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Part IV :

Algebras ~~& Coalgebras~~ in a Presheaf Category

Plan (tentative)

- Intro : syntax w/ var. binding
- presheaf
- The Yoneda lemma, end/coend formulas

- Goal Substitution by initiality

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§ 4.1 Intro

$\underline{\Sigma}$: alg. signature

$\underline{\text{init. } \Sigma\text{-alg}}$: {well-formed syntactic expr.}

interplay (via bialgebras) for structural operational semantics

F : behavior / transition type

F-coalg : state-based transition system

final F-coalg : {behaviors}

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But How about the syntax for λ-calculus?

$$t ::= \underset{V}{\overset{\cap}{x}} \mid t.t \mid \lambda x.t$$

+ where we need to suitably address

capture-avoiding substit.

⇒ Key: variable binders

$\boxed{\text{Ans.}}$ - λ-calculus terms should better be with an (explicit) variable context, i.e.

$$x_1 \vdash \lambda x_2.(x_1, x_2)$$

- and these form indeed an initial algebra ... in a presheaf category!!

Our goal

- Characterize syntax w/ var binders as an initial algebra
- And use initiality (i.e. induction) to define capture-avoiding substitution

References

- [Fiore, Plotkin, Turi, LICS '99]

  The first paper,
  not the easiest to read, but
  hopefully after this lecture
  you'll find it easy

- Subsequent papers by Fiore, Turi, Staton, ...
  - coalg. in a presheaf cat
    (We have no time to cover this)
  - second-order abstract syntax

- Related: nominal sets
  [Pitts, Gabbay, ...]
  - A non-categorical presentation

of what is essentially the same

- More precisely:

  a nominal set

  a sheaf wrt. the so-called

  atomic topology in $\mathbb{F}$

- [Mac Lane - Moerdijk,
  Sheaves for ...]

  Extensive treatment of presheaves

  (and much more!)

- [Fiore, Rough Notes on Presheaves]

  (Available on his website)

  A fabulous notes, hand-written.

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§4.2 Presheaves: Def. and First Examples

Presheaves are, in a sense, the first generalization of sets.

In the current notes we fix a base category to $\mathbb{F}$; but the most of what follows applies to a general setting.

<u>Def.</u> (The category $\mathbb{F}$)

<u>obj.</u> $0, 1, 2, 3, \ldots, n, \ldots$

$\boxed{\text{the finite set } \{0, 1, \ldots, n-1\}}$

<u>arr.</u> functions between them

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Therefore:

- $\mathbb{F}$ is the full subcategory of Sets consisting of objects 0, 1, 2, ...

- Concretely:

$\underline{\mathbb{F} \mid}$

$$\begin{tikzcd} 0 \arrow[r, "!"] & 1 \arrow[r, shift left, "\kappa_0"] \arrow[r, shift right=2, "\kappa_1"'] & 2 \arrow[l, bend right=40, "!"'] \arrow[r, shift left] \arrow[r, shift left=3, "!"] & 3 \arrow[l, shift left] \arrow[l, shift left=3, "!"] & \cdots \end{tikzcd}$$

- $\mathbb{F}$ is the free cocartesian category $\leftarrow$ $\boxed{\text{w/ finite coproducts}}$ with one generator 1

- That is: $\mathbb{F}$ is "generated" from $\left(\begin{array}{l} 0 \xrightarrow{!} 1 \\ 1 \xrightarrow{id} 1 \\ 2 \xrightarrow{!} 1 \end{array} \right)$ via $+$

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Def. (Presheaves)

$\mathbb{C}$: a category.

- A presheaf over $\mathbb{C}$ is a functor

$$P : \mathbb{C} \longrightarrow \mathbf{Sets}$$

$$\begin{array}{ccc} X & & PX \\ \downarrow f & \longmapsto & \downarrow Pf \\ Y & & PY \end{array}$$

- Presheaves and natural transf. between them $\left(\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "P"] \arrow[r, bend right, "Q"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{Sets} \end{tikzcd} \right)$ form a category $\mathbf{Sets}^{\mathbb{C}}$

Examples

- X : a topological space

  $\mathcal{O}(X)$: the set of open sets,
  with the inclusion order $\subseteq$,
  considered as a category

  $\mathcal{O}(X)^{op}$: its opposite category

$$\frac{\frac{U \longrightarrow V \text{ in } \mathcal{O}(X)^{op}}{}}{V \subseteq U}$$

Then
$P : \mathcal{O}(X)^{op} \longrightarrow \mathbf{Sets}$, a presheaf,

* assigns, to each open $U \subseteq X$, a set PU
* $V \subseteq U$ induces a restriction map $P_{UV} : PU \longrightarrow PV$

$\Longrightarrow$ Conventional notion of presheaf

- Conventionally,

sheaf = presheaf

+ the "patch-up" coherence condition

This also has a nice categorical generalization via the <u>Grothendieck topology</u>

We are interested in presheaves

$$P : \mathbb{F} \longrightarrow \mathbf{Sets},$$

examples of which are

terms in a context.

(or more generally 'elements')

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de Bruijn style

Def. λ-terms (w/ explicit context) are inductively defined by:

$$\frac{}{x_1, \ldots, x_n \vdash x_i} \text{(Var)} \quad (i \in [1, n])$$

$$\frac{x_1, \ldots, x_n \vdash t \qquad x_1, \ldots, x_n \vdash s}{x_1, \ldots, x_n \vdash ts} \text{(Appl.)}$$

$$\frac{x_1, \ldots, x_n, x_{n+1} \vdash t}{x_1, \ldots, x_n \vdash \lambda x_{n+1}. t} \text{(Abs.)}$$

modulo α-equivalence.

[**NB** The whole $\boxed{x_1, \ldots, x_n \vdash t}$ is a λ-term]

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'structural rules' for contexts

Lem. The following rules are admissible.

$$\frac{x_1, \ldots, x_n \vdash t}{x_1, \ldots, x_n, x_{n+1} \vdash t} \quad \text{(Weakening)}$$

$$\frac{x_1, \ldots, x_n \vdash t}{x_1, \ldots, x_n \vdash t[x_i/x_j, x_j/x_i]} \quad \text{(exchange)}$$

$$\frac{x_1, \ldots, x_n, x_{n+1} \vdash t}{x_1, \ldots, x_n \vdash t[x_n/x_{n+1}]} \quad \text{(Contraction)}$$

Exercise Prove this!

<u>Now</u> Define $\Lambda : \mathbb{F} \to \mathbf{Sets}$ by

- $\Lambda(n) := \left\{ \begin{array}{c} \lambda\text{-terms} \\ \text{w/ the context} \\ x_1, \dots, x_n \end{array} \right\}$

$$= \{ x_1, \dots, x_n \vdash t \}$$

- Given $n \xrightarrow{f} m$ in $\mathbb{F}$

$$\Lambda(n) \xrightarrow{\Lambda(f)} \Lambda(m)$$

$$\begin{pmatrix} x_1, \dots, x_n \\ \vdash t \end{pmatrix} \longmapsto \begin{array}{c} x_1, \dots, x_m \\ \vdash t\left[x_{f(i)} / x_i\right] \end{array}$$

For example,

$$\mathbb{F} \qquad 0 \xrightarrow{\;!\;} 1 \;\underset{\kappa_1}{\overset{\kappa_0}{\rightrightarrows}}\; 2 \;\; \cdots$$

weakening

$$\left(\vdash \lambda x_1. x_1\right) \mapsto \left(x_1 \vdash \lambda x_1. x_1\right)$$

$$\begin{pmatrix} x_1 \vdash \\ \lambda x_3. x_1(x_1, x_3) \end{pmatrix} \mapsfrom \begin{pmatrix} x_1, x_2 \vdash \\ \lambda x_3. x_1(x_2 x_3) \end{pmatrix}$$

contraction

Therefore:

(action of $\mathbb{F}$-arrows)

= (context manipulation, or structure maps)

weakening, contraction, exch.

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§4.3 Some General Facts on Presheaves

My mission here: Demonstrate my (own) working abstraction level

Presheaf as a generalization of set?

Def. $P : \mathbb{C} \to \mathbf{Sets}$.

- The category of elements of P is: (denoted by $\int P$)

obj. $(X \in \mathbb{C},\ x \in PX)$

arr. $(X, x) \xrightarrow{f} (Y, y)$ in $\int P$

$f : X \to Y$ in $\mathbb{C}$ s.t. $(Pf)(x) = y$

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- We have a <u>projection functor</u>

$$\begin{tikzcd}[row sep=small]
\Pi : \int P \arrow[rr] & & \mathcal{C} \\
(X, x) \arrow[d, "f"] & \longmapsto & X \arrow[d, "f"] \\
(Y, (Pf)(x)) & & Y
\end{tikzcd}$$

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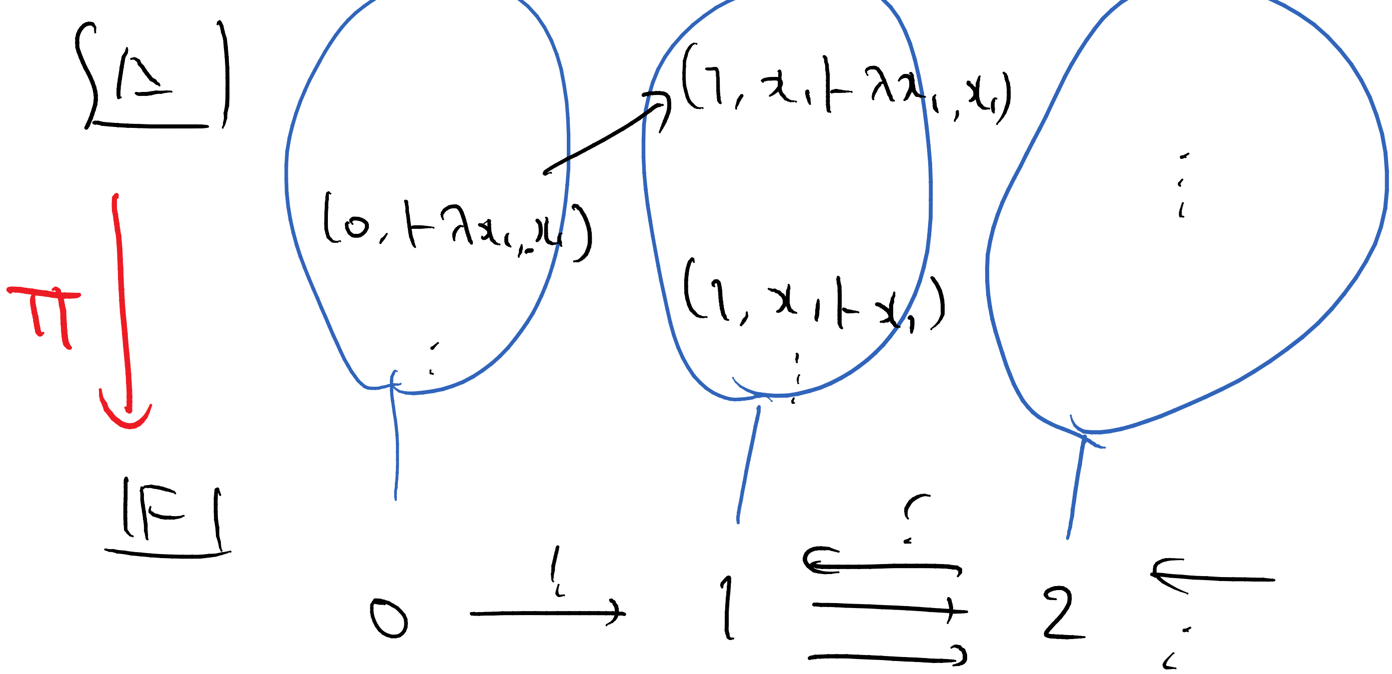
Therefore :

$\int \Delta$

$$\begin{array}{ccc} \boxed{(0, \ \vdash \lambda x_1. x_1)} & & \cdots \\ \Big\downarrow {\scriptstyle (0 \xrightarrow{i} 1)} & & \\ \boxed{(1, \ x_1 \vdash \lambda x_1. x_1)} & & \boxed{(1, \ x_1 \vdash x_1)} \\ \vdots & & \ddots \end{array}$$

Or :



Thus: "a presheaf is a set with a context"

Some 'nice' presheaves:

Def.

- Each obj. $X \in \mathbb{C}$ induces a presheaf

$$\begin{array}{ccccc} \mathbb{C}(X,-) & : & \mathbb{C} & \longrightarrow & \mathbf{Sets} \\ \| & & \left(\begin{array}{c} Y \\ g \downarrow \\ Z \end{array}\right) & \longmapsto & \left(\begin{array}{c} \mathbb{C}(X,Y) \\ \downarrow \mathbb{C}(X,g) \\ \mathbb{C}(X,Z) \end{array}\right) \\ yX & & & & \end{array}$$

"The Yoneda of X"

Exercise

What is $\mathbb{C}(X,g)$, precisely?

- This correspondence is functorial:

$$\mathbb{C}^{op} \xrightarrow{\;y\;} \mathbf{Sets}^{\mathbb{C}}$$

$$\begin{array}{ccc} \mathbb{C} \ni X & & yX \\ \downarrow f & \longmapsto & \uparrow yf \\ Y & & yY \end{array}$$

- $P \in \mathbf{Sets}^{\mathbb{C}}$ is said to be a <u>representable presheaf</u> if

$$P \cong yX \quad \text{for some } X \in \mathbb{C}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "P"] \arrow[r, bend right, "yX"'] \arrow[r, phantom, "\Downarrow \cong"] & \mathbf{Sets} \end{tikzcd}$$

Representable presheaves are prototypical; many results on presheaves reduce to the special case on representable presheaves.

This hinges on the following 'representation theorem':

Lem (The Yoneda lemma)

$P : \mathbb{C} \to \mathbf{Sets}$

$X \in \mathbb{C}$ (Hence $y(X = \mathbb{C}(X, -) : \mathbb{C} \to \mathbf{Sets}$)

Then

$$\underbrace{\mathrm{Nat}(\mathbb{C}(X, -), P)}_{\text{The set of nat. trans.}} \cong \underbrace{PX}_{P : \mathbb{C} \to \mathbf{Sets} \text{ so } PX \in \mathbf{Sets}}$$

$\mathrm{Nat}(\ldots)$: hom-set of $\mathbf{Sets}^{\mathbb{C}}$

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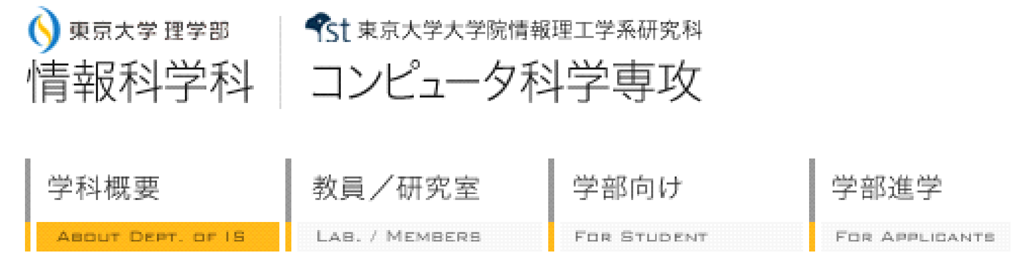
BTW:

Nobuo Yoneda (1930-1996)

米田 信夫

Prof. Emeritus,



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!!

In his days ...

Course "Theory of databases"

meant

"Intro. to Category Theory" ☺

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<u>Proof.</u> The bij. correspondence is :

$$\mathrm{Nat}(\mathbb{C}(X,-),P) \cong PX$$

$$\big(\alpha : \mathbb{C}(X,-) \Rightarrow P\big) \longmapsto \alpha_X(\mathrm{id}_X)$$

$$\begin{array}{ccc} \mathbb{C}(X,X) & \xrightarrow{\alpha_X} & PX \\ \rotatebox{90}{$\in$} & & \\ \mathrm{id}_X & & \end{array}$$

$$\left(\begin{array}{ccc} \mathbb{C}(X,Y) & \longrightarrow & PY \\ f & \longmapsto & (Pf)(x) \end{array}\right) \longleftarrow\!\shortmid \ x$$

$$\begin{array}{ccc} Pf : PX & \longrightarrow & PY \\ \rotatebox{90}{$\in$} & & \\ x & & \end{array}$$

Here the crucial diagram is the following naturality:

$$
\begin{array}{c|c}
\underline{\mathbb{C}} & \underline{\mathbf{Sets}} \\
\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd}
&
\begin{tikzcd}
\mathbb{C}(X,X) \arrow[r, "\alpha_X"] \arrow[d, "{\mathbb{C}(X,f)}"] & PX \arrow[d, "Pf"] \\
\mathbb{C}(X,Y) \arrow[r, "\alpha_Y"'] & PY
\end{tikzcd}
\end{array}
$$

which yields, for any $Y \in \mathbb{C}$ and $f \in \mathbb{C}(X,Y)$,

$$
\begin{aligned}
\alpha_Y(f) &= \alpha_Y(f \circ id_X) \\
&= (\alpha_Y \circ \mathbb{C}(X,f))(id_X) \\
&\underset{\circlearrowleft \text{ above}}{=} (Pf)\big(\underbrace{\alpha_X(id_X)}_{\in PX}\big)
\end{aligned}
$$

This is what matters

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<u>Cor.</u> $\mathcal{Y} : \mathbb{C}^{op} \longrightarrow \mathbf{Sets}^{\mathbb{C}}$ is full and faithful.

$F : \mathbb{C} \to \mathbb{D}$ induces

$$\mathbb{C}(X, Y) \xrightarrow{F_{X,Y}} \mathbb{D}(FX, FY)$$

- F is <u>full</u> if $F_{X,Y}$ is surj., $\forall X, Y$
- F is <u>faithful</u> if $F_{X,Y}$ is inj., $\forall X, Y$

<u>Proof.</u>

$$\mathbf{Sets}^{\mathbb{C}}(\mathcal{Y}X, \mathcal{Y}Y) = \mathrm{Nat}\left(\mathbb{C}(X, -), \mathbb{C}(Y, -)\right) \underset{\text{Yoneda}}{\cong} \mathbb{C}(Y, X) = \mathbb{C}^{op}(X, Y).$$

It is not hard to see that this correspondence (from right to left) is $\mathcal{Y}_{X,Y}$. ∎

Prop. $\mathbf{Sets}^{\mathbb{C}}$ is complete and cocomplete, with (co) limits computed in $\mathbf{Sets}$.

This means, e.g. $(P, Q \in \mathbf{Sets}^{\mathbb{C}})$

$$(P \underset{\substack{\uparrow \\ \times \text{ in } \mathbf{Sets}^{\mathbb{C}} \\ \text{prod.}}}{\times} Q)(\underset{\substack{\cap \\ \mathbb{C}}}{X}) = (PX) \underset{\substack{\uparrow \\ \times \text{ in } \mathbf{Sets} \\ \text{prod.}}}{\times} (PY)$$

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<u>Thm.</u> Each presheaf $P : \mathbb{C} \to \mathbf{Sets}$ is a colimit of representable presheaves. Specifically:

$$P \cong \mathrm{Colim}\left(\left(\int P\right)^{op} \xrightarrow{\pi^{op}} \mathbb{C}^{op} \xrightarrow{y} \mathbf{Sets}^{\mathbb{C}} \right)$$

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More concretely:

$$\begin{tikzcd}
\int P & \cdots & (X, x) \arrow[r, "f"] & (Y, (Pf)(x)) & \mathbb{C} \ f : X \to Y \\
(\int P)^{op} \arrow[d, "\pi^{op}"] & \cdots & (X, x) & (Y, (Pf)(x)) \arrow[l, "f"'] & \cdots \\
\mathbb{C}^{op} \arrow[d, "y"] & \cdots & X & Y \arrow[l, "f"'] & \cdots \\
\mathbf{Sets}^{\mathbb{C}} & \cdots & \mathbb{C}(X, -) \arrow[d, "\kappa_{(X,x)}"'] & \mathbb{C}(Y, -) \arrow[l, "\mathbb{C}(f, -)"'] \arrow[dl, "\kappa_{(Y, (Pf)(x))}"] & \cdots \\
 & & P & &
\end{tikzcd}$$

(colim)

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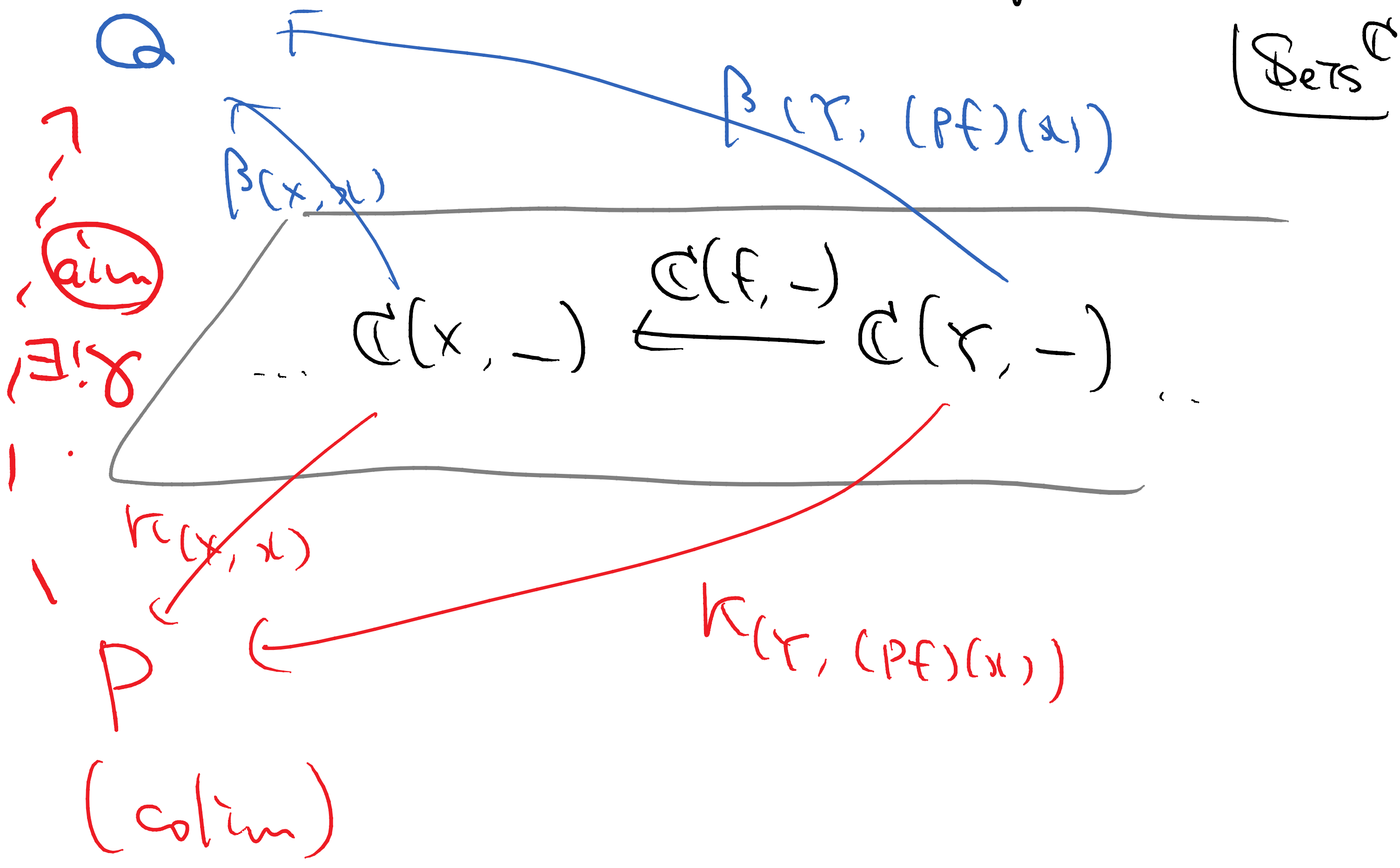
Proof. Assume a cocone; we show that $\exists!$ (mediating arrow).



Each nat. trans.

$$\beta(X, x) : \mathbb{C}(X, -) \to Q$$

corresponds to an elem.

$$q(X, x) \in QX$$

by Yoneda.

We define, for each $x \in \mathbb{C}$,

$$\gamma_x : PX \longrightarrow QX$$
$$\chi \longmapsto \xi(x, \chi)$$

We're done if we show

- $(\gamma_x : Px \to Qx)_{x \in \mathbb{C}}$

  consitutes a nat. trans.

-
$$\begin{tikzcd} \bullet & \\ & \bullet \arrow[ul, "\beta(x,\chi)"'] \arrow[dl, "\kappa(x,\chi)"] \\ \bullet \arrow[uu, "\gamma"] \arrow[ur, phantom, "//"] & \end{tikzcd}$$

- γ' is another mediating map $\Longrightarrow \gamma = \gamma'$

These are exercises

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§ 4.4 Ends & Coends

Usually employed as convenient tools in enriched category theory, I find ends and coends useful tools in the current context, too.

Highlights

- End / coend generalize limit / colimit
- Nat(F, G) is nicely an end
- This leads to the alternative presentations of the Yoneda lemma (end / coend forms)

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Def. (End)

$D : \mathbb{C}^{op} \times \mathbb{C} \longrightarrow \mathbb{D}$, a functor

Why?

A prototype:

$\mathbb{C}(-,+) : \mathbb{C}^{op} \times \mathbb{C} \longrightarrow \mathrm{Sets}$

An end is an obj. E

together with

$$\left(\pi_X : E \longrightarrow \mathbb{D}(X, X) \right)_{X \in \mathbb{C}}$$

the same!

s.t.

- π_X is "dinatural", meaning:

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$\mathbb{D}$

$$\begin{tikzcd}
 & E \arrow[dl, bend right, "\pi_X"'] \arrow[dr, bend left, "\pi_Y"] & \\
\mathbb{D}(X,X) \arrow[r, phantom, "="] \arrow[dr, "{\mathbb{D}(X,f)}"'] & & \mathbb{D}(Y,Y) \arrow[dl, "{\mathbb{D}(f,Y)}"] \\
 & \mathbb{D}(X,Y) &
\end{tikzcd}$$

$\mathbb{C}$

$$\begin{tikzcd}
X \arrow[d, "f"] \\
Y
\end{tikzcd}$$

Due to $\mathbb{D} : \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{D}$,
you get this nice
Cospan

$$\begin{tikzcd}
\mathbb{D}(X,X) \arrow[dr] & & \mathbb{D}(Y,Y) \arrow[dl] \\
 & \mathbb{D}(X,Y) &
\end{tikzcd}$$

$\downarrow$ cend

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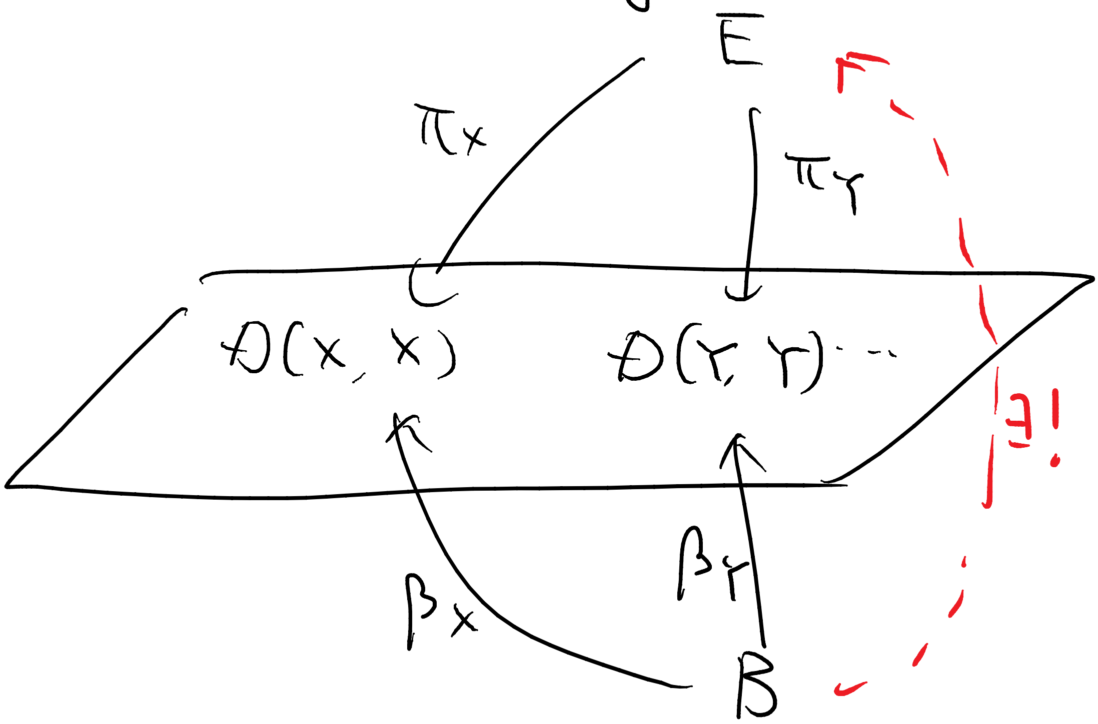
$\left(E \xrightarrow{\pi_X} D(X,X) \right)_{X \in \mathbb{C}}$ is universal among such:



We write the obj. E (of the end) as

$$\int_{X \in \mathbb{C}} D(X,X)$$

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- Ends generalize limits:

given a diagram $D : \mathbb{J} \to \mathbb{D}$,

we have

$$\begin{tikzcd} \mathbb{J}^{op} \times \mathbb{J} \arrow[r, "\pi_2"] \arrow[rr, bend right, "D'"'] & \mathbb{J} \arrow[r, "D"] & \mathbb{D} \end{tikzcd}$$

!!

and

$$\lim D = \lim_{I \in \mathbb{J}} D(I)$$

$$\cong \int_{I \in \mathbb{J}} D'(I, I)$$

(This is not hard to see)

$\Rightarrow$ Therefore we'll often use the end notation for limits too. $\left(\int_{I \in \mathbb{J}} D I = \lim D \right)$

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- Do ends exist? Informally:

  "as many ends as limits,"

  since

  * Given $D : \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{D}$,

  * We can construct a category $\mathbb{C}^{\S}$ and a functor $D^{\S} : \mathbb{C}^{\S} \to \mathbb{D}$

$\because$

$$\underline{\mathbb{C}} \quad x \xrightarrow{f} y$$

$$\underline{\mathbb{C}^{\S}} \quad \begin{tikzcd} id_x \arrow[dr] & & id_y \arrow[dl] \\ & f & \end{tikzcd}$$

See [Mac Lane CWM] for more details

s.t. $$\int_{I \in \mathbb{C}} D(I, I) \cong \lim D^{\S}$$

Notice that the sizes of $\mathbb{C}$ and $\mathbb{C}^{\S}$ are not very different

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Exercise Formulate the dual notion of coend. Hint: its dinaturality is

$$
\mathbb{C} \qquad
\begin{tikzcd}
X \arrow[d, "f"] \\
Y
\end{tikzcd}
\qquad\qquad
\mathbb{D} \qquad
\begin{tikzcd}
 & C & \\
D(X,X) \arrow[ur] \arrow[rr, phantom, "="] & & D(Y,Y) \arrow[ul] \\
 & D(Y,X) \arrow[ul, "D(f,X)"] \arrow[ur, "D(Y,f)"'] &
\end{tikzcd}
$$

We write a coend as

$$\int^{X \in \mathbb{C}} D(X,X)$$

← upper

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Prop. Given small $\mathbb{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathbb{D}$,

$$\mathrm{Nat}(F, G) \cong \int_{X \in \mathbb{C}} \mathbb{D}(FX, GX).$$

[Here: the functor wrt. which the end is taken is

$$\mathbb{C}^{op} \times \mathbb{C} \xrightarrow{F^{op} \times G} \mathbb{D}^{op} \times \mathbb{D} \xrightarrow{\mathbb{D}(-,+)} \mathbf{Sets}$$

[complete, thus the end exists]]

Proof. (Sketch)

As a limit is

a product + naturality

$$\begin{tikzcd} & \lim D \arrow[dl] \arrow[dr] & \\ DI \arrow[rr] & & DJ \end{tikzcd}$$

an end is

a product + dinaturality.

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Therefore

$$\int_{X \in \mathbb{C}} \mathbb{D}(FX, GX) \hookrightarrow \prod_{X \in \mathbb{C}} \mathbb{D}(FX, GX)$$

$$\ni \quad (\alpha_X : FX \to GX)_{X \in \mathbb{C}}$$

"only those $(\alpha_X)_X$ which are dinatural"

And the dinaturality here coincides with the naturality requirement

$$\mathbb{C} \quad \begin{array}{c} X \\ f \downarrow \\ Y \end{array} \qquad \mathbb{D} \quad \begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Ff"] \\ FY \arrow[r, "\alpha_Y"'] & GY \end{tikzcd}$$

$\boxed{\xi}$

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Lem. For the hom-functor

$$\mathbb{C}(-,+) : \mathbb{C}^{op} \times \mathbb{C} \to \mathbf{Set},$$

we have

$$\mathbb{C}\left(X, \int_{I \in \mathbb{I}} D(I,I)\right) \cong \int_{I \in \mathbb{I}} \mathbb{C}(X, D(I,I))$$

preserves ends (hence limits) in the positive position

$$\mathbb{C}\left(\int^{I \in \mathbb{I}} D(I,I), Y\right) \cong \int_{I \in \mathbb{I}} \mathbb{C}(D(I,I), Y)$$

turns a coend (in a neg pos.) to an end

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Proof. (Sketch)

$$\frac{\dfrac{X \longrightarrow \displaystyle\int_{I \in \mathbb{J}} \mathbb{D}(I, I)}{X \longrightarrow \mathbb{D}(I, I) \quad \text{for each } I \in \mathbb{J}\text{, dinatural}}}{\text{an elem. of } \displaystyle\int_I \mathbb{C}(X, \mathbb{D}(I, I))}$$

universality of an end

The other one is similar. $\square$

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This leads to the end/coend presentation of the Yoneda lemma:

$\boxed{\text{Lem}}$ $F : \mathbb{C} \to \mathrm{Sets}$.

$$\int_{\underline{X} \in \mathbb{C}} [\mathbb{C}(X, Y), FY] \cong FX.$$

function space

Proof.

$$(\mathrm{LHS}) \cong \mathrm{Nat}(\mathbb{C}(X, -), F) \underset{\text{Yoneda}}{\cong} FX.$$

The interpretation:

cancels out

$$\int_{X \in \mathbb{C}} [\mathbb{C}(X, \underline{Y}), F\underline{Y}] \cong FX.$$

negative

polarity

positive

The following coend form is as useful.

$\boxed{\text{Lem.}}$ $F : \mathbb{C} \to \mathbf{Sets}$

$$\int^{Y \in \mathbb{C}} \mathbb{C}(Y, X) \underset{\substack{\uparrow \\ \text{product} \\ \text{(copower)}}}{\cdot} FY \cong FX$$

<u>Proof.</u>

- $\mathcal{Y} : \mathbf{Sets}^{op} \longrightarrow \mathbf{Sets}^{\mathbf{Sets}}$ is full and faithful
  $S \longmapsto [S, \_]$
  (Yoneda)
- A full and faithful functor $G : \mathbb{C} \to \mathbb{D}$ reflects iso.
  (i.e. $GX \cong GY \Rightarrow X \cong Y$)

Therefore it suffices to show

$$\left[\int^{Y\in\mathcal{C}} \mathcal{C}(Y,X)\cdot FY,\ S\right]$$

$$\underset{\text{aim}}{\cong} [FX, S], \quad \text{for each } S.$$

(LHS)

$$\cong \int_{Y\in\mathcal{C}} [\mathcal{C}(Y,X)\cdot FY,\ S]$$

[hom. vs. (Co)end]

$$\frac{X\times Y\Rightarrow Z}{X\Rightarrow(Y\Rightarrow Z)}$$

$$\underset{\boxed{S\cdot(-)\dashv(-)^S}}{\cong} \int_{Y\in\mathcal{C}} [\mathcal{C}(\underline{Y},X),\ [F\underline{Y}, S]]$$

pos. neg.

Cancel

$$\underset{\text{Yoneda (end)}}{\cong} [FX, S] = (\text{RHS}).$$

$\square$

§4.5

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Ref.
[Fiore Plotkin Turi
LICS '99]

Type Constructors in $\mathbf{Sets}^{\mathbb{F}}$

We turn back to the original question

Our goal

- Characterize syntax w/ var binders as an initial algebra
- And use initiality (i.e. induction) to define capture-avoiding substitution

Recall: $\mathbb{F}$ is the category of nat. numbers and functions between them.

Sums, products $(+, \times)$

Recall: (co)limits in $\mathbf{Sets}^{\mathbb{C}}$ are computed pointwise in $\mathbf{Sets}$

Exponential

$P, Q : \mathbb{C} \to \mathbf{Sets}$.

The exponential $Q^P : \mathbb{C} \to \mathbf{Sets}$ is defined by adjunction:

$$\frac{P \times R \Rightarrow Q \quad \text{in } \mathbf{Sets}^{\mathbb{C}}}{R \Rightarrow Q^P \quad \text{in } \mathbf{Sets}^{\mathbb{C}}}$$

In a presheaf cat., Q^P has a nice concrete description:

$$\begin{aligned} (Q^P)(x) &\underset{\text{Yoneda}}{\cong} \mathbf{Sets}^{\mathbb{C}}(\mathbb{C}(x,-), Q^P) \\ (x \in \mathbb{C}) \quad &\underset{P\times - \dashv (-)^P}{\cong} \mathbf{Sets}^{\mathbb{C}}(P \times \mathbb{C}(x,-), Q) \end{aligned}$$

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These are enough to take care of

$$\frac{x_1, \ldots, x_n \vdash t \qquad x_1, \ldots, x_n \vdash s}{x_1, \ldots, x_n \vdash ts} \text{(Appl.)}$$

But how about λ-abstraction ??

change of contexts

$$\frac{x_1, \ldots, x_n, x_{n+1} \vdash t}{x_1, \ldots, x_n \vdash \lambda x_{n+1}.t} \text{(Abst.)}$$

It would be an operation

$$(x_1, \ldots, x_n, x_{n+1} \vdash ⊘) \mapsto (x_1, \ldots, x_n \vdash ⊘)$$

That is

$$\Delta(n+1) \longrightarrow \Delta(n)$$

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Therefore we consider

Context extension

$$\delta : \mathbf{Sets}^{\mathbb{F}} \longrightarrow \mathbf{Sets}^{\mathbb{F}}$$
$$P \longmapsto \delta P$$

with

$$\begin{array}{rccc} \delta P : & \mathbb{F} & \longrightarrow & \mathbf{Sets} \\ & n & & P(n+1) \\ & \downarrow f & \longmapsto & \downarrow P(f+1) \\ & m & & P(m+1) \end{array}$$

$(\delta P)(n) := P(n+1)$

Lem δ is indeed a functor

Proof.)
$$\frac{P \overset{\alpha}{\Rightarrow} Q}{\delta P \overset{\delta\alpha}{\Rightarrow} \delta Q},$$

$$\begin{array}{ccc} (\delta P)(n) & \xrightarrow{(\delta\alpha)_n} & (\delta Q)(n) \\ \parallel & \text{ii} & \parallel \\ P(n+1) & \xrightarrow[\alpha_{n+1}]{} & Q(n+1) \end{array}$$

証

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Variables   We want $V \in \mathbf{Sets}^{\mathbb{F}}$,

$$V(n) = \{x_1, \ldots, x_n\}$$

= more precisely

$$\left\{ \begin{array}{l} x_1, \ldots, x_n \vdash x_1 \\ x_1, \ldots, x_n \vdash x_2 \\ \vdots \\ x_1, \ldots, x_n \vdash x_n \end{array} \right\}$$

Def.

V is the inclusion functor $\mathbb{F} \hookrightarrow \mathbf{Sets}$

Lem.

$$U \cong \mathbb{F}(1, \_) : \mathbb{F} \to \mathbf{Sets}.$$

Proof.)

$$U(n) = n \xrightarrow{\cong} \mathbb{F}(1, n).$$

Naturality:

$\mathbb{F}$ | $\mathbf{Sets}$ |

$$\begin{tikzcd} n \arrow[d, "f"] \\ m \end{tikzcd} \qquad \begin{tikzcd} U(n) \arrow[r, "\cong"] \arrow[d, "U(f)"] \arrow[dr, phantom, "/\!/"] & \mathbb{F}(1, n) \arrow[d, "\mathbb{F}(1, f)"] \\ U(m) \arrow[r, "\cong"'] & \mathbb{F}(1, m) \end{tikzcd}$$

is obvious. $\square$

Summary We have, in $\mathbf{Sets}^{\mathbb{F}}$,

- $+$, $\times$
- Q^P, exponential,

$$(Q^P)(x) = \mathrm{Nat}\left(\mathbb{C}(x,\_)\times P,\, Q\right)$$

- $V \in \mathbf{Sets}^{\mathbb{F}}$, variables,
- $\delta : \mathbf{Sets}^{\mathbb{F}} \longrightarrow \mathbf{Sets}^{\mathbb{F}}$, ctxt extension

Lem. $\delta \cong (-)^{\mho}$

Proof.

$$(\delta P)(n) = P(n+1)$$

$$\underset{\text{Yoneda}}{\cong} \mathrm{Nat}(\mathbb{F}(n+1,-), P)$$

$$\cong \mathrm{Nat}(\mathbb{F}(n,-) \times \mathbb{F}(1,-), P)$$

Coend (hence coprod.) vs. hom-functor,

$$\mathbb{F}(n+1, m) \cong \mathbb{F}(n,m) \times \mathbb{F}(1,m)$$

$$\cong \mathrm{Nat}(\mathbb{F}(n,-) \times \mho, P)$$

$\mho \cong \mathbb{F}(1,-)$

$$\cong P^{\mho}.$$

$\boxtimes$

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Cor. δ has a left adjoint

Proof. $V \times (\_) \dashv (\_)^V \cong \delta$.

Lem. δ has a right adjoint, given by

$$\frac{\delta P \Rightarrow Q}{\overline{P \Rightarrow \langle V+1, Q \rangle}}$$

where

$$\langle V+1, Q \rangle : \mathbb{F} \longrightarrow \mathcal{S}ets$$

"the presheaf of operations"

$$\langle V+1, Q\rangle(n)$$
$$:= \mathrm{Nat}\left((V+1)^n, Q\right)$$
$$= \mathcal{S}ets^{\mathbb{F}}\left((V+1)^n, Q\right)$$

Proof. We fully exploit the (co)end machinery ☺

$$\mathrm{Nat}(P, \langle \mho+1, Q\rangle)$$

$$\cong \int_n [Pn, \langle \mho+1, Q\rangle(n)]$$

$$\cong \int_n [Pn, \mathrm{Nat}((\mho+1)^n, Q)]$$

$$\cong \int_n \left[Pn, \int_m [(\mho+1)^n(m), Qm]\right]$$

$$\cong \int_n \left[Pn, \int_m [((\mho+1)(m))^n, Qm]\right]$$

(limits are computed pointwise)

$$\cong \int_n \left[Pn, \int_m [[n, m+1], Qm]\right]$$

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$$\cong \int_n \int_m [Pn, [[n, m+1], Qm]]$$

(hom vs. end)

$$\cong \underbrace{\int_m \int_n}_{\text{ends are commutative (Fubini)}} [\underline{Pn \times [n, m+1]}, Qm]$$

adjunction
$[S, [T, U]]$
$\cong [S \times T, U]$

$$\cong \int_m \left[\int^n P\underline{n} \times [\underline{n}, m+1], Qm \right]$$

(hom vs. (co) end)

$$\cong \int_m [P(m+1), Qm]$$

(Yoneda, coend form)

$$\cong \int_m [(\delta P)m, Qm] \cong \mathrm{Nat}(\delta P, Q).$$

∎

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Thanks to 勝股さん

The underlying mathematical principle :

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "y"] \arrow[d, "F"'] & \mathrm{Sets}^{\mathbb{C}^{op}} = \hat{\mathbb{C}} \arrow[d, bend right=60, "F_!"'] \arrow[d, bend left=60, "F_*"] \\
\mathbb{D} \arrow[r, "y"'] & \mathrm{Sets}^{\mathbb{D}^{op}} \arrow[u, "F^*"]
\end{tikzcd}$$

$F_! \dashv F^* \dashv F_*$

- $F^* : \mathrm{Sets}^{\mathbb{D}^{op}} \longrightarrow \mathrm{Sets}^{\mathbb{C}^{op}}$

$$Q \longmapsto \left(\mathbb{C}^{op} \xrightarrow{F^{op}} \mathbb{D}^{op} \xrightarrow{Q} \mathrm{Sets} \right)$$

- $F_!(P) \cong F_!\left(\mathrm{Colim}\left(y \downarrow P \xrightarrow{\pi} \mathbb{C} \xrightarrow{y} \hat{\mathbb{C}} \right)\right)$

$$\underset{\varphi}{\cong} \mathrm{Colim}\left(y \downarrow P \xrightarrow{\pi} \mathbb{C} \xrightarrow{F_! \circ y} \hat{\mathbb{D}} \right)$$

$F_!$, left adj., pres. colim.

$$\underset{\varphi}{\cong} \mathrm{Colim}\left(y \downarrow P \xrightarrow{\pi} \mathbb{C} \xrightarrow{F} \mathbb{D} \xrightarrow{y} \hat{\mathbb{D}} \right).$$

required commutativity

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- $(F_*(P))(D)$

$$\underset{\text{Yoneda}}{\cong} \mathrm{Nat}(D(-, D), F_* P)$$

$$\underset{\substack{\text{required adjunction} \\ F^* \dashv F_*}}{\cong} \mathrm{Nat}((D(-, D)) \circ F, P)$$

$$= \mathrm{Nat}(D(F-, D), P)$$

- It is not hard to see that this is iso to the (misterious) right adj. 4 pages ago
- But I still don't get how [Fiore Plotkin Turi] got that concrete presentation...

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Anyway We have shown

$$\delta \cong (-)^{\nabla} : \mathbf{Sets}^{\mathbb{F}} \to \mathbf{Sets}^{\mathbb{F}}$$

has both left / right adjoints.

Therefore δ preserves limits and colimits.

§4.6 Functor for Binding Signatures

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<u>Recall</u> An alg. signature

$$\Sigma = (\Sigma_n)_{n \in \mathbb{N}}, \quad \text{or}$$

$$\Sigma = \left\{ \text{operations } \sigma, \text{ each w/ its arity } |\sigma| \in \mathbb{N} \right\}$$

Generalization of the "signature" of λ-calculus is a

<u>binding signature</u>, given by

[Plotkin]

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$$\Sigma = \left\{ \begin{array}{l} \text{oper. } \sigma, \text{ each w/} \\ \text{its arity} \\ |\sigma| = (n_1, n_2, \ldots, n_m) \end{array} \right\}$$

meaning :

- σ takes m arguments
- for each $i \in [1, m]$, σ binds n_i many var's of the i-th args.

For λ-calculus,

- app is of arity $(0, 0)$
- abst is ______ (1)

[We won't be working with general binding signatures, focusing on the λ-calculus case for simplicity]

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The <u>signature functor</u> for λ-calculus is

$$\Sigma_\lambda : \mathbf{Sets}^{\mathbb{F}} \longrightarrow \mathbf{Sets}^{\mathbb{F}}$$

$$P \longmapsto P \times P + \delta P$$

Thus

$$\Sigma_\lambda(P) : \mathbb{F} \longrightarrow \mathbf{Sets}$$

$$\left(\Sigma_\lambda(P)\right)(n)$$

$$= P(n) \times P(n) + (\delta P)(n)$$

$$= \underset{\left(\begin{array}{l} x_1, \ldots, x_n \vdash t, \\ x_1, \ldots, x_n \vdash s \end{array}\right)}{P(n) \times P(n)} + \underset{\left(\begin{array}{l} x_1, \ldots, x_n, x_{n+1} \\ \vdash t \end{array}\right)}{P(n+1)}$$

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Recall
$\Lambda(f)$: Context manipulation

Lem. $\Lambda \in \mathbf{Sets}^{\mathbb{F}}$,

$\Lambda(n) = \{$ λ-terms in the context $x_1, \dots, x_n \}$

Λ is a Σ_λ-algebra.

Proof.

$$\Lambda \times \Lambda + \delta\Lambda \overset{\text{aim}}{\Downarrow} \Lambda$$

alg. structure syntactically

$$\Lambda \times \Lambda \Rightarrow \Lambda \qquad \delta\Lambda \Rightarrow \Lambda$$

$$\Lambda(n) \times \Lambda(n) \longrightarrow \Lambda(n)$$

$$(x_1, \dots, x_n \vdash t,\ x_1, \dots, x_n \vdash s) \longmapsto (x_1, \dots, x_n \vdash \underline{ts})$$

$$\Lambda(n+1) \longrightarrow \Lambda(n)$$

$$(x_1, \dots, x_n, x_{n+1} \vdash t) \longmapsto (x_1, \dots, x_n \vdash \lambda x_{n+1}. t)$$

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Moreover: Λ is a free Σ_λ-alg. over $\overline{V}$. That is:

<u>Def.</u> $F : \mathbb{C} \to \mathbb{C}$, $X \in \mathbb{C}$

$\begin{array}{c} FA \\ \downarrow a \\ A \end{array}$ is a <u>free F-alg. over X</u>

if, for any alg. $\left(\begin{array}{c} FY \\ \downarrow b \\ Y \end{array}\right)$,

$$\frac{\dfrac{\left(\begin{array}{c} FA \\ \downarrow a \\ A \end{array}\right) \longrightarrow \left(\begin{array}{c} FY \\ \downarrow b \\ Y \end{array}\right) \text{ in } \mathrm{Alg}_F}{}}{X \longrightarrow Y \text{ in } \mathbb{C}}$$

<u>Exercise</u> Obtain $X \to A$.

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Lem. $F : \mathbb{C} \to \mathbb{C}$, $\mathbb{C}$: w/ coprod.
$X \in \mathbb{C}$

Define

$$X + F : \mathbb{C} \longrightarrow \mathbb{C}$$
$$Y \longmapsto X + FY$$

If $\left(\begin{array}{c} X + FA \\ \downarrow \alpha \\ A \end{array} \right)$ is an initial

$(X+F)$-algebra, then

$\left(\begin{array}{c} FA \\ \downarrow \kappa_2 \\ X + FA \\ \downarrow \alpha \\ A \end{array} \right)$ is a free F-alg. over X.

Proof. exercise.

Thm.

$$\begin{tikzcd} \Sigma_\lambda \Lambda \arrow[d, Rightarrow] \\ \Lambda \end{tikzcd}$$

is a free Σ_λ-algebra over V.

Proof. Given

$$\begin{tikzcd} V + \Sigma_\lambda P \arrow[d, Rightarrow] \\ P \end{tikzcd},$$

one constructs

$$\begin{tikzcd} V + \Sigma_\lambda \Lambda \arrow[r, dashed] \arrow[d, Rightarrow] & V + \Sigma_\lambda P \arrow[d, Rightarrow] \\ \Lambda \arrow[r, dashed, Rightarrow] & P \end{tikzcd}$$

concretely, by induction on the constr. of λ-terms in Λ.

3

§4.7 Substitution by Initiality

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Our final goal is to

- describe <u>capture-avoiding</u> substitution by initiality, and
- introduce the relevant tensor structure $\bullet$ on $\mathbf{Sets}^{\mathbb{F}}$.

We focus on simultaneous substitution

$$t\left[{}^{s_1}/_{x_1}, \ldots, {}^{s_m}/_{x_m}\right]$$

$$\left(\text{where } FV(t) = \{x_1, \ldots, x_m\}\right)$$

but using single-var. subst.

$$t\left[{}^{s}/_{x_{m+1}}\right] \quad \left(\begin{array}{l} FV(t) \\ = \{x_1, \ldots, x_m, x_{m+1}\} \end{array}\right)$$

is more or less the same.

(See [Fiore PT, LICS '99])

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The monoidal structure

Quickly: monoidal category?

$(\mathbb{C}, \otimes, I)$ with

"tensor" "tensor unit"
"monoidal unit"

- $\otimes : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$

operations

$$\left(\begin{array}{c} X \\ \downarrow f \\ Y \end{array}, \begin{array}{c} Z \\ \downarrow g \\ U \end{array} \right) \longmapsto \begin{array}{c} X \otimes Z \\ \downarrow f \otimes g \\ Y \otimes U \end{array}$$

$I \in \mathbb{C}$

- $I \otimes X \xrightarrow[\lambda_X]{\cong} X \xleftarrow[\rho_X]{\cong} X \otimes I$

equations

$$X \otimes (Y \otimes Z) \xrightarrow[\alpha_{X,Y,Z}]{\cong} (X \otimes Y) \otimes Z$$

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"alg. in which eq. axioms are satisfied only up-to $\cong$"

$\boxed{\text{Coherence conditions}}$

e.g.

- $$I \otimes I \underset{\rho_I}{\overset{\lambda_I}{\rightrightarrows}} I \qquad (\lambda_I = \rho_I)$$

$$\begin{array}{ccc}
X \otimes (Y \otimes (Z \otimes U)) & \xrightarrow{X \otimes \alpha} & X \otimes ((Y \otimes Z) \otimes U) \\
\downarrow \alpha_{X,Y,Z \otimes U} & & \\
(X \otimes Y) \otimes (Z \otimes U) & \text{"the pentagon"} = & \downarrow \alpha_{X, Y \otimes Z, U} \\
\downarrow \alpha_{X \otimes Y, Z, U} & & \\
((X \otimes Y) \otimes Z) \otimes U & \xleftarrow[\alpha_{X,Y,Z} \otimes U]{} & (X \otimes (Y \otimes Z)) \otimes U
\end{array}$$

- Examples
  - $\otimes = \times$ (In general, we don't have $X \otimes Y \xrightarrow{\pi} X$)
  - $\otimes = +$
  - $\otimes$: the usual tensor product in Vect
- Logically : Connectives (esp. $\otimes$)

- Logically : Connectives (esp. $\otimes$)
in linear logic

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Define

Our (highly nonsymmetric) tensor

$$\circledcirc : \mathbf{Sets}^{\mathbb{F}} \times \mathbf{Sets}^{\mathbb{F}} \longrightarrow \mathbf{Sets}^{\mathbb{F}}$$

negative

$$(P \circledcirc Q)(m) := \int^{n \in \mathbb{F}} \underbrace{P(n)}_{\text{positive}} \cdot \big(Q(m)\big)^{n}$$

$$= \Big(\coprod_{n} P(n) \cdot \big(Q(m)\big)^{n} \Big) \Big/ \simeq$$

coend / colim. by coprod. + coeq.

suitable equivalence

<u>idea</u>

$$\frac{x_1, \ldots, x_n \vdash t \qquad \begin{matrix} x_1, \ldots, x_m \vdash s_1 \\ \vdots \\ x_1, \ldots, x_m \vdash s_n \end{matrix}}{x_1, \ldots, x_m \vdash t[s_1/x_1, \ldots, s_n/x_n]}$$

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Its tensorial unit is $\underset{\text{variables}}{\Upsilon} : \mathbb{F} \to \mathrm{Sets}$!

Lem.

$$\Upsilon \circ P \cong P \cong P \circ \Upsilon$$

Proof.

$$(\Upsilon \circ P)(m) \underset{\boxed{\text{def. 56 } \Upsilon}}{=} \int^{n \in \mathbb{F}} \mathbb{F}(1, \underline{n}) \cdot (P(m))^{\underline{n}}$$

cancel

$$\underset{\boxed{\text{Yoneda}}}{\cong} (P(m))^1 \cong P(m).$$

$$(P \circ \Upsilon)(m) = \int^{n \in \mathbb{F}} P(n) \cdot (\Upsilon(m))^n$$

$$\underset{\boxed{\Upsilon(m) \cong m}}{\cong} \int^{n \in \mathbb{F}} P(n) \cdot [n, m]$$

$$\cong \int^{n \in \mathbb{F}} \mathbb{F}(\underline{n}, m) \cdot P(\underline{n})$$

$$\underset{\text{Yoneda}}{\cong} P(m). \quad \blacksquare$$

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Def. $(\mathbb{C}, \otimes, I)$: a monoidal cat.

A monoid in $(\mathbb{C}, \otimes, I)$ is

$X \in \mathbb{C}$ equipped with

$$X \otimes X \xrightarrow{m} X$$

$$I \xrightarrow{e} X$$

satisfying

$$\begin{tikzcd}
I \otimes X \arrow[r, "e \otimes X"] \arrow[dr, "\lambda_X"', "\cong"] & X \otimes X \arrow[d, "m"] & X \otimes I \arrow[l, "X \otimes e"'] \arrow[dl, "\rho_X", "\cong"'] \\
 & X &
\end{tikzcd}$$

unit law

$$\begin{tikzcd}
X \otimes (X \otimes X) \arrow[r, "\alpha", "\cong"'] \arrow[d, "X \otimes m"] & (X \otimes X) \otimes X \arrow[d, "m \otimes X"] \\
X \otimes X \arrow[dr, "m"'] & X \otimes X \arrow[d, "m"] \\
 & X
\end{tikzcd}$$

associativity

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The usual notion of monoid:

a monoid in $(\mathrm{Sets}, \times, 1)$

The point is: monoids can be defined more generally in any monoidal category!

Exercise

What is a monoid in $(\mathrm{Sets}, +, 0)$?

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We turn to monoids in

$$(\mathbf{Sets}^{\mathbb{F}}, \bullet, \underline{U})$$

- $P \bullet P \Rightarrow P$

$$\begin{array}{c} P \bullet P \\ \parallel \\ \int^n P(n) \cdot \big(P(\_)\big)^n \end{array}$$

- $\underline{U} \Rightarrow P$

$\Rightarrow$ These are presheaves w/ suitable

<u>(simultaneous) substitution structures</u>!

(The corresponding conventional notion: <u>clone</u> in universal algebra)

(Furthermore the same as <u>Lawvere theories</u>, thus <u>finitary monads</u>)

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$\boxed{\text{Our goal}}$

$\Lambda \in \mathbf{Sets}^{\mathbb{F}}$ carries a monoid in $(\mathbf{Sets}^{\mathbb{F}}, \bullet, \mathrm{V})$

$$\circ : \Lambda \bullet \Lambda \Longrightarrow \Lambda$$

$$\Lambda \bullet \Lambda = \int^{n} \Lambda(n) \cdot \big(\Lambda(-)\big)^{n}$$

$$\frac{x_1, \dots, x_n \vdash t \qquad \begin{array}{c} x_1, \dots, x_m \vdash s_1 \\ \vdots \\ x_1, \dots, x_m \vdash s_n \end{array}}{x_1, \dots, x_m \vdash t[{}^{s_1}/_{x_1}, \dots, {}^{s_n}/_{x_n}]}$$

Usually substitution is defined inductively on t, such as

$$t \equiv x_i \Rightarrow t[\vec{s}/\vec{x}] :\equiv s_i$$

$$t \equiv t_1 t_2 \Rightarrow t[\vec{s}/\vec{x}] :\equiv t_1[\vec{s}/\vec{x}]\, t_2[\vec{s}/\vec{x}]$$

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$\Rightarrow$ We want to define

$$\Lambda \bullet \Lambda \Rightarrow \Lambda$$

exploiting the initiality of

$$\bar{V} + \Sigma_\lambda \Lambda$$

$$\cong \Big\Downarrow \text{init}$$

$$\Lambda$$

!

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Lem. $(\mathbf{Sets}^{\mathbb{F}}, \bullet, \mathrm{U})$ is monoidal closed, that is, there is a functor

$$Q \multimap (\_) : \mathbf{Sets}^{\mathbb{F}} \to \mathbf{Sets}^{\mathbb{F}}$$

(for each $Q \in \mathbf{Sets}^{\mathbb{F}}$)

such that

$$\frac{P \bullet Q \Rightarrow R}{P \Rightarrow Q \multimap R}$$

cf. linear implication

Proof.

$Q \multimap R := \langle Q, R \rangle_{\bullet}$

presheaf of operations in [FiorePT]

$\langle Q, R \rangle : \mathbb{F} \to \mathbf{Sets}$

$$n \mapsto \mathrm{Nat}\left((Q(\_))^n, R\right)$$

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Then

$$\mathrm{Nat}(P \bullet Q, R)$$

$$\cong \int_m \left[\int^n P(n) \cdot (Q(m))^n ,\ R(m) \right]$$

$$\underset{\substack{\text{(co)end} \\ \text{vs. hom}}}{\cong} \int_m \int_n \left[P(n) \cdot (Q(m))^n , R(m) \right]$$

$$\cong \int_m \int_n \left[P(n), \left[(Q(m))^n , R(m) \right] \right]$$

$$\cong \int_n \left[P(n), \int_m \left[(Q(m))^n , R(m) \right] \right]$$

$$\cong \int_n \left[P(n), \mathrm{Nat}\left((Q(-))^n , R \right) \right]$$

$$\cong \mathrm{Nat}\left(P, \langle Q, R \rangle \right).$$

$\boxed{\xi}$

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We use this lemma and reduce our goal

$$\Lambda \cdot \Lambda \Rightarrow \Lambda$$

to

$$\Lambda \Rightarrow \langle \Lambda, \Lambda \rangle .$$

To use initiality

$$\begin{array}{c} \nabla + \Sigma_\lambda \Lambda \\ \text{init} \Downarrow \\ \Lambda \end{array} \quad \dashrightarrow \quad \langle \Lambda, \Lambda \rangle$$

What we need is an algebraic structure

$$\begin{array}{c} \nabla + \Sigma_\lambda \left(\langle \Lambda, \Lambda \rangle \right) \\ \Downarrow \\ \langle \Lambda, \Lambda \rangle \end{array}$$

That is

$$\nabla \Rightarrow \langle \Lambda, \Lambda \rangle$$

$$\Sigma_\lambda \left(\langle \Lambda, \Lambda \rangle \right) \Rightarrow \langle \Lambda, \Lambda \rangle$$

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The first one is immediate:

$$\begin{array}{c} \nabla \overset{aim}{\Longrightarrow} \langle \Delta , \Delta \rangle \\ \hline\hline \nabla \bullet \Delta \Longrightarrow \Delta \end{array}$$

$$\begin{array}{c} \nabla \bullet \Delta \\ \Downarrow \cong \\ \Delta \end{array}$$

$\boxed{\nabla \text{ is the unit for } \bullet}$

Thus we need

$$\Sigma_\lambda\big(\langle \Delta , \Delta \rangle\big) \Longrightarrow \langle \Delta , \Delta \rangle$$

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which is further equivalent to

$$\left(\frac{P\bullet Q \Rightarrow R}{P \Rightarrow \langle Q, R\rangle}\right)$$

$$\Sigma_\lambda(\langle\Lambda, \Lambda\rangle)\bullet\Lambda \Longrightarrow \Lambda$$

This we obtain as

$$\Big(\Sigma_\lambda(\langle\Lambda, \Lambda\rangle)\Big)\bullet\Lambda$$

$\Downarrow$ ① "strength-like" map

$$\Sigma_\lambda\big(\langle\Lambda, \Lambda\rangle\bullet\Lambda\big)$$

$\Downarrow$ ② $\Sigma_\lambda\big(\underbrace{\langle\Lambda, \Lambda\rangle\bullet\Lambda \xrightarrow{ev} \Lambda}\big)$

immediate from

$$\frac{P\bullet Q \Rightarrow R}{P \Rightarrow \langle Q, R\rangle}$$

(Exercise)

$$\Sigma_\lambda\ \Lambda$$

$\Downarrow$ init.

$$\Lambda.$$

① <u>Def.</u> $(\mathbb{C}, \otimes, I)$: a monoidal cat.

$F : \mathbb{C} \to \mathbb{C}$.

A <u>strength</u> for F is

a natural trans.

$$\left(st_{X,Y} : (FX) \otimes Y \longrightarrow F(X \otimes Y) \right)_{X,Y}$$

that is compatible with the monoidal structure.

e.g.

$$\begin{tikzcd} & FX & \\ FX \otimes I \arrow[ur, "\rho_{FX}"] \arrow[rr, "st_{X,I}"'] & & F(X \otimes I) \arrow[ul, "F\rho_X"'] \end{tikzcd}$$

Such a <u>strength</u> is the usual tool to get

$$(FX) \otimes Y \longrightarrow F(X \otimes Y)$$

For example,

<u>Lem.</u> Any $F: \mathbf{Sets} \to \mathbf{Sets}$ has a canonical strength

$$st: (FX) \times Y \longrightarrow F(X \times Y)$$

in $(\mathbf{Sets}, \times, 1)$

<u>Proof.</u>

$$\frac{FX \times Y \longrightarrow F(X \times Y)}{\dfrac{}{Y \longrightarrow [FX, F(X \times Y)]}}$$

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & {[FX, F(X \times Y)]} \\ {[X, X \times Y]} \arrow[ur, "\underbrace{F_{X, X \times Y}}_{\text{F's action on arrows}}"'] & \end{tikzcd}$$

$$V + (-)\times(-) + \delta(-)$$

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However, $\Sigma_\lambda : \mathbf{Sets}^{\mathbb{F}} \longrightarrow \mathbf{Sets}^{\mathbb{F}}$ //

itself is not strong (as far as I know), since δ is not known to be strong.

In [Fiore PT] it is shown that the lifting

$$\delta : V/\mathbf{Sets}^{\mathbb{F}} \longrightarrow V/\mathbf{Sets}^{\mathbb{F}}$$

$$(V \Rightarrow P) \longmapsto \begin{pmatrix} V \overset{\text{canonical}}{\Rightarrow} \delta V \\ \Rightarrow \delta P \end{pmatrix}$$

is indeed strong.

Here we just see its special case, for

$$\Sigma_\lambda(\langle \Delta, \Delta \rangle) \bullet \Delta$$

$$\Rightarrow \Sigma_\lambda(\langle \Delta, \Delta\rangle \bullet \Delta).$$

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We use here: $(-)\bullet\Lambda$ is left-adj., thus pres. coprod.

$$\Sigma_\lambda(\langle\Lambda,\Lambda\rangle)\bullet\Lambda \underset{\text{aim}}{\Longrightarrow} \Sigma_\lambda(\langle\Lambda,\Lambda\rangle\bullet\Lambda)$$

$$\| $$

$$(\langle\Lambda,\Lambda\rangle^2+\delta\langle\Lambda,\Lambda\rangle)\bullet\Lambda$$

$$\Downarrow \simeq$$

$$+(\langle\Lambda,\Lambda\rangle^2)\bullet\Lambda$$

$$+(\delta\langle\Lambda,\Lambda\rangle)\bullet\Lambda$$

1-1 $(\langle\Lambda,\Lambda\rangle^2)\bullet\Lambda \Rightarrow \Sigma_\lambda(\langle\Lambda,\Lambda\rangle\bullet\Lambda)$

1-2 $(\delta(\Lambda,\Lambda))\bullet\Lambda \Rightarrow \Sigma_\lambda(\langle\Lambda,\Lambda\rangle\bullet\Lambda)$

$$\| $$

$$(\langle\Lambda,\Lambda\rangle\bullet\Lambda)^2$$

$$+\ \delta(\langle\Lambda,\Lambda\rangle\bullet\Lambda)$$

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1-1

$$
\begin{tikzcd}
\langle \Delta , \Delta \rangle^2 \bullet \Delta \arrow[r, Rightarrow] \arrow[dr, Rightarrow, "\left\{ \begin{array}{l} \langle \Delta , \Delta \rangle^2 \bullet \Delta \xrightarrow{\pi_1 \bullet \Delta} \langle \Delta , \Delta \rangle \bullet \Delta , \\ \langle \Delta , \Delta \rangle^2 \bullet \Delta \xrightarrow{\pi_2 \bullet \Delta} \langle \Delta , \Delta \rangle \bullet \Delta \end{array} \right\}"'] & \Sigma_1 \left(\langle \Delta , \Delta \rangle \bullet \Delta \right) \\
& \left(\langle \Delta , \Delta \rangle \bullet \Delta \right)^2 \arrow[u, Rightarrow, "\kappa_1"']
\end{tikzcd}
$$

1-2

$$
\begin{array}{ccc}
\left(\delta\langle \Lambda,\Lambda\rangle\right)\bullet\Lambda & \Rightarrow & \Sigma_\lambda\left(\langle \Lambda,\Lambda\rangle\cdot\Lambda\right) \\
\| & & \Uparrow K_2 \\
\int^m \delta\langle \Lambda,\Lambda\rangle(m)\bullet\left(\Lambda(-)\right)^m & & \delta\left(\langle \Lambda,\Lambda\rangle\bullet\Lambda\right)
\end{array}
$$

yielded by

$$
\begin{array}{ccc}
\langle \Lambda,\Lambda\rangle(m+1)\bullet\left(\Lambda(n)\right)^m & \Rightarrow & \int^m \langle \Lambda,\Lambda\rangle(m)\cdot\left(\Lambda(n+1)\right)^m \\
& & \Uparrow K_{m+1} \\
& & \langle \Lambda,\Lambda\rangle(m+1)\cdot\left(\Lambda(n+1)\right)^{m+1}
\end{array}
$$

yielded by

$$
\left(\Lambda(n)\right)^m \longrightarrow \left(\Lambda(n+1)\right)^{m+1}
$$

$$
\begin{pmatrix}
x_1,\dots,x_n \vdash t_1, \\
\vdots \\
x_1,\dots,x_n \vdash t_m
\end{pmatrix}
\longmapsto
\begin{pmatrix}
x_1,\dots,x_n,\underline{x_{n+1}}_{\text{weakening}} \vdash t_1 \\
\vdots \\
x_1,\dots,x_n,x_{n+1} \vdash t_m \\
x_1,\dots,x_n,x_{n+1} \vdash x_n
\end{pmatrix}
$$

This concludes the constr. of the (simultaneous) substitution operation

$$\Delta \bullet \Delta \Rightarrow \Delta .$$

What is yet to come:

- Compatibility between subst. and alg. str.
  ("semantic substitution lemma")
  $\Rightarrow$ $\underline{\Sigma\text{-monoids}}$ (investigated by many, incl. Hamana)
- Coalg. in $\mathbf{Sets}^{\mathbb{F}}$
  $\Rightarrow$ for value / name passing calculi
- Bialg. in $\mathbf{Sets}^{\mathbb{F}}$
- HoRS, bialgebraically

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...

On the next occasion

Thanks
for your
attention!!