square$

---

Therefore we transformed a set of rules into an <u>abstract SOS rule</u> $\lambda : \Sigma F \Rightarrow F\Sigma$.

We shall now fully exploit this ...


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**Prop.** $\lambda$ lifts $\Sigma$ : Sets $\to$ Sets to

$$\Sigma_\lambda : \mathrm{Coalg}_F \longrightarrow \mathrm{Coalg}_F,$$

that is,

$$\begin{tikzcd} \mathrm{Coalg}_F \arrow[r, "\Sigma_\lambda"] \arrow[d, "\text{forgetful}"'] \arrow[dr, phantom, "="] & \mathrm{Coalg}_F \arrow[d] \\ \mathbf{Sets} \arrow[r, "\Sigma"'] & \mathbf{Sets} \end{tikzcd}$$

Concretely:

$$\mathrm{Coalg}_F \xrightarrow{\Sigma_\lambda} \mathrm{Coalg}_F$$

$$\begin{pmatrix} FX \\ \uparrow c \\ X \end{pmatrix} \longmapsto \begin{pmatrix} F\Sigma X \\ \uparrow \lambda_X \\ \Sigma F X \\ \uparrow \Sigma c \\ \Sigma X \end{pmatrix}$$

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**Exercise** Write down $\Sigma_\lambda$'s action on arrows (use naturality of $\lambda$)

**Proof.** Straightforward. $\square$

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A 2-categorical view:

- $$\begin{tikzcd} & \mathbb{C} \arrow[dr, "id"] \arrow[d, phantom, "\Downarrow"] & \\ \mathrm{Coalg}_F \arrow[ur, "forg."] \arrow[dr, "forget."'] & {} & \mathbb{C} \\ & \mathbb{C} \arrow[ur, "F"'] & \end{tikzcd}$$ $\boxed{\mathbf{Cat}}$ is an inserter.

- $$\begin{tikzcd} & \mathbb{C} \arrow[r, "\Sigma"] \arrow[dr, "id"] & \mathbb{C} \arrow[dr, "id"] \arrow[d, phantom, "="] & \\ \mathrm{Coalg}_F \arrow[ur, "forg."] \arrow[dr, "forg."'] \arrow[r, phantom, "\Downarrow"] & {} & \mathbb{C} \arrow[r, "\Sigma"] \arrow[d, phantom, "\Downarrow\lambda"] & \mathbb{C} \\ & \mathbb{C} \arrow[ur, "F"] \arrow[r, "\Sigma"'] & \mathbb{C} \arrow[ur, "F"'] & \end{tikzcd}$$

induces

$$\mathrm{Coalg}_F \overset{\Sigma_\lambda}{\dashrightarrow} \mathrm{Coalg}_F .$$

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Dually:

**Prop.** $\lambda$ lifts $F : \mathbf{Sets} \to \mathbf{Sets}$

to $F_\lambda : \mathrm{Alg}_\Sigma \longrightarrow \mathrm{Alg}_\Sigma$

$$\begin{pmatrix} \Sigma X \\ \downarrow a \\ X \end{pmatrix} \longmapsto \begin{pmatrix} \Sigma F X \\ \downarrow \lambda_X \\ F \Sigma X \\ \downarrow Fa \\ FX \end{pmatrix}$$

---

Therefore we obtained

$$\begin{tikzcd}
\mathrm{Coalg}_{F_\lambda} \arrow[d] & & \mathrm{Alg}_{\Sigma_\lambda} \arrow[d] \\
\mathrm{Alg}_\Sigma \arrow[loop left, "F_\lambda"] \arrow[dr] & & \mathrm{Coalg}_F \arrow[loop right, "\Sigma_\lambda"] \arrow[dl] \\
& \mathbf{Sets} \arrow[loop left, "F"] \arrow[loop right, "\Sigma"] &
\end{tikzcd}$$

$\lambda$

object:

$$\begin{pmatrix} F(\Sigma X) \\ \uparrow \cdots \\ \Sigma X \end{pmatrix}$$
$$\downarrow$$
$$\begin{pmatrix} FX \\ \uparrow \\ X \end{pmatrix},$$

a $\Sigma_\lambda$-coalg.

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Moreover:

**Def.** A <u>$\lambda$-bialgebra</u> is a pair

$$\begin{tikzcd} \Sigma X \arrow[d, "a"] \\ X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} FX \\ X \arrow[u, "c"'] \end{tikzcd}$$

(alg. & coalg. with the same carrier)

s.t.

$$\begin{tikzcd} \Sigma X \arrow[d, "a"] \arrow[rr, "\Sigma c"] & & \Sigma F X \arrow[d, "\lambda_X"] \\ X \arrow[d, "c"] & = & F \Sigma X \arrow[dll, "Fa"] \\ FX & & \end{tikzcd}$$

"The pentagon diagram"

A <u>map</u> of $\lambda$-bialg. is $f$ s.t.

$$\begin{tikzcd} \Sigma X \arrow[r, "\Sigma f"] \arrow[d] & \Sigma Y \arrow[d] \\ X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ FX \arrow[r, "Ff"] & FY \end{tikzcd}$$

$\Rightarrow$ $\lambda$-Bialg., the cat. of $\lambda$-bialg.

**Prop.** We have isomorphisms between categories:

$$\begin{tikzcd}[column sep=small] & \lambda\text{-}\mathbb{B}\mathrm{ialg} \arrow[dl, bend left, "\cong"] \arrow[dr, bend left, "\cong"] & \\ \mathrm{Alg}\,\Sigma_\lambda \arrow[ur, bend left] & & \mathrm{Coalg}\,F_\lambda \arrow[ul, bend left] \end{tikzcd}$$

**Proof.** Not hard. for example,

$$\left[\begin{tikzcd} (\Sigma X \arrow[r, "\Sigma c"] \arrow[d, "a"'] & \Sigma F X \arrow[r, "\lambda_X"] & F\Sigma X) \arrow[dl, "Fa"] \\ (X \arrow[r, "c"'] & FX) & \end{tikzcd}\right] \in \mathrm{Alg}\,\Sigma_\lambda$$

$$\longmapsto \left(\begin{tikzcd} \Sigma X \arrow[d, "a"] \\ X \arrow[d, "c"] \\ FX \end{tikzcd}\right)$$

(pentagon is obvious)

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We have seen this:
"What is an initial coalgebra?"

**Lem.** If $\mathbb{C}$ has an initial obj. $0$, for any functor $F : \mathbb{C} \to \mathbb{C}$

the coalgebra

$$\boxed{\mathbb{C}}\quad \begin{array}{c} F0 \\ \uparrow ! \\ 0 \\ \text{(initial)} \end{array}$$

is an initial $F$-coalg.

**Proof.**

$$\begin{tikzcd} F0 \arrow[r] & FX \\ 0 \arrow[u, "!"] \arrow[r, dashed, "!"'] \arrow[ur, phantom, "\vee"] & X \arrow[u, "c"'] \end{tikzcd}$$

trivial due to $0$: initial.

**Lem.** $\mathbb{C}$ has a final obj. $1$

$$\Longrightarrow \quad \begin{array}{c} F1 \\ \downarrow ! \\ 1 \end{array} \quad \text{is a final algebra.}$$

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<u>Thm.</u> Asm

$\Sigma$ : Sets $\to$ Sets has an initial algebra

$F$ : Sets $\to$ Sets has a final coalgebra

Then

- $\begin{array}{c}\Sigma A\\ \downarrow{\scriptstyle \mathrm{init}}\\ A\end{array}$ is canonically a $\lambda$-bialgebra.
- It is moreover an initial bialg.
- $\begin{array}{c}FZ\\ \uparrow{\scriptstyle \mathrm{final}}\\ Z\end{array}$ is canonically a $\lambda$-bialg.
- It is moreover a final $\lambda$-bialg.

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**Proof.** We apply the lemmas (2 pages ago) to

$$\begin{tikzcd}
\mathrm{Coalg}_{F_\lambda} \arrow[d] & & \mathrm{Alg}_{\Sigma_\lambda} \arrow[d] \\
\mathrm{Alg}_{\Sigma} \arrow[loop left, "F_\lambda"] \arrow[dr] & & \mathrm{Coalg}_{F} \arrow[loop below, "\Sigma_\lambda"] \arrow[dl] \\
& \mathbf{Sets} \arrow[loop left, "F"] \arrow[loop right, "\Sigma"] &
\end{tikzcd}$$

$\lambda$

$\exists$ initial obj.

$\exists$ final obj.

Therefore there are

- an init. obj. in $\mathrm{Alg}_{\Sigma_\lambda}$, and
- a final obj. in $\mathrm{Coalg}_{F_\lambda}$.

Now use

$$\begin{tikzcd}
& \lambda\text{-}\mathrm{Bialg} \arrow[dl, bend right, "\simeq"'] \arrow[dr, bend left, "\simeq"] & \\
\mathrm{Alg}_{\Sigma_\lambda} \arrow[ur, bend right] & & \mathrm{Coalg}_{F_\lambda} \arrow[ul, bend left]
\end{tikzcd}$$

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Concretely this is what we did on some 13 pages ago:

- $\Sigma(T_\Sigma 0)$ has an $F$-coalg. structure, by

$$\begin{array}{c}\Sigma(T_\Sigma 0)\\ \downarrow\\ T_\Sigma 0\end{array}$$

$$\begin{tikzcd}
\Sigma(T_\Sigma 0) \arrow[r, dashed] \arrow[dd, "\text{init.}"] & \Sigma(F(T_\Sigma 0)) \arrow[d, "\lambda_{T_\Sigma 0}"] \\
 & F(\Sigma(T_\Sigma 0)) \arrow[d, "F(\text{init.})"] \\
T_\Sigma 0 \arrow[r, dashed] & F(T_\Sigma 0)
\end{tikzcd}$$

!!

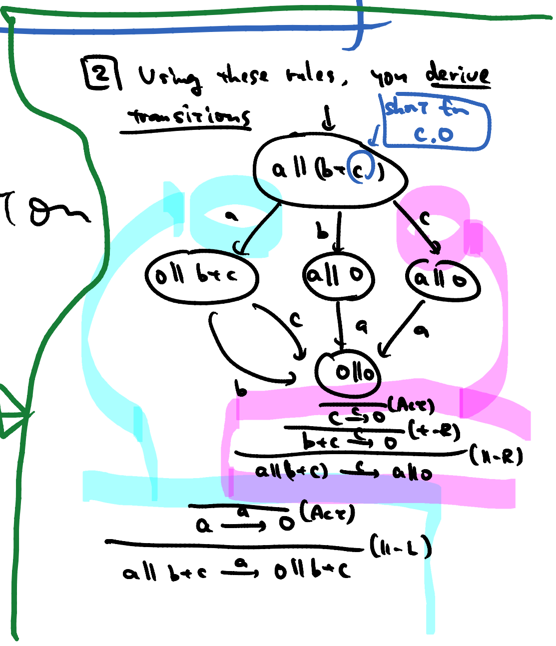
Trivial $F\lambda$-Coalg. str.

- Which can be used in

$$\begin{tikzcd}
F(T_\Sigma 0) \arrow[r, dashed] & F(L^\omega) \\
T_\Sigma 0 \arrow[u] \arrow[r, dashed, "[\![-]\!]"] & L^\omega \arrow[u, "\text{final}"]
\end{tikzcd}$$

Even More concretely

This "stream automaton structure" on terms is what we did in (Rule-based deriv. of transitions)



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However, we now see more:

- By the dual scheme we can
  * equip $\begin{array}{c} F(L^\omega) \\ \uparrow \\ L^\omega \end{array}$ with an alg. str. $\begin{array}{c} \Sigma(L^\omega) \\ \downarrow \\ L^\omega \end{array}$
  * and

$$\begin{tikzcd} \Sigma(T_\Sigma 0) \arrow[r, dashed] \arrow[d, "\mathrm{init}"] & \Sigma(L^\omega) \arrow[d] \\ T_\Sigma 0 \arrow[r, dashed] & L^\omega \end{tikzcd}$$

Note

On the last page: final coalg. semantics (operational)

on this page: initial alg. semantics (denotational)

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<u>Point</u> These two coincide!!

By

$$\begin{tikzcd}
\Sigma(T_\Sigma 0) \arrow[r, dashed] \arrow[d, "\cong"', "\mathrm{init}"] & \Sigma(L^\omega) \arrow[d] \\
T_\Sigma 0 \arrow[r, dashed, "[\![-]\!]"] \arrow[d] & L^\omega \arrow[d, "\cong"', "\mathrm{final}"] \\
F(T_\Sigma 0) \arrow[r, dashed] & F(L^\omega)
\end{tikzcd}$$

initial $\lambda$-bialg. $\qquad$ final $\lambda$-bialg.

$\because$) If $[\![-]\!]$ makes the above diagram commute, then in particular

$$\begin{tikzcd}
T_\Sigma 0 \arrow[r, dashed, "[\![-]\!]"] \arrow[d] & L^\omega \arrow[d, "\cong"', "\mathrm{final}"] \\
F(T_\Sigma 0) \arrow[r, dashed] & F(L^\omega)
\end{tikzcd}$$

thus this $[\![-]\!]$ is the same as $[\![-]\!]$ 2 pages ago.

# Compositionality

By $[\![-]\!]$ being an algebra hom., we immediately have

$$\begin{tikzcd}
\Sigma(T_\Sigma 0) \arrow[r, dashed] \arrow[d, "\cong"', "\mathrm{init}"] & \Sigma(L^\omega) \arrow[d] \\
T_\Sigma 0 \arrow[r, dashed, "{[\![-]\!]}"] \arrow[d] & L^\omega \arrow[d, "\cong"', "\mathrm{final}"] \\
F(T_\Sigma 0) \arrow[r, dashed] & F(L^\omega)
\end{tikzcd}$$

$$[\![f(t_1, \dots, t_n)]\!] = [f]\big([\![t_1]\!], \dots, [\![t_n]\!]\big)$$

The interpretation of $f \in \Sigma$ in
$$\begin{tikzcd}
\Sigma(L^\omega) \arrow[d] \\
L^\omega
\end{tikzcd}$$

What is crucial:

$[\![f(t_1, \dots, t_n)]\!]$ is a function on $[\![t_1]\!], \dots, [\![t_n]\!]$

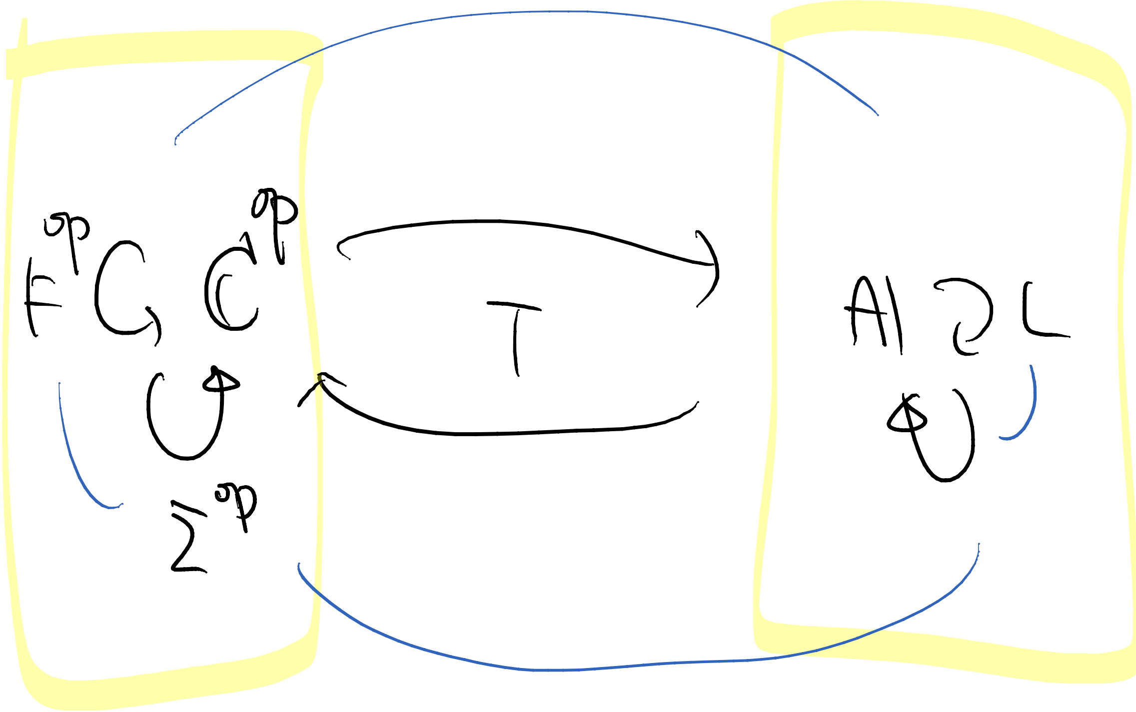
From this we have compositionality

$$\frac{t_1 \sim s_1 \quad \cdots \quad t_n \sim s_n}{f(t_1, \ldots, t_n) \sim f(s_1, \ldots, s_n)}$$

i.e. bisimilarity is a congruence.

final coalg. sem.

2012年8月1日
14:36



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# §3.3 Bialgebraic Modeling Beyond the Simple Setting

1. As we saw, an abstract SOS rule of the form

$$\Sigma F \Rightarrow F \Sigma$$

is very much restricted.
(For example, "zip" of two streams cannot be modeled)

More expressive formats of abstract SOS rules:

- $\Sigma(\underbrace{F \times id}_{\text{the cofree copointed functor over } F}) \Rightarrow F \underbrace{T_\Sigma}_{\text{the free monad over } \Sigma}$

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This corresponds to the well-known GSOS format.

- $$\Sigma \underbrace{F^{\infty}}_{\text{cofree comonad}} \Rightarrow F \underbrace{T_{\Sigma}}_{\text{free monad}}$$

[This canonically induces a <u>distributive law</u>

$$T_{\Sigma} F^{\infty} \Rightarrow F^{\infty} T_{\Sigma}$$
]

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$\boxed{2}$ For many functors $F$ / base categories $\mathbb{C}$

- For probabilistic systems: take $F$ that involves a distribution functor $\mathcal{D}$ [Bartels]
- Timed systems [Kick et al.]
- Continuous prob. sys. (with $\mathbb{C} = \mathbf{Meas}$) [Bacci, Miculan]
- For value-passing / name-passing calculi [Turi, Fiore, Staton, ...] (with $\mathbb{C}$: a presheaf cat.)