Chap. 2

Alg. & Coalg. in Sets, and Intro. to category theory

§2.1 System as Coalg.

http://www-mmm.is.s.u-tokyo.ac.jp/~ichiro/talks.html

ICHIRO HASUO

Dept CS, U Tokyo

ichiro@is.s.u-tokyo.ac.jp

- HOME
- CONTACT
- RESEARCH
- CV
- PAPERS
- TALKS
- TEACHING

— (For the general quantum computation & information cro
QIT 24, Tokyo Institute of Technology, Tokyo, Japan.
May 2011. Slides: [keynote | pdf]

▶ **The Microcosm Principle and Compositionality of GSOS-B Calculi**.
CALCO 2011, Winchester, UK.
September 2011. Slides: [keynote | pdf]

▶ **Generic Forward and Backward Simulations II: Probabilist**
CONCUR 2010, Paris, France.
September 2010. Slides: [keynote | pdf]

▶ **Theory of Coalgebra: Towards Mathematics of Systems**.
A gentle introduction to the theory of coalgebra, targeted at CS
Colloquium, Dept. of Computer Science, Univ. of Tokyo.
June 2010. Slides: [keynote | pdf]

▶ **Coalgebraic Representation Theory of Fractals**.
MFPS XXVI, Ottawa, Canada.
May 2010. Slides: [keynote | pdf]

▶ **Coalgebraic Components in a Many-Sorted Microcosm**.
CALCO 2009, Udine, Italy.
September 2009. Slides: [pptx | ppt | pdf]

P 17-21 (.key)

p. 22-53 (.pdf)

- A coalgebra is

$$\begin{tikzcd}\boxed{\cdots\ X \cdots} \\ X \arrow[u]\end{tikzcd}$$

Some set with X "occurring" in it

Let's write it as

$$\begin{tikzcd}FX \\ X \arrow[u]\end{tikzcd}$$

Here F is a "construction" that returns, given X, a set FX.

(

- We'd like F to apply, not only to sets $(X \longmapsto FX)$, but to functions $\left(\begin{array}{c} X \\ \downarrow f \\ Y \end{array} \longmapsto \begin{array}{c} FX \\ \downarrow Ff \\ FY \end{array}\right)$

* This is needed in

$$\begin{tikzcd}
(a, y) \arrow[rrr, mapsto] & & & (a, \mathsf{beh}(c)(y)) \\
& \Sigma \times X \arrow[r, dashed, "\Sigma \times \mathsf{beh}(c)"] & \Sigma \times \Sigma^{\mathbb{N}} & \\
& X \arrow[u, "c"] \arrow[r, dashed, "\mathsf{beh}(c)"'] & \Sigma^{\mathbb{N}} \arrow[u, "\mathrm{final}"'] & \\
x \arrow[uuu, mapsto] \arrow[rrr, mapsto] & & & \mathsf{beh}(c)(x) \arrow[uuu, mapsto]
\end{tikzcd}$$

- Such F is called a functor!

Def.

A functor (on Sets) is a correspondence

$$X \longmapsto FX \qquad \text{(on sets)}$$

$$\begin{array}{c} X \\ \downarrow f \\ Y \end{array} \longmapsto \begin{array}{c} FX \\ \downarrow Ff \\ FY \end{array} \qquad \text{(on functions)}$$

s.t.

- $$F\left(\begin{array}{c} X \\ \downarrow id \\ X \end{array}\right) = \left(\begin{array}{c} FX \\ \downarrow id \\ FX \end{array}\right)$$
- $$F\left(\begin{array}{c} X \\ \downarrow f \\ Y \\ \downarrow g \\ Z \end{array}\right) = \left(\begin{array}{c} FX \\ \downarrow Ff \\ FY \\ \downarrow Fg \\ FZ \end{array}\right)$$

$$\Big(F(g \circ f) = (Fg) \circ (Ff) \Big)$$

Examples

- $FX = L$ (The constant functor to a set L)

Exercise

Define $F\left(\begin{array}{c} X \\ \downarrow f \\ Y \end{array}\right)$ so that F is indeed a functor

- $FX = L \times X$
- $FX = X^2$
- $FX = 1 + X$

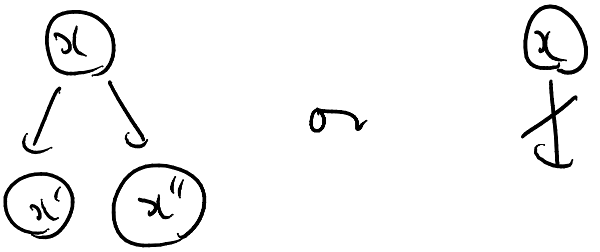
singleton, say $1 = \{1\}$ ↗ ↑ disjoint union

- $FX = \mathcal{P}(L \times X)$

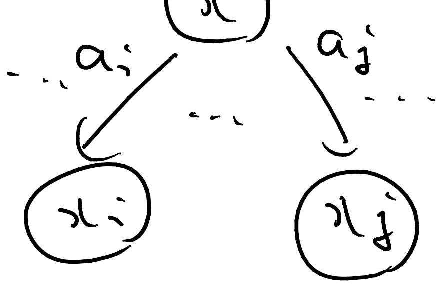
- $FX = 1 + X^2$
- $FX = (\Theta \times X)^I$

For such functors F, an F-coalgebra is such that:

- $F = 1 + (\_)^2$: a state either has two successors (left/right) or none ("terminates")



- $F = \mathcal{P}(L \times \_)$: each transition is L-labeled and a state has many of them (possibly none)

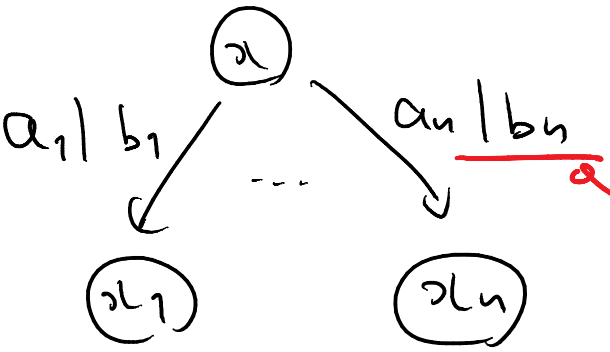


$\Longrightarrow$ LTS

labeled transition system

- $F = (\mathcal{O} \times \_)^{\mathcal{I}}$ ($\mathcal{I}, \mathcal{O}$: fixed sets)

Upon an input from $\mathcal{I}$, an output from $\mathcal{O}$ and the next state are determined $\Rightarrow$ "Mealy automaton"



when a_n is input, output is b_n

(Here $\mathcal{I} = \{a_1, \ldots, a_n\}$)

§2.2 Speaking w/ arrows

Time for rudimentary cat. theory.

BTW What is "category theory"?

Answer You should ask around!!

- Many answers, due to many ways to use it

  (Yes... CT is to use, at least currently)

- But probably what it is not is: the universal language for all the disciplines of science

  (There're many things CT is not good at)

- And we've asked around ☺
  * Adventures of Categories (RIMS, Kyoto U.)
  * 圏論の歩き方 (数セミ)

HOME > 雑誌 > 数学セミナー 2012年8月号 610号

数学セミナー

2012年8月号　通巻 610号

特集 1

数学ライブ2012

雑誌コード：05423　発刊日：2012.07.12
B5判　毎月12日発売　在庫あり

定価：税込 **980**円（本体価格 933円）

▣ 定期購読のお申込

▣ バックナンバー一覧

在庫があります。ショッピングカートがご利用になれます。　カートに入れる

内容紹介

数学の面白さ・奥深さを伝える公開講座を誌上で再現した「数学ライブ」が2 年ぶりに戻ってきました！　大学のオープンキャンパスやスーパーサイエンスハイスクールを通して、数学の講義風景を体感しながら興味深い数学の世界へと誘います。

目　次

画面の領域の取り込み日時: 2012/07/17 19:33

CT in CS

A few well-established usages

- For functional programming
  - * Type = obj.
    program = arr.
  - * category = "monoid with many obj."
- Algebra / coalgebra
  - * category : Sets and its variants

Rem. Don't let the following definitions (= "mathematical bureaucracy") lose your way. $\mathcal{C} = \mathbf{Sets}$ is always good enough.

Def. A category $\mathbb{C}$ is a tuple

$$(O, A, id, \circ)$$

- O : the collection of objects
- $A = \left(A(X,Y) \right)_{X,Y \in O}$

  the collection of arrows

  $$A(X,Y) = \{ X \xrightarrow{f} Y \}$$
- $id = \left(X \xrightarrow{id_X} X \right)_{X \in O}$,

  identity arrows
- $\circ = \left(\circ_{X,Y,Z} : A(X,Y) \times A(Y,Z) \to A(X,Z) \right)_{X,Y,Z \in O}$

  composition of arrows,

  $$\left(\begin{array}{c} X \\ \downarrow f \\ Y \end{array} , \begin{array}{c} Y \\ \downarrow g \\ Z \end{array} \right) \longmapsto \begin{array}{c} X \\ \downarrow g \circ f \\ Z \end{array}$$

(ctn'd) ↓

(↓ ctn'd)

Subject to the following cond.

- (Unit law)

$$\begin{tikzcd}
X \arrow[r, "id_X"] \arrow[dr, "f"', "="] & X \arrow[d, "f"] \arrow[dr, "f", "="'] & \\
& Y \arrow[r, "id_Y"'] & Y
\end{tikzcd}$$

- (Associativity)

Given $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} U$,

$$(h \circ g) \circ f = h \circ (g \circ f)$$

(refl. trans.)

- A preorder $(P, \leq)$ induces a category by

obj. $x \in P$

arr. $$\frac{x \longrightarrow y}{x \leq y}$$

'if and only if'
'in a 1-1 cor.)

A preorder :
a category with few arrows

- A monoid $(M, \bullet, e)$ induces a category by

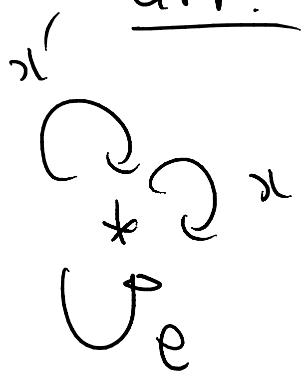
obj. A fresh symbol (say $*$)

arr. $x \in M$,

$$x' \circ x := x' \bullet y$$

comp. of arr.

multip. of a monoid



A monoid :

a category with <u>few</u> objects
(in fact
only one)

Examples

2012年7月17日
19:43

- **Sets** obj. a set

  arr. a function

"What is that category?" is usually answered like this.
(id, $\circ$ are then usually obvious)

- **Top** obj. a topological space

  arr. a conti. map

- **Mon** obj. a monoid

  arr. a monoid homomorphism

- **Meas** obj. a measurable space

  arr. a measurable func.

Def. A functor $F: \mathbb{C} \to \mathbb{D}$ is ...
(A straightfwd gener. of Sets-functor)

Examples

- $\mathbb{C}, \mathbb{D}$: preorders Then

$$\frac{F: \mathbb{C} \to \mathbb{D}, \text{ functor}}{\overline{\text{a monotone map}}}$$

- $\mathbb{C}, \mathbb{D}$: monoids Then

$$\frac{F: \mathbb{C} \to \mathbb{D}, \text{ functor}}{\overline{\text{a monoid homomorphism}}}$$

- $$\begin{tikzcd} \mathbf{Meas} \arrow[d, "U"] \\ \mathbf{Sets} \end{tikzcd}$$ The forgetful functor (Forgets measurable structures)

Speaking w/ arrows

- Not by elements
- Why? ⇒ generalization

(Real answer : since
it's COOL!!)

Injective function

Prop

A function $f : X \to Y$ is injective

(i.e. $f(x) = f(x') \Rightarrow x = x'$)

iff it is left-cancelable, that is,

$\forall Z \in \mathbf{Sets},\ \forall g, h : Z \rightrightarrows X,$

$$fg = fh \Rightarrow g = h$$

Also called mono

$$\begin{tikzcd} Z \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Proof.) Exercise!

For (If), notice that

$$\frac{x \in X}{1 \xrightarrow{x} X}$$

“is identified with”

“in a 1-1 cor. with”

□

Products

2012年7月17日
20:56

A binary product
$$X \times Y = \{ (x, y) \mid x \in X, y \in Y \}$$
is also characterized by arrows.

$\boxed{1}$ $X \times Y$ comes with two arrows

$$\begin{tikzcd}
 & X \times Y \arrow[ld, "\pi_1"'] \arrow[rd, "\pi_2"] & \\
X & (x, y) \arrow[ld, mapsto] \arrow[rd, mapsto] & Y \\
x & & y
\end{tikzcd}$$

$\boxed{2}$ It is universal among such:

that is,

- given any
$$\begin{tikzcd}
 & Z \arrow[ld, "f"'] \arrow[rd, "g"] & \\
X & & Y
\end{tikzcd},$$

- there exists a unique mediating map, i.e.

$$\begin{tikzcd}
 & Z \arrow[ld, "f"'] \arrow[rd, "g"] \arrow[dd, dashed, "\exists !"] & \\
X & & Y \\
 & X \times Y \arrow[lu, "\pi_1"] \arrow[ru, "\pi_2"'] &
\end{tikzcd}$$

$*$

($\dashrightarrow$ means "exists, and is unique")

$*$ Concretely, $(f, g) : Z \longrightarrow X \times Y$

$$z \longmapsto (f(z), g(z))$$

2012年7月17日
21:03

Def. A **product** of $X, Y \in \mathcal{C}$ is a triple

$$\Big(\underset{\underset{\mathcal{C}}{\cap}}{X \times Y},\ \underset{X\times Y \to X}{\pi_1},\ \underset{X \times Y \to Y}{\pi_2} \Big)$$

that is **universal** among $X \leftarrow \bullet \rightarrow Y$

means what's on the previous page

A product of X, Y need not be unique,
but...

Prop. A product of X, Y is unique up to commuting isomorphisms.

Proof. Asm. $X \xleftarrow{p} P \xrightarrow{q} Y$, $X \xleftarrow[p']{} P' \xrightarrow[q']{} Y$ both products.

$$\begin{tikzcd}
& P' \arrow[ld] \arrow[rd] \arrow[dd, dashed, "k"] & \\
X & & Y \\
& P \arrow[lu] \arrow[ru] &
\end{tikzcd}$$

product!

$$\begin{tikzcd}
& P \arrow[ld] \arrow[rd] \arrow[dd, dashed, "l"] & \\
X & & Y \\
& P' \arrow[lu] \arrow[ru] &
\end{tikzcd}$$

product!

Then

$$\begin{tikzcd}
& P \arrow[dl] \arrow[dr] \arrow[d, "l"] & \\
X & P' \arrow[l] \arrow[r] \arrow[d, "k"] & Y \\
& P \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

yields

$$\begin{tikzcd}
& P \arrow[dl] \arrow[dr] \arrow[d, "k \circ l"] & \\
X & & Y \\
& P \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

BTW,

$$\begin{tikzcd}
& P \arrow[dl] \arrow[dr] \arrow[d, "id"'] & \\
X & & Y \\
& P \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Thus, by the uniqueness of a mediating map, $k \circ l = id$.

Similarly $l \circ k = id$, therefore:

$$\begin{tikzcd}
& P \arrow[dl] \arrow[dr] \arrow[d, bend right, "l"'] \arrow[d, phantom, "\cong"] & \\
X & & Y \\
& P' \arrow[ul] \arrow[ur] \arrow[u, bend right, "k"'] &
\end{tikzcd}$$

∎

NB Not just $P \cong P'$,
but the isomorphisms
are the canonically
induced ones!

Discussions

The previous p'f is typical of CT.

- Use of universality
  (Exists and is unique)
  Mediating maps, "factors through"

- Constructions ($e.g.$ $X \times Y$ from X, Y)
  are characterized
  up-to iso.

- Everything is canonical
  [A limitation of CT]
  $\Rightarrow$ CT as an organization tool

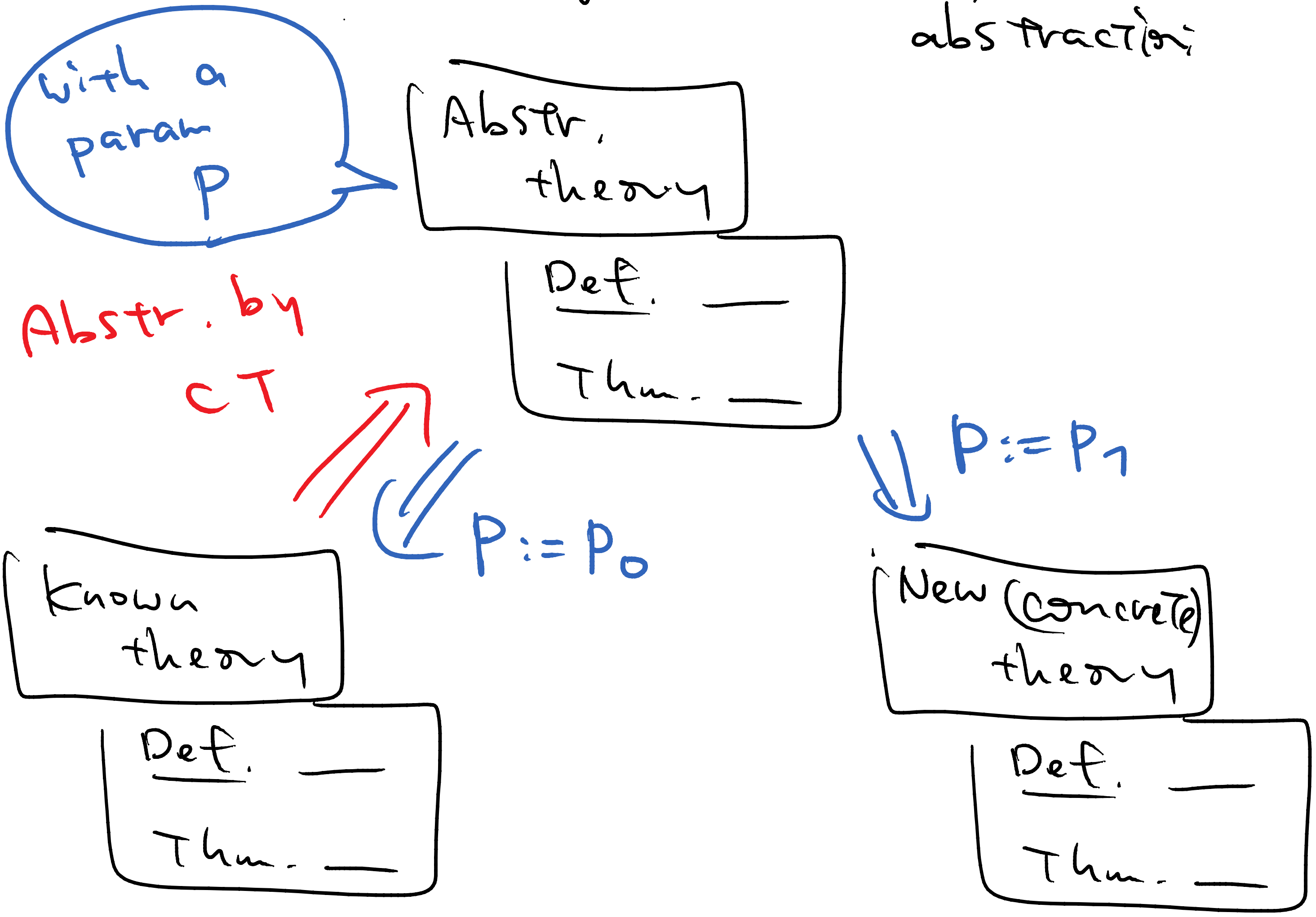
2012年7月17日
21:19

Intermission

An oft-heard criticism:

"CT just reproves stuffs already known"

- CT as a tool of organization/abstraction



- Indeed the new theory can be developed w/o CT
- But that's only in a retrospect...
  How can one think of such def./thm.?

2012年7月17日
21:25

Prop. In Sets,

(CT-)

$X \xleftarrow{p} C \xrightarrow{q} Y$ is a product of X, Y

$$\Longleftrightarrow \begin{tikzcd} & C \arrow[ld, "p"'] \arrow[rd, "q"] \arrow[d, "\exists ! \cong"] & \\ X & & Y \\ & X \times Y \arrow[lu, "\pi_1"] \arrow[ru, "\pi_2"'] \arrow[d, phantom, "\shortparallel"] & \\ & \{(x,y) \mid x \in X, y \in Y\} & \end{tikzcd}$$

Def. A final obj. in $\mathbb{C}$ is

$Z \in \mathbb{C}$ s.t. for each $X \in \mathbb{C}$,

$$X \overset{\exists !}{\dashrightarrow} Z$$

Exercise

- What is a final obj. in Sets?
- Prove that final objects are unique up to commuting isomorphisms. (Formulate the stmt. in precise terms!)

2012年7月17日
21:30

Duality In speaking by arrows, one can always *reverse* arrows!

Def. $A \in \mathbb{C}$ is *initial*

$$\underset{\text{def.}}{\iff} \quad \forall X \in \mathbb{C}. \quad A \overset{\exists!}{\dashrightarrow} X$$

Def. $X \xrightarrow[i]{} C \xleftarrow[j]{} Y$ is a *coproduct* if it is universal among $X \to \bullet \leftarrow Y$

(Cospan)

Prop. In Sets,

$$\begin{tikzcd} X \arrow[r, "\kappa_1"] & X+Y \arrow[d, phantom, "\shortparallel"] & Y \arrow[l, "\kappa_2"'] \\ & \{(1,x) \mid x \in X\} \cup \{(2,y) \mid y \in Y\} & \end{tikzcd}$$

$$x \longmapsto (1,x)$$

$$(2,y) \longmapsfrom y$$

is a coproduct.

§ 2.3 Limits & Colimits

2012年7月17日
21:34

We rely on these.

Def. (Equalizer)

Given $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ (parallel arrows)

an equalizer of f, g is

(E, i) s.t.

- $E \xrightarrow{i} X \underset{g}{\overset{f}{\rightrightarrows}} Y$
- $f \circ i = g \circ i$
- universal among such, i.e.
  - If $F \xrightarrow{j} X \underset{g}{\overset{f}{\rightrightarrows}} Y$, $fj = gj$
  - Then

$$\begin{tikzcd}
E \arrow[r, "i"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\
F \arrow[u, dashed, "\exists !"] \arrow[ur, "j"'] & &
\end{tikzcd}$$

We say "j factors through i"

Prop. In Sets,

$$\begin{tikzcd}\left\{x \in X \,\middle|\, \begin{array}{l} f(x) \\ = g(x) \end{array}\right\} \arrow[r, hook] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y\end{tikzcd}$$

is an equalizer.

Exercise Prove:

- In any category $\mathbb{C}$, if

$$\begin{tikzcd}E \arrow[r, "h"] & X \arrow[r, shift left] \arrow[r, shift right] & Y\end{tikzcd}$$

is an equalizer, then h is a mono.

(left-cancelable)

- $$\begin{tikzcd}\cdot \arrow[r, "h"] & \cdot \\ \cdot \arrow[u, "g"] \arrow[ur, "f"'] & \end{tikzcd}$$

g, h: mono $\Rightarrow$ f: mono

f: mono $\Rightarrow$ g: mono.

2012年7月17日
21:45

Def. (Coequalizer)

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "e"] \arrow[dr, "h"'] & C \arrow[d, dashed, "\exists!"] \\ & & Z \end{tikzcd}$$

① $ef = eg$

② for $\forall h, Z$ s.t. $hf = hg$,

③ there is a unique mediating arr.

Prop. In Sets: given $X \overset{f}{\underset{g}{\rightrightarrows}} Y$,

- $R := \{ (f(x), g(x)) \mid x \in X \} \subseteq Y^2$
- $\overline{R} \subseteq Y^2$: the equivalence closure of R
- Then

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\text{proj.}"] & Y/\overline{R} \end{tikzcd}$$

is a coequalizer.

Hence equalizers are for choosing elements

coequalizers are for quotienting elem.

2012年7月17日
21:48

Limits / Colimits

The definitions followed a common pattern:

- (co)product $\bullet \leftarrow \square \rightarrow \bullet$
- final / initial obj. $\square$
- (co)equalizer $\square \longrightarrow \bullet \rightrightarrows \bullet$

$\Longrightarrow$ Generalize!

Def. A 𝕁-diagram in ℂ is a functor

$$D : \mathbb{J} \longrightarrow \mathbb{C}$$

Examples

- $\mathbb{J} = (\bullet \quad \bullet)$ $\quad D = (X \quad Y)$

two-obj.
two-arr.
cat.

What are these arrows?

- $\mathbb{J} = \left(\bullet \rightrightarrows \bullet \right)$ $\quad D : \mathbb{J} \longrightarrow \mathbb{C}$

two-obj.
four-arr.

$$\begin{tikzcd}
\bullet \arrow[d, shift right] \arrow[d, shift left] & & X \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \\
\bullet \arrow[rr, phantom, "\longmapsto"] & & Y
\end{tikzcd}$$

- $\mathbb{J} = \left(\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \right)$ $\quad D : \mathbb{J} \longrightarrow \mathbb{C}$

$$\left(\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd} \right)$$

The prev. def. of diagram is (again) an instance of mathem. bureaucracy

(I don't mean bad...)

Def. (Limit)

A limit of a diagram $D: \mathbb{J} \to \mathbb{C}$

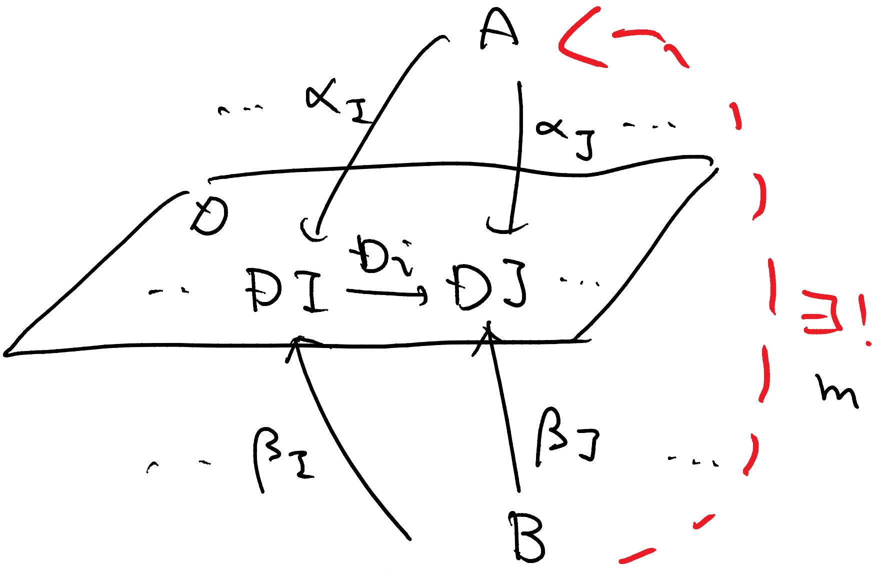
is

$$\Big(\underset{\mathbb{C}}{\underset{\cap}{A}},\ \ (\alpha_I : A \longrightarrow DI)_{I \in \mathbb{J}} \Big)$$

- such that, for each arr. $I \xrightarrow{i} J$ in $\mathbb{J}$,

$$\begin{tikzcd} & A \arrow[dl, bend right, "\alpha_I"'] \arrow[dr, bend left, "\alpha_J"] & \\ DI \arrow[rr, "Di"'] & /\!/ & DJ \end{tikzcd}$$

- Moreover, it is universal among such:



$\alpha_I \circ m$

$= \beta_I$,

$\forall I \in \mathbb{J}$

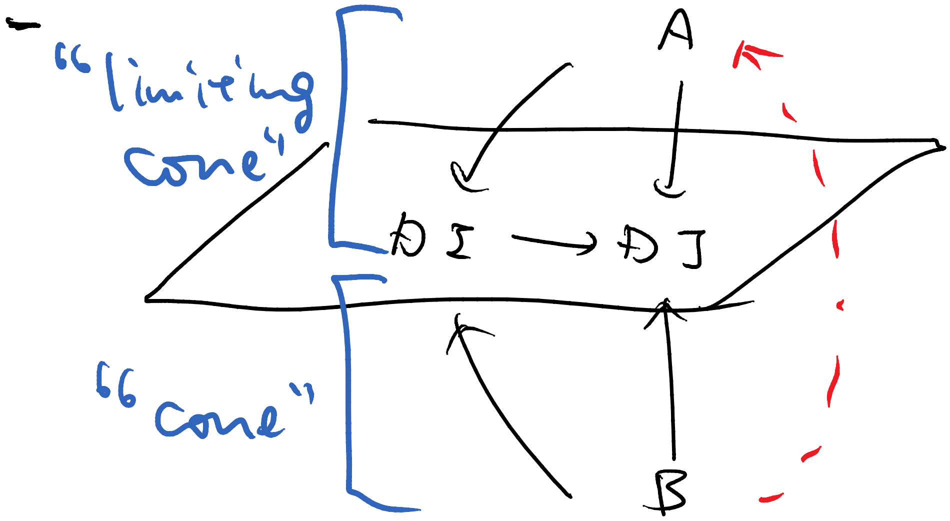
(β_I factors thru α_I)

We often write $\mathrm{Lim}\, D$ for A.

2012年7月18日
7:28

"Mathematics is about notations!"

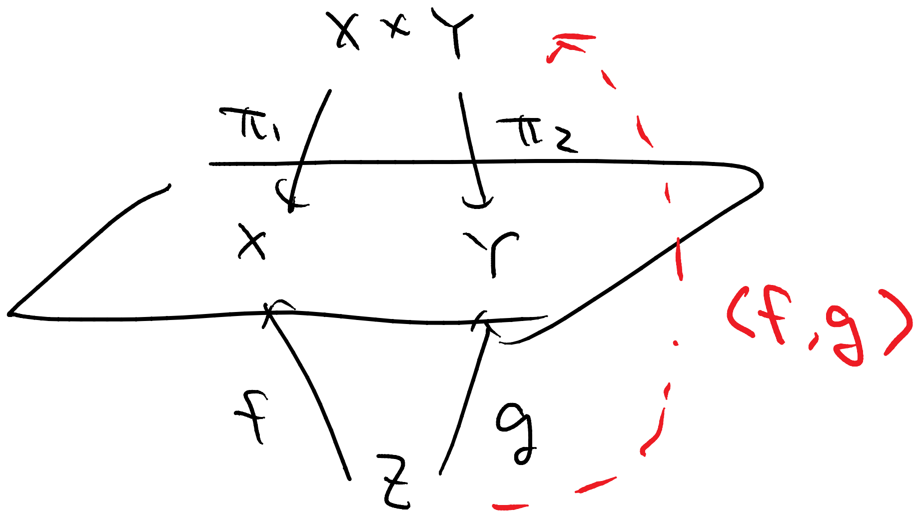
Two useful schematics



- "Double-line notation"

$$\begin{array}{c} B \longrightarrow DI, \quad \text{cone} \\ \hline\hline B \longrightarrow \mathrm{Lim}\, D \end{array}$$

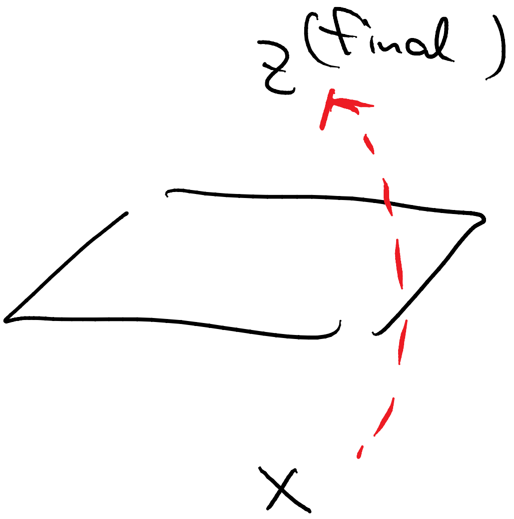
Products as limits



OR:

$$\frac{Z \longrightarrow X \qquad Z \longrightarrow Y}{Z \longrightarrow X \times Y}$$

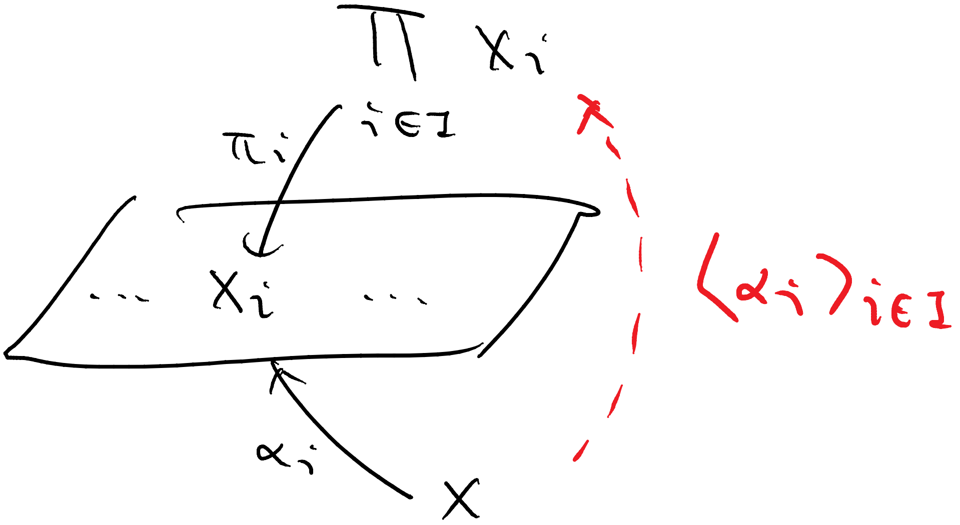
Final obj.



2012年7月18日
7:56

More generally:

<u>Def.</u> $(X_i)_{i\in I}$: an I-indexed family of $\mathbb{C}$-objects. Its <u>product</u> :



Notations for (co)products

$$\frac{X \xrightarrow{\alpha_i} X_i}{\overline{X \xrightarrow[\langle \alpha_i \rangle_i]{} \prod_{i} X_i}}$$

← tupling

$$\frac{X_i \xrightarrow{\alpha_i} X}{\overline{\coprod_{i} X_i \xrightarrow[{[\alpha_i]_i}]{} X}}$$

— cotupling

2012年7月18日
8:00

$$\frac{\dfrac{X \xrightarrow{f} Y}{X\times Z \xrightarrow{f\circ\pi_1} Y} \qquad \dfrac{Z \xrightarrow{g} U}{X\times Z \xrightarrow{g\circ\pi_2} U}}{\dfrac{}{X\times Z \xrightarrow[(f\circ\pi_1,\, g\circ\pi_2)]{} Y\times U}}$$

$$!!$$

$$f\times g$$

That is,

$$\begin{tikzcd}
 & X\times Z \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[ddl, bend right] \arrow[dd, dashed, "f\times g"] \arrow[ddr, bend left] & \\
X \arrow[d, "f"'] & & Z \arrow[d, "g"] \\
Y & Y\times U \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & U
\end{tikzcd}$$

Exercise

Define $X+Z \xrightarrow{f+g} Y+U$

via the universality of $+$

(i.e. using $[\ ,\ ]$ and κ_1, κ_2)

Exercise

- What are | a final obj. | in
  products
  equalizers

  a preorder as a category?

Recall

- $$\begin{array}{c} x \to y \\ \hline\hline x \le y \end{array}$$

- Characterization of inf's

$$\begin{array}{c} x \le y \qquad x \le z \\ \hline\hline x \le y \wedge z \end{array}$$

(↳) Therefore:
universality
= "the lease (or the biggest)
among …"
Witness $\underline{\text{ギリギリ感}}$
'Girigiri'

Exercise

Formulate the notion of colimit.

Our roadmap:

- initial alg. / final coalg.

$\Uparrow$ (special, important)

- constr. as a (co)limit of the "initial/final sequence" in Sets

Thus we'd like to know how

$$\boxed{\text{(co)limits look like concretely, in Sets}}$$

Towards that goal it's useful to exhibit

a general constr. of limits via products & equalizers

2012年7月18日
7:42

BTW ...

Exercise Exhibit a category which does <u>not</u> have products.

Prop. If a category $\mathbb{C}$ has

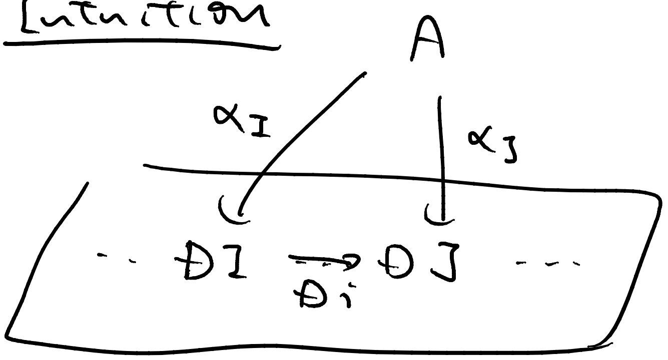
$\begin{cases} \text{products} \\ \text{equalizers,} \end{cases}$

then it has all limits.

The size issue which we'll circumvent

To be precise: for $D : \mathbb{I} \to \mathbb{C}$, if $\mathbb{C}$ has products of the size $|\mathrm{Arr}\,\mathbb{I}|$, then $\exists \lim D$

Intuition



- We need $\alpha_I, \alpha_J, \ldots$ that are like projections
  $\Rightarrow$ product!
- We also need commutativity
  $Di \circ \alpha_I = \alpha_J$
  $\Rightarrow$ Let an equalizer force it!

2012年7月18日
8:06

Proof.) Consider products

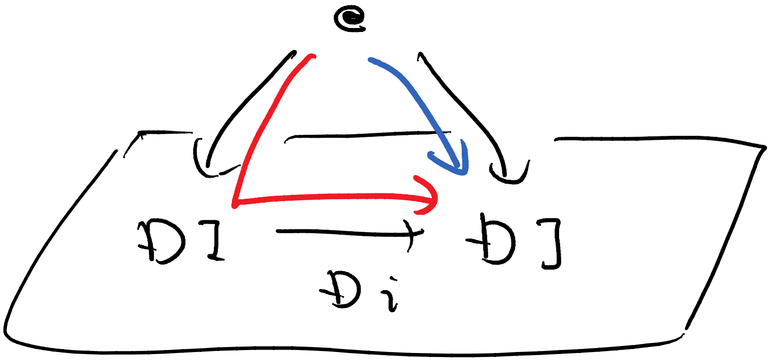
$\prod_{I \in \mathbb{J}} DI$ ← (products of "all objects")

$\prod_{i : \mathbb{J}\text{-arrow}} D(\mathrm{cod}(i))$ ← (prod. of cod. of "all arrows")

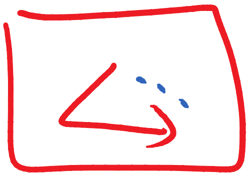
$I \xrightarrow{i} J$

domain codomain

Between these we have two arrows.
They correspond to



Specifically,



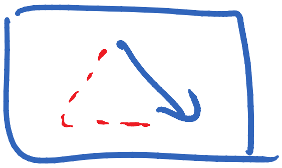
$$\prod_I DI \xrightarrow{f} \prod_i D(\mathrm{cod}(i))$$

'two-line' for prod

$$\overline{\underline{\hspace{12cm}}}$$

$$\begin{tikzcd} \prod_I DI \arrow[r] \arrow[d, "\pi_{\mathrm{dom}(i)}"'] \arrow[dr, phantom, "\parallel"] & D(\mathrm{cod}(i)) \\ D(\mathrm{dom}(i)) \arrow[ur, "Di"'] & \end{tikzcd}$$

for each i



$$\prod_I DI \xrightarrow{g} \prod_i D(\mathrm{cod}(i))$$

$$\overline{\underline{\hspace{12cm}}}$$

$$\prod_I DI \xrightarrow[\pi_{\mathrm{cod}(i)}]{} D(\mathrm{cod}(i))$$

for each i

Therefore

$$\begin{tikzcd} L \arrow[r, "e"] & \prod_I DI \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \prod_i D(\mathrm{cod}(i)) \end{tikzcd}$$

take equalizer

We claim this L is a limit.

$\boxed{1^\circ}$

$$\begin{tikzcd} L \arrow[r, hook, "e"] & \prod_I DI \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[d, "\pi_I"'] \arrow[dr, "\pi_J"] & & \prod_i D(\mathrm{cod}(i)) \\ & DI \arrow[r, "Di"'] & DJ & \end{tikzcd}$$

$\alpha_I := \pi_I \circ e$

■ $Di \circ \alpha_I = \alpha_J$

[∵ use that e is an equalizer

$\boxed{2^\circ}$

$$\begin{tikzcd} L \arrow[r, hook, "e"] & \prod_I DI \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d] \arrow[dr] & \bullet \\ & DI \arrow[r, "Di"] & DJ \\ & B \arrow[u, "\beta_I"] \arrow[ur, "\beta_J"'] \arrow[uu, dashed, bend right=60, "k"'] & \end{tikzcd}$$

Any cone $(B \xrightarrow[\beta_I]{} DI)_{I \in J}$ induces k in the above.

Since $fk = gk$, k factors thru e.

$\boxed{3^\circ}$ Uniqueness of the mediating map

[i.e

$$\begin{tikzcd}
 & L \arrow[dl] \arrow[dr] & \\
DI \arrow[rr, "Di"'] & & DJ \\
 & B \arrow[ul] \arrow[ur] \arrow[uu, bend right=70, "k"'] \arrow[uu, bend right=100, "k'"'] &
\end{tikzcd}$$

$\Rightarrow k = k'$]

$\Rightarrow$ Exercise!]

∎

Let's use this in Sets. Recall:

- products in Sets:
  (iso. to) set-theoretic products
- equalizers in Sets:

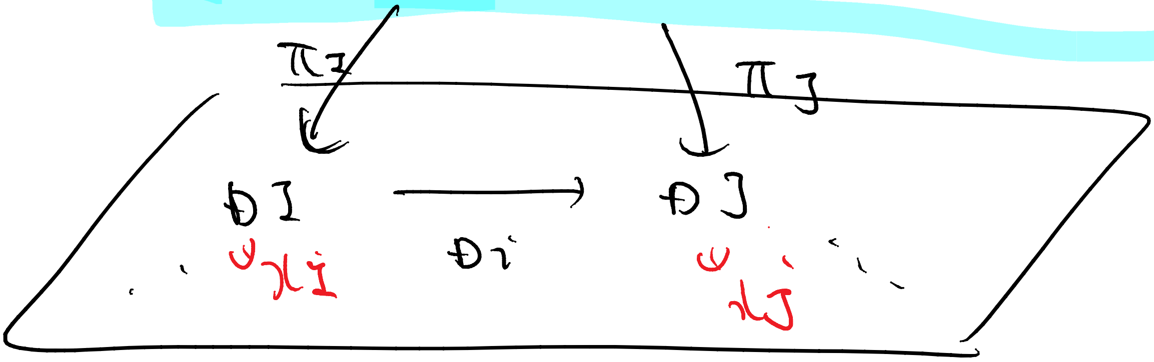
$$\{x \mid f(x)=g(x)\} \hookrightarrow X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

2012年7月18日
8:54

Prop. $D : \mathcal{I} \to \mathbf{Sets}$, a diagram.

$\mathrm{Lim}\, D$ is given by

$$\left\{ (x_I)_{I \in \mathcal{I}} \;\middle|\; \begin{array}{l} x_I \in DI \\ \forall i : I \to J \text{ in } \mathcal{I}, \\ (Di)(x_I) = x_J \end{array} \right\}$$



∎

That is,

$$\boxed{\text{product} \Longrightarrow \text{only “compatible” elements}}$$

Similarly:

Prop. $\mathbb{C}$ has $\left|\begin{array}{l}\text{coprod.}\\ \text{coequalizers}\end{array}\right|$

$\Rightarrow$ $\mathbb{C}$ has colimits. (modulo the size issue)

Prop. In Sets, colimits are given by:

$$\begin{tikzcd}
 & \boxed{\left(\coprod_{I\in\mathbb{J}} DI\right)/\sim} & \\
\cdots & DI \arrow[u, "\kappa_I"] \arrow[r, "Di"] & DJ \arrow[ul, "\kappa_J"'] & \cdots \\
 & x \arrow[u, phantom, sloped, "\in"] & (Di)(x) \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

where $\sim$ is the eq. rel. generated by

$$(x\in DI)\sim\left((Di)(x)\in DJ\right)$$

§2.4 Constr. of a final coalg.

2012年7月18日
9:20

Final coalgebras

- $$\begin{tikzcd} L \times L^{\omega} & & (a_0, a_1 a_2 \ldots) \\ L^{\omega} \arrow[u, "\langle hd, tl \rangle"'] & & a_0 a_1 a_2 \ldots \arrow[u, mapsto] \end{tikzcd}$$

- $L^{\infty} := L^{*} + L^{\omega} = \left\{ \begin{array}{l} \text{fin./infinite} \\ \text{str. over } L \end{array} \right\}$

- $$\begin{tikzcd} 1 + L \times L^{\infty} & \kappa_1(\perp) & \kappa_2((a, \alpha)) \\ L^{\infty} \arrow[u] & \varepsilon \arrow[u, mapsto] & a \cdot \alpha \arrow[u, mapsto] \\ & & \boxed{\text{cons}} \arrow[u] \end{tikzcd}$$

  $1 = \{\perp\}$

  (empty seq.)

  .

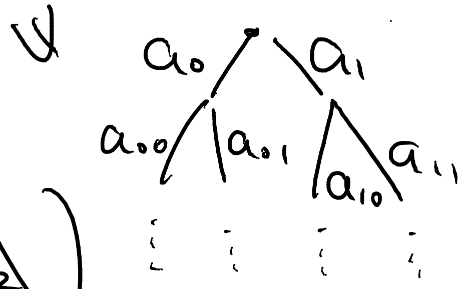
- $$\begin{tikzcd} 2 \times \left(\mathcal{P}(L^{*})\right)^{L} & \left(\begin{array}{c} \varepsilon \in U \, ? \, , \\ \lambda a \in L . \\ \partial_a U \end{array} \right) \\ \mathcal{P}(L^{*}) \arrow[u] & U \arrow[u, mapsto] \end{tikzcd}$$

2012年7月18日
12:52

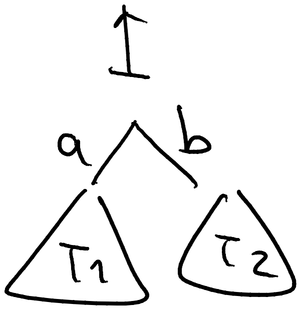
where $\partial_a U$ is the so-called Brzozowski derivative:

$$\partial_a U := \{\alpha \in L^* \mid a \cdot \alpha \in U\}$$

- $CBT_L :=$ { complete binary trees with edges labeled from L }



$$\left(L \times \underset{\substack{\uparrow \\ CBT_L}}{CBT_L}\right)^2 \quad (a, T_1, b, T_2)$$



2012年7月18日
9:38

Intuition

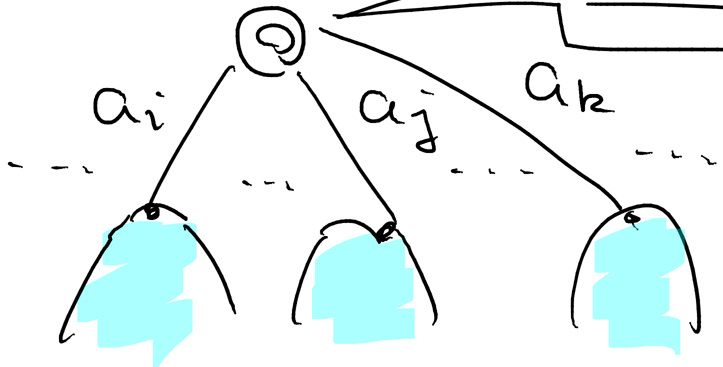
“datatype ~~constructors~~ destructors”

$\boxed{L\times\_}$

$\boxed{1+L\times\_}$

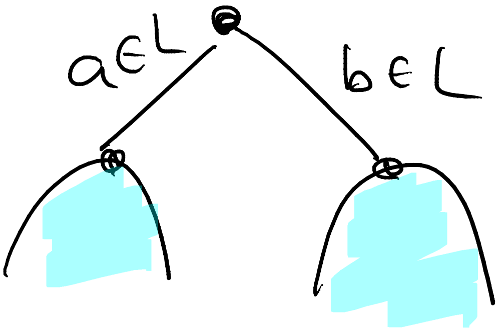
$\boxed{2\times(\_)^L}$

yes or no ; in fact terminal or nonterminal



$(L=\{a_i, a_j, \ldots\})$

$\boxed{(L\times\_)^2}$



$\boxed{Q}$ - Given a functor $F : \mathbf{Sets} \to \mathbf{Sets}$, how does a final F-coalg. look like?

- BTW, does a final coalg. exist?

Exercise Prove that, if $\mathbb{C}$ has an initial object, $F : \mathbb{C} \to \mathbb{C}$ has an initial coalgebra.
(Hence an initial coalg. is not very interesting)

Let's recall the definitions:

$\boxed{\text{Def.}}$ $F : \mathbb{C} \to \mathbb{C}$, a functor

- An F-coalgebra is a pair

$$\Big(\underset{\substack{\text{carrier}\\ \text{set}}}{X},\ \underset{\text{dynamics}}{c} : X \longrightarrow \underset{\substack{\text{transition}\\ \text{type}}}{FX}\Big)$$

$$\begin{tikzcd} FX \\ X \arrow[u, "c"'] \end{tikzcd}$$

- A morphism of F-coalg. is $f : X \to Y$ s.t.

$$\begin{tikzcd} FX \arrow[r, "Ff"] & FY \\ X \arrow[u, "c"] \arrow[r, "f"'] \arrow[ur, phantom, "="] & Y \arrow[u] \end{tikzcd}$$

Prop. F-coalg. and morphisms form a category:

$\mathbb{C}oalg_F$ obj. an F-coalg. $$\begin{tikzcd} FX \\ X \arrow[u, "c"'] \end{tikzcd}$$

arr. $$\left(\begin{tikzcd} FX \\ X \arrow[u, "c"] \end{tikzcd}\right) \xrightarrow{f} \left(\begin{tikzcd} FY \\ Y \arrow[u, "d"] \end{tikzcd}\right) \text{ in } \mathbb{C}oalg_F$$

---

f s.t. $$\begin{tikzcd} FX \arrow[r, "Ff"'] & FY \\ X \arrow[u, "c"] \arrow[r, "f"'] & Y \arrow[u] \end{tikzcd}$$

Def. A final F-coalg. is a final object in $\mathbb{C}oalg_F$.

Therefore:
$$\forall \left(\begin{tikzcd} FX \\ X \arrow[u, "c"] \end{tikzcd}\right) \overset{\exists !}{\dashrightarrow} \left(\begin{tikzcd} FZ \\ Z \arrow[u, "final"] \end{tikzcd}\right)$$

That is,
$$\begin{tikzcd} FX \arrow[r, dashed] & FZ \\ X \arrow[u, "\forall c"] \arrow[r, dashed] & Z \arrow[u, "final"'] \end{tikzcd} \quad \llcorner\mathbb{C}$$

BTW "Coalgebra" can mean many things.

- In many branches of mathematics:

  a comonoid obj. $X \to X \otimes X$

  (typically in Mod_R)

- For a comonad $M : \mathbb{C} \to \mathbb{C}$,
  (We'll see this later)

  an (Eilenberg-Moore) coalgebra

  is

  * (operation) $$\begin{tikzcd} MX \\ X \arrow[u, "c"] \end{tikzcd}$$

  * (equation) $$\begin{tikzcd} MX \arrow[r, "\varepsilon^M_X"] & X \\ X \arrow[u, "c"] \arrow[ur, equal] & \end{tikzcd}$$ $$\begin{tikzcd} MX \arrow[r, "\delta^M_X"] & M^2X \\ X \arrow[u, "c"] \arrow[r, "c"'] & MX \arrow[u, "Mc"'] \end{tikzcd}$$

- The current notion is often called functor coalgebra, or F-coalgebra.

(The same is true for algebras ...)

2012年7月19日
10:23

Back to $\boxed{Q}$: { Does a final coalg. exist?
How does it look?

One important lemma for "no-go thm":

<u>Lem</u> (Lambek's lemma)

$$\begin{tikzcd} FZ \\ Z \arrow[u, "\zeta"] \end{tikzcd}$$, a final F-coalg

$\Longrightarrow$ ζ is an isomorphism

Do it in the lecture

<u>Proof.</u> A nice ==exercise== ☺ Hints:

- $$\begin{tikzcd} F^2Z \\ FZ \arrow[u, "F\zeta"] \end{tikzcd}$$ is an F-coalg. too.
- Recall the def. of isomorphism; We need $FZ \xrightarrow[a]{} Z$ s.t. $\zeta \circ a = \mathrm{id}$, $a \circ \zeta = \mathrm{id}$

$$\begin{tikzcd} FZ \arrow[loop right, "id"] \arrow[d, bend right, "a"'] \\ Z \arrow[u, "\zeta"'] \arrow[loop right, "id"] \end{tikzcd}$$

☺

2012年7月19日
10:45

- You'll use the functoriality of F,
  i.e. $F(id) = id$
  $F(g \circ f) = (Fg) \circ (Ff)$ $\boxed{\xi}$

It is strongly recommended to solve this exercise, to check your understanding.

An immediate "no-go" consequence:

Prop. There is no final $\mathcal{P}$-coalg.

Proof. There is no iso.

$$\begin{tikzcd} \mathcal{P}X \\ X \arrow[u, "\cong"'] \end{tikzcd}$$

for any X

(Prf by a diagonal argument) $\blacksquare$

The (covariant) powerset functor

$$\begin{tikzcd} \mathcal{P} : \mathbf{Sets} \arrow[r] & \mathbf{Sets} \\ X \arrow[d, "f"'] \arrow[r, mapsto] & \mathcal{P}X \arrow[d, "\mathcal{P}f"] \\ Y & \mathcal{P}Y \end{tikzcd}$$

$(\mathcal{P}f)(U) = f[U]$ (direct image)

$\emptyset$

(direct image)

Proof. (Lambek's lemma)

(Aim
$$\begin{tikzcd} FZ \\ Z \arrow[u, "\zeta"', "\cong"] \end{tikzcd}$$
)

$\Uparrow$

(Aim
$$\begin{tikzcd} FZ \arrow[loop left, "id"] \arrow[r, shift left, "?"] & Z \arrow[l, shift left, "\zeta"] \arrow[loop right, "id"] \end{tikzcd}$$
)

$$\begin{tikzcd} F(FZ) \arrow[r, dashed, "Fe"] & FZ \\ FZ \arrow[u, "F\zeta"] \arrow[r, dashed, "e"'] & Z \arrow[u, "\zeta"', "\text{final}"] \end{tikzcd}$$

(Aim $\zeta \circ e = id$, $e \circ \zeta = id$)

$$\begin{tikzcd} FZ \arrow[r, "F\zeta"] & F(FZ) \arrow[r, "Fe"] \arrow[dr, phantom, "\text{(def. of } e)"] & FZ \\ Z \arrow[u, "\zeta"] \arrow[r, "\zeta"'] & FZ \arrow[u, "F\zeta"'] \arrow[r, "e"'] & Z \arrow[u, "\zeta"'] \end{tikzcd}$$

thus

$$\begin{tikzcd} FZ \arrow[r, "F(e\circ\zeta)"] & FZ \\ Z \arrow[u, "\zeta"'] \arrow[r, "e\circ\zeta"'] & Z \arrow[u, "\zeta"'] \end{tikzcd}$$

$$\begin{tikzcd} FZ \arrow[r, "F(id)"] & FZ \\ Z \arrow[u, "\zeta"] \arrow[r, "id"'] & Z \arrow[u, "\zeta"'] \end{tikzcd}$$

By the univ. of final coalg.

(uniqueness)

$e \circ \zeta = id$.

$$\begin{aligned} \zeta \circ e &= Fe \circ F\zeta \\ &= F(e \circ \zeta) = F(id) = id. \end{aligned}$$

∎

In fact: we only need some <u>bound</u> κ. For simplicity we're taking $\kappa = \aleph_0$

2012年7月19日
10:51

A lesson: the size matters.

We use a class of "small functors" that do have final coalg.

Def. A functor $F : \mathbf{Sets} \to \mathbf{Sets}$ is finitary if,

- for any set X and $t \in FX$,
- there exists a finite subset $X' \subseteq X$,
- s.t.

$$\begin{tikzcd} X' \arrow[r, hook, "i"] & X \end{tikzcd} \text{ inclusion}$$

$$\begin{tikzcd} FX' \arrow[r, hook, "Fi"] & FX \\ \exists t' \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & t \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

(This is a "bureaucratic" way of saying "t is already in FX'.")

2012年7月19日
10:58

This roughly means:

to form an element $t \in FX$,
you need only finitely many
elem. of X

Examples

- $\mathcal{P}$ is not finitary ($\exists\, U \in \mathcal{P}X$, infinite)
- $\mathcal{P}_{fin}$ (the <u>finite</u> powerset functor) is finitary
- $(-)^L$ (L is a fixed set) is
  - $*$ finitary if L is finite
  - $*$ not finitary if L is not.

2012年7月19日
11:00

Intermission

Note that the prev. def. is not really "categorical" — we speak with elements. ☹

A categorical definition:

Def. $F : \mathbb{C} \to \mathbb{D}$ is *finitary* if F preserves *filtered colimits*.

- colim. of a *filtered* $\mathbb{J}$

  $$\begin{tikzcd} * & I \arrow[dr] & & * \\ & & \exists K & \\ & J \arrow[ur] & & \end{tikzcd}$$

  $$\cdot \rightrightarrows \cdot \xrightarrow{\exists} \cdot$$

- Generalization of *directed sup.* in domain theory

Using this, we can talk about the *size* of an object $X \in \mathbb{C}$:

Def $X \in \mathbb{C}$ is *finitary* if the functor

$$\mathbb{C}(X, \_) : \mathbb{C} \to \mathbf{Sets}$$

is finitary.

References

- MacLane, CWM (for filtered colim.)
- Adamek, Rosicky
  "Locally presentable and ..."
  (Comprehensive ref., with its own style)

Prop. The two def. of "finitary functor" coincide in Sets.

Now we go on to exhibit a constr. of a final coalg. for F: finitary.

It's like a showcase of (Sets-oriented) categorical techniques — they're introduced when they're needed.

Def. $F : \mathbf{Sets} \to \mathbf{Sets}$,

The *final sequence* is the diagram

unique map to final 1

$$\begin{tikzcd} 1 & F1 \arrow[l, "!"'] & F^2 1 \arrow[l, "F!"'] & \cdots \arrow[l] \end{tikzcd}$$

[Precisely:

$$\mathbb{J} = \left(\begin{tikzcd} 0 \arrow[loop below] & 1 \arrow[l, "i_0^1"'] \arrow[loop below] & 2 \arrow[l, "i_1^2"'] \arrow[ll, bend right, "i_0^2"'] \arrow[loop below] & 3 \arrow[l] \arrow[ll, bend right, "i_1^3"'] \arrow[lll, bend left=40, "i_0^3"] & \cdots \arrow[l] \end{tikzcd} \right)$$

$D : \mathbb{J} \to \mathbf{Sets}$]

$(D : \mathbb{J} \to \mathbb{S}\mathrm{ets} \qquad i'_0)$

Intuition

(finitary)

$F = L \times (\_)$

Final coalg.:

$$\begin{tikzcd} L\times L^{\omega} \\ L^{\omega} \arrow[u, "\langle hd, tl\rangle", "\cong"'] \end{tikzcd}$$

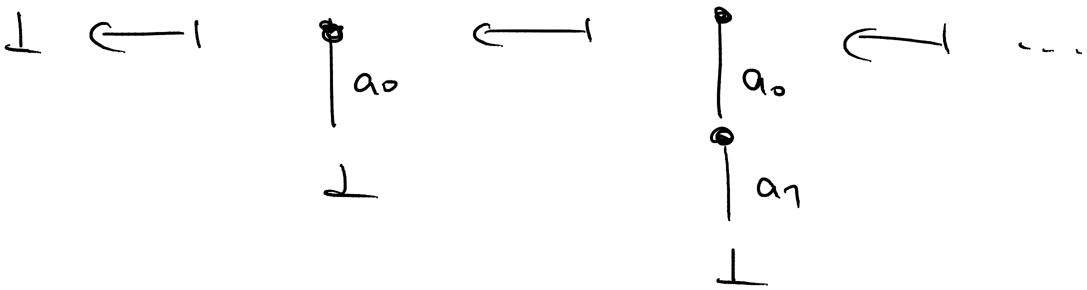
$$\begin{tikzcd} 1 & L\times 1 \arrow[l, "!"'] & L\times(L\times 1) \arrow[l, "L\times !"'] & \cdots \arrow[l] \\ \bot & (a_0,\bot) \arrow[l, mapsto] & (a_0,(a_1,\bot)) \arrow[l, mapsto] & \cdots \arrow[l, mapsto] \end{tikzcd}$$

suppressed

Thus:

$F^n 1 = L \times (L \times (\ldots \times (L \times 1) \ldots))$

as "the approx. up-to n steps"



We take a limit of this final seq.

BTW: Thm In Sets, a "small" diagram has a limit.
(Also a colimit)

cf nLab
"small complete category"

Recall also that a limit in Sets is given by the set of "compatible tuples". Therefore:

Prop. Let

$$B := \left\{ (t_0, t_1, t_2, \ldots) \;\middle|\; \begin{array}{l} t_i \in F^i 1, \\ \left(F^i(!)\right)(t_{i+1}) = t_i \end{array} \right\}$$

Then

$$\begin{tikzcd}
 & & & B \arrow[lll, bend right, "\pi_0"'] \arrow[ll, "\pi_1"'] \arrow[l, "\pi_2"] \\
1 & F1 \arrow[l, "!"'] & F^2 1 \arrow[l, "F!"'] & \cdots \arrow[l] \\
 & t_1 \arrow[u, phantom, "\rotatebox[origin=c]{90}{$\in$}"] & t_2 \arrow[u, phantom, "\rotatebox[origin=c]{90}{$\in$}"] \arrow[l, mapsto] &
\end{tikzcd}$$

is a limit.

2012年7月19日
12:15

Intuition If $F = L \times \_$,

$$B \ni \left(\bot, (a^1_0 \bot), (a^2_0\, a^2_1 \bot), (a^3_0\, a^3_1\, a^3_2 \bot), \ldots \right)$$

subject to the compatibility cond

$$\left[a^{i+1}_0\, a^{i+1}_1 \cdots \underbrace{a^{i+1}_i\ \bot}_{\text{suppress}} \xmapsto{F^i !} a^i_0\, a^i_1 \cdots a^i_{i-1}\ \underline{\bot} \right]$$

therefore

$$B \ni \left(\bot, (a_0 \bot), (a_0\, a_1 \bot), (a_0\, a_1\, a_2 \bot), \ldots \right)$$

$\Longrightarrow$ Looks like L^ω !!

But for a general F this is not quite enough ...

2012年7月19日
12:18

Our next goal: We prove that, by applying F, B doesn't grow. (I.e. we've come to 'saturation')

First, by applying F to the whole limit diagram we have

$$\begin{tikzcd}
F1 & F^2! \arrow[l, "F!"'] & \cdots \arrow[l, "F^2!"'] & \\
 & & & FB \arrow[ulll, bend left, "F\pi_0"] \arrow[ull, "F\pi_1"'] \arrow[ul, phantom, "\cdots"]
\end{tikzcd}$$

We can add the left-most obj. and obtain a cone over the final seq.

$$\begin{tikzcd}
1 & F1 \arrow[l, "!"'] & F^2! \arrow[l, "F!"'] & \cdots \arrow[l, "F^2!"'] & \\
 & & & & FB \arrow[ullll, bend left, "!"] \arrow[ulll, bend left, "F\pi_0"] \arrow[ull, "F\pi_1"'] \arrow[ul, phantom, "\cdots"]
\end{tikzcd}$$

2012年7月19日
12:22

By the universality of B (limit), we have a mediating map

$$\begin{tikzcd}
 & & & & B\ (\text{limit}) \arrow[dlll, "\pi_i"'] \arrow[dll, "\pi_{i+1}"] \\
\cdots & F^i 1 \arrow[l] & F^{i+1} 1 \arrow[l, "F^i !"'] & \cdots \arrow[l] & \\
 & & & & FB \arrow[ulll, "F\pi_{i-1}"] \arrow[ull, "F\pi_i"'] \arrow[uu, dashed, "b"']
\end{tikzcd}$$

We claim that b is monic

(hence FB is no bigger than B)

(We'll use that F is finitary.)

Lem. Thus induced b is monic.

Proof. Let $t_0, t_1 \in FB$, and $b(t_0) = b(t_1)$. $\left(\begin{array}{c}\underline{\text{Aim}}\\ t_0 = t_1\end{array}\right)$

Since F is finitary, there is $B' \subseteq_{\text{fin.}} B$, $t_0', t_1' \in FB'$

s.t. $(Fj)(t_0') = t_0$
$(Fj)(t_1') = t_1$.

$$\begin{tikzcd} B' \arrow[r, hook, "j"] & B \\ FB' \arrow[r, hook, "Fj"] & FB \\ t_0' \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & t_0 \arrow[u, phantom, sloped, "\in"] \\ t_1' \arrow[r, mapsto] & t_1 \end{tikzcd}$$

$\left(\begin{array}{c}\text{Aim}\\ t_0' = t_1'\end{array}\right)$

Exercise This is not precisely the def. of finitary functor. Prove this follows.

$$\begin{tikzcd}
& & & & B' \arrow[d, hook, "j\ (\mathrm{ind.})"] \\
& & & & B \arrow[lld, "\pi_{i+1}"'] \\
\cdots & F^{i}1 \arrow[l] & F^{i+1}1 \arrow[l] & \cdots \arrow[l] & \\
& & & & FB \arrow[uu, bend right, "b"'] \arrow[llu, "F\pi_i"] \\
& & & & FB' \ni t_0', t_1' \arrow[u, hook, "Fj"]
\end{tikzcd}$$

- $F(\pi_i \circ j)(t_0') = F(\pi_i \circ j)(t_1'), \ \forall i$

$$\because \ (\mathrm{LHS}) \underset{\substack{\uparrow \\ \text{above diagram} \\ \text{(i.e. def. of } b)}}{=} (\pi_{i+1} \circ b \circ Fj)(t_0')$$

$$= \pi_{i+1}(b(t_0))$$

$$\underset{\text{asmp.}}{=} \pi_{i+1}(b(t_1))$$

$$= \cdots = (\mathrm{RHS})$$

Now we use the following Sets-specific fact.

Sublem. $F : \mathbf{Sets} \longrightarrow \mathbf{Sets}$ preserves monos w/ a nonempty domain.

Proof. In Sets, a mono is either

- w/ the empty domain

$$\emptyset \xrightarrow{m} X,$$ or

- w/ a left inverse.

$$\begin{tikzcd} X \arrow[loop left, "id"] \arrow[r, bend right, "m"'] & Y \arrow[l, bend right, "\exists e"'] \end{tikzcd} \quad \left(\text{i.e. } e \circ m = id\right)$$

cf. Adamek. presentation of Set functors.

In the latter case,

$$\begin{tikzcd} FX \arrow[loop left, "id"] \arrow[r, bend right, "Fm"'] & FY \arrow[l, bend right, "Fe"'] \end{tikzcd}$$

$$Fe \circ Fm = F(e \circ m) = F\,id = id.$$

Thus Fm has a left inverse, hence is a mono. (Exercise)

(An arrow with a left inverse is called a split mono.)

☑

Therefore we are done if we show that for some $i \in \mathbb{N}$,

$$B' \xhookrightarrow{j} B \xrightarrow{\pi_i} F^i 1 \quad \text{is monic}$$

(∵ Then $F(\pi_i \circ j)$ is monic, hence by 2 pages before, $t_0' = t_i'$)

By def. of B, each element of B' is of the form

$$(x_i)_{i \in \mathbb{N}}, \quad x_i \in F^i 1$$

with

$$(x_i)_i = (x'_i)_i$$

$$\overset{\text{def.}}{\iff} \quad x_i = x'_i \quad \text{for } \forall i \in \mathbb{N}.$$

Therefore if $(x_i)_i \neq (x'_i)_i$, there is $i \in \mathbb{N}$ s.t. $x_i \neq x'_i$.

Since B' is <u>finite</u>, there is a large enough $i_0 \in \mathbb{N}$ s.t.

$$\forall (x_i)_i, (x'_i)_i \in B',$$

$$(x_i)_i \neq (x'_i)_i \Rightarrow x_{i_0} \neq x'_{i_0},$$

For such i_0,

$$B' \hookrightarrow B \xrightarrow[\pi_{i_0}]{} F^{i_0}] \text{ is}$$

monic. $\boxed{/\!/}$

Lem. 4-page ago

2012年7月19日
13:36

Back to the picture:

$$\begin{tikzcd}
& & & & B \arrow[dlll, bend right, "\pi_i"'] \arrow[dll, "\pi_{i+1}"'] \\
\cdots & F^i 1 \arrow[l] & F^{i+1} 1 \arrow[l, "F^i !"'] & \cdots \arrow[l] & \\
& & & & FB \arrow[ulll, bend left, "F\pi_{i-1}"] \arrow[ull, "F\pi_i"'] \arrow[uu, dashed, bend right, "b"']
\end{tikzcd}$$

We proved b is monic,
thus in Sets we can take its
left inverse

$$\begin{tikzcd}
FB \arrow[loop left, "id"] \arrow[d, bend left, "b"] \\
B \arrow[u, "c"]
\end{tikzcd}$$

Coalgebra!!

We're almost done ... the remaining
is to <u>quotient</u> B

2012年7月19日
13:40

In other words:

- B contains all possible F-behaviors
- But it can still have redundancy ...

$$\left.\begin{array}{l} b \\ b' \end{array}\right. \in B \text{ w/ the same behavior}$$

Lem. An F-coalg. $\begin{array}{c} FX \\ \uparrow c \\ X \end{array}$ induces a canonical cone over the final sequence.

Proof.

$$\begin{tikzcd}
{} & {} & {} & {} & B \arrow[llll, bend right] \arrow[lll, bend right] \arrow[ll] \\
1 & F1 \arrow[l] & F^2 1 \arrow[l] & \cdots \arrow[l] & {} \\
{} & FX \arrow[u, "F!"] & FX \arrow[u, "F\gamma_1"'] & {} & {} \\
{} & {} & X \arrow[ul, "c"] \arrow[u, "c"'] \arrow[uull, bend left, "!"] \arrow[uuuull, dashed, bend right, "\overline{c}"'] & {} & {}
\end{tikzcd}$$

$\gamma_0 := !$

$\gamma_{i+1} := (F\gamma_i) \circ c$

Thus by the universality of B we have

$$X \overset{\overline{c}}{\dashrightarrow} B$$

$\left(\textcircled{$x_0$} \xrightarrow{a_1} \textcircled{x_1} \xrightarrow{a_2} \textcircled{x_2} \xrightarrow{a_3} \cdots \right)$

BTW The cone $(\gamma_i)_{i \in \mathbb{N}}$ induced by $\left(\begin{smallmatrix} FX \\ \uparrow c \\ X \end{smallmatrix} \right)$: e.g. when $F = L \times \_$

$$\begin{tikzcd}
 & & & \bullet \arrow[d, dash, "a_1"] & \\
 & & \bullet \arrow[d, dash, "a_1"] & \bullet \arrow[d, dash, "a_2"] & \cdots \\
 & \bullet \arrow[d, dash, "a_1"] & \bullet \arrow[d, dash, "a_2"] & \bullet \arrow[d, dash, "a_3"] & \\
\bot & \bot & \bot & \bot & \\
1 & F1 \arrow[l, "!"'] & F^2 1 \arrow[l] & F^3 1 \arrow[l] & \cdots \arrow[l] \\
 & & & F^3 X \arrow[u, "F^3 !"'] & (a_1,(a_2,(a_3, \cdots x_3))) \arrow[l, phantom, "\ni"] \\
 & & F^2 X \arrow[uu, "F^2 !"] \arrow[ur, "F^2 c"] & (a_1,(a_2,x_2)) \arrow[l, phantom, "\ni"] & \\
 & & FX \arrow[uuul, "F!"] \arrow[u, "Fc"'] & & \\
 & & (a_1,x_1) \arrow[u, phantom, sloped, "\in"] & X \arrow[ul, "c"] \arrow[uuuulll, bend left, "\gamma_0 := !"] & \\
 & & & x \arrow[u, phantom, sloped, "\in"] &
\end{tikzcd}$$

Lem. Thus induced $\bar{c}$ is a coalg. morphism

$$\begin{tikzcd}
FX \arrow[r, "F\bar{c}"] & FB \\
X \arrow[u, "c"] \arrow[r, "\bar{c}"'] & B \arrow[u, "e"']
\end{tikzcd}$$

2012年7月20日
0:57

Proof.

We prove

$$b \circ F\bar{c} \circ c = \bar{c}$$

by universality.

(Uniqueness of a mediating map.)

Aim

$$\begin{tikzcd} FX \arrow[r, "F\bar{c}"] & FB \arrow[loop right, "id"] \arrow[d, bend left, "b"] \\ X \arrow[u, "c"] \arrow[r, "\bar{c}"'] \arrow[ur, phantom, "//"] & B \arrow[u, "e"] \end{tikzcd}$$

$\Uparrow$

Suffices to show :

$$b \circ F\bar{c} \circ c = \bar{c}$$

Then

$$F\bar{c} \circ c = e \circ b \circ F\bar{c} \circ c$$
$(e \circ b = id)$
$$= e \circ \bar{c}$$

By def. of $\bar{c}$,

$$\begin{tikzcd} & & & B \ (\mathrm{limit}) \arrow[dll, bend right, "\pi_i"'] \\ \cdots & F^i 1 \arrow[l] & F^{i+1} 1 \arrow[l] & \cdots \arrow[l] \\ & & & X \arrow[ull, bend left, "\gamma_i"] \arrow[uu, dashed, "\bar{c}"'] \end{tikzcd}$$

2012年7月20日
1:07

$$\begin{tikzcd}
 & & & & B \text{ (limit)} \arrow[dlll, bend right, "\pi_i"'] \\
\cdots & F^i 1 \arrow[l] & F^{i+1} 1 \arrow[l] & \cdots \arrow[l] & \\
 & & & & FB \arrow[u, "b"'] \arrow[ulll, bend left, "F\pi_{i-1}"] \\
 & & & & Fx \arrow[u, "F\bar{c}"'] \arrow[uulll, bend left, "F\gamma_{i-1}"] \\
 & & & & x \arrow[u, "c"'] \arrow[uuulll, bend left, "\gamma_i"]
\end{tikzcd}$$

$$\gamma_i = (F\gamma_{i-1}) \circ c \quad \left(\text{Def. of } \gamma_i\right)$$

$$= F(\pi_{i-1} \circ \bar{c}) \circ c \quad \left(\text{Def. of } \bar{c}\right)$$

$$= \pi_i \circ b \circ F\bar{c} \circ c \quad \left(\text{Def. of } b\right)$$

Thus $b \circ F\bar{c} \circ c$ is also a mediating map.

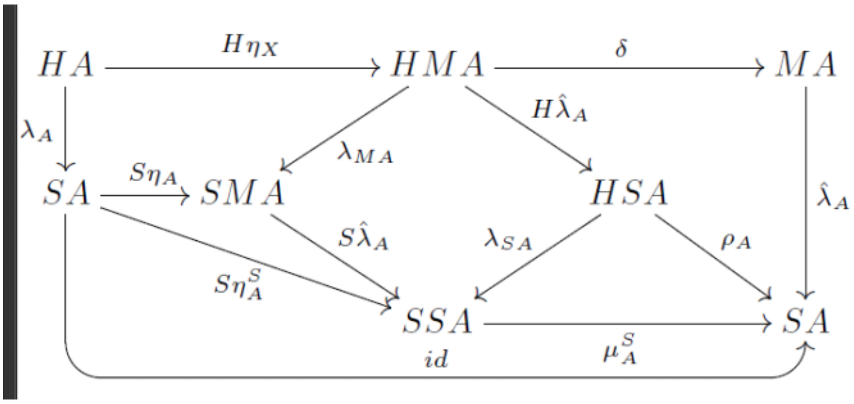
$$\therefore \quad b \circ F\bar{c} \circ c = \bar{c}. \qquad \blacksquare$$

2012年7月20日
1:11

<u>BTW</u> A lesson: in CT,

Diagrammatic reasoning (by commutativity) is <u>not always</u> superior to equational reasoning (by =).

画面の領域の取り込み日時: 2012/07/20 1:17



$$\begin{tikzcd}
HA \arrow[rr, "H\eta_X"] \arrow[d, "\lambda_A"'] & & HMA \arrow[rr, "\delta"] \arrow[dl, "\lambda_{MA}"] \arrow[dr, "H\hat{\lambda}_A"] & & MA \arrow[dd, "\hat{\lambda}_A"] \\
SA \arrow[r, "S\eta_A"] \arrow[drr, "S\eta^S_A"'] \arrow[drrrr, bend right, "id"'] & SMA \arrow[dr, "S\hat{\lambda}_A"] & & HSA \arrow[dl, "\lambda_{SA}"'] \arrow[dr, "\rho_A"] & \\
 & & SSA \arrow[rr, "\mu^S_A"'] & & SA
\end{tikzcd}$$

2012年7月19日
13:44

Thus : $\begin{matrix} FB \\ \uparrow e \\ B \end{matrix}$ is a *weakly* *final* coalgebra.

$$\begin{pmatrix} FX \\ \uparrow c \\ X \end{pmatrix} \xrightarrow{\text{exists,}} \begin{pmatrix} FB \\ \uparrow e \\ B \end{pmatrix}$$

but not nec. unique

Next goal

We quotient $\begin{pmatrix} FB \\ \uparrow e \\ B \end{pmatrix}$ to get a *proper* final coalgebra

(This will also take time ...)

2012年7月20日
1:18

Def. Fix $F : \mathbf{Sets} \to \mathbf{Sets}$,

$$\begin{tikzcd} FX \\ X \arrow[u, "c"] \end{tikzcd}$$: an F-coalg.

$\approx \subseteq X^2$ ("F-behavioral equivalence")

is defined by

$x \approx x' \underset{\text{def.}}{\iff}$ for some coalg. $\begin{pmatrix} FY \\ \uparrow d \\ Y \end{pmatrix}$

and a mor.

$$\begin{pmatrix} FX \\ \uparrow c \\ X \end{pmatrix} \xrightarrow{f} \begin{pmatrix} FY \\ \uparrow d \\ Y \end{pmatrix},$$

$$f(x) = f(x')$$

Intuition

A coalg. mor. is a "beh.-preserving map"

Rem. Immediate generalization:

$$\left(\begin{matrix} FX \\ c \uparrow \\ X \end{matrix} , \begin{matrix} x \\ \rotatebox{90}{$\in$} \\ X \end{matrix} \right) \approx \left(\begin{matrix} FX' \\ c' \uparrow \\ X' \end{matrix} , \begin{matrix} x' \\ \rotatebox{90}{$\in$} \\ X' \end{matrix} \right)$$

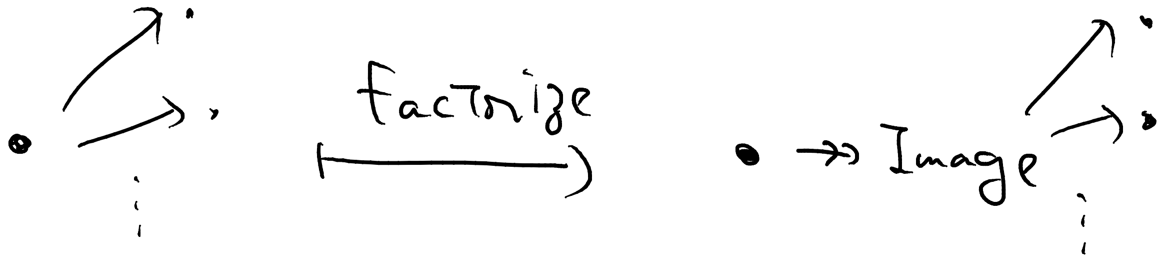
2012年7月20日
1:23

We quotient $\begin{matrix} FB \\ \uparrow e \\ B \end{matrix}$ (The weakly final coalg. we've obtained) by $\approx$. That is,

$$\begin{tikzcd} F(B/\approx) \\ B/\approx \arrow[u, "e/\approx"'] \end{tikzcd}$$

An obvious question: is $e/\approx$ well-dfd.?

- A full-blown answer is by a <u>factorization structure on sources</u>;



Ⓒⓕ Adamek, Herrlich, Strecker, "The Joy of Cats" (Textbook, now on the web)

- Here we use a bit more concrete (elementwise) arguments

- Here we use a bit more concrete (elementwise) arguments

Some categorical machinery.

A well-known result [e.g. in Barr & Wells, TTT]

Thm. T: a monad on $\mathbb{C}$

Then $$\begin{tikzcd}\mathbb{C}^T \arrow[d, "U"] \\ \mathbb{C}\end{tikzcd}$$ creates limits.

A result on Eilenberg-Moore algebras

It doesn't matter you don't understand the statement. What we're proving now is the same result for F-coalgebras.

2012年7月20日
1:50

Thm. $F : \mathbb{C} \to \mathbb{C}$ ($\mathbb{C}$ is not nec. Sets)

$D : \mathbb{J} \to \mathrm{Coalg}_F$, a diagram

like

$$\begin{tikzcd} FX_I & & FX_J & & FX_K \\ X_I \arrow[u, "c_I"] \arrow[rr, "D_i"] & & X_J \arrow[u, "c_J"] & & X_K \arrow[u, "c_K"] \end{tikzcd}$$

Consider

$$\mathrm{Colim}\left(\mathbb{J} \xrightarrow{D} \mathrm{Coalg}_F \xrightarrow{U} \mathbb{C}\right)$$

$$\begin{tikzcd} & & \mathrm{Colim}(UD) \\ X_I \arrow[urr, bend left, "\kappa_I"] \arrow[r, "D_i"] & X_J \arrow[ur, "\kappa_J"] & X_K \arrow[u, "\kappa_K"'] \end{tikzcd}$$

Then:

- $\mathrm{Colim}(UD)$ has a canonical F-coalg. structure.
- It is moreover a colimit of D.

Remark This is what is meant by

$$\begin{tikzcd} \text{“} \mathbb{C}\mathrm{oalg}_F \arrow[d, "U"] & \begin{pmatrix} FX \\ \uparrow c \\ X \end{pmatrix} \arrow[d, mapsto] \\ \mathbb{C} & X \end{tikzcd} \quad \text{creates colimits”}$$

,

that is,

colimits in $\mathbb{C}\mathrm{oalg}_F$ are

“computed in $\mathbb{C}$”.

Proof. | Let us write

$$\mathbb{D}I = \begin{pmatrix} FX_I \\ \uparrow c_I \\ X_I \end{pmatrix} \text{ for } I \in \mathbb{I}$$

$$\left(\text{Thus } \quad U\mathbb{D}I = X_I \right)$$

So

$$\begin{tikzcd} & & \mathrm{Colim}\, U\mathbb{D} \text{ (colimit)} \\ \mathbb{C} \; \cdots & X_I \arrow[ur, bend left, "K_I"] \arrow[r, "\mathbb{D}i"] & X_J \arrow[u, bend left, "K_J"'] & \cdots \end{tikzcd}$$

2012年7月20日
1:56

We have another cocone

$\mathcal{C}$

$$\begin{tikzcd}
\cdots & X_I \arrow[r, "D_i"] \arrow[d, "c_I"'] & X_J \arrow[d, "c_J"] & \cdots & \\
\cdots & FX_I \arrow[r, "FD_i"] \arrow[rrrd, "F\kappa_I"'] & FX_J \arrow[rrd, "F\kappa_J"] & \cdots & \\
 & & & & F(\mathrm{Colim}\, UD)
\end{tikzcd}$$

The prev. cocone, after appl. of F

Thus

$$\begin{tikzcd}
 & & & & \mathrm{Colim}\, UD \arrow[dd, dashed, "(\mathrm{colim}\, c)"] \\
\cdots & X_I \arrow[r, "D_i"] \arrow[rrru, "\kappa_I"] \arrow[rrrd, "F\kappa_I \circ c_I"'] & X_J \arrow[rru, "\kappa_J"] \arrow[rrd, "F\kappa_J \circ c_J"] & \cdots & \\
 & & & & F(\mathrm{Colim}\, UD)
\end{tikzcd}$$

This is the canonical F-coalg str on

0. Colin VD.

2012年7月20日
2:02

The fact that

$$\begin{tikzcd}
 & & & \begin{pmatrix} F(\mathrm{Colim}\,UD) \\ \uparrow \\ \mathrm{Colim}\,UD \end{pmatrix} \\
\cdots & \begin{pmatrix} Fx_I \\ \uparrow c_I \\ x_I \end{pmatrix} \arrow[r] \arrow[urr] & \begin{pmatrix} Fx_J \\ \uparrow c_J \\ x_J \end{pmatrix} \arrow[ur] & \cdots
\end{tikzcd}$$

is a colimit is straightfwd.
(Exercise) Q

Thm, 3 pages ago

Corollary $F : \mathbf{Sets} \to \mathbf{Sets}$

$\mathbf{Coalg}_F$ has all small colimits. (is cocomplete)

Let us also be prepared with a categorical view on <u>image factorization</u>

<u>Sets</u> |

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\text{surjection (epi)}"'] & & Y \\
& \mathrm{Im}(f) \arrow[ur, "\text{injection (mono)}"'] \arrow[d, equal] & \\
& \{f(x) \mid x \in X\} \subseteq Y &
\end{tikzcd}$$

Exercise

- Define epi arrows as right-cancelable ones
- Show that in Sets, epi $\Longleftrightarrow$ surjective

$\Uparrow$ AC

split epi ($\exists$ right inverse)

Prop. (Diagonal fill-in)

In Sets, if we have

$$\begin{tikzcd} X \arrow[r, two heads, "e"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ Z \arrow[r, tail, "m"'] & U \end{tikzcd}$$

- a comm. diagram
- w/ epi and mono in suitable places

then

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[d] & Y \arrow[d] \arrow[dl, dashed, "\exists!" near start, "h"] \\ Z \arrow[r, tail] & U \end{tikzcd}$$

i.e. $\exists! h$ s.t. $he = f$ and $mh = g$

Generalizing (Surj, Inj) in Sets, there is a notion of factorization structure (E, M) in a category $\mathbb{C}$.

Diagonal fill-in is an important property of such (E, M) ⒸⒻ Adamek HS Joy of Cats

2012年7月20日
2:12

Proof. Factorize f, g :

$$\begin{tikzcd}
X \arrow[r, two heads, "e"] \arrow[d, two heads, "e_f"] \arrow[dd, bend right, "f"'] & Y \arrow[d, two heads, "e_g"'] \arrow[dd, bend left, "g"] \\
\mathrm{Im}(f) \arrow[d, tail, "m_f"] & \mathrm{Im}(g) \arrow[d, tail, "m_g"'] \\
Z \arrow[r, tail, "m"'] & U
\end{tikzcd}$$

$\boxed{\mathrm{Sets}}$

Therefore

$$X \twoheadrightarrow \mathrm{Im}(f) \rightarrowtail U$$

$$X \twoheadrightarrow \mathrm{Im}(g) \rightarrowtail U$$

are image factorizations of the same function $mf = ge$

Thus

$$\begin{tikzcd}
 & \mathrm{Im}(f) \arrow[dr, tail, "m \circ m_f"] \arrow[dd, "\cong"] & \\
X \arrow[ur, two heads] \arrow[dr, two heads] & & U \\
 & \mathrm{Im}(g) \arrow[ur, tail, "m_g"'] &
\end{tikzcd}$$

That is,

$$\begin{tikzcd}
X \arrow[r, two heads, "e"] \arrow[d, two heads, "e_f"] \arrow[dd, bend right=60, "f"'] & Y \arrow[d, two heads, "e_g"'] \arrow[dd, bend left=60, "g"] \\
\mathrm{Im}(f) \arrow[d, tail, "m_f"] & \mathrm{Im}(g) \arrow[l, "k"', "\cong"] \arrow[d, tail, "m_g"'] \\
Z \arrow[r, tail, "m"'] & U
\end{tikzcd}$$

A fill-in is obtained as

$h := m_f \circ k \circ e_g$.

Uniqueness of a fill-in is an exercise. ☒

Hint: - Commutativity
- m is mono (left-cancelable)

2012年7月20日
2:29

Not a big restriction!

- $FX = \emptyset$ for some $X \neq \emptyset$
  $\Rightarrow$ FX is everywhere $\emptyset$
- See Adamek, Presentation of Set functors

Corollary

$F : \mathbf{Sets} \to \mathbf{Sets}$. Asm $FX \neq \emptyset$ for any $X \in \mathbf{Sets}$

$$\begin{tikzcd}
\left(\begin{smallmatrix} FX \\ \uparrow \\ X \end{smallmatrix}\right) \arrow[r, "f"] & \left(\begin{smallmatrix} FY \\ \uparrow \\ Y \end{smallmatrix}\right)
\end{tikzcd}$$

Then the image $\mathrm{Im}(f)$ has a canonical F-coalg. str. with

$$\begin{tikzcd}
\left(\begin{smallmatrix} FX \\ \uparrow \\ X \end{smallmatrix}\right) \arrow[r, two heads] & \left(\begin{smallmatrix} F(\mathrm{Im}(f)) \\ \uparrow \\ \mathrm{Im}(f) \end{smallmatrix}\right) \arrow[r, tail] & \left(\begin{smallmatrix} FY \\ \uparrow \\ Y \end{smallmatrix}\right)
\end{tikzcd}$$

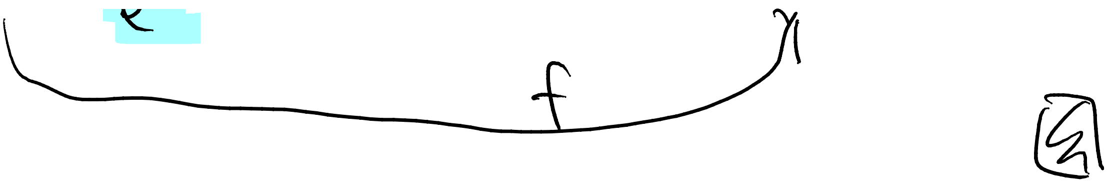
Proof.

Fm : mono

Sets functors pres. monos w/ nonempty domain

$$\begin{tikzcd}
FX \arrow[r, two heads, "Fe"] & F(\mathrm{Im}(f)) \arrow[r, tail, "Fm"] & FY \\
X \arrow[u] \arrow[r, two heads, "e"'] \arrow[rr, bend right, "f"'] & \mathrm{Im}(f) \arrow[u, dashed] \arrow[r, tail, "m"'] & Y \arrow[u]
\end{tikzcd}$$

diagonal fill-in


e
f
x1

Finally :

Lem. $F : \mathbf{Sets} \to \mathbf{Sets}$, $\begin{matrix} FX \\ c\uparrow \\ X \end{matrix}$: F-coalg.

$\approx \subseteq X^2$, behavioral equivalence.

Then There is a unique coalg. str. on $X/\approx$ s.t.

$$\begin{tikzcd} FX \arrow[r, two heads, "Fp"] & F(X/\approx) \\ X \arrow[u, "c"] \arrow[r, two heads, "p"'] \arrow[ur, phantom, "="] & X/\approx \arrow[u, "c/\approx"'] \end{tikzcd}$$

ae (projection)

Proof

- Uniqueness of $c/\approx$: obvious from p : epi, that is

$$kp = k'p \quad (= Fp \circ c)$$

$$\underset{p:\text{epi}}{\Longrightarrow} k = k'$$

Define

$$(c/\sim)([x]) := (Fp)(c(x))$$

We check this is well-defined.

Asm. $x \sim x'$, witnessed by

$$\begin{tikzcd} \begin{pmatrix} FX \\ \uparrow c \\ X \end{pmatrix} \arrow[r, "f"] & \begin{pmatrix} FY \\ \uparrow d \\ Y \end{pmatrix} \end{tikzcd}, \quad f(x) = f(x')$$

$\left(\dfrac{\text{aim}}{(Fp)(c(x)) = (Fp)(c(x'))} \right)$

- First we factorize:

$$\begin{tikzcd} \begin{pmatrix} FX \\ c\uparrow \\ X \end{pmatrix} \arrow[r, two heads, "e_f"] & \begin{pmatrix} FI \\ \uparrow g \\ I \end{pmatrix} \arrow[r, tail, "m_f"] & \begin{pmatrix} FY \\ \uparrow d \\ Y \end{pmatrix} \end{tikzcd}$$

(The Corollary 2 pages ago)

- By $f(x) = f(x')$,

$$e_f(x) = e_f(x')$$

Therefore

$$\begin{array}{ccc} (Fe_f \circ c)(x) & = & (g \circ e_f)(x) \\ \circledast \ \| & & \| \\ (Fe_f \circ c)(x') & = & (g \circ e_f)(x') \end{array}$$

$$(Fe_f \circ c)(x') = (g \circ e_f)(x')$$

- By def. of $\approx$,

$$\begin{tikzcd} X \arrow[r, two heads, "ef"] \arrow[dr, two heads, "p"'] & I \arrow[d, "q"] \\ & X/\approx \end{tikzcd}$$

$\Leftarrow$

Idea

- ef : identification by f
- p : id. by f and other coalg. mor.

Exercise

Show q is necessarily an epi. (Easy. The mono ver. was already presented)

Therefore

$$\begin{tikzcd} FX \arrow[r, two heads, "Fef"] \arrow[dr, two heads, "Fp"', "\text{☺}"] & FI \arrow[d, "Fq"] \\ & F(X/\approx) \end{tikzcd}$$

- Putting the above together we have

$$\begin{tikzcd} (Fp\circ c)(x) \arrow[r, phantom, "\overset{\text{☺}}{=}"] & (Fq\circ Fef\circ c)(x) \arrow[d, equal, "\star"] \\ (Fp\circ c)(x') \arrow[r, phantom, "\underset{\text{☺}}{=}"] & (Fq\circ Fef\circ c)(x') \end{tikzcd}$$

∎

(Lem. 2 pages ago)

2012年7月20日
2:48

This answers the question 14 pages ago:

"$$\begin{pmatrix} FB \\ \uparrow e \\ B \end{pmatrix} \xmapsto{\text{quotient}} \begin{matrix} F(B/\approx) \\ \uparrow e/\approx \\ B/\approx \end{matrix}$$

Is $e/\approx$ well-dfd.?"

(In fact we have shown that one can always quotient a coalg. modulo $\approx$. This is <u>minimization</u> of an automaton)

2012年7月20日
2:51

Def. A Coalg. $\begin{array}{c}FX\\ \uparrow c\\ X\end{array}$ is simple

if $x \approx x' \Longrightarrow x = x'$.

Lem.

1) $\begin{array}{c}F(X/\approx)\\ \uparrow\\ X/\approx\end{array}$ is simple.

2) Let $\left(\begin{array}{c}FX\\ \uparrow c\\ X\end{array}\right)$, $\left(\begin{array}{c}FY\\ \uparrow d\\ Y\end{array}\right)$: Coalg's,

$\left(\begin{array}{c}FY\\ \uparrow d\\ Y\end{array}\right)$: simple.

Then there's at most one mor.

$$\left(\begin{array}{c}FX\\ \uparrow c\\ X\end{array}\right) \longrightarrow \left(\begin{array}{c}FY\\ \uparrow d\\ Y\end{array}\right)$$

Proof 1) is easy.

2012年7月20日
2:54

For 2), assume there are

$$\begin{tikzcd} \left(\begin{matrix}FX\\ \uparrow c\\ X\end{matrix}\right) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \left(\begin{matrix}FY\\ \uparrow d\\ Y\end{matrix}\right) \end{tikzcd}$$

with $f \neq g$ (i.e. $f(x) \neq g(x)$ for some x)

What we do is to take a coequalizer (Recall: $\mathbb{C}\mathrm{oalg}_F$ is cocomplete with colimits computed in $\mathbb{S}\mathrm{ets}$)

$$\begin{tikzcd} \left(\begin{matrix}FX\\ \uparrow c\\ X\end{matrix}\right) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \left(\begin{matrix}FY\\ \uparrow d\\ Y\end{matrix}\right) \arrow[r, "e"] & \left(\begin{matrix}FU\\ \uparrow k\\ U\end{matrix}\right) \end{tikzcd}$$

Then $ef = eg$.

Thus $(e \circ f)(x) = (e \circ g)(x)$

Therefore $f(x) \approx g(x)$

But $f(x) \neq g(x)$ by asmp. This contradicts d being simple. □

2012年7月20日
2:59

Finally

Thm. $F : \mathbf{Sets} \longrightarrow \mathbf{Sets}$, finitary.

Then there is a final F-coalg.

Proof | (We summarize the constr. we've used)

$\boxed{1}$

$$\begin{tikzcd}
 & & & B \text{ (limit)} \arrow[lll, bend right, "\pi_0"'] \arrow[ll, bend right, "\pi_1"'] \\
1 & F1 \arrow[l] & \cdots \arrow[l] & \\
 & & & FB \arrow[uu, dashed, "b"'] \arrow[ll, bend left, "F\pi_0"] \arrow[lll, bend left, "!"]
\end{tikzcd}$$

b is mono (using F: finitary)

$$\Longrightarrow \begin{tikzcd}
B \arrow[d, bend right, "e"'] \\
FB \arrow[u, bend right, "b"'] \arrow[loop right, "id"]
\end{tikzcd}$$

∃ existence
× uniqueness

This $\begin{matrix} FB \\ \uparrow \\ B \end{matrix}$ is weakly final

$\boxed{2}$

$$\begin{tikzcd} FB \arrow[r, two heads] & F(B/\approx) \\ B \arrow[u] \arrow[r, two heads] & B/\approx \arrow[u, dashed] \end{tikzcd}$$

Quotient modulo beh. eq. $\approx$

$\boxed{3}$ By Lem. 3 pages ago,

$$\left(\begin{array}{c} FX \\ \uparrow \\ X \end{array}\right) \xrightarrow[\text{one}]{\text{at most}} \left(\begin{array}{c} F(B/\approx) \\ \uparrow \\ B/\approx \end{array}\right)$$

Combined w/ weak finality

of $\begin{array}{c} FB \\ \uparrow \\ B \end{array}$, we have finality

of $\begin{array}{c} F(B/\approx) \\ \uparrow \\ B/\approx \end{array}$.

▨

Exercise

- Show that $\tilde{\sim}$ is indeed an equiv. rel.

[Hint Transit. is nontrivial
You use a pushout, a special type of a colimit.]

2012年7月20日
12:16

By the way: if F preserves a limit of the final sequence, then that gives us a final F-coalg. much more easily.

$$\begin{tikzcd}
 & & & B \text{ (limit)} \arrow[dll, bend right, "\pi_i"'] \\
\cdots & F^i 1 \arrow[l] & F^{i+1} 1 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

$\Longrightarrow$ F pres. limit

$$\begin{tikzcd}
 & & & & FB \text{ (limit)} \arrow[dll, bend right, "F\pi_i"'] \\
F1 & \cdots \arrow[l] & F^{i+1} 1 \arrow[l] & F^{i+2} 1 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Putting these together,

$$\begin{tikzcd}
 & & B \text{ (limit)} \arrow[dl, bend right, "\pi_i"'] \arrow[dd, dashed, bend left] \\
\cdots & F^i 1 \arrow[l] & \cdots \arrow[l] \\
 & & FB \text{ (limit)} \arrow[ul, bend left, "F\pi_{i-1}"] \arrow[uu, dashed, bend left, "\cong"]
\end{tikzcd}$$

Prop.

$$\begin{tikzcd}
 & & & B\ (\text{limit}) \arrow[dlll, bend right, "\pi_0"'] \arrow[dll, bend right, "\pi_1"] \quad \cdots \\
1 & F1 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Asm. F preserves the limit B.
Then B canonically carries a final F-coalg.

Proof.) It is easy to see;

$$\begin{array}{c}
\text{a cone over } 1 \xleftarrow{!} F1 \xleftarrow{F!} F^2 1 \leftarrow \cdots \\
\hline\hline
\text{a cone over } F1 \xleftarrow{F!} F^2 1 \leftarrow \cdots
\end{array}$$

(Since 1 is final)

Therefore the limiting cone

$$\begin{tikzcd}
 & & & FB \arrow[dlll, bend right, "F\pi_0"'] \arrow[dll, bend right, "F\pi_1"] \quad \cdots \\
F1 & F^2 1 \arrow[l, "F!"'] & \cdots \arrow[l]
\end{tikzcd}$$

(Since F pres. the limit B)

is indeed a limit of the final seq.

The following sublemma easily yields the claim.

Sublem. $\begin{pmatrix} FX \\ \uparrow c \\ X \end{pmatrix}$: F-coalg.

$f : X \to B$ **Then**

$$\begin{tikzcd}
FX \arrow[r, "Ff"] & FB \\
X \arrow[u, "c"] \arrow[r, "f"'] \arrow[ur, phantom, "="] & B \arrow[u, "\beta", "\cong"']
\end{tikzcd}
\iff
\begin{tikzcd}
 & & & B \arrow[dll, bend right, "\pi_i"'] \\
\cdots & F^i 1 \arrow[l] & \cdots \arrow[l] & {} \\
 & & & X \arrow[ull, bend left, "\gamma_i"] \arrow[uu, bend right, "f"'] \arrow[uu, phantom, "\|"]
\end{tikzcd}$$

Here $(\gamma_i : X \to F^i 1)_{i \in \mathbb{N}}$ is the cone induced by c as before.

$$\begin{tikzcd}
1 & F1 \arrow[l] & F^2 1 \arrow[l] & \cdots \arrow[l] \\
 & FX \arrow[u, "F!"] & FX \arrow[u, "F\gamma_1"'] & \\
 & & X \arrow[uull, bend left, "!"] \arrow[ul, "c"] \arrow[u, "c"'] &
\end{tikzcd}$$

$\gamma_0 := !$

$\gamma_{i+1} := (F\gamma_i) \circ c$

2012年7月24日
20:28

Proof.

[⇒]

By ind. on $i \in \mathbb{N}$.

$$\begin{tikzcd}
 & & B \arrow[dll, "\pi_i"'] \arrow[ddl, "\pi_{i+1}"] \arrow[d, "\beta"] \\
 & & FB \arrow[dl, "F\pi_i"] \\
F^i 1 & F^{i+1} 1 \arrow[l, "F^i !"'] & \\
 & & FX \arrow[ul, "F\gamma_i"] \arrow[uu, bend right, "Ff"'] \\
 & X \arrow[uul, bend left, "\gamma_i"] \arrow[uu, "\gamma_{i+1}"] \arrow[ur, "c"'] \arrow[uuuur, bend right=60, "f"'] &
\end{tikzcd}$$

$\underline{i=0}$ $F^i 1 = 1$ is final, thus obvious.

$\underline{i+1}$

$$\begin{aligned}
\gamma_{i+1} &= F\gamma_i \circ c && (\text{Def. of } \gamma_{i+1}) \\
&= F(\pi_i \circ f) \circ c && (\text{Ind. Hyp.}) \\
&= F\pi_i \circ \underline{Ff \circ c} \\
&= \underline{F\pi_i \circ \beta \circ f} && (\text{By asmp.}) \\
&= \pi_{i+1} \circ f && \left(\begin{array}{l}\text{By def. of} \\ \beta \text{ as a mediat.} \\ \text{map}\end{array}\right)
\end{aligned}$$

OK.

[$\Leftarrow$] We use the universality of FB (limit). It suffices to show that

$$\begin{tikzcd}
 & & & FB\ (\mathrm{limit}) \arrow[dll, bend right, "F\pi_i"'] \\
\cdots & F^{i+1}1 \arrow[l] & \cdots \arrow[l] & \\
 & & & X \arrow[ull, bend left, "\gamma_{i+1}"] \arrow[uu, bend left=10, "\beta\circ f"] \arrow[uu, bend right=10, "Ff\circ c"']
\end{tikzcd}$$

are both mediating maps. This is easy. ▨

Exercise In the setting F: finitary, point out why the above proof does not work (and hence we need an additional step of quotienting a weakly final $\left(\begin{array}{c}FB\\ \uparrow\\ B\end{array}\right)$).

Then why all this complicated business of

$$\begin{tikzcd} FB \\ B \arrow[u, "b"] \end{tikzcd}$$

be a mono?

$\Longrightarrow$ $\boxed{\text{Ans.}}$ There are many finitary F; not so many of limit-preserving F

called "continuous functor"

2012年7月20日
12:26

For example,

Prop. Let $\mathcal{F}$ be a family of Sets-endofunctors inductively defined by

the constant functor $\begin{pmatrix} X \\ \downarrow f \\ Y \end{pmatrix} \longmapsto \begin{pmatrix} C \\ \downarrow \mathrm{id} \\ C \end{pmatrix}$

D : a finite set

$$\underset{\substack{\uparrow \\ \mathcal{F}}}{F} ::= \mathrm{Id} \mid \underline{C} \mid F \times F \mid F + F \mid F^{D} \mid \underline{\mathcal{P}_{\mathrm{fin}}}(F\_)$$

the finite powerset functor

Then every $F \in \mathcal{F}$ is finitary.

Proof By induction on $F \in \mathcal{F}$. $\square$

Some limit-preserving functors:

- "limit-like", such as $\_ \times \_$ (limits 'commute' — see Mac Lane CWM)
- Right adjoint, like $A \times \_ \dashv (\_)^{A}$ (A standard result)

$F(\mathrm{Lim}\,D) \cong \mathrm{Lim}(FD)$

2012年7月20日
12:42

Proof sketch) Let $L \dashv F$.

We show, for $\mathcal{J} \xrightarrow{D} \mathbb{C} \xrightarrow{F} \mathcal{D}$

$F(\mathrm{Lim}\,D)$ is a limit of FD.

univ. of $\mathrm{Lim}\,FD$!

$$\begin{array}{c l} X \longrightarrow FDI, \ \text{cone} & \\ \hline\hline & L \dashv F \\ LX \longrightarrow DI, \ \text{cone} & \\ \hline\hline & \text{Univ. of } \mathrm{Lim}\,D \\ LX \longrightarrow \lim D & \\ \hline\hline & L \dashv F \\ X \longrightarrow F(\mathrm{Lim}\,D) & \end{array}$$

$$\begin{tikzcd} FB \arrow[r, dashed] & F(B/\sim) \\ B \arrow[u, "e"] \arrow[r, dashed] & B/\sim \arrow[u] \end{tikzcd}$$

By the way The "quotienting" part can be described in more abstract terms (Thanks to S. Katsumata)

Thm (Freyd) (See Mac Lane, CWM)

Asm. $\mathbb{C}$ has colimits.

$B \in \mathbb{C}$, weakly final

Then the colimit

$$\begin{tikzcd} B \arrow[r, shift left=2] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] \arrow[dr, bend right] & B \arrow[d, "e"] \\ & C \end{tikzcd}$$

All the arrows $B \to B$

This is a "big coequalizer"

(colimit)

gives a final object C.

Proof.

- For any $C \xrightarrow{f} B$,

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[dr, "id"'] \arrow[dr, phantom, shift left=2, "="] & B \arrow[d, "e"] \\ & C \end{tikzcd}$$

$\because$ We have $efe = e$, since

$$\begin{tikzcd} B \arrow[r, shift left, "id_B"] \arrow[r, shift right, "f \circ e"'] & B \arrow[r, "e"'] & C \end{tikzcd}$$

$\vdots$ (colim.)

Now

$$\begin{tikzcd} B \arrow[r, shift left] \arrow[r, shift right] & B \arrow[r, "e"] \arrow[dr, "efe"'] & C \arrow[d, shift right, "ef"'] \arrow[d, shift left, "id"] \\ & & C \end{tikzcd}$$

(colim)

By the universality, $ef = id$.

[This is like coequalizer is an epi, which is dual to an equalizer being a mono.]

Let $X \underset{h}{\overset{g}{\rightrightarrows}} C$. $\left(\begin{array}{c}\text{Aim} \\ g = h\end{array}\right)$

Take a coequalizer:

$$\begin{tikzcd}X \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \arrow[r, "q"] & D\end{tikzcd}$$

Then

$$\begin{tikzcd} & & & \text{weakly final} \arrow[d] & \\ X \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \arrow[r, "q"] \arrow[rrr, bend right, "\shortparallel\ (\text{prev. page})", "\text{id}"'] & D \arrow[r, "\exists i"] & B \arrow[r, "e"] & C\end{tikzcd}$$

Thus

$$g = e i \underline{q g}$$

$$= \underline{e i q} h \quad (\because q : \text{coequalizer})$$

$$= h.$$

It is obvious that

$$\forall X \xrightarrow{\exists} C \quad \left(\begin{array}{l}\text{Take} \\ X \xrightarrow{\exists} B \xrightarrow{e} C\end{array}\right)$$

□

2012年7月24日
22:40

Maybe Yoneda lemma is another example ...

This is among (not too many) nontrivial results in CT (itself), and is the essential part of Freyd's adjoint functor theorem.

The proof might not be too intuitive — it'd help if you imagine

- $\mathbb{C}$ = $\mathbf{Coalg}_F$
- An arrow = beh.-pres. map
- A final coalg. = a fully abstract domain wrt. $\simeq$

§2.5 Coalgebraic bisimulation

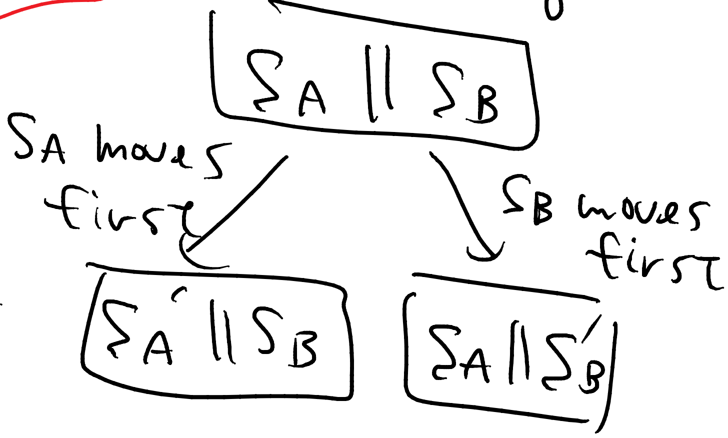
In this section:
We define the notion of F-bisimilarity and discuss its relationship with $\approx$ (beh. eq.) and a final coalgebra

BTW What is a bisimulation?

- Yields bisimilarity, a well-est. notion of equivalence for branching / concurrent systems.

In fact there're many notions of equiv. (the van Glabbeek spectrum) Bisimilarity is the finest of those

concurrent ⇒ branching:



- A basis for the "coinductive p'fs" (as they're usually called) for infinite/coinductive data types

(We've seen an example, for
$$\mathrm{alt}(\alpha, \alpha) = \alpha$$)

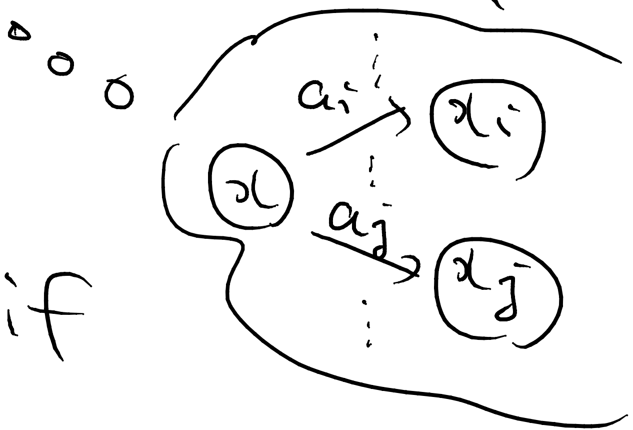
- The notion is due to Park; advocated by Milner.

2012年7月20日
12:57

Labeled trans. sys.

First, the conventional definition:

Def. Let $\begin{array}{c}\mathcal{P}(L\times X)\\ \uparrow \\ X\end{array}$ be an LTS



$R \subseteq X\times X$ is a **bisimulation** if

- $x_1 R x_2$, $x_1 \xrightarrow{a} x_1'$

  $\Longrightarrow \exists x_2'$ s.t. $\begin{cases} x_2 \xrightarrow{a} x_2' \\ x_1' R x_2' \end{cases}$

- $x_1 R x_2$, $x_2 \xrightarrow{a} x_2'$

  $\Longrightarrow \exists x_1'$ s.t. $\begin{cases} x_1 \xrightarrow{a} x_1' \\ x_1' R x_2 \end{cases}$

That is,

$$\begin{tikzcd} x_1 \arrow[r, "a"] \arrow[d, dash, dotted, "R"'] & x_1' \arrow[d, dash, dotted, "R"] \\ {} & {} \end{tikzcd}$$

(and vice versa)

$$\begin{tikzcd} {} \arrow[d, dash, dashed, "R"'] & {} \arrow[d, dash, dashed, "R"] \\ x_2 \arrow[r, "a"'] & \exists\, y_2 \end{tikzcd}$$

(vice versa)

Def.

$$\begin{tikzcd} \mathcal{P}(L\times X) \\ X \arrow[u] \end{tikzcd} \quad : \text{ an LTS}$$

$x_1, x_2 \in X$

x_1 and x_2 are bisimilar

$\underset{\text{def.}}{\iff}$ $\exists R$. bisimulation s.t.

$x_1 \, R \, x_2$

“witness”

Straightfwd results:

Prop.
- Bisimilarity is an equivalence rel.
- Bisimilarity itself is a bisimulation

It is also easy to consider bisim between different LTSs. $\left(\begin{tikzcd} FX \\ X \arrow[u] \end{tikzcd} \text{ and } \begin{tikzcd} FY \\ Y \arrow[u] \end{tikzcd} \right)$

:

Exercise

For stream automata, formulate the notion of bisimulation.

There are notions of bisim. for many different types of systems — notably probabilistic systems.

(But the def's themselves are often puzzling and not enlightening)

⟹ Coalgebraic def. offers a unified view!

("For many different types of sys"

⟹ "For many $F : \mathbf{Sets} \to \mathbf{Sets}$")

Def. $F : \mathbf{Sets} \to \mathbf{Sets}$,

$\begin{pmatrix} FX \\ \uparrow c \\ X \end{pmatrix}, \begin{pmatrix} FY \\ \uparrow d \\ Y \end{pmatrix}$: F-coalgebras

$R \subseteq X \times Y$ is an F-bisimulation

$\overset{\text{def}}{\Longleftrightarrow}$ R has an F-coalg. str.

$$\begin{tikzcd} FX & FR \arrow[l, "F\pi_1"'] \arrow[r, "F\pi_2"] & FY \\ X \arrow[u, "c"] \arrow[ru, phantom, "//"] & R \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[u, "r"] \arrow[ru, phantom, "="] & Y \arrow[u, "d"'] \end{tikzcd}$$

s.t.

Indeed:

2012年7月20日
13:09

Prop. For $F = \mathcal{P}(L \times \_)$,

R is a coalgebraic bisimulation

$\Longleftrightarrow$ R is a conventional bisimulation.

Proof. Easy. Very roughly:

$$\begin{tikzcd}
(a,x') & (a,(x',y')) \arrow[l, mapsto] \arrow[r, mapsto] & (a,y') \\
x \arrow[u, mapsto] \arrow[ur, phantom, "="] & (x,y) \arrow[l, mapsto] \arrow[r, mapsto] \arrow[u, mapsto] \arrow[ur, phantom, "="] & y \arrow[u, mapsto] \\
& R \arrow[u, phantom, sloped, "\in"] &
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[r, "a"] \arrow[d, dotted, no head, "R"'] & x' \arrow[d, dotted, no head, "R"] \\
y \arrow[r, "a"'] & y'
\end{tikzcd}$$

④

2012年7月20日
13:12

Now observe:

- A coalg. bisimulation is

$$\begin{tikzcd}
 & \begin{pmatrix} FR \\ \uparrow r \\ R \end{pmatrix} \arrow[dl] \arrow[dr] & \\
\begin{pmatrix} FX \\ \uparrow c \\ X \end{pmatrix} & & \begin{pmatrix} FY \\ \uparrow d \\ Y \end{pmatrix}
\end{tikzcd}$$

<u>Span</u>

- observational equivalence $\approx$ is defined by

$$\begin{tikzcd}
\begin{pmatrix} FX \\ c\uparrow \\ X \end{pmatrix} \arrow[dr, "f"'] & & \begin{pmatrix} FY \\ d\uparrow \\ Y \end{pmatrix} \arrow[dl, "g"] \\
 & \begin{pmatrix} FU \\ \uparrow k \\ U \end{pmatrix} &
\end{tikzcd}$$

<u>Cospan</u>

$$\left(\text{i.e. } \left(x, \begin{matrix} FX \\ \uparrow c \\ X \end{matrix}\right) \approx \left(y, \begin{matrix} FY \\ \uparrow d \\ Y \end{matrix}\right) \underset{\text{def.}}{\iff} \exists \begin{pmatrix} FU \\ \uparrow k \\ U \end{pmatrix}, f, g. \; f(x) = g(y) \right)$$

The former (bisim.) is well-est.; but the latter behaves more nicely in a categorical setting:

Prop. If a final coalg. $\left(\begin{array}{c} FZ \\ \cong\uparrow\zeta \\ Z \end{array}\right)$ exists, it is a *fully abstract domain* wrt. $\approx$, that is,

$$\left(x, \begin{array}{c} FX \\ \uparrow c \\ X \end{array}\right) \approx \left(y, \begin{array}{c} FY \\ \uparrow d \\ Y \end{array}\right)$$

$$\iff \begin{tikzcd} \left(\begin{array}{c} FX \\ \uparrow c \\ X \end{array}\right) \arrow[r, dashed, "\bar{c}"'] & \left(\begin{array}{c} FZ \\ \cong\uparrow\zeta \\ Z \end{array}\right) & \left(\begin{array}{c} FY \\ \uparrow d \\ Y \end{array}\right) \arrow[l, dashed, "\bar{d}"] \end{tikzcd}$$

$$\bar{c}(x) = \bar{d}(y)$$

Proof. $[\Leftarrow]$ Obvious.

2012年7月20日
13:18

[⇒]

$$\begin{tikzcd}
\left(\begin{smallmatrix} FX \\ \uparrow \\ X \end{smallmatrix}\right) \arrow[dr, "\exists\ \exists"] & & \left(\begin{smallmatrix} FY \\ \uparrow \\ Y \end{smallmatrix}\right) \arrow[dl, "\exists"'] \\
& \left(\begin{smallmatrix} FU \\ \uparrow \\ U \end{smallmatrix}\right) \arrow[d, dashed] & \\
& \left(\begin{smallmatrix} FZ \\ \cong\uparrow\zeta \\ Z \end{smallmatrix}\right) &
\end{tikzcd}$$

▨

So the question: how can we reconcile

bisim. vs. $\simeq_\zeta$

In fact they do coincide for many F!

2012年7月20日
13:19

Thm. Coalg. bisimilarity implies beh. eq.

Proof) Asm.

$$\begin{tikzcd} & \left(\begin{array}{c} FR \\ \uparrow \\ R \end{array}\right) \arrow[dl] \arrow[dr] & \\ \left(\begin{array}{c} FX \\ \uparrow \\ X \end{array}\right) & & \left(\begin{array}{c} FY \\ \uparrow \\ Y \end{array}\right) \end{tikzcd} \quad ☺ ,$$

$(x, y) \in R$. (i.e. x, y are bisimilar)

Recall that $\mathbf{Coalg}_F$ has colimits (computed in $\mathbf{Sets}$),

So we take a colim. of ☺ above.

$$\begin{tikzcd} & \left(\begin{array}{c} FR \\ \uparrow \\ R \end{array}\right) \arrow[dl] \arrow[dr] \arrow[dd, "\kappa_3"] & \\ \left(\begin{array}{c} FX \\ \uparrow \\ X \end{array}\right) \arrow[dr, "\kappa_1"'] & & \left(\begin{array}{c} FY \\ \uparrow \\ Y \end{array}\right) \arrow[dl, "\kappa_2"] \\ & \left(\begin{array}{c} FU \\ \uparrow \\ U \end{array}\right) & \end{tikzcd}$$

2012年7月20日
13:23

Now $\kappa_1(x) = \kappa_3(x,y)$

$= \kappa_2(y)$.

Thus $x \approx y$. $\square$

Thm If F weakly preserves pullbacks

then

$\approx$ implies bisimilarity.

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd} \Rightarrow \begin{tikzcd} F\cdot \arrow[r] \arrow[d] & FY \arrow[d] \\ FX \arrow[r] & FZ \end{tikzcd}$$

is a weak pullback

A cond. satisfied by many functors

Proof.

$$\begin{tikzcd} & F\bar{U} \arrow[dl] \arrow[rr] & & FY \arrow[dl] \\ FX \arrow[rr] & & FU & \\ & \bar{U} \arrow[dl] \arrow[rr] \arrow[uu, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dl] \arrow[uu] \\ X \arrow[rr] \arrow[uu] & & U \arrow[uu] & \end{tikzcd}$$

① Take a pullback

② Apply F to the pullback

③ $F\bar{U}$ is a weak pullback, so $\exists$ a mediating map $\square$

Exercise

We used a relaxed notion of coalg. bisimulation:

A bisim. is a span

$$\begin{pmatrix} FX \\ \uparrow \\ X \end{pmatrix} \xleftarrow{f} \begin{pmatrix} FU \\ \uparrow \\ U \end{pmatrix} \xrightarrow{g} \begin{pmatrix} FY \\ \uparrow \\ Y \end{pmatrix}$$

(Notice that U need not be a subset of $X \times Y$)

Prove that the bisimilarity according to this definition is the same as the original one.

Hint
- $$\begin{tikzcd} U \arrow[rr, "{(f,g)}"] \arrow[dr, two heads] & & X \times Y \\ & \bullet \arrow[ur, tail] & \end{tikzcd}$$

  Image factorization
- Equip the image with a coalg. str.

Hint

- $$\begin{tikzcd} U \arrow[rr, "{(f,g)}"] \arrow[dr, two heads] & & X \times Y \\ & \bullet \arrow[ur, tail] & \end{tikzcd}$$

Image factorization

- Equip the image with a coalg. str.

§ 2.6 Algebra & Initial Algebra

2012年7月18日
9:38

<u>Algebra</u> is the categorical dual of coalgebra:

<u>Def.</u> $F : \mathbb{C} \to \mathbb{C}$

- An <u>F-algebra</u> is $\begin{pmatrix} FX \\ \downarrow a \\ X \end{pmatrix}$
- The category $\mathbf{Alg}_F$ of F-alg.:

<u>obj.</u> $\begin{pmatrix} FX \\ \downarrow a \\ X \end{pmatrix}$, F-alg.

<u>arr.</u>

$$\begin{array}{c} \begin{pmatrix} FX \\ \downarrow a \\ X \end{pmatrix} \xrightarrow{f} \begin{pmatrix} FY \\ \downarrow b \\ Y \end{pmatrix} \text{ in } \mathbf{Alg}_F \\ \hline\hline f : X \to Y \text{ in } \mathbb{C} \text{ s.t.} \end{array}$$

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "a"'] \arrow[dr, phantom, "="] & FY \arrow[d, "b"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

2012年7月25日
8:38

But what are their examples/use in CS?

$\Longrightarrow$ $\boxed{\text{Ans.}}$ [The ADJ group, J. Goguen]

- A syntactic specification (e.g. by a BNF notat.)

$\Longrightarrow$ $F : \mathbf{Sets} \to \mathbf{Sets}$

- $$\left(\begin{tikzcd} FA \arrow[d, "\cong"', "\text{initial}"] \\ A \end{tikzcd}\right)$$: the set of **well-formed expressions**

Def. An algebraic signature is an $\mathbb{N}$-indexed family of sets:

$$\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$$

$\sigma \in \Sigma_n$ is called an n-ary operation.

Examples

- $\Sigma_0 = \{0\}$ $\Sigma_1 = \{s\}$

  $\Sigma_2 = \{+, \times\}$ $\Sigma_3 = \Sigma_4 = \cdots = \emptyset$

  (For PA)

- $\Sigma_0 = \{e\}$ $\Sigma_1 = \{(\_)^{-1}\}$

  $\Sigma_2 = \{\cdot\}$

2012年7月25日
8:56

- In general, an alg. signat. Σ is written as a (one-sorted) BNF notation;

$$t ::= \sigma(\underbrace{t, t, \ldots, t}_{n})$$

Or more concretely:

$$t ::= \sigma^0 \mid \sigma^1(t) \mid \sigma^2(t, t) \mid \sigma^3(t, t, t) \mid \ldots$$

① “Syntactic spec. as an alg. sign.”

Rem. Usual notions of algebra (groups, rings, monoids, ...) are determined by

- an alg. signature Σ
- a set of equational axioms E

together: 'algebraic specification'

Here we're only speaking of the former.

The categorical machinary for dealing with both is:

- monads & their Eilenberg-Moore alg.
- Lawvere theories

Def. A Σ-alg. is a set X together with

$$\underbrace{[\![\sigma]\!]}_{\text{`interpretation' of } \sigma} : X^n \longrightarrow X \quad \text{for each } \sigma \in \Sigma_n$$

Lem. An alg. sign. Σ induces a functor $F_\Sigma : \mathbf{Sets} \to \mathbf{Sets}$,

$$F_\Sigma X := \coprod_{n \in \mathbb{N}} \coprod_{\sigma \in \Sigma_n} X^n$$

$$= \coprod_{\sigma \in \Sigma} X^{\underbrace{|\sigma|}_{\text{the arity of } \sigma}}$$

w/ its obvious action on arrows.

2012年7月25日
8:52

Prop. Σ-algebras are in a bijective correspondence with F_Σ-algebras.

(Moreover: an isomorphism of categories:

$$\Sigma\text{-}\mathbf{Alg} \cong \mathbf{Alg}_{F_\Sigma}$$
)

Proof)

Aim →

$$\begin{array}{c} \begin{array}{c} X^{|\sigma|} \\ {\scriptstyle [\![\sigma]\!]}\downarrow \\ X \end{array} \quad \text{for each } \sigma\in\Sigma \\ \hline\hline \begin{array}{c} \coprod_{\sigma\in\Sigma} X^{|\sigma|} \\ \downarrow{\scriptstyle a} \\ X \end{array} \end{array}$$

$[\downarrow]$ is obtained by cotupling

$$\left[\,[\![\sigma]\!]\,\right]_{\sigma\in\Sigma}$$

$[\uparrow]$ is by

$$[\![\sigma]\!] := a \circ \kappa_\sigma$$ ∎

Recall:

$$\begin{tikzcd} X \arrow[r, "\kappa_1"] \arrow[dr, "f"'] & X+Y \arrow[d, dashed, "{[f,g]}"] & Y \arrow[l, "\kappa_2"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

In what follows, an F_Σ-alg. is often simply called a Σ-alg.

An initial algebra (as the dual of a final coalg.) plays an important role. It is an init. obj. in Alg_F ; thus

Def. $\begin{array}{c} FA \\ \alpha \big\downarrow \\ A \end{array}$ is initial

$\underset{\text{def.}}{\iff}$ for any F-alg. $\begin{array}{c} FX \\ \big\downarrow a \\ X \end{array}$,

$$\left(\begin{array}{c} FA \\ \big\downarrow \alpha \\ A \end{array}\right) \overset{\exists !}{\dashrightarrow} \left(\begin{array}{c} FX \\ \big\downarrow a \\ X \end{array}\right)$$

2012年7月25日
9:02

Examples

- $F = 1 + (\_)$

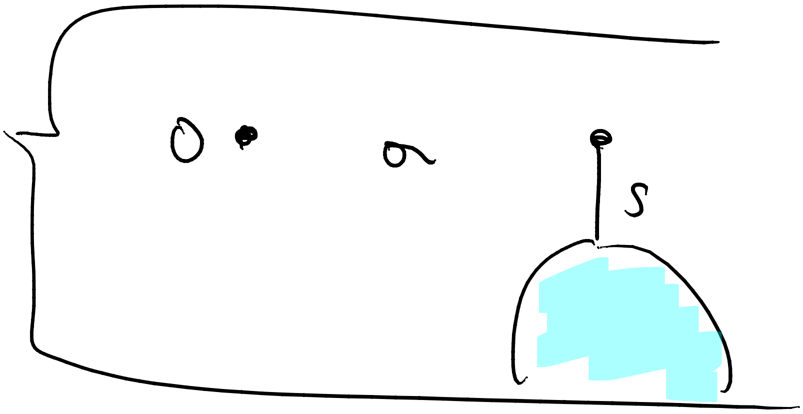
[i.e. $F = F_\Sigma$ with $\Sigma_0 = \{0\}$, $\Sigma_1 = \{s\}$, $\Sigma_2 = \cdots = \emptyset$

Then

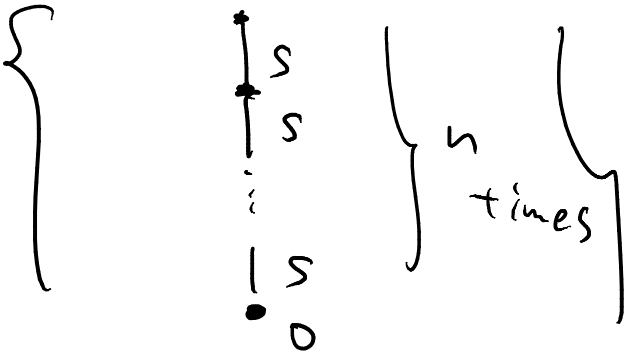
$$\begin{tikzcd}
1+\mathbb{N} \arrow[d, "{[0,s]}"'] & \bot \in 1 \arrow[d, mapsto] & n \in \mathbb{N} \arrow[d, mapsto] \\
\mathbb{N} & 0 & n+1
\end{tikzcd}$$

is initial. Pictorially:

$F = 1 + (\_)$



$\mathbb{N} =$



2012年7月25日
9:06

- In general, an initial F_Σ-alg. is given by

$$\begin{tikzcd} \coprod_{\sigma\in\Sigma} (T_\Sigma 0)^{|\sigma|} \arrow[d] & (t_1,\dots,t_{|\sigma|}) \arrow[d, mapsto] \\ T_\Sigma 0 & \sigma(t_1,\dots,t_{|\sigma|}) \end{tikzcd}$$

{The set of well-formed Σ-terms with var's from 0 (i.e. no variables)}

- If $F0 \cong 0$, then

$$\left(\begin{tikzcd} F0 \arrow[d, "\cong"] \\ 0 \end{tikzcd}\right) \text{ is an init. alg.}$$

2012年7月25日
9:08

initial alg.		final coalg.
datatype constructor	F functor	datatype destructor
fin.-depth trees	element	fin. & infinite-depth trees
inductive datatype, well-founded		coinductive datatype, non-well-founded

Lem. (Lambek)

$$\begin{tikzcd} FA \arrow[d, "\alpha"] \\ A \end{tikzcd}$$ is initial $\Rightarrow$ α is an iso.

(Easy)

2012年7月25日
19:20

$\boxed{Q}$ [- Does an initial F-alg. exist?

- How does it look like?]

⇒ The answer is by the dual of the final coalg. case, i.e.

<u>colimit of an initial sequence</u>!

We work again with finitary F.

<u>Def.</u> An <u>initial F-sequence</u> is the diagram

<u>$\mathbb{C}$</u>|

$$\begin{tikzcd} 0 \arrow[r, "!"] & F0 \arrow[r, "F!"] & F^2 0 \arrow[r] & \cdots \end{tikzcd}$$

(initial)

Prop. $F : \mathbf{Sets} \to \mathbf{Sets}$, finitary. Then

Sets

$$\begin{tikzcd}
 & & & & A\ (\text{colim.}) \\
0 \arrow[r, "!"] \arrow[urrrr, bend left, "\kappa_0"] & F0 \arrow[r, "F!"] \arrow[urrr, bend left, "\kappa_1"] & F^2 0 \arrow[r] \arrow[urr, bend left, "\kappa_2"] & \cdots & \cdots
\end{tikzcd}$$

canonically induces an initial F-algebra.

Proof.) A categorical characterization of F: finitary is: F preserves filtered colimits. The diagram

$$\begin{tikzcd}
0 \arrow[r, "!"] & F0 \arrow[r, "F!"] & F^2 0 \arrow[r] & \cdots
\end{tikzcd}$$

is filtered; thus we have

$$\begin{tikzcd}
 & & & & A\ (\text{colim.}) \arrow[d, shift left, "\cong"] \\
0 \arrow[r, "!"] \arrow[urrrr, bend left, "\kappa_0"] \arrow[drrrr, bend right, "!"'] & F0 \arrow[r, "F!"] \arrow[urrr, bend left, "\kappa_1"] \arrow[drrr, bend right, "F\kappa_0"'] & F^2 0 \arrow[r] \arrow[urr, bend left, "\kappa_2"] \arrow[drr, bend right, "F\kappa_2"'] & \cdots & \cdots \\
 & & & & FA\ (\text{colim.}) \arrow[u, shift left]
\end{tikzcd}$$

2012年7月26日
0:17

The fact that

$$\begin{tikzcd} FA \arrow[d, "\cong"] \\ A \end{tikzcd}$$

is an initial algebra is proved much like the prev. case that, when F preserves a suitable limit,

$$\begin{tikzcd}
 & & & B \arrow[dlll, bend right] \arrow[dll, bend right] \arrow[dd, bend right, red] \arrow[dd, phantom, "\textcolor{red}{\cong}"] \\
1 & F1 \arrow[l] & \cdots \arrow[l] & \\
 & & & FB \arrow[ulll, bend left] \arrow[ull, bend left] \arrow[uu, bend right, red]
\end{tikzcd}$$

yields a final coalg.

$\square$

2012年7月26日
0:19

Let us exhibit the situation concretely.

$F = L\times \_$ is not interesting since

$L\times 0 \cong 0$ thus

$\left(\begin{array}{c} L\times 0 \\ \downarrow\cong \\ 0 \end{array}\right)$ is an initial algebra.

Let $F = 1 + L\times \_$.

$F^i 0$: $\{$terms of depth $\leq i\}$

$$\begin{tikzcd}
0 \arrow[r] & F0 \arrow[r, "F!"] \arrow[d, equal] & F^2 0 \arrow[r] \arrow[d, equal] & F^3 0 \arrow[dd, "\cong"] & \\
& \underline{1}+\underline{L}\times\underline{0} \arrow[d, "\cong"] & \underline{1}+L\times\underline{(1+L\times 0)} \arrow[d, "\cong"] & & \\
& 1 & 1+L & 1+L+L^2 & \\
& \perp \arrow[r, mapsto] & \perp \arrow[r, mapsto] & \perp \arrow[r, mapsto] & \cdots \\
& & a_0\perp \arrow[r, mapsto] & a_0\perp \arrow[r, mapsto] & \cdots \\
& & & a_0 a_1\perp \arrow[r, mapsto] & \cdots
\end{tikzcd}$$

2012年7月26日
0:23

Therefore:

$$F^i 0 = \left\{ \begin{array}{l} \text{terms of “depth”} \\ \text{up-to } i \end{array} \right\}$$

Recall that a $\underbrace{\text{colimit}}_{\text{coproduct + coequalizer}}$ in Sets is concretely given by

$$\begin{tikzcd}
 & & C \;(\text{colim}) \\
DI \arrow[r, "Di"] \arrow[urr, bend left, "\kappa_I"] & DJ \arrow[ur, bend left, "\kappa_J"'] & \\
x \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & (Di)(x) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] &
\end{tikzcd}$$

$$C = \coprod_{I \in \mathbb{J}} DI \Big/ \sim$$

with $\sim$ gener. by

$$(x, DI) \sim \big((Di)(x), DJ\big)$$

2012年7月26日
0:26

Therefore $(F = 1 + L \times \_)$

$$\begin{tikzcd}
 & & & & & A\ (\text{colim}) \\
0 \arrow[r] \arrow[urrrrr, bend left] & F0 \arrow[r] \arrow[urrrr, bend left] \arrow[d, equal] & F^2 0 \arrow[r] \arrow[urrr, bend left] \arrow[d, equal] & F^3 0 \arrow[r] \arrow[urr, bend left] \arrow[d, equal] & \cdots \\
 & 1 & 1+L & 1+L+L^2 & \\
 & \perp \arrow[r, mapsto] & \perp \arrow[r, mapsto] & \perp \arrow[r, mapsto] & \cdots \\
 & & a_0\perp \arrow[r, mapsto] & a_0\perp \arrow[r, mapsto] & \cdots \\
 & & & a_0a_1\perp \arrow[r, mapsto] & \cdots
\end{tikzcd}$$

$$A \cong 1 + L + L^2 + L^3 + \cdots$$

$$= L^*.$$

$\therefore$

$$\begin{tikzcd}
1 + L \times L^* \arrow[d, "\cong"', "\text{init.}"] & \perp \arrow[d, mapsto] & (a, \alpha) \arrow[d, mapsto] \\
L^* & \varepsilon & a.\alpha
\end{tikzcd}$$

(the empty seq.)

Hopefully the next result now seems trivial to you:

Prop. Σ: an alg. signature.

$$\begin{tikzcd} F_\Sigma(T_\Sigma 0) \arrow[d] \\ T_\Sigma 0 \end{tikzcd}$$

$\left(\text{where } T_\Sigma 0 = \left\{\begin{array}{l}\Sigma\text{-terms} \\ \text{w/ no} \\ \text{var's}\end{array}\right\}\right)$

is an initial alg. $\boxed{1}$

This is what is meant by:

Syntactic specific. —	functor F
(Well-formed) expressions —	initial F-alg.